CHAPTER 0 Preliminaries

0.1 Concepts Review

1. rational numbers
2. dense
3. If not Q then not P.
4. theorems

Problem Set 0.1

1. $$4-2(8-11)+6=4-2(-3)+6 = 4+6+6=16$$

2. $$3\left[2-4\left(7-12\right)\right]=3\left[2-4(-5)\right] = 3\left[2+20\right]=3(22)=66$$

3. $$-4[5(-3+12-4)+2(13-7)] = -4[5(5)+2(6)]=-4[25+12] = -4(37)=-148$$

4. $$5\left[-1(7+12-16)+4\right]+2 = 5\left[-1(3)+4\right]+2=5\left(-3+4\right)+2 = 5\left(1\right)+2=5+2=7$$

5. $$\frac{5}{7}-\frac{1}{13}=\frac{65}{91}-\frac{7}{91}=\frac{58}{91}$$

6. $$\frac{3}{4-7}+\frac{3}{21}-\frac{1}{6}=\frac{3}{-3}+\frac{3}{21}-\frac{1}{6} = -\frac{42}{42}+\frac{6}{42}-\frac{7}{42}=-\frac{43}{42}$$

7. $$\frac{1}{3}\left[\frac{1}{2}\left(\frac{1}{4}-\frac{1}{3}\right)+\frac{1}{6}\right]=\frac{1}{3}\left[\frac{1}{2}\left(\frac{3-4}{12}\right)+\frac{1}{6}\right] = \frac{1}{3}\left[\frac{1}{2}\left(-\frac{1}{12}\right)+\frac{1}{6}\right] = \frac{1}{3}\left[-\frac{1}{24}+\frac{4}{24}\right] = \frac{1}{3}\left(\frac{3}{24}\right)=\frac{1}{24}$$

8. $$-\frac{1}{3}\left[\frac{2}{5}-\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)\right]=-\frac{1}{3\left[\frac{2}{5}-\frac{1}{2}\left(\frac{5}{15}-\frac{3}{15}\right)\right]}$$
$$=-\frac{1}{3}\left[\frac{2}{5}-\frac{1}{2}\left(\frac{2}{15}\right)\right]=-\frac{1}{3}\left[\frac{2}{5}-\frac{1}{15}\right] = -\frac{1}{3}\left(\frac{6}{15}-\frac{1}{15}\right)=-\frac{1}{3}\left(\frac{5}{15}\right)=-\frac{1}{9}$$

9. $$\frac{14}{21}\left(\frac{2}{5-\frac{1}{3}}\right)^2=\frac{14}{21}\left(\frac{2}{\frac{14}{3}}\right)^2=\frac{14}{21}\left(\frac{6}{14}\right)^2 = \frac{14}{21}\left(\frac{3}{7}\right)^2=\frac{2}{3}\left(\frac{9}{49}\right)=\frac{6}{49}$$

10. $$\frac{\left(\frac{2}{7}-5\right)}{\left(1-\frac{1}{7}\right)}=\frac{\left(\frac{2}{7}-\frac{35}{7}\right)}{\left(\frac{7}{7}-\frac{1}{7}\right)}=\frac{\left(-\frac{33}{7}\right)}{\left(\frac{6}{7}\right)}=-\frac{33}{6}=-\frac{11}{2}$$

11. $$\frac{\frac{11}{7}-\frac{12}{21}}{\frac{11}{7}+\frac{12}{21}}=\frac{\frac{11}{7}-\frac{4}{7}}{\frac{11}{7}+\frac{4}{7}}=\frac{\frac{7}{7}}{\frac{15}{7}}=\frac{7}{15}$$

12. $$\frac{\frac{1}{2}-\frac{3}{4}+\frac{7}{8}}{\frac{1}{2}+\frac{3}{4}-\frac{7}{8}}=\frac{\frac{4}{8}-\frac{6}{8}+\frac{7}{8}}{\frac{4}{8}+\frac{6}{8}-\frac{7}{8}}=\frac{\frac{5}{8}}{\frac{3}{8}}=\frac{5}{3}$$

13. $$1-\frac{1}{1+\frac{1}{2}}=1-\frac{1}{\frac{3}{2}}=1-\frac{2}{3}=\frac{3}{3}-\frac{2}{3}=\frac{1}{3}$$

14. $$2+\frac{3}{1+\frac{5}{2}}=2+\frac{3}{\frac{2}{2}-\frac{5}{2}}=2+\frac{3}{\frac{7}{2}} = 2+\frac{6}{7}=\frac{14}{7}+\frac{6}{7}=\frac{20}{7}$$

15. $$\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)=\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2 = 5-3=2$$

16. $\left(\sqrt{5}-\sqrt{3}\right)^2=\left(\sqrt{5}\right)^2-2\left(\sqrt{5}\right)\left(\sqrt{3}\right)+\left(\sqrt{3}\right)^2$
$=5-2\sqrt{15}+3=8-2\sqrt{15}$

17. $(3x-4)(x+1)=3x^2+3x-4x-4$
$=3x^2-x-4$

18. $(2x-3)^2=(2x-3)(2x-3)$
$=4x^2-6x-6x+9$
$=4x^2-12x+9$

19. $(3x-9)(2x+1)=6x^2+3x-18x-9$
$=6x^2-15x-9$

20. $(4x-11)(3x-7)=12x^2-28x-33x+77$
$=12x^2-61x+77$

21. $(3t^2-t+1)^2=(3t^2-t+1)(3t^2-t+1)$
$=9t^4-3t^3+3t^2-3t^3+t^2-t+3t^2-t+1$
$=9t^4-6t^3+7t^2-2t+1$

22. $(2t+3)^3=(2t+3)(2t+3)(2t+3)$
$=(4t^2+12t+9)(2t+3)$
$=8t^3+12t^2+24t^2+36t+18t+27$
$=8t^3+36t^2+54t+27$

23. $\dfrac{x^2-4}{x-2}=\dfrac{(x-2)(x+2)}{x-2}=x+2,\ x\neq 2$

24. $\dfrac{x^2-x-6}{x-3}=\dfrac{(x-3)(x+2)}{(x-3)}=x+2,\ x\neq 3$

25. $\dfrac{t^2-4t-21}{t+3}=\dfrac{(t+3)(t-7)}{t+3}=t-7,\ t\neq -3$

26. $\dfrac{2x-2x^2}{x^3-2x^2+x}=\dfrac{2x(1-x)}{x(x^2-2x+1)}$
$=\dfrac{-2x(x-1)}{x(x-1)(x-1)}$
$=-\dfrac{2}{x-1}$

27. $\dfrac{12}{x^2+2x}+\dfrac{4}{x}+\dfrac{2}{x+2}$
$=\dfrac{12}{x(x+2)}+\dfrac{4(x+2)}{x(x+2)}+\dfrac{2x}{x(x+2)}$
$=\dfrac{12+4x+8+2x}{x(x+2)}=\dfrac{6x+20}{x(x+2)}$
$=\dfrac{2(3x+10)}{x(x+2)}$

28. $\dfrac{2}{6y-2}+\dfrac{y}{9y^2-1}$
$=\dfrac{2}{2(3y-1)}+\dfrac{y}{(3y+1)(3y-1)}$
$=\dfrac{2(3y+1)}{2(3y+1)(3y-1)}+\dfrac{2y}{2(3y+1)(3y-1)}$
$=\dfrac{6y+2+2y}{2(3y+1)(3y-1)}=\dfrac{8y+2}{2(3y+1)(3y-1)}$
$=\dfrac{2(4y+1)}{2(3y+1)(3y-1)}=\dfrac{4y+1}{(3y+1)(3y-1)}$

29. **a.** $0\cdot 0=0$ **b.** $\dfrac{0}{0}$ is undefined.

c. $\dfrac{0}{17}=0$ **d.** $\dfrac{3}{0}$ is undefined.

e. $0^5=0$ **f.** $17^0=1$

30. If $\dfrac{0}{0}=a$, then $0=0\cdot a$, but this is meaningless because a could be any real number. No single value satisfies $\dfrac{0}{0}=a$.

31.
$$\begin{array}{r} .08\overline{3} \\ 12\overline{)1.000} \\ \underline{96} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

32.
$$7\overline{)2.000000} = .\overline{285714}$$
$$\begin{array}{l} \underline{1\,4} \\ \;\;60 \\ \;\;\underline{56} \\ \;\;\;\;40 \\ \;\;\;\;\underline{35} \\ \;\;\;\;\;\;50 \\ \;\;\;\;\;\;\underline{49} \\ \;\;\;\;\;\;\;\;10 \\ \;\;\;\;\;\;\;\;\;\underline{7} \\ \;\;\;\;\;\;\;\;\;30 \\ \;\;\;\;\;\;\;\;\;\underline{28} \\ \;\;\;\;\;\;\;\;\;\;\;2 \end{array}$$

33.
$$21\overline{)\;3.000000} = .\overline{142857}$$
$$\begin{array}{l} \underline{2\,1} \\ \;\;90 \\ \;\;\underline{84} \\ \;\;\;\;60 \\ \;\;\;\;\underline{42} \\ \;\;\;\;180 \\ \;\;\;\;\underline{168} \\ \;\;\;\;\;\;120 \\ \;\;\;\;\;\;\underline{105} \\ \;\;\;\;\;\;\;\;150 \\ \;\;\;\;\;\;\;\;\underline{147} \\ \;\;\;\;\;\;\;\;\;\;\;3 \end{array}$$

34.
$$17\overline{)5.000000...} = .294117... \to 0.\overline{2941176470588235}$$
$$\begin{array}{l} \underline{3\,4} \\ \;160 \\ \;\underline{153} \\ \;\;\;\;70 \\ \;\;\;\;\underline{68} \\ \;\;\;\;\;\;20 \\ \;\;\;\;\;\;\underline{17} \\ \;\;\;\;\;\;\;\;30 \\ \;\;\;\;\;\;\;\;\underline{17} \\ \;\;\;\;\;\;\;\;130 \\ \;\;\;\;\;\;\;\;\underline{119} \\ \;\;\;\;\;\;\;\;\;\;11 \\ \;\;\;\;\;\;\;\;\;\;\;\vdots \end{array}$$

35.
$$3\overline{)11.0} = 3.\overline{6}$$
$$\begin{array}{l} \underline{9} \\ \;20 \\ \;\underline{18} \\ \;\;\;2 \end{array}$$

36.
$$13\overline{)11.000000} = .\overline{846153}$$
$$\begin{array}{l} \underline{10\,4} \\ \;\;\;\;60 \\ \;\;\;\;\underline{52} \\ \;\;\;\;\;\;80 \\ \;\;\;\;\;\;\underline{78} \\ \;\;\;\;\;\;\;\;20 \\ \;\;\;\;\;\;\;\;\underline{13} \\ \;\;\;\;\;\;\;\;\;\;70 \\ \;\;\;\;\;\;\;\;\;\;\underline{65} \\ \;\;\;\;\;\;\;\;\;\;\;\;50 \\ \;\;\;\;\;\;\;\;\;\;\;\;\underline{39} \\ \;\;\;\;\;\;\;\;\;\;\;\;11 \end{array}$$

37. $x = 0.123123123...$
$$\begin{aligned} 1000x &= 123.123123... \\ \underline{\quad x} &\underline{= 0.123123...} \\ 999x &= 123 \\ x &= \frac{123}{999} = \frac{41}{333} \end{aligned}$$

38. $x = 0.217171717\ldots$
$$\begin{aligned} 1000x &= 217.171717... \\ \underline{\quad 10x} &\underline{= 2.171717...} \\ 990x &= 215 \\ x &= \frac{215}{990} = \frac{43}{198} \end{aligned}$$

39. $x = 2.56565656...$
$$\begin{aligned} 100x &= 256.565656... \\ \underline{\quad x} &\underline{= 2.565656...} \\ 99x &= 254 \\ x &= \frac{254}{99} \end{aligned}$$

40. $x = 3.929292\ldots$
$$\begin{aligned} 100x &= 392.929292... \\ \underline{\quad x} &\underline{= 3.929292...} \\ 99x &= 389 \\ x &= \frac{389}{99} \end{aligned}$$

41. $x = 0.199999...$
$$\begin{aligned} 100x &= 19.99999... \\ 10x &= 1.99999... \\ \hline 90x &= 18 \\ x &= \frac{18}{90} = \frac{1}{5} \end{aligned}$$

42. $x = 0.399999\ldots$
$$\begin{aligned} 100x &= 39.99999... \\ 10x &= 3.99999... \\ \hline 90x &= 36 \\ x &= \frac{36}{90} = \frac{2}{5} \end{aligned}$$

43. Those rational numbers that can be expressed by a terminating decimal followed by zeros.

44. $\frac{p}{q} = p\left(\frac{1}{q}\right)$, so we only need to look at $\frac{1}{q}$. If $q = 2^n \cdot 5^m$, then $\frac{1}{q} = \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{5}\right)^m = (0.5)^n (0.2)^m$. The product of any number of terminating decimals is also a terminating decimal, so $(0.5)^n$ and $(0.2)^m$, and hence their product, $\frac{1}{q}$, is a terminating decimal. Thus $\frac{p}{q}$ has a terminating decimal expansion.

45. Answers will vary. Possible answer: 0.000001, $\frac{1}{\pi^{12}} \approx 0.0000010819...$

46. Smallest positive integer: 1; There is no smallest positive rational or irrational number.

47. Answers will vary. Possible answer: 3.14159101001...

48. There is no real number between 0.9999… (repeating 9's) and 1. 0.9999… and 1 represent the *same* real number.

49. Irrational

50. Answers will vary. Possible answers: $-\pi$ and π, $-\sqrt{2}$ and $\sqrt{2}$

51. $(\sqrt{3}+1)^3 \approx 20.39230485$

52. $\left(\sqrt{2}-\sqrt{3}\right)^4 \approx 0.0102051443$

53. $\sqrt[4]{1.123} - \sqrt[3]{1.09} \approx 0.00028307388$

54. $(3.1415)^{-1/2} \approx 0.5641979034$

55. $\sqrt{8.9\pi^2+1} - 3\pi \approx 0.000691744752$

56. $\sqrt[4]{(6\pi^2-2)\pi} \approx 3.661591807$

57. Let a and b be real numbers with $a < b$. Let n be a natural number that satisfies $1/n < b-a$. Let $S = \{k : k/n > b\}$. Since a nonempty set of integers that is bounded below contains a least element, there is a $k_0 \in S$ such that $k_0/n > b$ but $(k_0 - 1)/n \le b$. Then
$$\frac{k_0-1}{n} = \frac{k_0}{n} - \frac{1}{n} > b - \frac{1}{n} > a$$
Thus, $a < \frac{k_0-1}{n} \le b$. If $\frac{k_0-1}{n} < b$, then choose $r = \frac{k_0-1}{n}$. Otherwise, choose $r = \frac{k_0-2}{n}$.

Note that $a < b - \frac{1}{n} < r$.

Given $a < b$, choose r so that $a < r_1 < b$. Then choose r_2, r_3 so that $a < r_2 < r_1 < r_3 < b$, and so on.

58. Answers will vary. Possible answer: $\approx 120 \text{ in}^3$

59. $r = 4000 \text{ mi} \times 5280 \frac{\text{ft}}{\text{mi}} = 21{,}120{,}000 \text{ ft}$
$$\begin{aligned} \text{equator} &= 2\pi r = 2\pi(21{,}120{,}000) \\ &\approx 132{,}700{,}874 \text{ ft} \end{aligned}$$

60. Answers will vary. Possible answer:
$$70\frac{\text{beats}}{\text{min}} \times 60\frac{\text{min}}{\text{hr}} \times 24\frac{\text{hr}}{\text{day}} \times 365\frac{\text{day}}{\text{year}} \times 20 \text{ yr}$$
$= 735{,}840{,}000$ beats

61. $V = \pi r^2 h = \pi\left(\frac{16}{2}\cdot 12\right)^2 (270 \cdot 12)$
$\approx 93{,}807{,}453.98 \text{ in.}^3$

volume of one board foot (in inches):
$1 \times 12 \times 12 = 144 \text{ in.}^3$

number of board feet:
$$\frac{93{,}807{,}453.98}{144} \approx 651{,}441 \text{ board ft}$$

62. $V = \pi(8.004)^2(270) - \pi(8)^2(270) \approx 54.3 \text{ ft.}^3$

63. a. If I stay home from work today then it rains. If I do not stay home from work, then it does not rain.

b. If the candidate will be hired then she meets all the qualifications. If the candidate will not be hired then she does not meet all the qualifications.

64. a. If I pass the course, then I got an A on the final exam. If I did not pass the course, thn I did not get an A on the final exam.

b. If I take off next week, then I finished my research paper. If I do not take off next week, then I did not finish my research paper.

65. a. If a triangle is a right triangle, then $a^2 + b^2 = c^2$. If a triangle is not a right triangle, then $a^2 + b^2 \neq c^2$.

b. If the measure of angle *ABC* is greater than 0° and less than 90°, it is acute. If the measure of angle *ABC* is less than 0° or greater than 90°, then it is not acute.

66. a. If angle *ABC* is an acute angle, then its measure is 45°. If angle *ABC* is not an acute angle, then its measure is not 45°.

b. If $a^2 < b^2$ then $a < b$. If $a^2 \geq b^2$ then $a \geq b$.

67. a. The statement, converse, and contrapositive are all true.

b. The statement, converse, and contrapositive are all true.

68. a. The statement and contrapositive are true. The converse is false.

b. The statement, converse, and contrapositive are all false.

69. a. Some isosceles triangles are not equilateral. The negation is true.

b. All real numbers are integers. The original statement is true.

c. Some natural number is larger than its square. The original statement is true.

70. a. Some natural number is not rational. The original statement is true.

b. Every circle has area less than or equal to 9π. The original statement is true.

c. Some real number is less than or equal to its square. The negation is true.

71. a. True; If x is positive, then x^2 is positive.

b. False; Take $x = -2$. Then $x^2 > 0$ but $x < 0$.

c. False; Take $x = \frac{1}{2}$. Then $x^2 = \frac{1}{4} < x$

d. True; Let x be any number. Take $y = x^2 + 1$. Then $y > x^2$.

e. True; Let y be any positive number. Take $x = \frac{y}{2}$. Then $0 < x < y$.

72. a. True; $x + (-x) < x + 1 + (-x) : 0 < 1$

b. False; There are infinitely many prime numbers.

c. True; Let x be any number. Take $y = \frac{1}{x} + 1$. Then $y > \frac{1}{x}$.

d. True; $1/n$ can be made arbitrarily close to 0.

e. True; $1/2^n$ can be made arbitrarily close to 0.

73. a. If n is odd, then there is an integer k such that $n = 2k + 1$. Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

b. Prove the contrapositive. Suppose n is even. Then there is an integer k such that $n = 2k$. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Thus n^2 is even.

74. Parts (a) and (b) prove that n is odd if and only if n^2 is odd.

75. a. $243 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

b. $124 = 4 \cdot 31 = 2 \cdot 2 \cdot 31$ or $2^2 \cdot 31$

c. $5100 = 2 \cdot 2550 = 2 \cdot 2 \cdot 1275$
$= 2 \cdot 2 \cdot 3 \cdot 425 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 85$
$= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 17$ or $2^2 \cdot 3 \cdot 5^2 \cdot 17$

76. For example, let $A = b \cdot c^2 \cdot d^3$; then $A^2 = b^2 \cdot c^4 \cdot d^6$, so the square of the number is the product of primes which occur an even number of times.

77. $\sqrt{2} = \frac{p}{q}; 2 = \frac{p^2}{q^2}; 2q^2 = p^2$; Since the prime factors of p^2 must occur an even number of times, $2q^2$ would not be valid and $\frac{p}{q} = \sqrt{2}$ must be irrational.

78. $\sqrt{3} = \frac{p}{q};\ 3 = \frac{p^2}{q^2};\ 3q^2 = p^2$; Since the prime factors of p^2 must occur an even number of times, $3q^2$ would not be valid and $\frac{p}{q} = \sqrt{3}$ must be irrational.

79. Let a, b, p, and q be natural numbers, so $\frac{a}{b}$ and $\frac{p}{q}$ are rational. $\frac{a}{b} + \frac{p}{q} = \frac{aq + bp}{bq}$ This sum is the quotient of natural numbers, so it is also rational.

80. Assume a is irrational, $\frac{p}{q} \neq 0$ is rational, and $a \cdot \frac{p}{q} = \frac{r}{s}$ is rational. Then $a = \frac{q \cdot r}{p \cdot s}$ is rational, which is a contradiction.

81. a. $-\sqrt{9} = -3$; rational

b. $0.375 = \frac{3}{8}$; rational

c. $(3\sqrt{2})(5\sqrt{2}) = 15\sqrt{4} = 30$; rational

d. $(1 + \sqrt{3})^2 = 1 + 2\sqrt{3} + 3 = 4 + 2\sqrt{3}$; irrational

82. a. –2 **b.** –2

c. $x = 2.4444...;$

$$\begin{array}{r} 10x = 24.4444... \\ x = 2.4444... \\ \hline 9x = 22 \end{array}$$

$$x = \frac{22}{9}$$

d. 1

e. $n = 1{:}\ x = 0,\ n = 2{:}\ x = \frac{3}{2},\ n = 3{:}\ x = -\frac{2}{3},$
$n = 4{:}\ x = \frac{5}{4}$
The upper bound is $\frac{3}{2}$.

f. $\sqrt{2}$

83. a. Answers will vary. Possible answer: An example
is $S = \{x : x^2 < 5, x \text{ a rational number}\}$.
Here the least upper bound is $\sqrt{5}$, which is real but irrational.

b. True

0.2 Concepts Review

1. $[-1, 5); (-\infty, -2]$

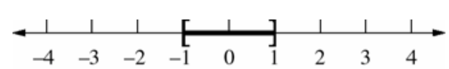
2. $b > 0; b < 0$

3. (b) and (c)

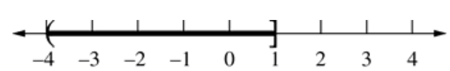
4. $-1 \le x \le 5$

Problem Set 0.2

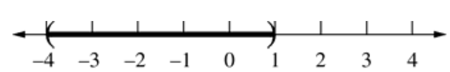
1. a.



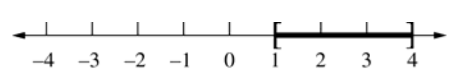
b.

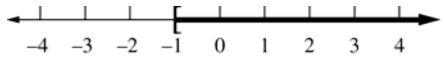


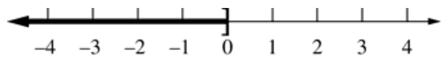
c.



d.



e. 

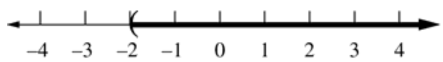
f. 

2. **a.** $(2,7)$ **b.** $[-3,4)$

c. $(-\infty,-2]$ **d.** $[-1,3]$

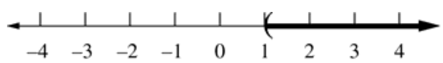
3. $x-7<2x-5$

$-2<x;(-2,\infty)$



4. $3x-5<4x-6$

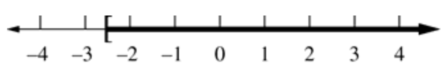
$1<x;(1,\infty)$



5. $7x-2\le 9x+3$

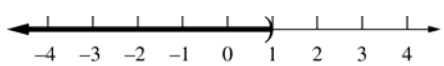
$-5\le 2x$

$x\ge -\frac{5}{2};\left[-\frac{5}{2},\infty\right)$



6. $5x-3>6x-4$

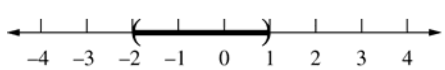
$1>x;(-\infty,1)$



7. $-4<3x+2<5$

$-6<3x<3$

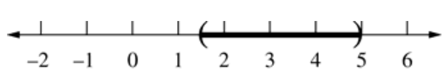
$-2<x<1;(-2,-1)$



8. $-3<4x-9<11$

$6<4x<20$

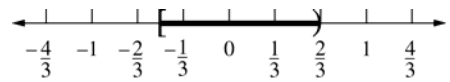
$\frac{3}{2}<x<5;\left(\frac{3}{2},5\right)$



9. $-3<1-6x\le 4$

$-4<-6x\le 3$

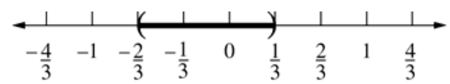
$\frac{2}{3}>x\ge -\frac{1}{2};\left[-\frac{1}{2},\frac{2}{3}\right)$



10. $4<5-3x<7$

$-1<-3x<2$

$\frac{1}{3}>x>-\frac{2}{3};\ \left(-\frac{2}{3},\frac{1}{3}\right)$



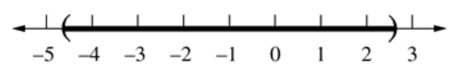
11. $x^2+2x-12<0;$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(-12)}}{2(1)}=\frac{-2\pm\sqrt{52}}{2}$$

$$=-1\pm\sqrt{13}$$

$$\left[x-\left(-1+\sqrt{13}\right)\right]\left[x-\left(-1-\sqrt{13}\right)\right]<0;$$

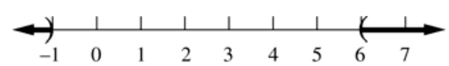
$\left(-1-\sqrt{13},-1+\sqrt{13}\right)$



12. $x^2-5x-6>0$

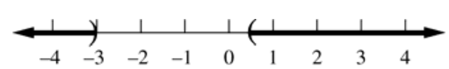
$(x+1)(x-6)>0;$

$(-\infty,-1)\cup(6,\infty)$



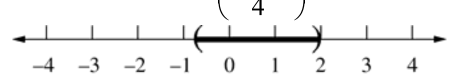
13. $2x^2+5x-3>0;\ (2x-1)(x+3)>0;$

$(-\infty,-3)\cup\left(\frac{1}{2},\infty\right)$

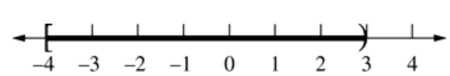


14. $4x^2-5x-6<0$

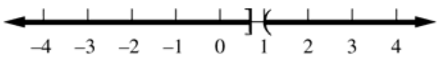
$(4x+3)(x-2)<0;\left(-\frac{3}{4},2\right)$



15. $\frac{x+4}{x-3}\le 0;\quad [-4,3)$



16. $\frac{3x-2}{x-1} \ge 0; \left(-\infty, \frac{2}{3}\right] \cup (1, \infty)$

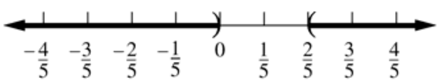


17. $\frac{2}{x} < 5$

$\frac{2}{x} - 5 < 0$

$\frac{2-5x}{x} < 0;$

$(-\infty, 0) \cup \left(\frac{2}{5}, \infty\right)$

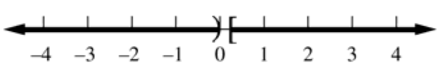


18. $\frac{7}{4x} \le 7$

$\frac{7}{4x} - 7 \le 0$

$\frac{7-28x}{4x} \le 0;$

$(-\infty, 0) \cup \left[\frac{1}{4}, \infty\right)$

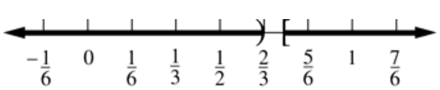


19. $\frac{1}{3x-2} \le 4$

$\frac{1}{3x-2} - 4 \le 0$

$\frac{1-4(3x-2)}{3x-2} \le 0$

$\frac{9-12x}{3x-2} \le 0; \left(-\infty, \frac{2}{3}\right) \cup \left[\frac{3}{4}, \infty\right)$

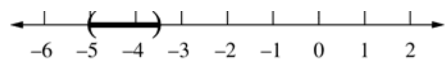


20. $\frac{3}{x+5} > 2$

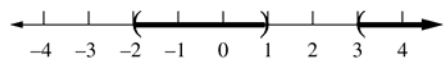
$\frac{3}{x+5} - 2 > 0$

$\frac{3-2(x+5)}{x+5} > 0$

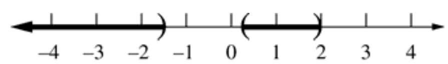
$\frac{-2x-7}{x+5} > 0; \left(-5, -\frac{7}{2}\right)$



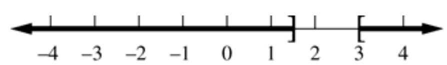
21. $(x+2)(x-1)(x-3) > 0; (-2, 1) \cup (3, 8)$



22. $(2x+3)(3x-1)(x-2) < 0; \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{1}{3}, 2\right)$

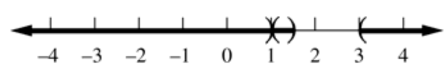


23. $(2x-3)(x-1)^2(x-3) \ge 0; \left(-\infty, \frac{3}{2}\right] \cup [3, \infty)$



24. $(2x-3)(x-1)^2(x-3) > 0;$

$(-\infty, 1) \cup \left(1, \frac{3}{2}\right) \cup (3, \infty)$

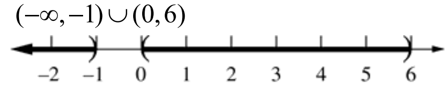


25. $x^3 - 5x^2 - 6x < 0$

$x(x^2 - 5x - 6) < 0$

$x(x+1)(x-6) < 0;$

$(-\infty, -1) \cup (0, 6)$

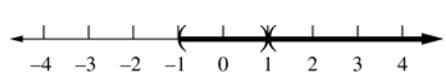


26. $x^3 - x^2 - x + 1 > 0$

$(x^2 - 1)(x-1) > 0$

$(x+1)(x-1)^2 > 0;$

$(-1, 1) \cup (1, \infty)$



27. **a.** False. **b.** True.

c. False.

28. a. True. **b.** True.

c. False.

29. a. $\Rightarrow$ Let $a < b$, so $ab < b^2$. Also, $a^2 < ab$. Thus, $a^2 < ab < b^2$ and $a^2 < b^2$. $\Leftarrow$ Let $a^2 < b^2$, so $a \neq b$ Then

$0 < (a-b)^2 = a^2 - 2ab + b^2$
$< b^2 - 2ab + b^2 = 2b(b-a)$

Since $b > 0$, we can divide by $2b$ to get $b - a > 0$.

b. We can divide or multiply an inequality by any positive number.

$a < b \Leftrightarrow \frac{a}{b} < 1 \Leftrightarrow \frac{1}{b} < \frac{1}{a}$.

30. (b) and (c) are true.
(a) is false: Take $a = -1, b = 1$.
(d) is false: if $a \le b$, then $-a \ge -b$.

31. a. $3x + 7 > 1$ and $2x + 1 < 3$
$3x > -6$ and $2x < 2$
$x > -2$ and $x < 1$; $(-2, 1)$

b. $3x + 7 > 1$ and $2x + 1 > -4$
$3x > -6$ and $2x > -5$
$x > -2$ and $x > -\frac{5}{2}$; $(-2, \infty)$

c. $3x + 7 > 1$ and $2x + 1 < -4$
$x > -2$ and $x < -\frac{5}{2}$; $\varnothing$

32. a. $2x - 7 > 1$ or $2x + 1 < 3$
$2x > 8$ or $2x < 2$
$x > 4$ or $x < 1$
$(-\infty, 1) \cup (4, \infty)$

b. $2x - 7 \le 1$ or $2x + 1 < 3$
$2x \le 8$ or $2x < 2$
$x \le 4$ or $x < 1$
$(-\infty, 4]$

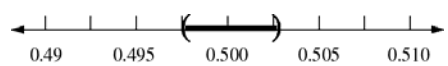
c. $2x - 7 \le 1$ or $2x + 1 > 3$
$2x \le 8$ or $2x > 2$
$x \le 4$ or $x > 1$
$(-\infty, \infty)$

33. a. $(x+1)(x^2 + 2x - 7) \ge x^2 - 1$
$x^3 + 3x^2 - 5x - 7 \ge x^2 - 1$
$x^3 + 2x^2 - 5x - 6 \ge 0$
$(x+3)(x+1)(x-2) \ge 0$
$[-3, -1] \cup [2, \infty)$

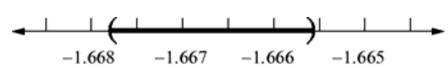
b. $x^4 - 2x^2 \ge 8$
$x^4 - 2x^2 - 8 \ge 0$
$(x^2 - 4)(x^2 + 2) \ge 0$
$(x^2 + 2)(x + 2)(x - 2) \ge 0$
$(-\infty, -2] \cup [2, \infty)$

c. $(x^2 + 1)^2 - 7(x^2 + 1) + 10 < 0$
$[(x^2 + 1) - 5][(x^2 + 1) - 2] < 0$
$(x^2 - 4)(x^2 - 1) < 0$
$(x+2)(x+1)(x-1)(x-2) < 0$
$(-2, -1) \cup (1, 2)$

34. a. $1.99 < \frac{1}{x} < 2.01$
$1.99x < 1 < 2.01x$
$1.99x < 1$ and $1 < 2.01x$
$x < \frac{1}{1.99}$ and $x > \frac{1}{2.01}$
$\frac{1}{2.01} < x < \frac{1}{1.99}$
$\left(\frac{1}{2.01}, \frac{1}{1.99}\right)$



b. $2.99 < \frac{1}{x+2} < 3.01$
$2.99(x+2) < 1 < 3.01(x+2)$
$2.99x + 5.98 < 1$ and $1 < 3.01x + 6.02$
$x < \frac{-4.98}{2.99}$ and $x > \frac{-5.02}{3.01}$
$-\frac{5.02}{3.01} < x < -\frac{4.98}{2.99}$
$\left(-\frac{5.02}{3.01}, -\frac{4.98}{2.99}\right)$



35. $|x - 2| \ge 5$;
$x - 2 \le -5$ or $x - 2 \ge 5$
$x \le -3$ or $x \ge 7$
$(-\infty, -3] \cup [7, \infty)$

36. $|x+2|<1;$

$-1<x+2<1$

$-3<x<-1$

$(-3,-1)$

37. $|4x+5|\le 10;$

$-10\le 4x+5\le 10$

$-15\le 4x\le 5$

$-\frac{15}{4}\le x\le \frac{5}{4}; \left[-\frac{15}{4},\frac{5}{4}\right]$

38. $|2x-1|>2;$

$2x-1<-2$ or $2x-1>2$

$2x<-1$ or $2x>3;$

$x<-\frac{1}{2}$ or $x>\frac{3}{2}, \left(-\infty,-\frac{1}{2}\right)\cup\left(\frac{3}{2},\infty\right)$

39. $\left|\frac{2x}{7}-5\right|\ge 7$

$\frac{2x}{7}-5\le -7$ or $\frac{2x}{7}-5\ge 7$

$\frac{2x}{7}\le -2$ or $\frac{2x}{7}\ge 12$

$x\le -7$ or $x\ge 42;$

$(-\infty,-7]\cup[42,\infty)$

40. $\left|\frac{x}{4}+1\right|<1$

$-1<\frac{x}{4}+1<1$

$-2<\frac{x}{4}<0;$

$-8<x<0;\ (-8,0)$

41. $|5x-6|>1;$

$5x-6<-1$ or $5x-6>1$

$5x<5$ or $5x>7$

$x<1$ or $x>\frac{7}{5}; (-\infty,1)\cup\left(\frac{7}{5},\infty\right)$

42. $|2x-7|>3;$

$2x-7<-3$ or $2x-7>3$

$2x<4$ or $2x>10$

$x<2$ or $x>5;\ (-\infty,2)\cup(5,\infty)$

43. $\left|\frac{1}{x}-3\right|>6;$

$\frac{1}{x}-3<-6$ or $\frac{1}{x}-3>6$

$\frac{1}{x}+3<0$ or $\frac{1}{x}-9>0$

$\frac{1+3x}{x}<0$ or $\frac{1-9x}{x}>0;$

$\left(-\frac{1}{3},0\right)\cup\left(0,\frac{1}{9}\right)$

44. $\left|2+\frac{5}{x}\right|>1;$

$2+\frac{5}{x}<-1$ or $2+\frac{5}{x}>1$

$3+\frac{5}{x}<0$ or $1+\frac{5}{x}>0$

$\frac{3x+5}{x}<0$ or $\frac{x+5}{x}>0;$

$(-\infty,-5)\cup\left(-\frac{5}{3},0\right)\cup(0,\infty)$

45. $x^2-3x-4\ge 0;$

$x=\frac{3\pm\sqrt{(-3)^2-4(1)(-4)}}{2(1)}=\frac{3\pm 5}{2}=-1,4$

$(x+1)(x-4)=0; (-\infty,-1]\cup[4,\infty)$

46. $x^2-4x+4\le 0;\ x=\frac{4\pm\sqrt{(-4)^2-4(1)(4)}}{2(1)}=2$

$(x-2)(x-2)\le 0; x=2$

47. $3x^2+17x-6>0;$

$x=\frac{-17\pm\sqrt{(17)^2-4(3)(-6)}}{2(3)}=\frac{-17\pm 19}{6}=-6,\frac{1}{3}$

$(3x-1)(x+6)>0;\ (-\infty,-6)\cup\left(\frac{1}{3},\infty\right)$

48. $14x^2+11x-15\le 0;$

$x=\frac{-11\pm\sqrt{(11)^2-4(14)(-15)}}{2(14)}=\frac{-11\pm 31}{28}$

$x=-\frac{3}{2},\frac{5}{7}$

$\left(x+\frac{3}{2}\right)\left(x-\frac{5}{7}\right)\le 0; \left[-\frac{3}{2},\frac{5}{7}\right]$

49. $|x-3|<0.5\Rightarrow 5|x-3|<5(0.5)\Rightarrow |5x-15|<2.5$

50. $|x+2|<0.3\Rightarrow 4|x+2|<4(0.3)\Rightarrow |4x+18|<1.2$

51. $|x-2|<\frac{\varepsilon}{6}\Rightarrow 6|x-2|<\varepsilon\Rightarrow|6x-12|<\varepsilon$

52. $|x+4|<\frac{\varepsilon}{2}\Rightarrow 2|x+4|<\varepsilon\Rightarrow|2x+8|<\varepsilon$

53. $|3x-15|<\varepsilon\Rightarrow|3(x-5)|<\varepsilon$

$\Rightarrow 3|x-5|<\varepsilon$

$\Rightarrow |x-5|<\frac{\varepsilon}{3};\delta=\frac{\varepsilon}{3}$

54. $|4x-8|<\varepsilon\Rightarrow|4(x-2)|<\varepsilon$

$\Rightarrow 4|x-2|<\varepsilon$

$\Rightarrow |x-2|<\frac{\varepsilon}{4};\delta=\frac{\varepsilon}{4}$

55. $|6x+36|<\varepsilon\Rightarrow|6(x+6)|<\varepsilon$

$\Rightarrow 6|x+6|<\varepsilon$

$\Rightarrow |x+6|<\frac{\varepsilon}{6};\delta=\frac{\varepsilon}{6}$

56. $|5x+25|<\varepsilon\Rightarrow|5(x+5)|<\varepsilon$

$\Rightarrow 5|x+5|<\varepsilon$

$\Rightarrow |x+5|<\frac{\varepsilon}{5};\delta=\frac{\varepsilon}{5}$

57. $C=\pi d$

$|C-10|\le 0.02$

$|\pi d-10|\le 0.02$

$\left|\pi\left(d-\frac{10}{\pi}\right)\right|\le 0.02$

$\left|d-\frac{10}{\pi}\right|\le\frac{0.02}{\pi}\approx 0.0064$

We must measure the diameter to an accuracy of 0.0064 in.

58. $|C-50|\le 1.5, \left|\frac{5}{9}(F-32)-50\right|\le 1.5;$

$\frac{5}{9}|(F-32)-90|\le 1.5$

$|F-122|\le 2.7$

We are allowed an error of 2.7° F.

59.

$|x-1|<2|x-3|$

$|x-1|<|2x-6|$

$(x-1)^2<(2x-6)^2$

$x^2-2x+1<4x^2-24x+36$

$3x^2-22x+35>0$

$(3x-7)(x-5)>0;$

$\left(-\infty,\frac{7}{3}\right)\cup(5,\infty)$

60. $|2x-1|\ge|x+1|$

$(2x-1)^2\ge(x+1)^2$

$4x^2-4x+1\ge x^2+2x+1$

$3x^2-6x\ge 0$

$3x(x-2)\ge 0$

$(-\infty,0]\cup[2,\infty)$

61.

$2|2x-3|<|x+10|$

$|4x-6|<|x+10|$

$(4x-6)^2<(x+10)^2$

$16x^2-48x+36<x^2+20x+100$

$15x^2-68x-64<0$

$(5x+4)(3x-16)<0;$

$\left(-\frac{4}{5},\frac{16}{3}\right)$

62.

$|3x-1|<2|x+6|$

$|3x-1|<|2x+12|$

$(3x-1)^2<(2x+12)^2$

$9x^2-6x+1<4x^2+48x+144$

$5x^2-54x-143<0$

$(5x+11)(x-13)<0$

$\left(-\frac{11}{5},13\right)$

63. $|x| < |y| \Rightarrow |x||x| \le |x||y|$ and $|x||y| < |y||y|$ Order property: $x < y \Leftrightarrow xz < yz$ when z is positive.

$\Rightarrow |x|^2 < |y|^2$ Transitivity

$\Rightarrow x^2 < y^2$ $\left(|x|^2 = x^2\right)$

Conversely,

$x^2 < y^2 \Rightarrow |x|^2 < |y|^2$ $\left(x^2 = |x|^2\right)$

$\Rightarrow |x|^2 - |y|^2 < 0$ Subtract $|y|^2$ from each side.

$\Rightarrow (|x| - |y|)(|x| + |y|) < 0$ Factor the difference of two squares.

$\Rightarrow |x| - |y| < 0$ This is the only factor that can be negative.

$\Rightarrow |x| < |y|$ Add $|y|$ to each side.

64. $0 < a < b \Rightarrow a = \left(\sqrt{a}\right)^2$ *and* $b = \left(\sqrt{b}\right)^2$, so

$\left(\sqrt{a}\right)^2 < \left(\sqrt{b}\right)^2$, and, by Problem 63,

$\left|\sqrt{a}\right| < \left|\sqrt{b}\right| \Rightarrow \sqrt{a} < \sqrt{b}$.

65. **a.** $|a - b| = |a + (-b)| \le |a| + |-b| = |a| + |b|$

b. $|a - b| \ge \big||a| - |b|\big| \ge |a| - |b|$ Use Property 4 of absolute values.

c. $|a + b + c| = |(a + b) + c| \le |a + b| + |c|$

$\le |a| + |b| + |c|$

66. $\left|\frac{1}{x^2+3} - \frac{1}{|x|+2}\right| = \left|\frac{1}{x^2+3} + \left(-\frac{1}{|x|+2}\right)\right|$

$\le \left|\frac{1}{x^2+3}\right| + \left|-\frac{1}{|x|+2}\right|$

$= \left|\frac{1}{x^2+3}\right| + \left|\frac{1}{|x|+2}\right|$

$= \frac{1}{x^2+3} + \frac{1}{|x|+2}$

by the Triangular Inequality, and since

$x^2 + 3 > 0,\ |x| + 2 > 0 \Rightarrow \frac{1}{x^2+3} > 0, \frac{1}{|x|+2} > 0.$

$x^2 + 3 \ge 3$ and $|x| + 2 \ge 2$, so

$\frac{1}{x^2+3} \le \frac{1}{3}$ and $\frac{1}{|x|+2} \le \frac{1}{2}$, thus,

$\frac{1}{x^2+3} + \frac{1}{|x|+2} \le \frac{1}{3} + \frac{1}{2}$

67. $\left|\frac{x-2}{x^2+9}\right| = \left|\frac{x+(-2)}{x^2+9}\right|$

$\left|\frac{x-2}{x^2+9}\right| \le \left|\frac{x}{x^2+9}\right| + \left|\frac{-2}{x^2+9}\right|$

$\left|\frac{x-2}{x^2+9}\right| \le \frac{|x|}{x^2+9} + \frac{2}{x^2+9} = \frac{|x|+2}{x^2+9}$

Since $x^2 + 9 \ge 9, \frac{1}{x^2+9} \le \frac{1}{9}$

$\frac{|x|+2}{x^2+9} \le \frac{|x|+2}{9}$

$\left|\frac{x-2}{x^2+9}\right| \le \frac{|x|+2}{9}$

68. $|x| \le 2 \Rightarrow \left|x^2 + 2x + 7\right| \le \left|x^2\right| + |2x| + 7$

$\le 4 + 4 + 7 = 15$

and $\left|x^2 + 1\right| \ge 1$ so $\frac{1}{x^2+1} \le 1.$

Thus, $\left|\frac{x^2+2x+7}{x^2+1}\right| = \left|x^2 + 2x + 7\right|\left|\frac{1}{x^2+1}\right|$

$\le 15 \cdot 1 = 15$

69. $\left|x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{1}{16}\right|$

$\le \left|x^4\right| + \frac{1}{2}\left|x^3\right| + \frac{1}{4}\left|x^2\right| + \frac{1}{8}|x| + \frac{1}{16}$

$\le 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ since $|x| \le 1$.

So $\left|x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{1}{16}\right| \le 1.9375 < 2.$

70. **a.**
$$\begin{aligned} x &< x^2 \\ x - x^2 &< 0 \\ x(1-x) &< 0 \\ x < 0 &\text{ or } x > 1 \end{aligned}$$

b.
$$\begin{aligned} x^2 &< x \\ x^2 - x &< 0 \\ x(x-1) &< 0 \\ 0 < x &< 1 \end{aligned}$$

71. $a \neq 0 \Rightarrow$

$$0 \le \left(a - \frac{1}{a}\right)^2 = a^2 - 2 + \frac{1}{a^2}$$

so, $2 \le a^2 + \frac{1}{a^2}$ or $a^2 + \frac{1}{a^2} \ge 2$.

72. $a < b$

$a + a < a + b$ and $a + b < b + b$

$2a < a + b < 2b$

$a < \frac{a+b}{2} < b$

73. $0 < a < b$

$a^2 < ab$ and $ab < b^2$

$a^2 < ab < b^2$

$a < \sqrt{ab} < b$

74. $\sqrt{ab} \le \frac{1}{2}(a+b) \Leftrightarrow ab \le \frac{1}{4}\left(a^2 + 2ab + b^2\right)$

$\Leftrightarrow 0 \le \frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2 = \frac{1}{4}\left(a^2 - 2ab + b^2\right)$

$\Leftrightarrow 0 \le \frac{1}{4}(a-b)^2$ which is always true.

75. For a rectangle the area is *ab*, while for a square the area is $a^2 = \left(\frac{a+b}{2}\right)^2$. From Problem 74, $\sqrt{ab} \le \frac{1}{2}(a+b) \Leftrightarrow ab \le \left(\frac{a+b}{2}\right)^2$ so the square has the largest area.

76. $1 + x + x^2 + x^3 + \ldots + x^{99} \le 0;$

$(-\infty, -1]$

77. $\frac{1}{R} \le \frac{1}{10} + \frac{1}{20} + \frac{1}{30}$

$\frac{1}{R} \le \frac{6+3+2}{60}$

$\frac{1}{R} \le \frac{11}{60}$

$R \ge \frac{60}{11}$

$\frac{1}{R} \ge \frac{1}{20} + \frac{1}{30} + \frac{1}{40}$

$\frac{1}{R} \ge \frac{6+4+3}{120}$

$R \le \frac{120}{13}$

Thus, $\frac{60}{11} \le R \le \frac{120}{13}$

78. $A = 4\pi r^2; A = 4\pi(10)^2 = 400\pi$

$\left|4\pi r^2 - 400\pi\right| < 0.01$

$4\pi\left|r^2 - 100\right| < 0.01$

$\left|r^2 - 100\right| < \frac{0.01}{4\pi}$

$-\frac{0.01}{4\pi} < r^2 - 100 < \frac{0.01}{4\pi}$

$\sqrt{100 - \frac{0.01}{4\pi}} < r < \sqrt{100 + \frac{0.01}{4\pi}}$

$\delta \approx 0.00004$ in

0.3 Concepts Review

1. $\sqrt{(x+2)^2 + (y-3)^2}$

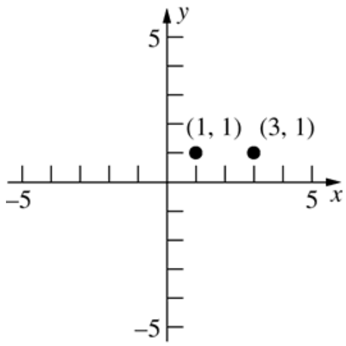
2. $(x + 4)^2 + (y – 2)^2 = 25$

3. $\left(\frac{-2+5}{2}, \frac{3+7}{2}\right) = (1.5, 5)$

4. $\frac{d-b}{c-a}$

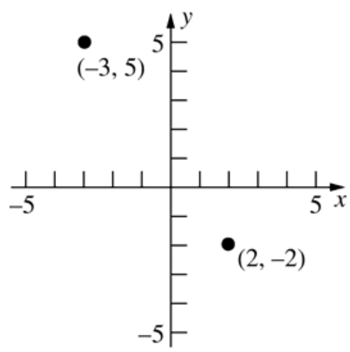
Problem Set 0.3

1.



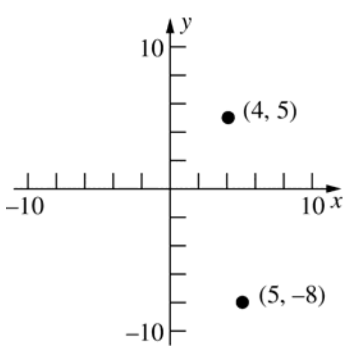
$d = \sqrt{(3-1)^2 + (1-1)^2} = \sqrt{4} = 2$

2.



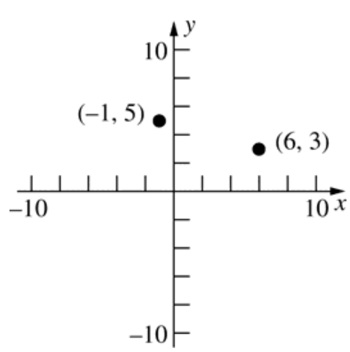
$d = \sqrt{(-3-2)^2 + (5+2)^2} = \sqrt{74} \approx 8.60$

3.



$d = \sqrt{(4-5)^2 + (5+8)^2} = \sqrt{170} \approx 13.04$

4.



$d = \sqrt{(-1-6)^2 + (5-3)^2} = \sqrt{49+4} = \sqrt{53}$
≈ 7.28

5. $d_1 = \sqrt{(5+2)^2 + (3-4)^2} = \sqrt{49+1} = \sqrt{50}$
$d_2 = \sqrt{(5-10)^2 + (3-8)^2} = \sqrt{25+25} = \sqrt{50}$
$d_3 = \sqrt{(-2-10)^2 + (4-8)^2}$
$= \sqrt{144+16} = \sqrt{160}$
$d_1 = d_2$ so the triangle is isosceles.

6. $a = \sqrt{(2-4)^2 + (-4-0)^2} = \sqrt{4+16} = \sqrt{20}$
$b = \sqrt{(4-8)^2 + (0+2)^2} = \sqrt{16+4} = \sqrt{20}$
$c = \sqrt{(2-8)^2 + (-4+2)^2} = \sqrt{36+4} = \sqrt{40}$
$a^2 + b^2 = c^2$, so the triangle is a right triangle.

7. (–1, –1), (–1, 3); (7, –1), (7, 3); (1, 1), (5, 1)

8. $\sqrt{(x-3)^2 + (0-1)^2} = \sqrt{(x-6)^2 + (0-4)^2}$;
$$x^2 - 6x + 10 = x^2 - 12x + 52$$
$$6x = 42$$
$$x = 7 \Rightarrow (7, 0)$$

9. $\left(\frac{-2+4}{2}, \frac{-2+3}{2}\right) = \left(1, \frac{1}{2}\right)$;
$$d = \sqrt{(1+2)^2 + \left(\frac{1}{2} - 3\right)^2} = \sqrt{9 + \frac{25}{4}} \approx 3.91$$

10. midpoint of $AB = \left(\frac{1+2}{2}, \frac{3+6}{2}\right) = \left(\frac{3}{2}, \frac{9}{2}\right)$
midpoint of $CD = \left(\frac{4+3}{2}, \frac{7+4}{2}\right) = \left(\frac{7}{2}, \frac{11}{2}\right)$
$$d = \sqrt{\left(\frac{3}{2} - \frac{7}{2}\right)^2 + \left(\frac{9}{2} - \frac{11}{2}\right)^2}$$
$$= \sqrt{4+1} = \sqrt{5} \approx 2.24$$

11 $(x-1)^2 + (y-1)^2 = 1$

12. $(x+2)^2 + (y-3)^2 = 4^2$
$(x+2)^2 + (y-3)^2 = 16$

13. $(x-2)^2 + (y+1)^2 = r^2$
$(5-2)^2 + (3+1)^2 = r^2$
$r^2 = 9 + 16 = 25$
$(x-2)^2 + (y+1)^2 = 25$

14. $(x-4)^2+(y-3)^2=r^2$

$(6-4)^2+(2-3)^2=r^2$

$r^2=4+1=5$

$(x-4)^2+(y-3)^2=5$

15. center $=\left(\frac{1+3}{2},\frac{3+7}{2}\right)=(2,5)$

radius $=\frac{1}{2}\sqrt{(1-3)^2+(3-7)^2}=\frac{1}{2}\sqrt{4+16}$

$=\frac{1}{2}\sqrt{20}=\sqrt{5}$

$(x-2)^2+(y-5)^2=5$

16. Since the circle is tangent to the x-axis, $r=4$.

$(x-3)^2+(y-4)^2=16$

17. $x^2+2x+10+y^2-6y-10=0$

$x^2+2x+y^2-6y=0$

$(x^2+2x+1)+(y^2-6y+9)=1+9$

$(x+1)^2+(y-3)^2=10$

center $=(-1,3)$; radius $=\sqrt{10}$

18. $x^2+y^2-6y=16$

$x^2+(y^2-6y+9)=16+9$

$x^2+(y-3)^2=25$

center $=(0,3)$; radius $=5$

19. $x^2+y^2-12x+35=0$

$x^2-12x+y^2=-35$

$(x^2-12x+36)+y^2=-35+36$

$(x-6)^2+y^2=1$

center $=(6,0)$; radius $=1$

20. $x^2+y^2-10x+10y=0$

$(x^2-10x+25)+(y^2+10y+25)=25+25$

$(x-5)^2+(y+5)^2=50$

center $=(5,-5)$; radius $=\sqrt{50}=5\sqrt{2}$

21. $4x^2+16x+15+4y^2+6y=0$

$4(x^2+4x+4)+4\left(y^2+\frac{3}{2}y+\frac{9}{16}\right)=-15+16+\frac{9}{4}$

$4(x+2)^2+4\left(y+\frac{3}{4}\right)^2=\frac{13}{4}$

$(x+2)^2+\left(y+\frac{3}{4}\right)^2=\frac{13}{16}$

center $=\left(-2,-\frac{3}{4}\right)$; radius $=\frac{\sqrt{13}}{4}$

22. $4x^2+16x+\frac{105}{16}+4y^2+3y=0$

$4(x^2+4x+4)+4\left(y^2+\frac{3}{4}y+\frac{9}{64}\right)$

$=-\frac{105}{16}+16+\frac{9}{16}$

$4(x+2)^2+4\left(y+\frac{3}{8}\right)^2=10$

$(x+2)^2+\left(y+\frac{3}{8}\right)^2=\frac{5}{2}$

center $=\left(-2,-\frac{3}{8}\right)$; radius $=\sqrt{\frac{5}{2}}=\frac{\sqrt{10}}{2}$

23. $\frac{2-1}{2-1}=1$

24. $\frac{7-5}{4-3}=2$

25. $\frac{-6-3}{-5-2}=\frac{9}{7}$

26. $\frac{-6+4}{0-2}=1$

27. $\frac{5-0}{0-3}=-\frac{5}{3}$

28. $\frac{6-0}{0+6}=1$

29. $y-2=-1(x-2)$

$y-2=-x+2$

$x+y-4=0$

30. $y-4=-1(x-3)$

$y-4=-x+3$

$x+y-7=0$

31. $y=2x+3$

$2x-y+3=0$

32. $y=0x+5$

$0x+y-5=0$

33. $m = \dfrac{8-3}{4-2} = \dfrac{5}{2}$;

$$y - 3 = \frac{5}{2}(x-2)$$
$$2y - 6 = 5x - 10$$
$$5x - 2y - 4 = 0$$

34. $m = \dfrac{2-1}{8-4} = \dfrac{1}{4}$;

$$y - 1 = \frac{1}{4}(x-4)$$
$$4y - 4 = x - 4$$
$$x - 4y + 0 = 0$$

35. $3y = -2x + 1$; $y = -\dfrac{2}{3}x + \dfrac{1}{3}$; slope $= -\dfrac{2}{3}$;

y-intercept $= \dfrac{1}{3}$

36. $-4y = 5x - 6$

$$y = -\frac{5}{4}x + \frac{3}{2}$$

slope $= -\dfrac{5}{4}$; y-intercept $= \dfrac{3}{2}$

37. $6 - 2y = 10x - 2$

$$-2y = 10x - 8$$
$$y = -5x + 4;$$

slope $= -5$; y-intercept $= 4$

38. $4x + 5y = -20$

$$5y = -4x - 20$$
$$y = -\frac{4}{5}x - 4$$

slope $= -\dfrac{4}{5}$; y-intercept $= -4$

39. **a.** $m = 2$;

$$y + 3 = 2(x-3)$$
$$y = 2x - 9$$

b. $m = -\dfrac{1}{2}$;

$$y + 3 = -\frac{1}{2}(x-3)$$
$$y = -\frac{1}{2}x - \frac{3}{2}$$

c. $2x + 3y = 6$

$$3y = -2x + 6$$
$$y = -\frac{2}{3}x + 2;$$

$m = -\dfrac{2}{3}$;

$$y + 3 = -\frac{2}{3}(x-3)$$
$$y = -\frac{2}{3}x - 1$$

d. $m = \dfrac{3}{2}$;

$$y + 3 = \frac{3}{2}(x-3)$$
$$y = \frac{3}{2}x - \frac{15}{2}$$

e. $m = \dfrac{-1-2}{3+1} = -\dfrac{3}{4}$;

$$y + 3 = -\frac{3}{4}(x-3)$$
$$y = -\frac{3}{4}x - \frac{3}{4}$$

f. $x = 3$ **g.** $y = -3$

40. **a.** $3x + cy = 5$

$$3(3) + c(1) = 5$$
$$c = -4$$

b. $c = 0$

c. $2x + y = -1$

$$y = -2x - 1$$

$m = -2$;

$3x + cy = 5$

$$cy = -3x + 5$$
$$y = -\frac{3}{c}x + \frac{5}{c}$$
$$-2 = -\frac{3}{c}$$
$$c = \frac{3}{2}$$

d. c must be the same as the coefficient of x, so $c = 3$.

e. $y-2=3(x+3)$;

perpendicular slope $=-\frac{1}{3}$;

$$-\frac{1}{3}=-\frac{3}{c}$$
$$c=9$$

41. $m=\frac{3}{2}$;

$$y+1=\frac{3}{2}(x+2)$$
$$y=\frac{3}{2}x+2$$

42. a. $m=2$;

$$kx-3y=10$$
$$-3y=-kx+10$$
$$y=\frac{k}{3}x-\frac{10}{3}$$
$$\frac{k}{3}=2; k=6$$

b. $m=-\frac{1}{2}$;

$$\frac{k}{3}=-\frac{1}{2}$$
$$k=-\frac{3}{2}$$

c. $2x+3y=6$

$$3y=-2x+6$$
$$y=-\frac{2}{3}x+2;$$
$$m=\frac{3}{2};\ \frac{k}{3}=\frac{3}{2};\ k=\frac{9}{2}$$

43. $y = 3(3) - 1 = 8$; (3, 9) is above the line.

44. $(a,0),(0,b);\ m=\frac{b-0}{0-a}=-\frac{b}{a}$

$$y=-\frac{b}{a}x+b;\ \frac{bx}{a}+y=b; \frac{x}{a}+\frac{y}{b}=1$$

45. $2x+3y=4$

$-3x+y=5$

$$\begin{array}{l} 2x+3y=4 \\ \underline{9x-3y=-15} \\ 11x \quad =-11 \end{array}$$
$$x=-1$$
$$-3(-1)+y=5$$
$$y=2$$

Point of intersection: (–1, 2)

$$3y=-2x+4$$
$$y=-\frac{2}{3}x+\frac{4}{3}$$
$$m=\frac{3}{2}$$
$$y-2=\frac{3}{2}(x+1)$$
$$y=\frac{3}{2}x+\frac{7}{2}$$

46. $4x-5y=8$

$2x+y=-10$

$$\begin{array}{l} 4x-5y=8 \\ \underline{-4x-2y=20} \\ -7y=28 \end{array}$$
$$y=-4$$
$$4x-5(-4)=8$$
$$4x=-12$$
$$x=-3$$

Point of intersection: $(-3,-4)$;

$$4x-5y=8$$
$$-5y=-4x+8$$
$$y=\frac{4}{5}x-\frac{8}{5}$$
$$m=-\frac{5}{4}$$
$$y+4=-\frac{5}{4}(x+3)$$
$$y=-\frac{5}{4}x-\frac{31}{4}$$

47. $3x-4y=5$
$2x+3y=9$

$$\begin{array}{l} 9x-12y=15 \\ \underline{8x+12y=36} \\ 17x \quad\;\; =51 \\ \quad\;\; x=3 \end{array}$$

$$\begin{aligned} 3(3)-4y&=5 \\ -4y&=-4 \\ y&=1 \end{aligned}$$

Point of intersection: (3, 1); $3x-4y=5$;

$$\begin{aligned} -4y&=-3x+5 \\ y&=\frac{3}{4}x-\frac{5}{4} \end{aligned}$$

$$m=-\frac{4}{3}$$

$$\begin{aligned} y-1&=-\frac{4}{3}(x-3) \\ y&=-\frac{4}{3}x+5 \end{aligned}$$

48. $5x-2y=5$
$\underline{2x+3y=6}$

$$\begin{array}{l} 15x-6y=15 \\ \underline{\;4x+6y=12} \\ 19x \quad\;\; =27 \end{array}$$

$$x=\frac{27}{19}$$

$$\begin{aligned} 2\left(\frac{27}{19}\right)+3y&=6 \\ 3y&=\frac{60}{19} \\ y&=\frac{20}{19} \end{aligned}$$

Point of intersection: $\left(\frac{27}{19},\frac{20}{19}\right)$;

$$\begin{aligned} 5x-2y&=5 \\ -2y&=-5x+5 \\ y&=\frac{5}{2}x-\frac{5}{2} \end{aligned}$$

$$m=-\frac{2}{5}$$

$$\begin{aligned} y-\frac{20}{19}&=-\frac{2}{5}\left(x-\frac{27}{19}\right) \\ y&=-\frac{2}{5}x+\frac{54}{95}+\frac{20}{19} \\ y&=-\frac{2}{5}x+\frac{154}{95} \end{aligned}$$

49. center: $\left(\frac{2+6}{2},\frac{-1+3}{2}\right)=(4,1)$

midpoint $=\left(\frac{2+6}{2},\frac{3+3}{2}\right)=(4,3)$

inscribed circle: radius $=\sqrt{(4-4)^2+(1-3)^2}$
$=\sqrt{4}=2$

$(x-4)^2+(y-1)^2=4$

circumscribed circle:

radius $=\sqrt{(4-2)^2+(1-3)^2}=\sqrt{8}$

$(x-4)^2+(y-1)^2=8$

50. The radius of each circle is $\sqrt{16}=4$. The centers are $(1,-2)$ and $(-9,10)$. The length of the belt is the sum of half the circumference of the first circle, half the circumference of the second circle, and twice the distance between their centers.

$$\begin{aligned} L&=\frac{1}{2}\cdot 2\pi(4)+\frac{1}{2}\cdot 2\pi(4)+2\sqrt{(1+9)^2+(-2-10)^2} \\ &=8\pi+2\sqrt{100+144} \\ &\approx 56.37 \end{aligned}$$

51. Put the vertex of the right angle at the origin with the other vertices at $(a, 0)$ and $(0, b)$. The midpoint of the hypotenuse is $\left(\frac{a}{2},\frac{b}{2}\right)$. The distances from the vertices are

$$\sqrt{\left(a-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\frac{1}{2}\sqrt{a^2+b^2},$$

$$\sqrt{\left(0-\frac{a}{2}\right)^2+\left(b-\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\frac{1}{2}\sqrt{a^2+b^2}, \text{ and}$$

$$\sqrt{\left(0-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\frac{1}{2}\sqrt{a^2+b^2},$$

which are all the same.

52. From Problem 51, the midpoint of the hypotenuse, $(4,3,)$, is equidistant from the vertices. This is the center of the circle. The radius is $\sqrt{16+9}=5$. The equation of the circle is

$(x-4)^2+(y-3)^2=25.$

53. $x^2 + y^2 - 4x - 2y - 11 = 0$

$(x^2 - 4x + 4) + (y^2 - 2y + 1) = 11 + 4 + 1$

$(x-2)^2 + (y-1)^2 = 16$

$x^2 + y^2 + 20x - 12y + 72 = 0$

$(x^2 + 20x + 100) + (y^2 - 12y + 36)$
$= -72 + 100 + 36$

$(x+10)^2 + (y-6)^2 = 64$

center of first circle: (2, 1)
center of second circle: (–10, 6)

$d = \sqrt{(2+10)^2 + (1-6)^2} = \sqrt{144+25}$
$= \sqrt{169} = 13$

However, the radii only sum to 4 + 8 = 12, so the circles must not intersect if the distance between their centers is 13.

54. $x^2 + ax + y^2 + by + c = 0$

$\left(x^2 + ax + \frac{a^2}{4}\right) + \left(y^2 + by + \frac{b^2}{4}\right)$
$= -c + \frac{a^2}{4} + \frac{b^2}{4}$

$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2 + b^2 - 4c}{4}$

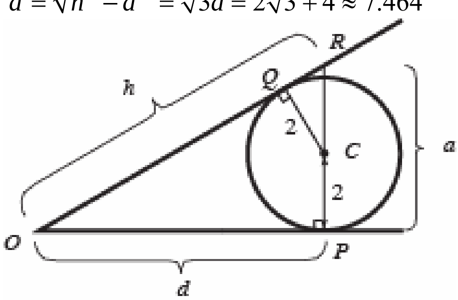
$\frac{a^2 + b^2 - 4c}{4} > 0 \Rightarrow a^2 + b^2 > 4c$

55. Label the points *C*, *P*, *Q*, and *R* as shown in the figure below. Let $d = |OP|$, $h = |OR|$, and $a = |PR|$. Triangles ΔOPR and ΔCQR are similar because each contains a right angle and they share angle $\angle QRC$. For an angle of 30°, $\frac{d}{h} = \frac{\sqrt{3}}{2}$ and $\frac{a}{h} = \frac{1}{2} \Rightarrow h = 2a$. Using a property of similar triangles, $|QC|/|RC| = \sqrt{3}/2$,

$\frac{2}{a-2} = \frac{\sqrt{3}}{2} \quad \rightarrow \quad a = 2 + \frac{4}{\sqrt{3}}$

By the Pythagorean Theorem, we have

$d = \sqrt{h^2 - a^2} = \sqrt{3}a = 2\sqrt{3} + 4 \approx 7.464$



56. The equations of the two circles are

$(x-R)^2 + (y-R)^2 = R^2$

$(x-r)^2 + (y-r)^2 = r^2$

Let (a, a) denote the point where the two circles touch. This point must satisfy

$(a-R)^2 + (a-R)^2 = R^2$

$(a-R)^2 = \frac{R^2}{2}$

$a = \left(1 \pm \frac{\sqrt{2}}{2}\right)R$

Since $a < R$, $a = \left(1 - \frac{\sqrt{2}}{2}\right)R.$

At the same time, the point where the two circles touch must satisfy

$(a-r)^2 + (a-r)^2 = r^2$

$a = \left(1 \pm \frac{\sqrt{2}}{2}\right)r$

Since $a > r$, $a = \left(1 + \frac{\sqrt{2}}{2}\right)r.$

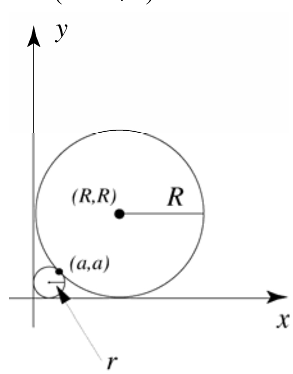
Equating the two expressions for *a* yields

$\left(1 - \frac{\sqrt{2}}{2}\right)R = \left(1 + \frac{\sqrt{2}}{2}\right)r$

$r = \frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}R = \frac{\left(1 - \frac{\sqrt{2}}{2}\right)^2}{\left(1 + \frac{\sqrt{2}}{2}\right)\left(1 - \frac{\sqrt{2}}{2}\right)}R$

$r = \frac{1 - \sqrt{2} + \frac{1}{2}}{1 - \frac{1}{2}}R$

$r = (3 - 2\sqrt{2})R \approx 0.1716R$



57. Refer to figure 15 in the text. Given ine l_1 with slope *m*, draw $\triangle ABC$ with vertical and horizontal sides *m*, 1.
Line l_2 is obtained from l_1 by rotating it around the point *A* by $90°$ counter-clockwise. Triangle ABC is rotated into triangle AED. We read off

$$\text{slope of } l_2 = \frac{1}{-m} = -\frac{1}{m}.$$

58. $2\sqrt{(x-1)^2+(y-1)^2} = \sqrt{(x-3)^2+(y-4)^2}$

$4(x^2-2x+1+y^2-2y+1)$
$= x^2-6x+9+y^2-8y+16$

$3x^2-2x+3y^2 = 9+16-4-4;$

$3x^2-2x+3y^2 = 17; x^2-\frac{2}{3}x+y^2 = \frac{17}{3};$

$\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+y^2 = \frac{17}{3}+\frac{1}{9}$

$\left(x-\frac{1}{3}\right)^2+y^2 = \frac{52}{9}$

center: $\left(\frac{1}{3},0\right)$; radius: $\left(\frac{\sqrt{52}}{3}\right)$

59. Let *a*, *b*, and *c* be the lengths of the sides of the right triangle, with *c* the length of the hypotenuse. Then the Pythagorean Theorem says that $a^2+b^2=c^2$

Thus, $\frac{\pi a^2}{8}+\frac{\pi b^2}{8}=\frac{\pi c^2}{8}$ or

$\frac{1}{2}\pi\left(\frac{a}{2}\right)^2+\frac{1}{2}\pi\left(\frac{b}{2}\right)^2=\frac{1}{2}\pi\left(\frac{c}{2}\right)^2$

$\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$ is the area of a semicircle with diameter *x*, so the circles on the legs of the triangle have total area equal to the area of the semicircle on the hypotenuse.

From $a^2+b^2=c^2$,

$\frac{\sqrt{3}}{4}a^2+\frac{\sqrt{3}}{4}b^2=\frac{\sqrt{3}}{4}c^2$

$\frac{\sqrt{3}}{4}x^2$ is the area of an equilateral triangle with sides of length *x*, so the equilateral triangles on the legs of the right triangle have total area equal to the area of the equilateral triangle on the hypotenuse of the right triangle.

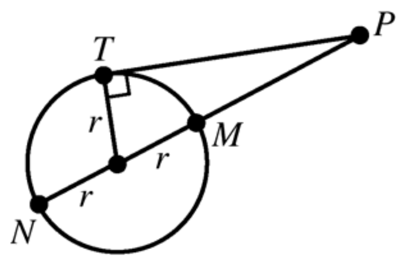
60. See the figure below. The angle at *T* is a right angle, so the Pythagorean Theorem gives

$(PM+r)^2=(PT)^2+r^2$

$\Leftrightarrow (PM)^2+2rPM+r^2=(PT)^2+r^2$

$\Leftrightarrow PM(PM+2r)=(PT)^2$

$PM+2r=PN$ so this gives $(PM)(PN)=(PT)^2$



61. The lengths *A*, *B*, and *C* are the same as the corresponding distances between the centers of the circles:

$A=\sqrt{(-2)^2+(8)^2}=\sqrt{68}\approx 8.2$

$B=\sqrt{(6)^2+(8)^2}=\sqrt{100}=10$

$C=\sqrt{(8)^2+(0)^2}=\sqrt{64}=8$

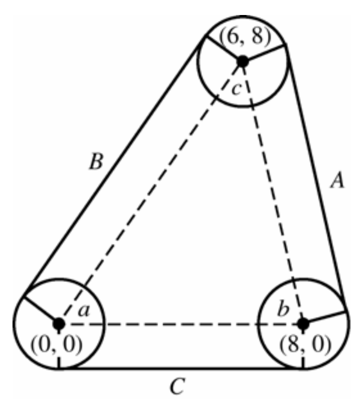
Each circle has radius 2, so the part of the belt around the wheels is

$2(2\pi-\text{a}-\pi)+2(2\pi-\text{b}-\pi)+2(2\pi-\text{c}-\pi)$

$= 2[3\pi - (a+b+c)] = 2(2\pi) = 4\pi$

Since $a+b+c=\pi$, the sum of the angles of a triangle.

The length of the belt is $\approx 8.2+10+8+4\pi$
≈ 38.8 units.



62 As in Problems 50 and 61, the curved portions of the belt have total length $2\pi r$. The lengths of the straight portions will be the same as the lengths of the sides. The belt will have length $2\pi r+d_1+d_2+\ldots+d_n$.

63. $A = 3, B = 4, C = -6$

$$d = \frac{|3(-3)+4(2)+(-6)|}{\sqrt{(3)^2+(4)^2}} = \frac{7}{5}$$

64. $A = 2, B = -2, C = 4$

$$d = \frac{|2(4)-2(-1)+4)|}{\sqrt{(2)^2+(2)^2}} = \frac{14}{\sqrt{8}} = \frac{7\sqrt{2}}{2}$$

65. $A = 12, B = -5, C = 1$

$$d = \frac{|12(-2)-5(-1)+1|}{\sqrt{(12)^2+(-5)^2}} = \frac{18}{13}$$

66. $A = 2, B = -1, C = -5$

$$d = \frac{|2(3)-1(-1)-5|}{\sqrt{(2)^2+(-1)^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

67. $2x + 4(0) = 5$

$$x = \frac{5}{2}$$

$$d = \frac{\left|2\left(\frac{5}{2}\right)+4(0)-7\right|}{\sqrt{(2)^2+(4)^2}} = \frac{2}{\sqrt{20}} = \frac{\sqrt{5}}{5}$$

68. $7(0) - 5y = -1$

$$y = \frac{1}{5}$$

$$d = \frac{\left|7(0)-5\left(\frac{1}{5}\right)-6\right|}{\sqrt{(7)^2+(-5)^2}} = \frac{7}{\sqrt{74}} = \frac{7\sqrt{74}}{74}$$

69. $m = \frac{-2-3}{1+2} = -\frac{5}{3}$; $m = \frac{3}{5}$; passes through

$$\left(\frac{-2+1}{2}, \frac{3-2}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$y - \frac{1}{2} = \frac{3}{5}\left(x + \frac{1}{2}\right)$$

$$y = \frac{3}{5}x + \frac{4}{5}$$

70. $m = \frac{0-4}{2-0} = -2$; $m = \frac{1}{2}$; passes through

$$\left(\frac{0+2}{2}, \frac{4+0}{2}\right) = (1, 2)$$

$$y - 2 = \frac{1}{2}(x-1)$$

$$y = \frac{1}{2}x + \frac{3}{2}$$

$m = \frac{6-0}{4-2} = 3$; $m = -\frac{1}{3}$; passes through

$$\left(\frac{2+4}{2}, \frac{0+6}{2}\right) = (3, 3)$$

$$y - 3 = -\frac{1}{3}(x-3)$$

$$y = -\frac{1}{3}x + 4$$

$$\frac{1}{2}x + \frac{3}{2} = -\frac{1}{3}x + 4$$

$$\frac{5}{6}x = \frac{5}{2}$$

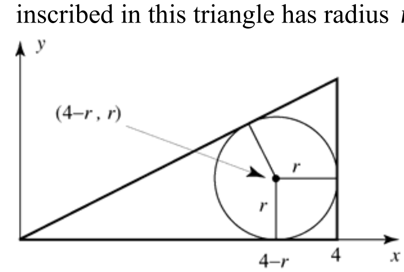
$$x = 3$$

$$y = \frac{1}{2}(3) + \frac{3}{2} = 3$$

center = (3, 3)

71. Let the origin be at the vertex as shown in the figure below. The center of the circle is then $(4-r, r)$, so it has equation $(x-(4-r))^2 + (y-r)^2 = r^2$. Along the side of length 5, the y-coordinate is always $\frac{3}{4}$ times the x-coordinate. Thus, we need to find the value of r for which there is exactly one x-solution to $(x-4+r)^2 + \left(\frac{3}{4}x - r\right)^2 = r^2$.

Solving for x in this equation gives $x = \frac{16}{25}\left(16 - r \pm \sqrt{24\left(-r^2+7r-6\right)}\right)$. There is exactly one solution when $-r^2 + 7r - 6 = 0$, that is, when $r = 1$ or $r = 6$. The root $r = 6$ is extraneous. Thus, the largest circle that can be inscribed in this triangle has radius $r = 1$.



72. The line tangent to the circle at (a,b) will be perpendicular to the line through (a,b) and the center of the circle, which is $(0,0)$. The line through (a,b) and $(0,0)$ has slope

$$m=\frac{0-b}{0-a}=\frac{b}{a}; ax+by=r^2 \Rightarrow y=-\frac{a}{b}x+\frac{r^2}{b}$$

so $ax+by=r^2$ has slope $-\frac{a}{b}$ and is perpendicular to the line through (a,b) and $(0,0)$, so it is tangent to the circle at (a,b).

73. $12a + 0b = 36$
$a = 3$
$3^2+b^2=36$
$b=\pm 3\sqrt{3}$
$3x-3\sqrt{3}y=36$
$x-\sqrt{3}y=12$
$3x+3\sqrt{3}y=36$
$x+\sqrt{3}y=12$

74. Use the formula given for problems 63-66, for $(x,y)=(0,0)$.

$A=m, B=-1, C=B-b; (0,0)$

$$d=\frac{|m(0)-1(0)+B-b|}{\sqrt{m^2+(-1)^2}}=\frac{|B-b|}{\sqrt{m^2+1}}$$

75. The midpoint of the side from (0, 0) to (a, 0) is

$$\left(\frac{0+a}{2},\frac{0+0}{2}\right)=\left(\frac{a}{2},0\right)$$

The midpoint of the side from (0, 0) to (b, c) is

$$\left(\frac{0+b}{2},\frac{0+c}{2}\right)=\left(\frac{b}{2},\frac{c}{2}\right)$$

$$m_1=\frac{c-0}{b-a}=\frac{c}{b-a}$$

$$m_2=\frac{\frac{c}{2}-0}{\frac{b}{2}-\frac{a}{2}}=\frac{c}{b-a}; m_1=m_2$$

76. See the figure below. The midpoints of the sides are

$$P\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right), Q\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right),$$

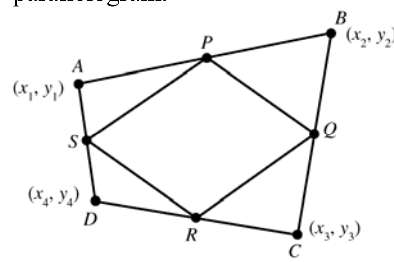
$$R\left(\frac{x_3+x_4}{2},\frac{y_3+y_4}{2}\right), \text{ and}$$

$$S\left(\frac{x_1+x_4}{2},\frac{y_1+y_4}{2}\right).$$

The slope of PS is

$$\frac{\frac{1}{2}[y_1+y_4-(y_1+y_2)]}{\frac{1}{2}[x_1+x_4-(x_1+x_2)]}=\frac{y_4-y_2}{x_4-x_2}.$$ The slope of

QR is $$\frac{\frac{1}{2}[y_3+y_4-(y_2+y_3)]}{\frac{1}{2}[x_3+x_4-(x_2+x_3)]}=\frac{y_4-y_2}{x_4-x_2}.$$ Thus PS and QR are parallel. The slopes of SR and PQ are both $\frac{y_3-y_1}{x_3-x_1}$, so $PQRS$ is a parallelogram.



77. $x^2+(y-6)^2=25$; passes through (3, 2)
tangent line: $3x - 4y = 1$
The dirt hits the wall at $y = 8$.

0.4 Concepts Review

1. y-axis

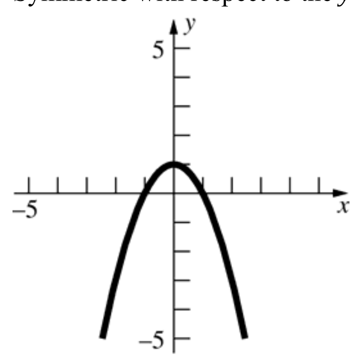
2. $(4,-2)$

3. 8; –2, 1, 4

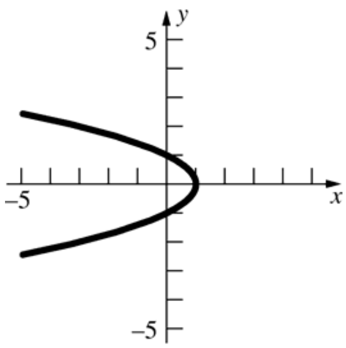
4. line; parabola

Problem Set 0.4

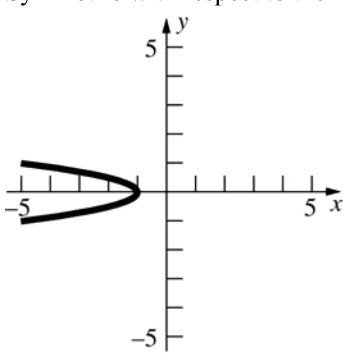
1. $y = -x^2 + 1$; y-intercept = 1; $y = (1 + x)(1 - x)$;
x-intercepts = –1, 1
Symmetric with respect to the y-axis



2. $x = -y^2 + 1$; y-intercepts $= -1, 1$;
x-intercept $= 1$.
Symmetric with respect to the x-axis.



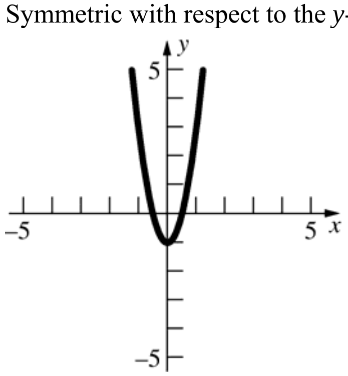
3. $x = -4y^2 - 1$; x-intercept $= -1$
Symmetric with respect to the x-axis



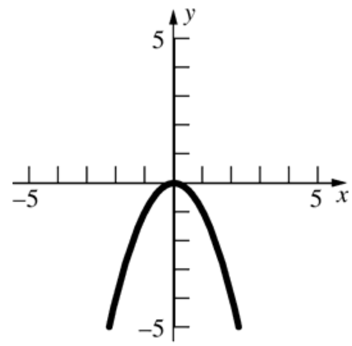
4. $y = 4x^2 - 1$; y-intercept $= -1$

$y = (2x+1)(2x-1)$; x-intercepts $= -\frac{1}{2}, \frac{1}{2}$

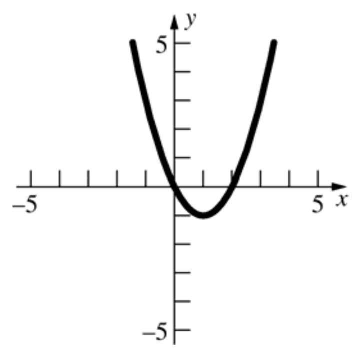
Symmetric with respect to the y-axis.



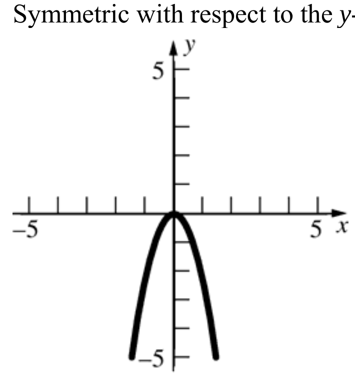
5. $x^2 + y = 0$; $y = -x^2$
x-intercept $= 0$, y-intercept $= 0$
Symmetric with respect to the y-axis



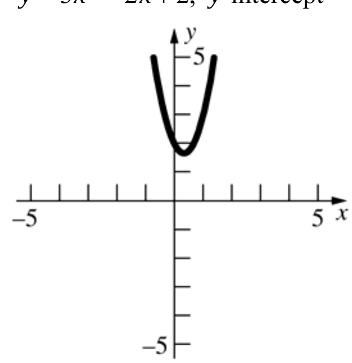
6. $y = x^2 - 2x$; y-intercept $= 0$
$y = x(2-x)$; x-intercepts $= 0, 2$



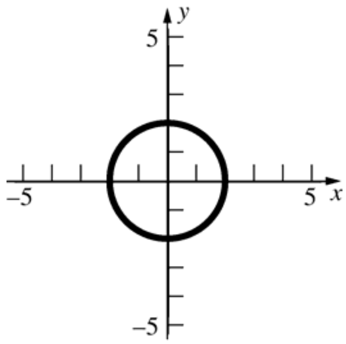
7. $7x^2 + 3y = 0$; $3y = -7x^2$; $y = -\frac{7}{3}x^2$
x-intercept $= 0$, y-intercept $= 0$
Symmetric with respect to the y-axis



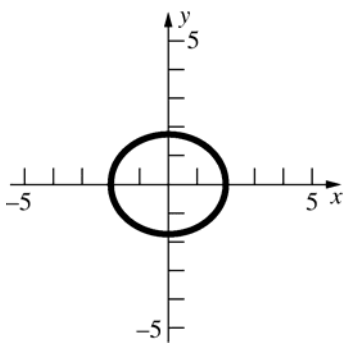
8. $y = 3x^2 - 2x + 2$; y-intercept $= 2$



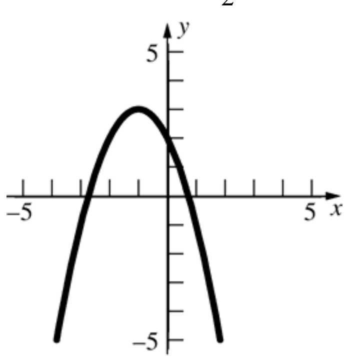
9. $x^2 + y^2 = 4$
x-intercepts = -2, 2; y-intercepts = -2, 2
Symmetric with respect to the x-axis, y-axis, and origin



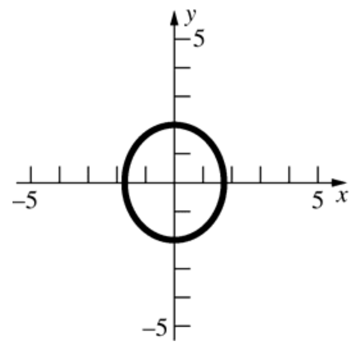
10. $3x^2 + 4y^2 = 12$; y-intercepts $= -\sqrt{3}, \sqrt{3}$
x-intercepts $= -2, 2$
Symmetric with respect to the x-axis, y-axis, and origin



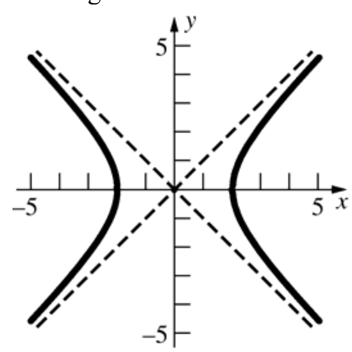
11. $y = -x^2 - 2x + 2$: y-intercept = 2
$x\text{-intercepts} = \dfrac{2 \pm \sqrt{4+8}}{-2} = \dfrac{2 \pm 2\sqrt{3}}{-2} = -1 \pm \sqrt{3}$



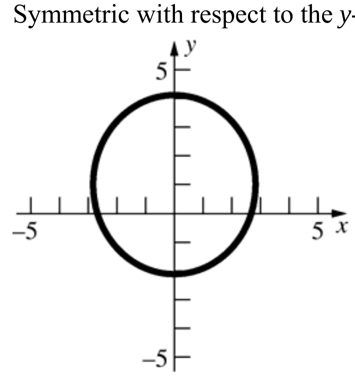
12. $4x^2 + 3y^2 = 12$; y-intercepts $= -2, 2$
x-intercepts $= -\sqrt{3}, \sqrt{3}$
Symmetric with respect to the x-axis, y-axis, and origin



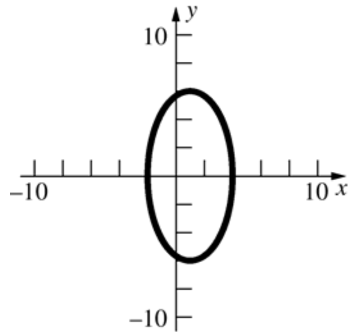
13. $x^2 - y^2 = 4$
x-intercept = -2, 2
Symmetric with respect to the x-axis, y-axis, and origin



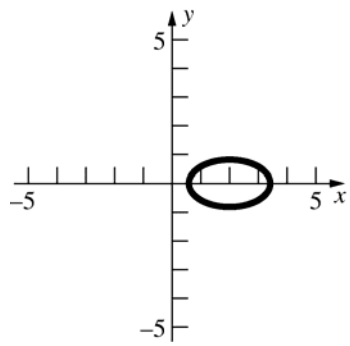
14. $x^2 + (y-1)^2 = 9$; y-intercepts $= -2, 4$
x-intercepts $= -2\sqrt{2}, 2\sqrt{2}$
Symmetric with respect to the y-axis



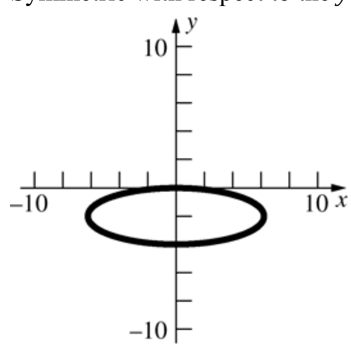
15. $4(x-1)^2 + y^2 = 36$;
y-intercepts $= \pm\sqrt{32} = \pm 4\sqrt{2}$
x-intercepts = –2, 4
Symmetric with respect to the x-axis



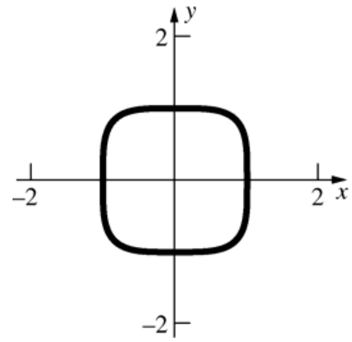
16. $x^2 - 4x + 3y^2 = -2$
x-intercepts $= 2 \pm \sqrt{2}$
Symmetric with respect to the x-axis



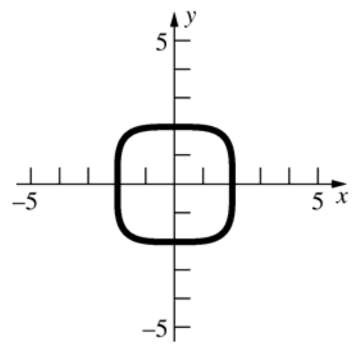
17. $x^2 + 9(y+2)^2 = 36$; y-intercepts = –4, 0
x-intercept = 0
Symmetric with respect to the y-axis



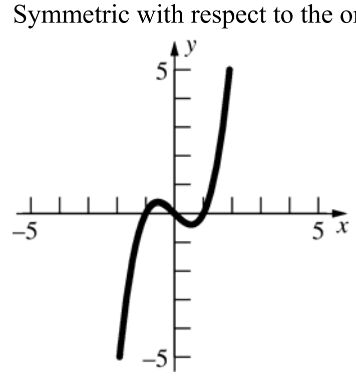
18. $x^4 + y^4 = 1$; y-intercepts $= -1, 1$
x-intercepts $= -1, 1$
Symmetric with respect to the x-axis, y-axis, and origin



19. $x^4 + y^4 = 16$; y-intercepts $= -2, 2$
x-intercepts $= -2, 2$
Symmetric with respect to the y-axis, x-axis and origin

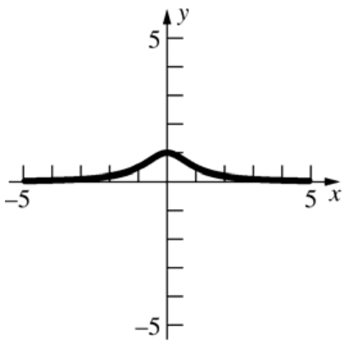


20. $y = x^3 - x$; y-intercepts = 0;
$y = x(x^2 - 1) = x(x+1)(x-1)$;
x-intercepts = –1, 0, 1
Symmetric with respect to the origin



21. $y = \dfrac{1}{x^2+1}$; y-intercept = 1

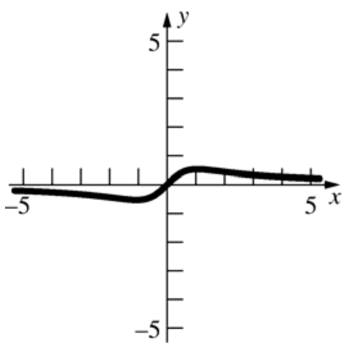
Symmetric with respect to the y-axis



22. $y = \dfrac{x}{x^2+1}$; y-intercept = 0

x-intercept = 0

Symmetric with respect to the origin



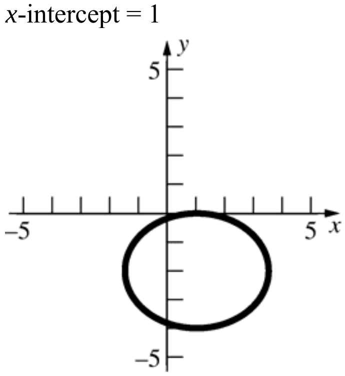
23. $2x^2 - 4x + 3y^2 + 12y = -2$

$2(x^2 - 2x + 1) + 3(y^2 + 4y + 4) = -2 + 2 + 12$

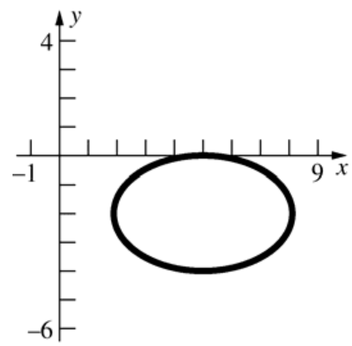
$2(x-1)^2 + 3(y+2)^2 = 12$

y-intercepts = $-2 \pm \dfrac{\sqrt{30}}{3}$

x-intercept = 1

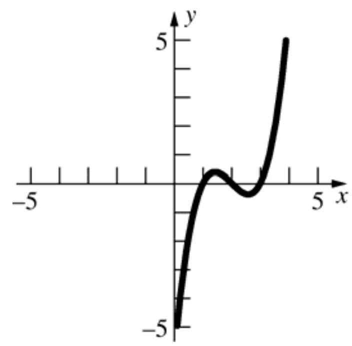


24. $4(x-5)^2 + 9(y+2)^2 = 36$; x-intercept = 5



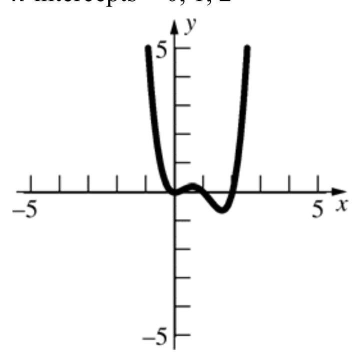
25. $y = (x-1)(x-2)(x-3)$; y-intercept = –6

x-intercepts = 1, 2, 3



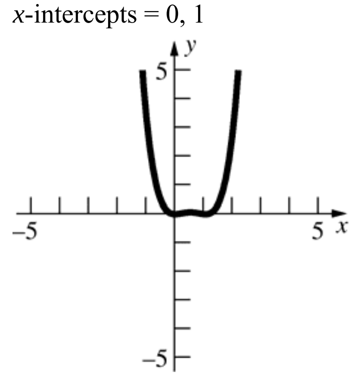
26. $y = x^2(x-1)(x-2)$; y-intercept = 0

x-intercepts = 0, 1, 2

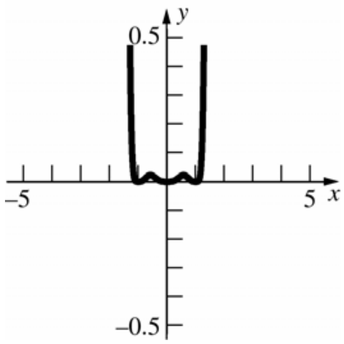


27. $y = x^2(x-1)^2$; y-intercept = 0

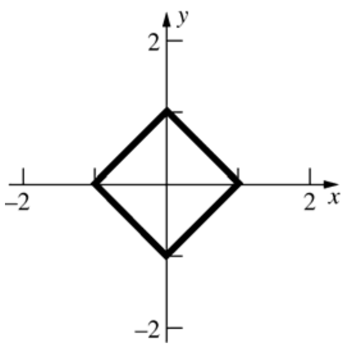
x-intercepts = 0, 1



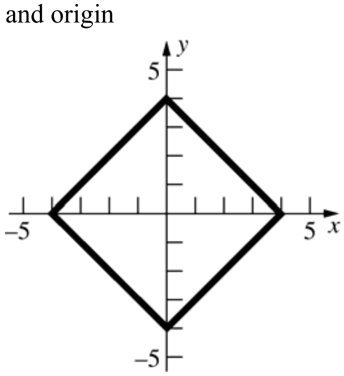
28. $y = x^4(x-1)^4(x+1)^4$; y-intercept = 0
x-intercepts $= -1, 0, 1$
Symmetric with respect to the y-axis



29. $|x| + |y| = 1$; y-intercepts = –1, 1;
x-intercepts = –1, 1
Symmetric with respect to the x-axis, y-axis and origin

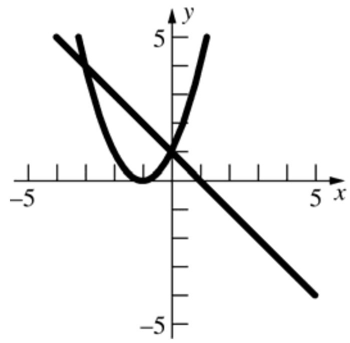


30. $|x| + |y| = 4$; y-intercepts = –4, 4;
x-intercepts = –4, 4
Symmetric with respect to the x-axis, y-axis and origin

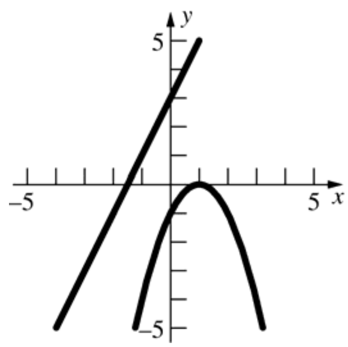


31.
$$-x + 1 = (x+1)^2$$
$$-x + 1 = x^2 + 2x + 1$$
$$x^2 + 3x = 0$$
$$x(x+3) = 0$$
$$x = 0, -3$$

Intersection points: (0, 1) and (–3, 4)

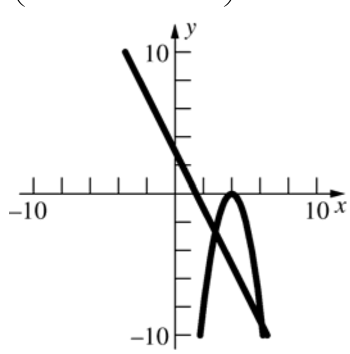


32.
$$2x + 3 = -(x-1)^2$$
$$2x + 3 = -x^2 + 2x - 1$$
$$x^2 + 4 = 0$$
No points of intersection



33.
$$-2x + 3 = -2(x-4)^2$$
$$-2x + 3 = -2x^2 + 16x - 32$$
$$2x^2 - 18x + 35 = 0$$
$$x = \frac{18 \pm \sqrt{324 - 280}}{4} = \frac{18 \pm 2\sqrt{11}}{4} = \frac{9 \pm \sqrt{11}}{2};$$

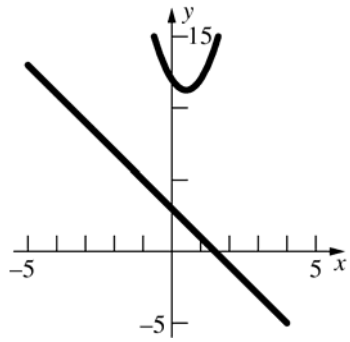
Intersection points: $\left(\frac{9-\sqrt{11}}{2}, -6 + \sqrt{11}\right)$, $\left(\frac{9+\sqrt{11}}{2}, -6 - \sqrt{11}\right)$



34. $-2x+3=3x^2-3x+12$

$3x^2-x+9=0$

No points of intersection

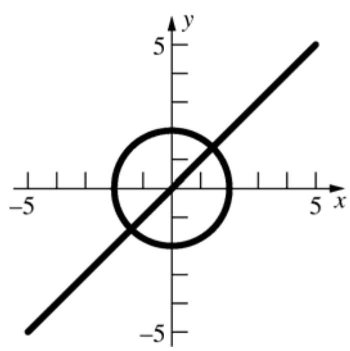


35. $x^2+x^2=4$

$x^2=2$

$x=\pm\sqrt{2}$

Intersection points: $\left(-\sqrt{2},-\sqrt{2}\right),\left(\sqrt{2},\sqrt{2}\right)$



36. $2x^2+3(x-1)^2=12$

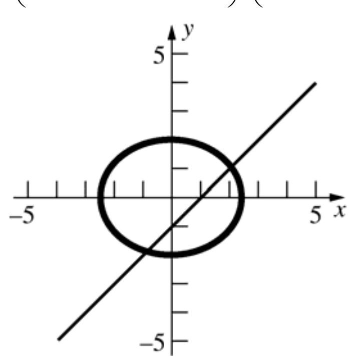
$2x^2+3x^2-6x+3=12$

$5x^2-6x-9=0$

$x=\dfrac{6\pm\sqrt{36+180}}{10}=\dfrac{6\pm6\sqrt{6}}{10}=\dfrac{3\pm3\sqrt{6}}{5}$

Intersection points:

$\left(\dfrac{3-3\sqrt{6}}{5},\dfrac{-2-3\sqrt{6}}{5}\right),\left(\dfrac{3+3\sqrt{6}}{5},\dfrac{-2+3\sqrt{6}}{5}\right)$



37. $y=3x+1$

$x^2+2x+(3x+1)^2=15$

$x^2+2x+9x^2+6x+1=15$

$10x^2+8x-14=0$

$2(5x^2+4x-7)=0$

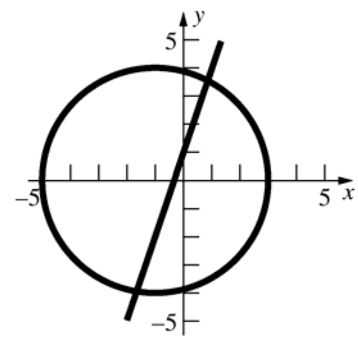
$x=\dfrac{-2\pm\sqrt{39}}{5}\approx-1.65, 0.85$

Intersection points:

$\left(\dfrac{-2-\sqrt{39}}{5},\dfrac{-1-3\sqrt{39}}{5}\right)$ and

$\left(\dfrac{-2+\sqrt{39}}{5},\dfrac{-1+3\sqrt{39}}{5}\right)$

[or roughly (–1.65, –3.95) and (0.85, 3.55)]



38. $x^2+(4x+3)^2=81$

$x^2+16x^2+24x+9=81$

$17x^2+24x-72=0$

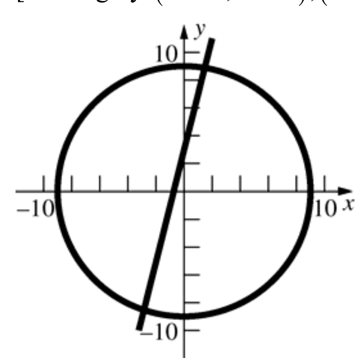
$x=\dfrac{-12\pm\sqrt{38}}{17}\approx-2.88,\ 1.47$

Intersection points:

$\left(\dfrac{-12-\sqrt{38}}{17},\dfrac{3-24\sqrt{38}}{17}\right)$ and

$\left(\dfrac{-12+\sqrt{38}}{17},\dfrac{3+24\sqrt{38}}{17}\right)$

[or roughly $\left(-2.88,-8.52\right),\left(1.47,8.88\right)$]



39. **a.** $y = x^2$; (2)

b. $ax^3 + bx^2 + cx + d$, with $a > 0$: (1)

c. $ax^3 + bx^2 + cx + d$, with $a < 0$: (3)

d. $y = ax^3$, with $a > 0$: (4)

40. $x^2 + y^2 = 13; (-2,-3), (-2,3), (2,-3), (2,3)$

$d_1 = \sqrt{(2+2)^2 + (-3+3)^2} = 4$

$d_2 = \sqrt{(2+2)^2 + (-3-3)^2} = \sqrt{52} = 2\sqrt{13}$

$d_3 = \sqrt{(2-2)^2 + (3+3)^2} = 6$

Three such distances.

41. $x^2 + 2x + y^2 - 2y = 20;\ \left(-2, 1+\sqrt{21}\right),$ $\left(-2, 1-\sqrt{21}\right), \left(2, 1+\sqrt{13}\right), \left(2, 1-\sqrt{13}\right)$

$d_1 = \sqrt{(-2-2)^2 + \left[1+\sqrt{21} - \left(1+\sqrt{13}\right)\right]^2}$

$= \sqrt{16 + \left(\sqrt{21} - \sqrt{13}\right)^2}$

$= \sqrt{50 - 2\sqrt{273}} \approx 4.12$

$d_2 = \sqrt{(-2-2)^2 + \left[1+\sqrt{21} - \left(1-\sqrt{13}\right)\right]^2}$

$= \sqrt{16 + \left(\sqrt{21} + \sqrt{13}\right)^2}$

$= \sqrt{50 + 2\sqrt{273}} \approx 9.11$

$d_3 = \sqrt{(-2+2)^2 + \left[1+\sqrt{21} - \left(1-\sqrt{21}\right)\right]^2}$

$= \sqrt{0 + \left(\sqrt{21} + \sqrt{21}\right)^2} = \sqrt{\left(2\sqrt{21}\right)^2}$

$= 2\sqrt{21} \approx 9.17$

$d_4 = \sqrt{(-2-2)^2 + \left[1-\sqrt{21} - (1+\sqrt{13})\right]^2}$

$= \sqrt{16 + \left(-\sqrt{21} - \sqrt{13}\right)^2}$

$= \sqrt{50 + 2\sqrt{273}} \approx 9.11$

$d_5 = \sqrt{(-2-2)^2 + \left[1-\sqrt{21} - \left(1-\sqrt{13}\right)\right]^2}$

$= \sqrt{16 + \left(\sqrt{13} - \sqrt{21}\right)^2}$

$= \sqrt{50 - 2\sqrt{273}} \approx 4.12$

$d_6 = \sqrt{(2-2)^2 + \left[1+\sqrt{13} - \left(1-\sqrt{13}\right)\right]^2}$

$= \sqrt{0 + \left(\sqrt{13} + \sqrt{13}\right)^2} = \sqrt{\left(2\sqrt{13}\right)^2}$

$= 2\sqrt{13} \approx 7.21$

Four such distances ($d_2 = d_4$ and $d_1 = d_5$).

0.5 Concepts Review

1. domain; range

2. $f(2u) = 3(2u)^2 = 12u^2$; $f(x+h) = 3(x+h)^2$

3. asymptote

4. even; odd; *y*-axis; origin

Problem Set 0.5

1. **a.** $f(1) = 1 - 1^2 = 0$

b. $f(-2) = 1 - (-2)^2 = -3$

c. $f(0) = 1 - 0^2 = 1$

d. $f(k) = 1 - k^2$

e. $f(-5) = 1 - (-5)^2 = -24$

f. $f\left(\frac{1}{4}\right) = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$

g. $f(1+h) = 1 - (1+h)^2 = -2h - h^2$

h. $f(1+h) - f(1) = -2h - h^2 - 0 = -2h - h^2$

i. $f(2+h) - f(2) = 1 - (2+h)^2 + 3$
$= -4h - h^2$

2. **a.** $F(1) = 1^3 + 3 \cdot 1 = 4$

b. $F(\sqrt{2}) = (\sqrt{2})^3 + 3(\sqrt{2}) = 2\sqrt{2} + 3\sqrt{2}$
$= 5\sqrt{2}$

c. $F\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right) = \frac{1}{64} + \frac{3}{4} = \frac{49}{64}$

d. $F(1+h)=(1+h)^3+3(1+h)$
$=1+3h+3h^2+h^3+3+3h$
$=4+6h+3h^2+h^3$

e. $F(1+h)-1=3+6h+3h^2+h^3$

f. $F(2+h)-F(2)$
$=(2+h)^3+3(2+h)-\left[2^3-3(2)\right]$
$=8+12h+6h^2+h^3+6+3h-14$
$=15h+6h^2+h^3$

3. a. $G(0)=\frac{1}{0-1}=-1$

b. $G(0.999)=\frac{1}{0.999-1}=-1000$

c. $G(1.01)=\frac{1}{1.01-1}=100$

d. $G(y^2)=\frac{1}{y^2-1}$

e. $G(-x)=\frac{1}{-x-1}=-\frac{1}{x+1}$

f. $G\left(\frac{1}{x^2}\right)=\frac{1}{\frac{1}{x^2}-1}=\frac{x^2}{1-x^2}$

4. a. $\Phi(1)=\frac{1+1^2}{\sqrt{1}}=2$

b. $\Phi(-t)=\frac{-t+(-t)^2}{\sqrt{-t}}=\frac{t^2-t}{\sqrt{-t}}$

c. $\Phi\left(\frac{1}{2}\right)=\frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\sqrt{\frac{1}{2}}}=\frac{\frac{3}{4}}{\sqrt{\frac{1}{2}}}\approx 1.06$

d. $\Phi(u+1)=\frac{(u+1)+(u+1)^2}{\sqrt{u+1}}=\frac{u^2+3u+2}{\sqrt{u+1}}$

e. $\Phi(x^2)=\frac{(x^2)+(x^2)^2}{\sqrt{x^2}}=\frac{x^2+x^4}{|x|}$

f. $\Phi(x^2+x)=\frac{(x^2+x)+(x^2+x)^2}{\sqrt{x^2+x}}$
$=\frac{x^4+2x^3+2x^2+x}{\sqrt{x^2+x}}$

5. a. $f(0.25)=\frac{1}{\sqrt{0.25-3}}=\frac{1}{\sqrt{-2.75}}$ is not defined

b. $f(x)=\frac{1}{\sqrt{\pi-3}}\approx 2.658$

c. $f(3+\sqrt{2})=\frac{1}{\sqrt{3+\sqrt{2}-3}}=\frac{1}{\sqrt{\sqrt{2}}}$
$=2^{-0.25}\approx 0.841$

6. a. $f(0.79)=\frac{\sqrt{(0.79)^2+9}}{0.79-\sqrt{3}}\approx -3.293$

b. $f(12.26)=\frac{\sqrt{(12.26)^2+9}}{12.26-\sqrt{3}}\approx 1.199$

c. $f(\sqrt{3})=\frac{\sqrt{(\sqrt{3})^2+9}}{\sqrt{3}-\sqrt{3}}$; undefined

7. a. $x^2+y^2=1$
$y^2=1-x^2$
$y=\pm\sqrt{1-x^2}$; not a function

b. $xy+y+x=1$
$y(x+1)=1-x$
$y=\frac{1-x}{x+1}$; $f(x)=\frac{1-x}{x+1}$

c. $x=\sqrt{2y+1}$
$x^2=2y+1$
$y=\frac{x^2-1}{2}$; $f(x)=\frac{x^2-1}{2}$

d. $x=\frac{y}{y+1}$
$xy+x=y$
$x=y-xy$
$x=y(1-x)$
$y=\frac{x}{1-x}$; $f(x)=\frac{x}{1-x}$

8. The graphs on the left are not graphs of functions, the graphs on the right are graphs of functions.

9. $$\frac{f(a+h)-f(a)}{h}=\frac{[2(a+h)^2-1]-(2a^2-1)}{h}$$
$$=\frac{4ah+2h^2}{h}=4a+2h$$

10. $$\frac{F(a+h)-F(a)}{h}=\frac{4(a+h)^3-4a^3}{h}$$
$$=\frac{4a^3+12a^2h+12ah^2+4h^3-4a^3}{h}$$
$$=\frac{12a^2h+12ah^2+4h^3}{h}$$
$$=12a^2+12ah+4h^2$$

11. $$\frac{g(x+h)-g(x)}{h}=\frac{\frac{3}{x+h-2}-\frac{3}{x-2}}{h}$$
$$=\frac{\frac{3x-6-3x-3h+6}{x^2-4x+hx-2h+4}}{h}$$
$$=\frac{-3h}{h(x^2-4x+hx-2h+4)}$$
$$=-\frac{3}{x^2-4x+hx-2h+4}$$

12. $$\frac{G(a+h)-G(a)}{h}=\frac{\frac{a+h}{a+h+4}-\frac{a}{a+4}}{h}$$
$$=\frac{\frac{a^2+4a+ah+4h-a^2-ah-4a}{a^2+8a+ah+4h+16}}{h}$$
$$=\frac{4h}{h(a^2+8a+ah+4h+16)}$$
$$=\frac{4}{a^2+8a+ah+4h+16}$$

13. a. $F(z)=\sqrt{2z+3}$

$2z+3\ \geq\ 0;\ z\geq -\frac{3}{2}$

Domain: $\left\{z\in\mathbb{R}: z\geq -\frac{3}{2}\right\}$

b. $g(v)=\frac{1}{4v-1}$

$4v-1=0;\ v=\frac{1}{4}$

Domain: $\left\{v\in\mathbb{R}: v\neq \frac{1}{4}\right\}$

c. $\psi(x)=\sqrt{x^2-9}$

$x^2-9\geq 0;\ \ x^2\geq 9;\ |x|\geq 3$

Domain: $\{x\in\mathbb{R}: |x|\geq 3\}$

d. $H(y)=-\sqrt{625-y^4}$

$625-y^4\geq 0; 625\geq y^4; |y|\leq 5$

Domain: $\{y\in\mathbb{R}: |y|\leq 5\}$

14. a. $f(x)=\frac{4-x^2}{x^2-x-6}=\frac{4-x^2}{(x-3)(x+2)}$

Domain: $\{x\in\mathbb{R}: x\neq -2, 3\}$

b. $G(y)=\sqrt{(y+1)^{-1}}$

$\frac{1}{y+1}\geq 0;\ y>-1$

Domain: $\{y\in\mathbb{R}: y>-1\}$

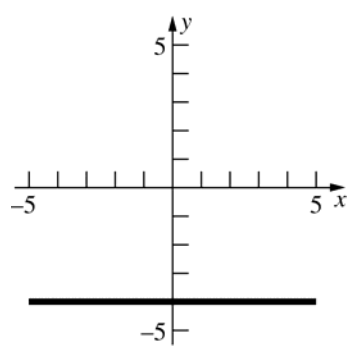
c. $\phi(u)=|2u+3|$

Domain: $\mathbb{R}$ (all real numbers)

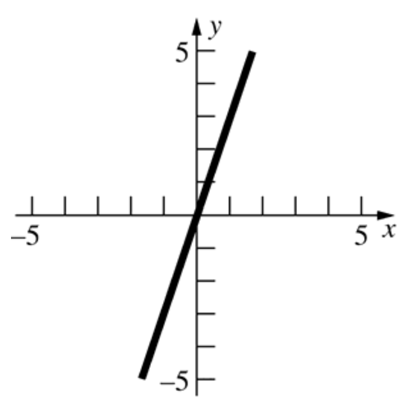
d. $F(t)=t^{2/3}-4$

Domain: $\mathbb{R}$ (all real numbers)

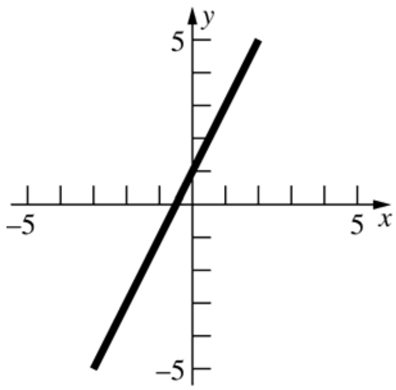
15. $f(x)=-4$; $f(-x)=-4$; even function



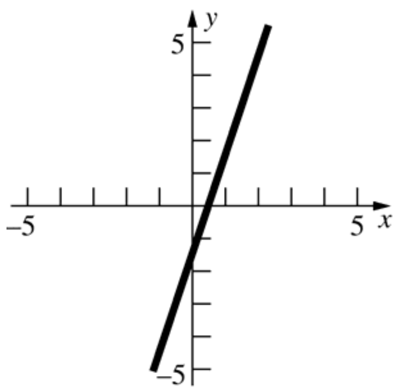
16. $f(x)=3x$; $f(-x)=-3x$; odd function



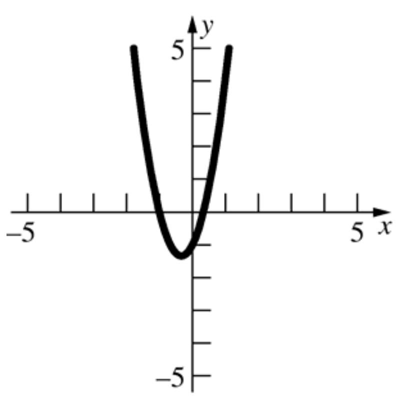
17. $F(x) = 2x + 1; F(-x) = -2x + 1$; neither



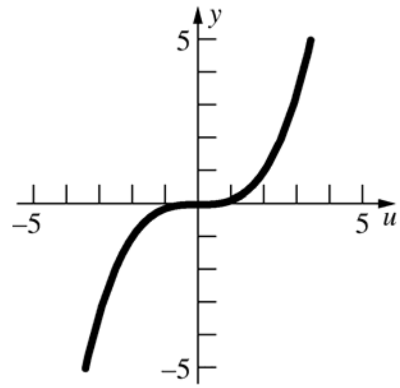
18. $F(x) = 3x - \sqrt{2}; F(-x) = -3x - \sqrt{2}$; neither



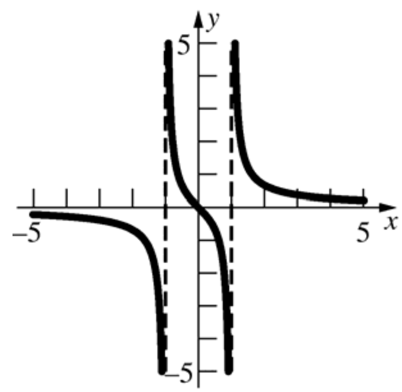
19. $g(x) = 3x^2 + 2x - 1; g(-x) = 3x^2 - 2x - 1$; neither



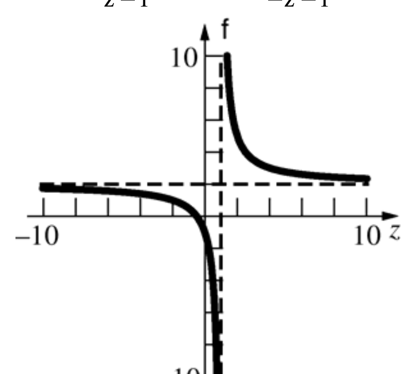
20. $g(u) = \frac{u^3}{8}; g(-u) = -\frac{u^3}{8}$; odd function



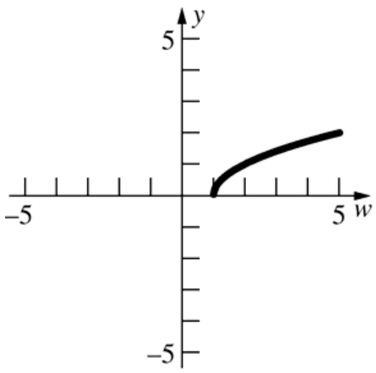
21. $g(x) = \frac{x}{x^2 - 1}; g(-x) = \frac{-x}{x^2 - 1}$; odd



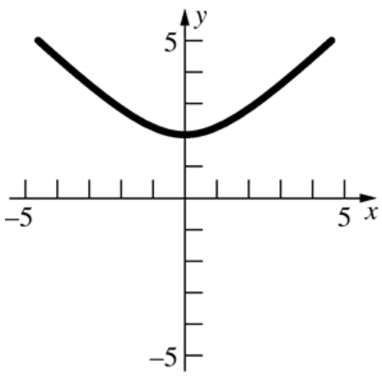
22. $\phi(z) = \frac{2z + 1}{z - 1}; \phi(-z) = \frac{-2z + 1}{-z - 1}$; neither



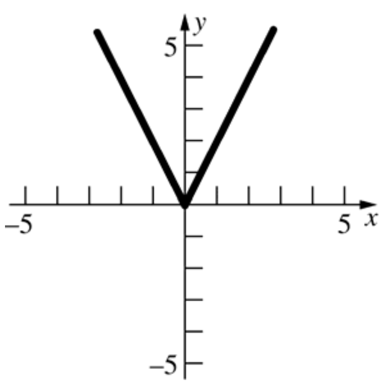
23. $f(w) = \sqrt{w-1};\ f(-w) = \sqrt{-w-1};$ neither



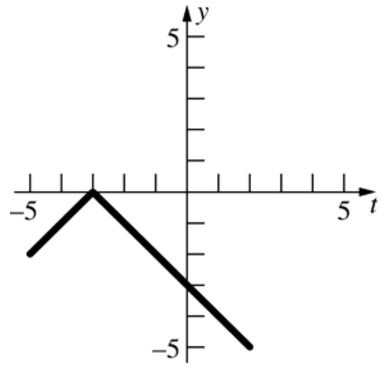
24. $h(x) = \sqrt{x^2+4};\ h(-x) = \sqrt{x^2+4};$ even function



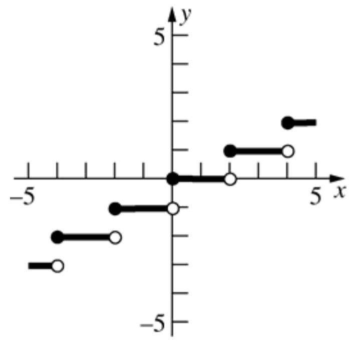
25. $f(x) = |2x|;\ f(-x) = |-2x| = |2x|;$ even function



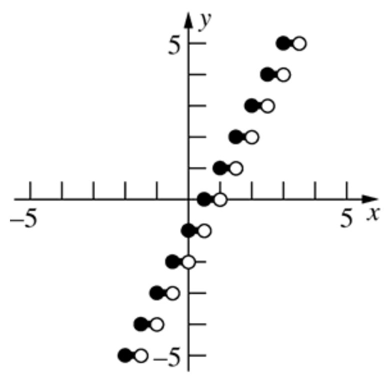
26. $F(t) = -|t+3|;\ F(-t) = -|-t+3|;$ neither



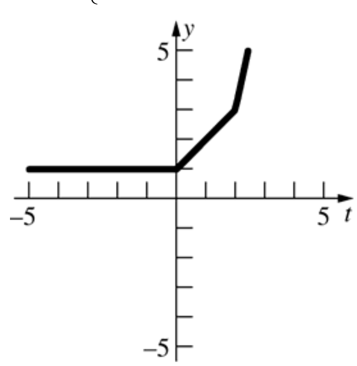
27. $g(x) = \left[\!\left[\frac{x}{2}\right]\!\right];\ g(-x) = \left[\!\left[-\frac{x}{2}\right]\!\right];$ neither



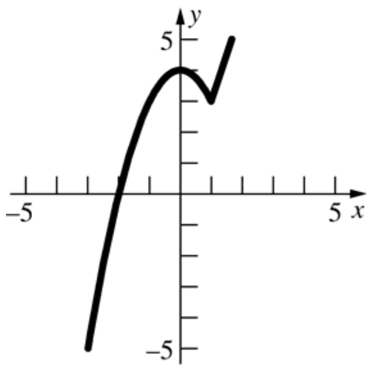
28. $G(x) = [\![2x-1]\!];\ G(-x) = [\![-2x+1]\!];$ neither



29. $g(t) = \begin{cases} 1 & \text{if } t \le 0 \\ t+1 & \text{if } 0 < t < 2 \\ t^2-1 & \text{if } t \ge 2 \end{cases}$ neither



30. $h(x)=\begin{cases}-x^2+4 & \text{if } x\le 1\\ 3x & \text{if } x>1\end{cases}$ neither



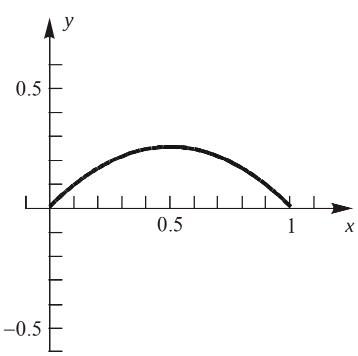
31. $T(x) = 5000 + 805x$
Domain: $\{x \in \text{integers}: 0 \le x \le 100\}$
$u(x)=\dfrac{T(x)}{x}=\dfrac{5000}{x}+805$
Domain: $\{x \in \text{integers}: 0 < x \le 100\}$

32. **a.** $P(x)=6x-(400+5\sqrt{x(x-4)})$
$=6x-400-5\sqrt{x(x-4)}$

b. $P(200)\approx -190$; $P(1000)\approx 610$

c. ABC breaks even when $P(x) = 0$;
$6x-400-5\sqrt{x(x-4)}=0;\ x\approx 390$

33. $E(x)=x-x^2$



$\dfrac{1}{2}$ exceeds its square by the maximum amount.

34. Each side has length $\dfrac{p}{3}$. The height of the triangle is $\dfrac{\sqrt{3}p}{6}$.
$A(p)=\dfrac{1}{2}\left(\dfrac{p}{3}\right)\left(\dfrac{\sqrt{3}p}{6}\right)=\dfrac{\sqrt{3}p^2}{36}$

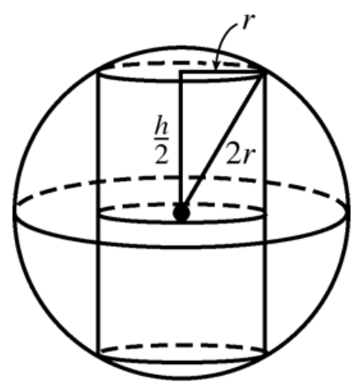
35. Let y denote the length of the other leg. Then
$$x^2+y^2=h^2$$
$$y^2=h^2-x^2$$
$$y=\sqrt{h^2-x^2}$$
$$L(x)=\sqrt{h^2-x^2}$$

36. The area is
$A(x)=\dfrac{1}{2}\text{base}\times\text{height}=\dfrac{1}{2}x\sqrt{h^2-x^2}$

37. **a.** $E(x) = 24 + 0.40x$

b. $120 = 24 + 0.40x$
$0.40x = 96$; $x = 240$ mi

38. The volume of the cylinder is $\pi r^2 h$, where h is the height of the cylinder. From the figure,
$r^2+\left(\dfrac{h}{2}\right)^2=(2r)^2;\ \dfrac{h^2}{4}=3r^2;$
$h=\sqrt{12r^2}=2r\sqrt{3}.$
$V(r)=\pi r^2(2r\sqrt{3})=2\pi r^3\sqrt{3}$



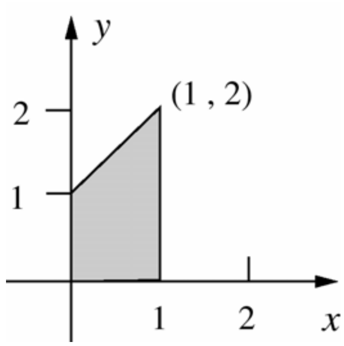
39. The area of the two semicircular ends is $\dfrac{\pi d^2}{4}$.

The length of each parallel side is $\dfrac{1-\pi d}{2}$.

$A(d)=\dfrac{\pi d^2}{4}+d\left(\dfrac{1-\pi d}{2}\right)=\dfrac{\pi d^2}{4}+\dfrac{d-\pi d^2}{2}$
$=\dfrac{2d-\pi d^2}{4}$

Since the track is one mile long, $\pi d < 1$, so $d<\dfrac{1}{\pi}$. Domain: $\left\{d\in\mathbb{R}: 0<d<\dfrac{1}{\pi}\right\}$

40. a. $A(1) = 1(1) + \frac{1}{2}(1)(2-1) = \frac{3}{2}$

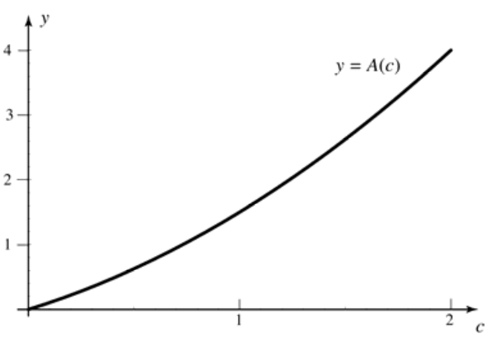


b. $A(2) = 2(1) + \frac{1}{2}(2)(3-1) = 4$

c. $A(0) = 0$

d. $A(c) = c(1) + \frac{1}{2}(c)(c+1-1) = \frac{1}{2}c^2 + c$

e.



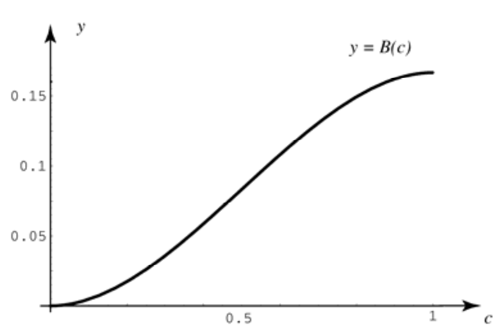
f. Domain: $\{c \in \mathbb{R} : c \geq 0\}$

Range: $\{y \in \mathbb{R} : y \geq 0\}$

41. a. $B(0) = 0$

b. $B\left(\frac{1}{2}\right) = \frac{1}{2}B(1) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

c.



42. a. $f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y)$

b. $f(x+y) = (x+y)^2 = x^2 + 2xy + y^2$
$\neq f(x) + f(y)$

c. $f(x + y) = 2(x + y) + 1 = 2x + 2y + 1$
$\neq f(x) + f(y)$

d. $f(x + y) = -3(x + y) = -3x - 3y = f(x) + f(y)$

43. For any x, $x + 0 = x$, so
$f(x) = f(x + 0) = f(x) + f(0)$, hence $f(0) = 0$.
Let m be the value of $f(1)$. For p in N,
$p = p \cdot 1 = 1 + 1 + \ldots + 1$, so
$f(p) = f(1 + 1 + \ldots + 1) = f(1) + f(1) + \ldots + f(1)$
$= pf(1) = pm$.

$1 = p\left(\frac{1}{p}\right) = \frac{1}{p} + \frac{1}{p} + \ldots + \frac{1}{p}$, so

$m = f(1) = f\left(\frac{1}{p} + \frac{1}{p} + \ldots + \frac{1}{p}\right)$

$= f\left(\frac{1}{p}\right) + f\left(\frac{1}{p}\right) + \ldots + f\left(\frac{1}{p}\right) = pf\left(\frac{1}{p}\right)$,

hence $f\left(\frac{1}{p}\right) = \frac{m}{p}$. Any rational number can be written as $\frac{p}{q}$ with p, q in N.

$\frac{p}{q} = p\left(\frac{1}{q}\right) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q}$,

so $f\left(\frac{p}{q}\right) = f\left(\frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q}\right)$

$= f\left(\frac{1}{q}\right) + f\left(\frac{1}{q}\right) + \ldots + f\left(\frac{1}{q}\right)$

$= pf\left(\frac{1}{q}\right) = p\left(\frac{m}{q}\right) = m\left(\frac{p}{q}\right)$

44. The player has run $10t$ feet after t seconds. He reaches first base when $t = 9$, second base when $t = 18$, third base when $t = 27$, and home plate when $t = 36$. The player is $10t - 90$ feet from first base when $9 \le t \le 18$, hence $\sqrt{90^2 + (10t-90)^2}$ feet from home plate. The player is $10t - 180$ feet from second base when $18 \le t \le 27$, thus he is $90 - (10t - 180) = 270 - 10t$ feet from third base and $\sqrt{90^2 + (270-10t)^2}$ feet from home plate. The player is $10t - 270$ feet from third base when $27 \le t \le 36$, thus he is $90 - (10t - 270) = 360 - 10t$ feet from home plate.

a. $$s = \begin{cases} 10t & \text{if } 0 \le t \le 9 \\ \sqrt{90^2 + (10t-90)^2} & \text{if } 9 < t \le 18 \\ \sqrt{90^2 + (270-10t)^2} & \text{if } 18 < t \le 27 \\ 360 - 10t & \text{if } 27 < t \le 36 \end{cases}$$

b. $$s = \begin{cases} 180 - |180 - 10t| & \text{if } 0 \le t \le 9 \\ & \text{or } 27 < t \le 36 \\ \sqrt{90^2 + (10t-90)^2} & \text{if } 9 < t \le 18 \\ \sqrt{90^2 + (270-10t)^2} & \text{if } 18 < t \le 27 \end{cases}$$

45. a. $f(1.38) \approx 0.2994$
$f(4.12) \approx 3.6852$

b.

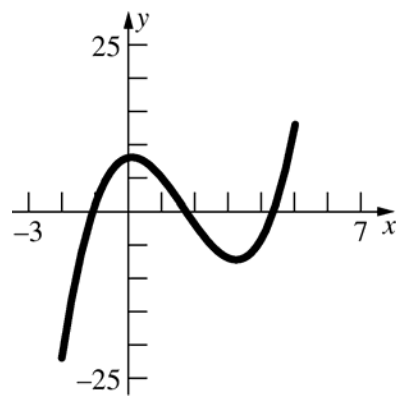
x	$f(x)$
–4	–4.05
–3	–3.1538
–2	–2.375
–1	–1.8
0	–1.25
1	–0.2
2	1.125
3	2.3846
4	3.55

46. a. $f(1.38) \approx -76.8204$
$f(4.12) \approx 6.7508$

b.

x	$f(x)$
–4	–6.1902
–3	0.4118
–2	13.7651
–1	9.9579
0	0
1	–7.3369
2	–17.7388
3	–0.4521
4	4.4378

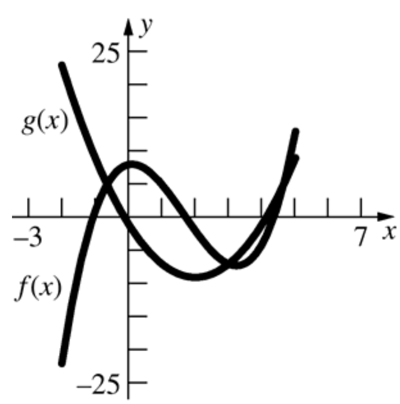
47.



a. Range: $\{y \in \mathbb{R}: -22 \le y \le 13\}$

b. $f(x) = 0$ when $x \approx -1.1, 1.7, 4.3$
$f(x) \ge 0$ on $[-1.1, 1.7] \cup [4.3, 5]$

48.

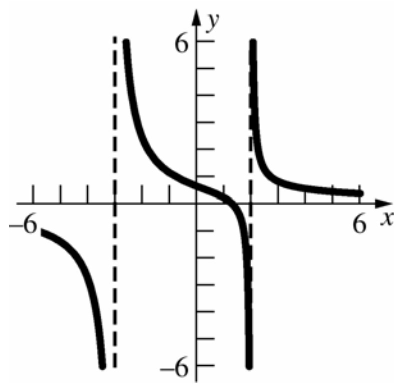


a. $f(x) = g(x)$ at $x \approx -0.6, 3.0, 4.6$

b. $f(x) \geq g(x)$ on $[-0.6, 3.0] \cup [4.6, 5]$

c. $|f(x)-g(x)|$
$= |x^3 - 5x^2 + x + 8 - 2x^2 + 8x + 1|$
$= |x^3 - 7x^2 + 9x + 9|$
Largest value $|f(-2) - g(-2)| = 45$

49.



a. x-intercept: $3x - 4 = 0$; $x = \frac{4}{3}$

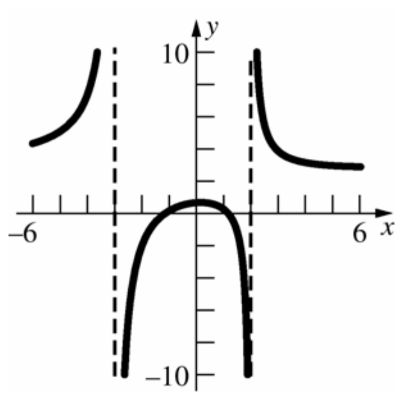
y-intercept: $\frac{3 \cdot 0 - 4}{0^2 + 0 - 6} = \frac{2}{3}$

b. $\mathbb{R}$

c. $x^2 + x - 6 = 0$; $(x + 3)(x - 2) = 0$
Vertical asymptotes at $x = -3$, $x = 2$

d. Horizontal asymptote at $y = 0$

50.



a. x-intercepts:
$3x^2 - 4 = 0$; $x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2\sqrt{3}}{3}$
y-intercept: $\frac{2}{3}$

b. On $[-6,-3)$, g increases from $g(-6) = \frac{13}{3} \approx 4.3333$ to ∞. On $(2,6]$, g decreased from ∞ to $\frac{26}{9} \approx 2.8889$. On $(-3,2)$ the maximum occurs around $x = 0.1451$ with value 0.6748. Thus, the range is $(-\infty, 0.6748] \cup [2.8889, \infty)$.

c. $x^2 + x - 6 = 0$; $(x + 3)(x - 2) = 0$
Vertical asymptotes at $x = -3$, $x = 2$

d. Horizontal asymptote at $y = 3$

0.6 Concepts Review

1. $(x^2 + 1)^3$

2. $f(g(x))$

3. 2; left

4. a quotient of two polynomial functions

Problem Set 0.6

1. a. $(f + g)(2) = (2 + 3) + 2^2 = 9$

b. $(f \cdot g)(0) = (0 + 3)(0^2) = 0$

c. $(g/f)(3) = \frac{3^2}{3+3} = \frac{9}{6} = \frac{3}{2}$

d. $(f \circ g)(1) = f(1^2) = 1 + 3 = 4$

e. $(g \circ f)(1) = g(1+3) = 4^2 = 16$

f. $(g \circ f)(-8) = g(-8+3) = (-5)^2 = 25$

2. a. $(f - g)(2) = (2^2 + 2) - \frac{2}{2+3} = 6 - \frac{2}{5} = \frac{28}{5}$

b. $(f/g)(1) = \frac{1^2 + 1}{\frac{2}{1+3}} = \frac{2}{\frac{2}{4}} = 4$

c. $g^2(3) = \left[\frac{2}{3+3}\right]^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

d. $(f\circ g)(1)=f\left(\frac{2}{1+3}\right)=\left(\frac{1}{2}\right)^2+\frac{1}{2}=\frac{3}{4}$

e. $(g\circ f)(1)=g(1^2+1)=\frac{2}{2+3}=\frac{2}{5}$

f. $(g\circ g)(3)=g\left(\frac{2}{3+3}\right)=\frac{2}{\frac{1}{3}+3}=\frac{2}{\frac{10}{3}}=\frac{3}{5}$

3. a. $(\Phi+\Psi)(t)=t^3+1+\frac{1}{t}$

b. $(\Phi\circ\Psi)(r)=\Phi\left(\frac{1}{r}\right)=\left(\frac{1}{r}\right)^3+1=\frac{1}{r^3}+1$

c. $(\Psi\circ\Phi)(r)=\Psi(r^3+1)=\frac{1}{r^3+1}$

d. $\Phi^3(z)=(z^3+1)^3$

e. $(\Phi-\Psi)(5t)=[(5t)^3+1]-\frac{1}{5t}$

$=125t^3+1-\frac{1}{5t}$

f. $((\Phi-\Psi)\circ\Psi)(t)=(\Phi-\Psi)\left(\frac{1}{t}\right)$

$=\left(\frac{1}{t}\right)^3+1-\frac{1}{\frac{1}{t}}=\frac{1}{t^3}+1-t$

4. a. $(f\cdot g)(x)=\frac{2\sqrt{x^2-1}}{x}$

Domain: $(-\infty,\ -1]\cup[1,\ \infty)$

b. $f^4(x)+g^4(x)=\left(\sqrt{x^2-1}\right)^4+\left(\frac{2}{x}\right)^4$

$=(x^2-1)^2+\frac{16}{x^4}$

Domain: $(-\infty,\ 0)\cup(0,\ \infty)$

c. $(f\circ g)(x)=f\left(\frac{2}{x}\right)=\sqrt{\left(\frac{2}{x}\right)^2-1}=\sqrt{\frac{4}{x^2}-1}$

Domain: $[-2, 0)\cup(0, 2]$

d. $(g\circ f)(x)=g\left(\sqrt{x^2-1}\right)=\frac{2}{\sqrt{x^2-1}}$

Domain: $(-\infty, -1)\cup(1,\ \infty)$

5. $(f\circ g)(x)=f\left(|1+x|\right)=\sqrt{|1+x|^2-4}$

$=\sqrt{x^2+2x-3}$

$(g\circ f)(x)=g\left(\sqrt{x^2-4}\right)=\left|1+\sqrt{x^2-4}\right|$

$=1+\sqrt{x^2-4}$

6. $g^3(x)=(x^2+1)^3=(x^4+2x^2+1)(x^2+1)$

$=x^6+3x^4+3x^2+1$

$(g\circ g\circ g)(x)=(g\circ g)(x^2+1)$

$=g[(x^2+1)^2+1]=g(x^4+2x^2+2)$

$=(x^4+2x^2+2)^2+1$

$=x^8+4x^6+8x^4+8x^2+5$

7. $g(3.141)\approx 1.188$

8. $g(2.03)\approx 0.000205$

9. $\left[g^2(\pi)-g(\pi)\right]^{1/3}=\left[(11-7\pi)^2-|11-7\pi|\right]^{1/3}$

≈ 4.789

10. $[g^3(\pi)-g(\pi)]^{1/3}=[(6\pi-11)^3-(6\pi-11)]^{1/3}$

≈ 7.807

11. a. $g(x)=\sqrt{x},\ f(x)=x+7$

b. $g(x)=x^{15},\quad f(x)=x^2+x$

12. a. $f(x)=\frac{2}{x^3},\ g(x)=x^2+x+1$

b. $f(x)=\frac{1}{x},\ g(x)=x^3+3x$

13. $p=f\circ g\circ h$ if $f(x)=1/x$, $g(x)=\sqrt{x}$,

$h(x)=x^2+1$

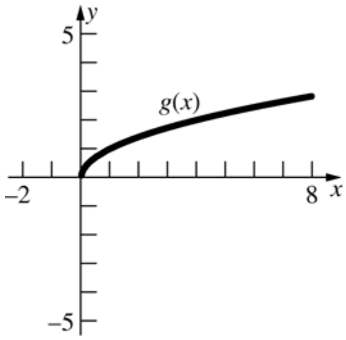
$p=f\circ g\circ h$ if $f(x)=1/\sqrt{x}$, $g(x)=x+1$,

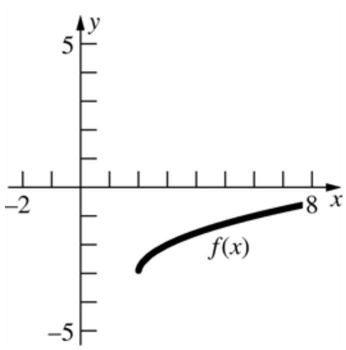
$h(x)=x^2$

14. $p=f\circ g\circ h\circ l$ if $f(x)=1/x$, $g(x)=\sqrt{x}$,

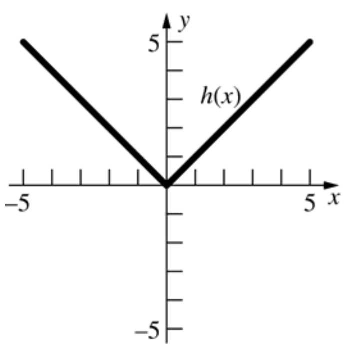
$h(x)=x+1$, $l(x)=x^2$

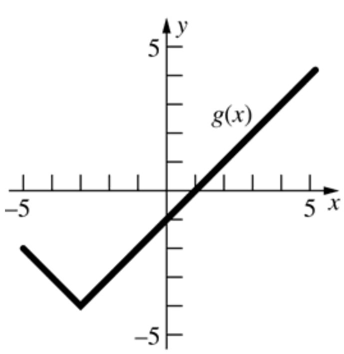
15. Translate the graph of $g(x) = \sqrt{x}$ to the right 2 units and down 3 units.



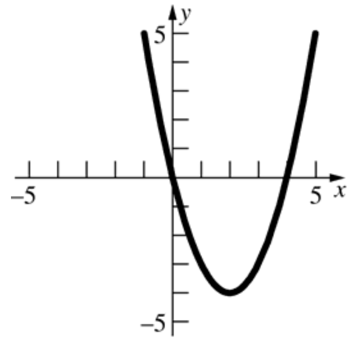


16. Translate the graph of $h(x) = |x|$ to the left 3 units and down 4 units.

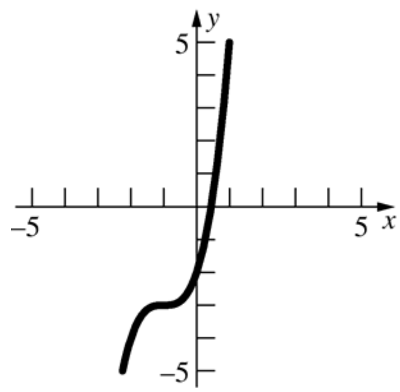




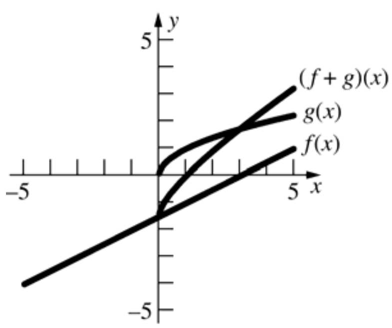
17. Translate the graph of $y = x^2$ to the right 2 units and down 4 units.



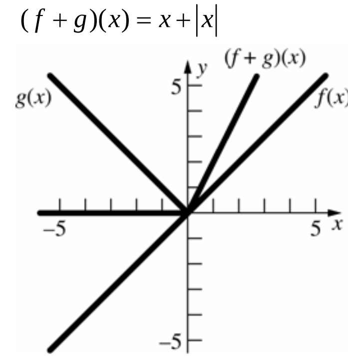
18. Translate the graph of $y = x^3$ to the left 1 unit and down 3 units.



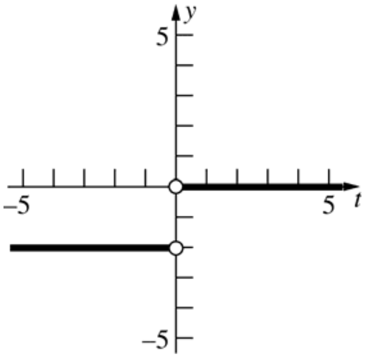
19. $(f+g)(x) = \dfrac{x-3}{2} + \sqrt{x}$



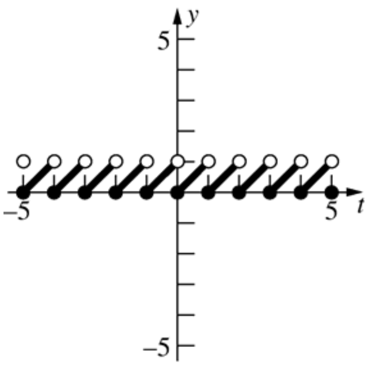
20. $(f+g)(x) = x + |x|$



21. $F(t) = \frac{|t| - t}{t}$



22. $G(t) = t - [\![t]\!]$



23. **a.** Even;
$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x)$
$= (f+g)(x)$ if f and g are both even functions.

b. Odd;
$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x)$
$= -(f+g)(x)$ if f and g are both odd functions.

c. Even;
$(f \cdot g)(-x) = [f(-x)][g(-x)]$
$= [f(x)][g(x)] = (f \cdot g)(x)$
if f and g are both even functions.

d. Even;
$(f \cdot g)(-x) = [f(-x)][g(-x)]$
$= [-f(x)][-g(x)] = [f(x)][g(x)]$
$= (f \cdot g)(x)$
if f and g are both odd functions.

e. Odd;
$(f \cdot g)(-x) = [f(-x)][g(-x)]$
$= [f(x)][-g(x)] = -[f(x)][g(x)]$
$= -(f \cdot g)(x)$
if f is an even function and g is an odd function.

24. **a.** $F(x) - F(-x)$ is odd because
$F(-x) - F(x) = -[F(x) - F(-x)]$

b. $F(x) + F(-x)$ is even because
$F(-x) + F(-(-x)) = F(-x) + F(x)$
$= F(x) + F(-x)$

c. $\frac{F(x) - F(-x)}{2}$ is odd and $\frac{F(x) + F(-x)}{2}$ is even.

$$\frac{F(x) - F(-x)}{2} + \frac{F(x) + F(-x)}{2} = \frac{2F(x)}{2} = F(x)$$

25. Not every polynomial of even degree is an even function. For example $f(x) = x^2 + x$ is neither even nor odd. Not every polynomial of odd degree is an odd function. For example $g(x) = x^3 + x^2$ is neither even nor odd.

26. **a.** Neither

b. PF

c. RF

d. PF

e. RF

f. Neither

27. **a.** $P = \sqrt{29 - 3(2 + \sqrt{t}) + (2 + \sqrt{t})^2}$
$= \sqrt{t + \sqrt{t} + 27}$

b. When $t = 15$, $P = \sqrt{15 + \sqrt{15} + 27} \approx 6.773$

28. $R(t) = (120 + 2t + 3t^2)(6000 + 700t)$
$= 2100t^3 + 19{,}400t^2 + 96{,}000t + 720{,}000$

29. $$D(t) = \begin{cases} 400t & \text{if } 0 < t < 1 \\ \sqrt{(400t)^2 + [300(t-1)]^2} & \text{if } t \ge 1 \end{cases}$$

$$D(t) = \begin{cases} 400t & \text{if } 0 < t < 1 \\ \sqrt{250{,}000t^2 - 180{,}000t + 90{,}000} & \text{if } t \ge 1 \end{cases}$$

30. $D(2.5) \approx 1097$ mi

31. $f(f(x)) = f\left(\frac{ax+b}{cx-a}\right) = \frac{a\left(\frac{ax+b}{cx-a}\right)+b}{c\left(\frac{ax+b}{cx-a}\right)-a}$

$= \frac{a^2x+ab+bcx-ab}{acx+bc-acx+a^2} = \frac{x(a^2+bc)}{a^2+bc} = x$

If $a^2+bc=0$, $f(f(x))$ is undefined, while if $x = \frac{a}{c}$, $f(x)$ is undefined.

32. $f(f(f(x))) = f\left(f\left(\frac{x-3}{x+1}\right)\right) = f\left(\frac{\frac{x-3}{x+1}-3}{\frac{x-3}{x+1}+1}\right)$

$= f\left(\frac{x-3-3x-3}{x-3+x+1}\right) = f\left(\frac{-2x-6}{2x-2}\right) = f\left(\frac{-x-3}{x-1}\right)$

$= \frac{\frac{-x-3}{x-1}-3}{\frac{-x-3}{x-1}+1} = \frac{-x-3-3x+3}{-x-3+x-1} = \frac{-4x}{-4} = x$

If $x = -1$, $f(x)$ is undefined, while if $x = 1$, $f(f(x))$ is undefined.

33. a. $f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{\frac{1}{x}-1} = \frac{1}{1-x}$

b. $f(f(x)) = f\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}$

$= \frac{x}{x-x+1} = x$

c. $f\left(\frac{1}{f(x)}\right) = f\left(\frac{x-1}{x}\right) = \frac{\frac{x-1}{x}}{\frac{x-1}{x}-1} = \frac{x-1}{x-1-x}$

$= 1-x$

34. a. $f(1/x) = \frac{1/x}{\sqrt{1/x}-1} = \frac{1}{\sqrt{x}-x}$

b. $f(f(x)) = f(x/(\sqrt{x}-1)) = \frac{x/(\sqrt{x}-1)}{\sqrt{\frac{x}{\sqrt{x}-1}}-1}$

$= \frac{x}{\sqrt{x(\sqrt{x}-1)}+1-\sqrt{x}}$

35. $(f_1 \circ (f_2 \circ f_3))(x) = f_1((f_2 \circ f_3)(x))$
$= f_1(f_2(f_3(x)))$

$((f_1 \circ f_2) \circ f_3)(x) = (f_1 \circ f_2)(f_3(x))$
$= f_1(f_2(f_3(x)))$
$= (f_1 \circ (f_2 \circ f_3))(x)$

36. $f_1(f_1(x)) = x;$

$f_1(f_2(x)) = \frac{1}{x};$

$f_1(f_3(x)) = 1-x;$

$f_1(f_4(x)) = \frac{1}{1-x};$

$f_1(f_5(x)) = \frac{x-1}{x};$

$f_1(f_6(x)) = \frac{x}{x-1};$

$f_2(f_1(x)) = \frac{1}{x};$

$f_2(f_2(x)) = \frac{1}{\frac{1}{x}} = x;$

$f_2(f_3(x)) = \frac{1}{1-x};$

$f_2(f_4(x)) = \frac{1}{\frac{1}{1-x}} = 1-x;$

$f_2(f_5(x)) = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1};$

$f_2(f_6(x)) = \frac{1}{\frac{x}{x-1}} = \frac{x-1}{x};$

$f_3(f_1(x)) = 1-x;$

$f_3(f_2(x)) = 1-\frac{1}{x} = \frac{x-1}{x};$

$f_3(f_3(x)) = 1-(1-x) = x;$

$f_3(f_4(x)) = 1-\frac{1}{1-x} = \frac{x}{x-1};$

$f_3(f_5(x)) = 1-\frac{x-1}{x} = \frac{1}{x};$

$f_3(f_6(x)) = 1-\frac{x}{x-1} = \frac{1}{1-x};$

$f_4(f_1(x)) = \frac{1}{1-x};$

$f_4(f_2(x)) = \frac{1}{1-\frac{1}{x}} = \frac{x}{x-1};$

$f_4(f_3(x)) = \frac{1}{1-(1-x)} = \frac{1}{x};$

$f_4(f_4(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x};$

$f_4(f_5(x)) = \frac{1}{1-\frac{x-1}{x}} = \frac{x}{x-(x-1)} = x;$

$f_4(f_6(x)) = \frac{1}{1-\frac{x}{x-1}} = \frac{x-1}{x-1-x} = 1-x;$

$$f_5(f_1(x)) = \frac{x-1}{x};$$

$$f_5(f_2(x)) = \frac{\frac{1}{x}-1}{\frac{1}{x}} = 1-x;$$

$$f_5(f_3(x)) = \frac{1-x-1}{1-x} = \frac{x}{x-1};$$

$$f_5(f_4(x)) = \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}} = \frac{1-(1-x)}{1} = x;$$

$$f_5(f_5(x)) = \frac{\frac{x-1}{x}-1}{\frac{x-1}{x}} = \frac{x-1-x}{x-1} = \frac{1}{1-x};$$

$$f_5(f_6(x)) = \frac{\frac{x}{x-1}-1}{\frac{x}{x-1}} = \frac{x-(x-1)}{x} = \frac{1}{x};$$

$$f_6(f_1(x)) = \frac{x}{x-1};$$

$$f_6(f_2(x)) = \frac{\frac{1}{x}}{\frac{1}{x}-1} = \frac{1}{1-x};$$

$$f_6(f_3(x)) = \frac{1-x}{1-x-1} = \frac{x-1}{x};$$

$$f_6(f_4(x)) = \frac{\frac{1}{1-x}}{\frac{1}{1-x}-1} = \frac{1}{1-(1-x)} = \frac{1}{x};$$

$$f_6(f_5(x)) = \frac{\frac{x-1}{x}}{\frac{x-1}{x}-1} = \frac{x-1}{x-1-x} = 1-x;$$

$$f_6(f_6(x)) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-(x-1)} = x$$

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_5	f_1	f_6	f_2	f_4
f_4	f_4	f_6	f_2	f_5	f_1	f_3
f_5	f_5	f_3	f_6	f_1	f_4	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

a. $f_3 \circ f_3 \circ f_3 \circ f_3 \circ f_3$
$= ((((f_3 \circ f_3) \circ f_3) \circ f_3) \circ f_3)$
$= (((f_1 \circ f_3) \circ f_3) \circ f_3)$
$= ((f_3 \circ f_3) \circ f_3)$
$= f_1 \circ f_3 = f_3$

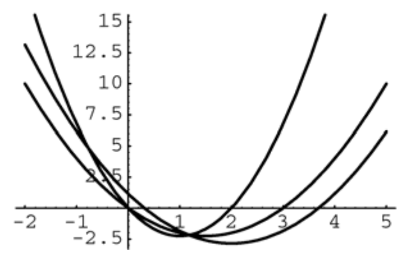
b. $f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 \circ f_6$
$= (((((f_1 \circ f_2) \circ f_3) \circ f_4) \circ f_5) \circ f_6)$
$= ((((f_2 \circ f_3) \circ f_4) \circ f_5) \circ f_6)$
$= (f_4 \circ f_4) \circ (f_5 \circ f_6)$
$= f_5 \circ f_2 = f_3$

c. If $F \circ f_6 = f_1$, then $F = f_6$.

d. If $G \circ f_3 \circ f_6 = f_1$, then $G \circ f_4 = f_1$ so $G = f_5$.

e. If $f_2 \circ f_5 \circ H = f_5$, then $f_6 \circ H = f_5$ so $H = f_3$.

37.



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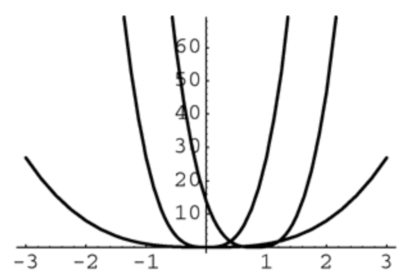

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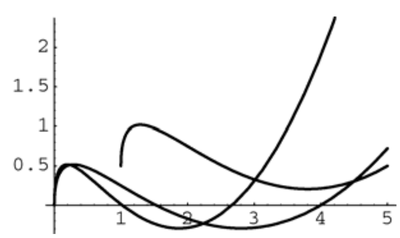

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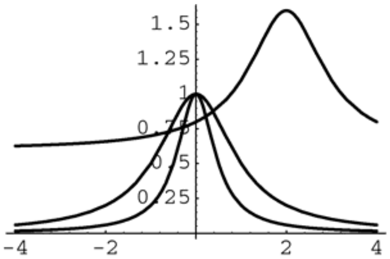
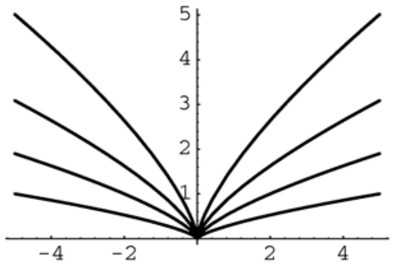
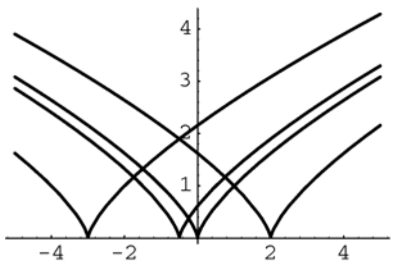


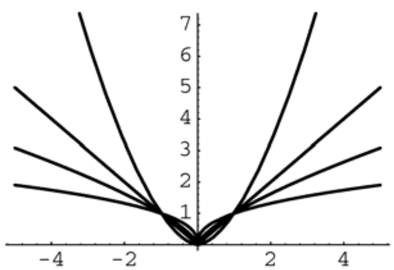
41. a.



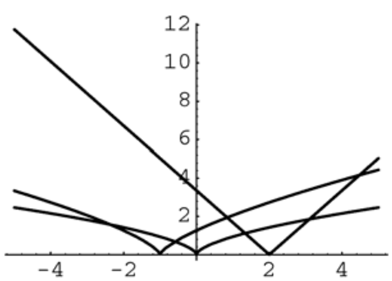
b.



c.



42.



0.7 Concepts Review

1. $(-\infty, \infty)$; $[-1, 1]$

2. 2π; 2π; π

3. odd; even

4. $r = \sqrt{(-4)^2 + 3^2} = 5; \cos\theta = \dfrac{x}{r} = -\dfrac{4}{5}$

Problem Set 0.7

1. a. $30\left(\dfrac{\pi}{180}\right) = \dfrac{\pi}{6}$

b. $45\left(\dfrac{\pi}{180}\right) = \dfrac{\pi}{4}$

c. $-60\left(\dfrac{\pi}{180}\right) = -\dfrac{\pi}{3}$

d. $240\left(\dfrac{\pi}{180}\right) = \dfrac{4\pi}{3}$

e. $-370\left(\dfrac{\pi}{180}\right) = -\dfrac{37\pi}{18}$

f. $10\left(\dfrac{\pi}{180}\right) = \dfrac{\pi}{18}$

2. a. $\dfrac{7}{6}\pi\left(\dfrac{180}{\pi}\right) = 210°$

b. $\dfrac{3}{4}\pi\left(\dfrac{180}{\pi}\right) = 135°$

c. $-\dfrac{1}{3}\pi\left(\dfrac{180}{\pi}\right) = -60°$

d. $\dfrac{4}{3}\pi\left(\dfrac{180}{\pi}\right) = 240°$

e. $-\dfrac{35}{18}\pi\left(\dfrac{180}{\pi}\right) = -350°$

f. $\dfrac{3}{18}\pi\left(\dfrac{180}{\pi}\right) = 30°$

3. a. $33.3\left(\dfrac{\pi}{180}\right) \approx 0.5812$

b. $46\left(\dfrac{\pi}{180}\right) \approx 0.8029$

c. $-66.6\left(\dfrac{\pi}{180}\right) \approx -1.1624$

d. $240.11\left(\dfrac{\pi}{180}\right) \approx 4.1907$

e. $-369\left(\dfrac{\pi}{180}\right) \approx -6.4403$

f. $11\left(\dfrac{\pi}{180}\right) \approx 0.1920$

4. a. $3.141\left(\frac{180}{\pi}\right) \approx 180°$

b. $6.28\left(\frac{180}{\pi}\right) \approx 359.8°$

c. $5.00\left(\frac{180}{\pi}\right) \approx 286.5°$

d. $0.001\left(\frac{180}{\pi}\right) \approx 0.057°$

e. $-0.1\left(\frac{180}{\pi}\right) \approx -5.73°$

f. $36.0\left(\frac{180}{\pi}\right) \approx 2062.6°$

5. a. $\frac{56.4 \tan 34.2°}{\sin 34.1°} \approx 68.37$

b. $\frac{5.34 \tan 21.3°}{\sin 3.1° + \cot 23.5°} \approx 0.8845$

c. $\tan(0.452) \approx 0.4855$

d. $\sin(-0.361) \approx -0.3532$

6. a. $\frac{234.1 \sin(1.56)}{\cos(0.34)} \approx 248.3$

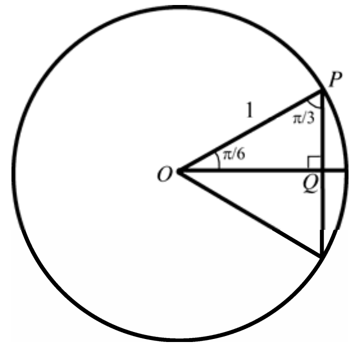
b. $\sin^2(2.51) + \sqrt{\cos(0.51)} \approx 1.2828$

7. a. $\frac{56.3 \tan 34.2°}{\sin 56.1°} \approx 46.097$

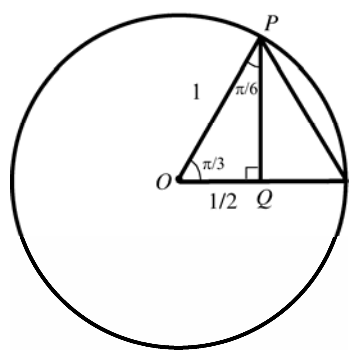
b. $\left(\frac{\sin 35°}{\sin 26° + \cos 26°}\right)^3 \approx 0.0789$

8. Referring to Figure 2, it is clear that $\sin 0 = 0$ and $\cos 0 = 1$. If the angle is $\pi/6$, then the triangle in the figure below is equilateral. Thus, $|PQ| = \frac{1}{2}|OP| = \frac{1}{2}$. This implies that $\sin\frac{\pi}{6} = \frac{1}{2}$. By the Pythagorean Identity, $\cos^2\frac{\pi}{6} = 1 - \sin^2\frac{\pi}{6} = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.

Thus



$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. The results $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ were derived in the text.
If the angle is $\pi/3$ then the triangle in the figure below is equilateral. Thus $\cos\frac{\pi}{3} = \frac{1}{2}$ and by the Pythagorean Identity, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.



Referring to Figure 2, it is clear that $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$. The rest of the values are obtained using the same kind of reasoning in the second quadrant.

9. a. $\tan\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{\sqrt{3}}{3}$

b. $\sec(\pi) = \frac{1}{\cos(\pi)} = -1$

c. $\sec\left(\frac{3\pi}{4}\right) = \frac{1}{\cos\left(\frac{3\pi}{4}\right)} = -\sqrt{2}$

d. $\csc\left(\frac{\pi}{2}\right) = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$

e. $\cot\left(\frac{\pi}{4}\right) = \frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = 1$

f. $\tan\left(-\frac{\pi}{4}\right) = \frac{\sin\left(-\frac{\pi}{4}\right)}{\cos\left(-\frac{\pi}{4}\right)} = -1$

10. a. $\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \sqrt{3}$

b. $\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2$

c. $\cot\left(\frac{\pi}{3}\right) = \frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{3}$

d. $\csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$

e. $\tan\left(-\frac{\pi}{6}\right) = \frac{\sin\left(-\frac{\pi}{6}\right)}{\cos\left(-\frac{\pi}{6}\right)} = -\frac{\sqrt{3}}{3}$

f. $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

11. a. $(1+\sin z)(1-\sin z) = 1-\sin^2 z$

$= \cos^2 z = \frac{1}{\sec^2 z}$

b. $(\sec t - 1)(\sec t + 1) = \sec^2 t - 1 = \tan^2 t$

c. $\sec t - \sin t \tan t = \frac{1}{\cos t} - \frac{\sin^2 t}{\cos t}$

$= \frac{1-\sin^2 t}{\cos t} = \frac{\cos^2 t}{\cos t} = \cos t$

d. $\frac{\sec^2 t - 1}{\sec^2 t} = \frac{\tan^2 t}{\sec^2 t} = \frac{\frac{\sin^2 t}{\cos^2 t}}{\frac{1}{\cos^2 t}} = \sin^2 t$

12. a. $\sin^2 v + \frac{1}{\sec^2 v} = \sin^2 v + \cos^2 v = 1$

b. $\cos 3t = \cos(2t + t) = \cos 2t \cos t - \sin 2t \sin t$

$= (2\cos^2 t - 1)\cos t - 2\sin^2 t \cos t$

$= 2\cos^3 t - \cos t - 2(1-\cos^2 t)\cos t$

$= 2\cos^3 t - \cos t - 2\cos t + 2\cos^3 t$

$= 4\cos^3 t - 3\cos t$

c. $\sin 4x = \sin[2(2x)] = 2\sin 2x \cos 2x$

$= 2(2\sin x \cos x)(2\cos^2 x - 1)$

$= 2(4\sin x \cos^3 x - 2\sin x \cos x)$

$= 8\sin x \cos^3 x - 4\sin x \cos x$

d. $(1+\cos\theta)(1-\cos\theta) = 1-\cos^2\theta = \sin^2\theta$

13. a. $\frac{\sin u}{\csc u} + \frac{\cos u}{\sec u} = \sin^2 u + \cos^2 u = 1$

b. $(1-\cos^2 x)(1+\cot^2 x) = (\sin^2 x)(\csc^2 x)$

$= \sin^2 x\left(\frac{1}{\sin^2 x}\right) = 1$

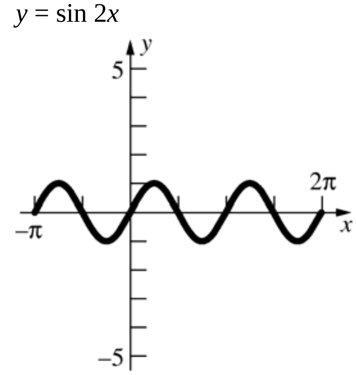
c. $\sin t(\csc t - \sin t) = \sin t\left(\frac{1}{\sin t} - \sin t\right)$

$= 1 - \sin^2 t = \cos^2 t$

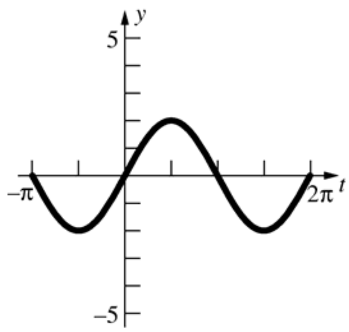
d. $\frac{1-\csc^2 t}{\csc^2 t} = -\frac{\cot^2 t}{\csc^2 t} = -\frac{\frac{\cos^2 t}{\sin^2 t}}{\frac{1}{\sin^2 t}}$

$= -\cos^2 t = -\frac{1}{\sec^2 t}$

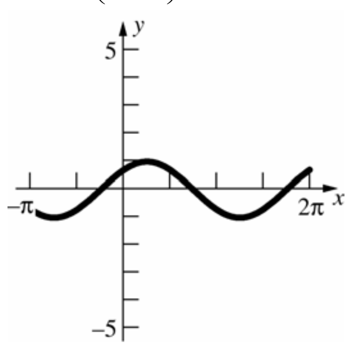
14. a. $y = \sin 2x$



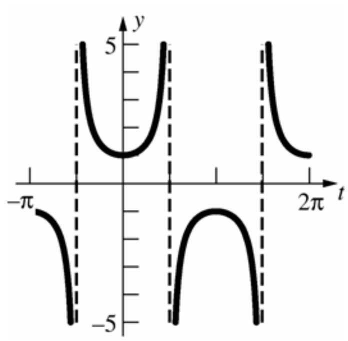
b. $y = 2\sin t$



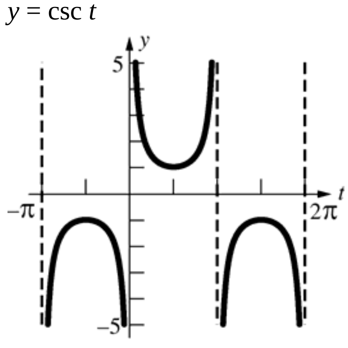
c. $y = \cos\left(x - \dfrac{\pi}{4}\right)$



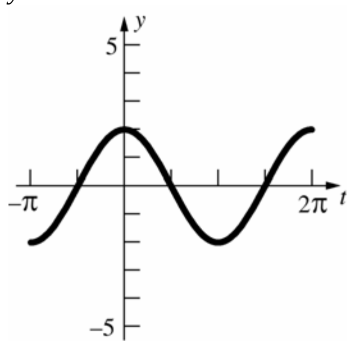
d. $y = \sec t$



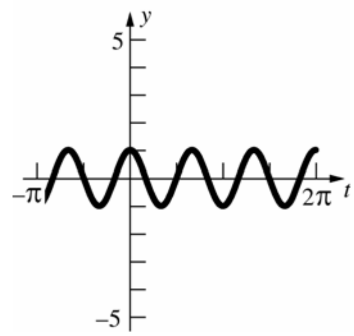
15. a. $y = \csc t$



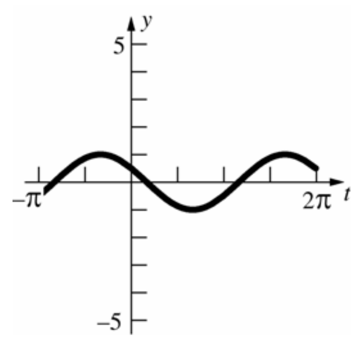
b. $y = 2 \cos t$



c. $y = \cos 3t$

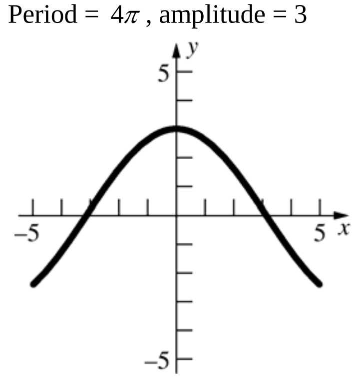


d. $y = \cos\left(t + \dfrac{\pi}{3}\right)$

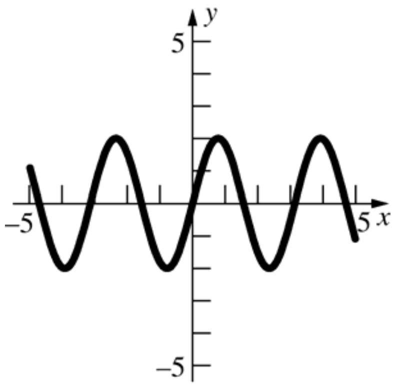


16. $y = 3\cos\dfrac{x}{2}$

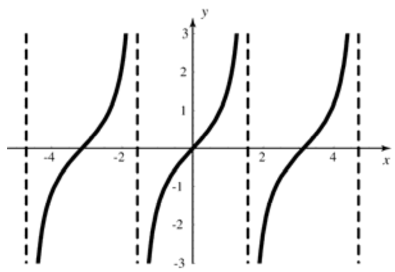
Period = 4π , amplitude = 3



17. $y = 2\sin 2x$
Period = π, amplitude = 2

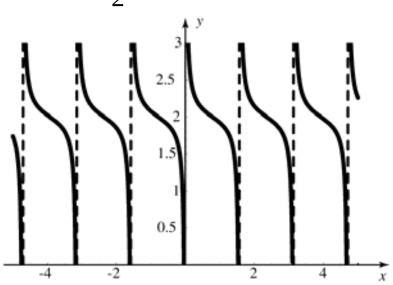


18. $y = \tan x$
Period = π

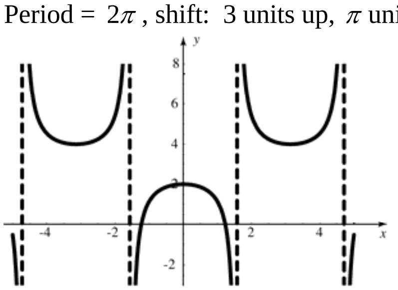


19. $y = 2 + \frac{1}{6}\cot(2x)$

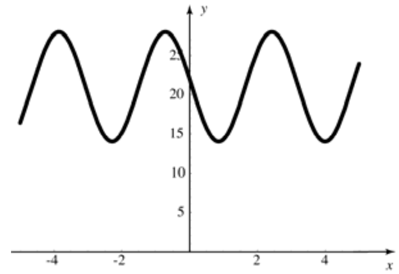
Period = $\frac{\pi}{2}$, shift: 2 units up



20. $y = 3 + \sec(x - \pi)$
Period = 2π, shift: 3 units up, π units right

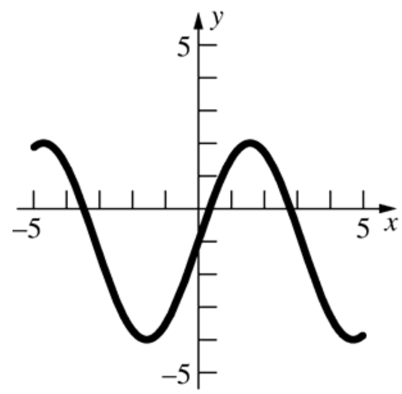


21. $y = 21 + 7\sin(2x + 3)$
Period = π, amplitude = 7, shift: 21 units up, $\frac{3}{2}$ units left



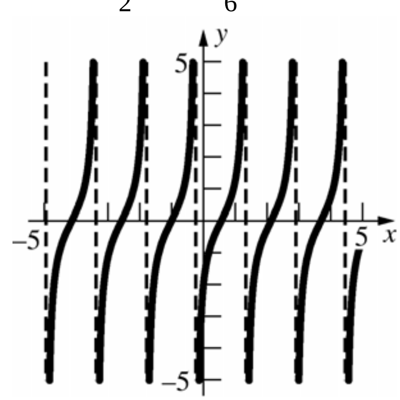
22. $y = 3\cos\left(x - \frac{\pi}{2}\right) - 1$

Period = 2π, amplitude = 3, shifts: $\frac{\pi}{2}$ units right and 1 unit down.



23. $y = \tan\left(2x - \frac{\pi}{3}\right)$

Period = $\frac{\pi}{2}$, shift: $\frac{\pi}{6}$ units right



24. **a.** and **g.**: $y = \sin\left(x + \frac{\pi}{2}\right) = \cos x = -\cos(\pi - x)$

b. and **e.**: $y = \cos\left(x + \frac{\pi}{2}\right) = \sin(x + \pi)$

$= -\sin(\pi - x)$

c. and **f.**: $y = \cos\left(x - \frac{\pi}{2}\right) = \sin x$

$= -\sin(x + \pi)$

d. and **h.**: $y = \sin\left(x - \frac{\pi}{2}\right) = \cos(x + \pi)$

$= \cos(x - \pi)$

25. **a.** $-t \sin(-t) = t \sin t$; even

b. $\sin^2(-t) = \sin^2 t$; even

c. $\csc(-t) = \frac{1}{\sin(-t)} = -\csc t$; odd

d. $|\sin(-t)| = |-\sin t| = |\sin t|$; even

e. $\sin(\cos(-t)) = \sin(\cos t)$; even

f. $-x + \sin(-x) = -x - \sin x = -(x + \sin x)$; odd

26. **a.** $\cot(-t) + \sin(-t) = -\cot t - \sin t$
$= -(\cot t + \sin t)$; odd

b. $\sin^3(-t) = -\sin^3 t$; odd

c. $\sec(-t) = \frac{1}{\cos(-t)} = \sec t$; even

d. $\sqrt{\sin^4(-t)} = \sqrt{\sin^4 t}$; even

e. $\cos(\sin(-t)) = \cos(-\sin t) = \cos(\sin t)$; even

f. $(-x)^2 + \sin(-x) = x^2 - \sin x$; neither

27. $\cos^2 \frac{\pi}{3} = \left(\cos\frac{\pi}{3}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

28. $\sin^2 \frac{\pi}{6} = \left(\sin\frac{\pi}{6}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

29. $\sin^3 \frac{\pi}{6} = \left(\sin\frac{\pi}{6}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

30. $\cos^2 \frac{\pi}{12} = \frac{1 + \cos 2\left(\frac{\pi}{12}\right)}{2} = \frac{1 + \cos\frac{\pi}{6}}{2} = \frac{1 + \frac{\sqrt{3}}{2}}{2}$

$= \frac{2 + \sqrt{3}}{4}$

31. $\sin^2 \frac{\pi}{8} = \frac{1 - \cos 2\left(\frac{\pi}{8}\right)}{2} = \frac{1 - \cos\frac{\pi}{4}}{2} = \frac{1 - \frac{\sqrt{2}}{2}}{2}$

$= \frac{2 - \sqrt{2}}{4}$

32. **a.** $\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y)$
$= \sin x \cos y - \cos x \sin y$

b. $\cos(x - y) = \cos x \cos(-y) - \sin x \sin(-y)$
$= \cos x \cos y + \sin x \sin y$

c. $\tan(x - y) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)}$

$= \frac{\tan x - \tan y}{1 + \tan x \tan y}$

33. $\tan(t + \pi) = \frac{\tan t + \tan \pi}{1 - \tan t \tan \pi} = \frac{\tan t + 0}{1 - (\tan t)(0)}$

$= \tan t$

34. $\cos(x - \pi) = \cos x \cos(-\pi) - \sin x \sin(-\pi)$

$= -\cos x - 0 \cdot \sin x = -\cos x$

35. $s = rt = (2.5 \text{ ft})(2\pi \text{ rad}) = 5\pi$ ft, so the tire goes 5π feet per revolution, or $\frac{1}{5\pi}$ revolutions per foot.

$\left(\frac{1}{5\pi}\frac{\text{rev}}{\text{ft}}\right)\left(60\frac{\text{mi}}{\text{hr}}\right)\left(\frac{1}{60}\frac{\text{hr}}{\text{min}}\right)\left(5280\frac{\text{ft}}{\text{mi}}\right)$

≈ 336 rev/min

36. $s = rt = (2 \text{ ft})(150 \text{ rev})(2\pi \text{ rad/rev}) \approx 1885$ ft

37. $r_1 t_1 = r_2 t_2;\ 6(2\pi)t_1 = 8(2\pi)(21)$

$t_1 = 28$ rev/sec

38. $\Delta y = \sin\alpha$ and $\Delta x = \cos\alpha$

$m = \frac{\Delta y}{\Delta x} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

39. **a.** $\tan\alpha = \sqrt{3}$

$\alpha = \dfrac{\pi}{3}$

b. $\sqrt{3}x + 3y = 6$

$3y = -\sqrt{3}x + 6$

$y = -\dfrac{\sqrt{3}}{3}x + 2;\ m = -\dfrac{\sqrt{3}}{3}$

$\tan\alpha = -\dfrac{\sqrt{3}}{3}$

$\alpha = \dfrac{5\pi}{6}$

40. $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

$$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 + \tan(-\theta_1)}{1 - \tan\theta_2\tan(-\theta_1)}$$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

41. **a.** $\tan\theta = \dfrac{3-2}{1+3(2)} = \dfrac{1}{7}$

$\theta \approx 0.1419$

b. $\tan\theta = \dfrac{-1-\frac{1}{2}}{1+\left(\frac{1}{2}\right)(-1)} = -3$

$\theta \approx 1.8925$

c. $2x - 6y = 12 \qquad 2x + y = 0$

$-6y = -2x + 12y = -2x$

$y = \dfrac{1}{3}x - 2$

$m_1 = \dfrac{1}{3},\ m_2 = -2$

$\tan\theta = \dfrac{-2-\frac{1}{3}}{1+\left(\frac{1}{3}\right)(-2)} = -7;\ \theta \approx 1.7127$

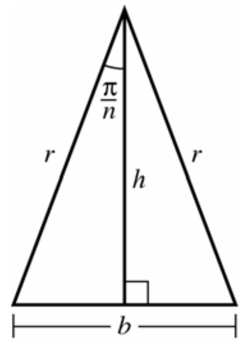
42. Recall that the area of the circle is πr^2. The measure of the vertex angle of the circle is 2π. Observe that the ratios of the vertex angles must equal the ratios of the areas. Thus,

$\dfrac{t}{2\pi} = \dfrac{A}{\pi r^2}$, so

$A = \dfrac{1}{2}r^2 t$.

43. $A = \dfrac{1}{2}(2)(5)^2 = 25\text{cm}^2$

44. Divide the polygon into n isosceles triangles by drawing lines from the center of the circle to the corners of the polygon. If the base of each triangle is on the perimeter of the polygon, then the angle opposite each base has measure $\dfrac{2\pi}{n}$. Bisect this angle to divide the triangle into two right triangles (See figure).



$\sin\dfrac{\pi}{n} = \dfrac{b}{2r}$ so $b = 2r\sin\dfrac{\pi}{n}$ and $\cos\dfrac{\pi}{n} = \dfrac{h}{r}$ so

$h = r\cos\dfrac{\pi}{n}$.

$P = nb = 2rn\sin\dfrac{\pi}{n}$

$A = n\left(\dfrac{1}{2}bh\right) = nr^2\cos\dfrac{\pi}{n}\sin\dfrac{\pi}{n}$

45. The base of the triangle is the side opposite the angle t. Then the base has length $2r\sin\dfrac{t}{2}$ (similar to Problem 44). The radius of the semicircle is $r\sin\dfrac{t}{2}$ and the height of the triangle is $r\cos\dfrac{t}{2}$.

$$A = \frac{1}{2}\left(2r\sin\frac{t}{2}\right)\left(r\cos\frac{t}{2}\right) + \frac{\pi}{2}\left(r\sin\frac{t}{2}\right)^2$$

$$= r^2\sin\frac{t}{2}\cos\frac{t}{2} + \frac{\pi r^2}{2}\sin^2\frac{t}{2}$$

46. $\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cos\frac{x}{16}$

$=\frac{1}{2}\left[\cos\frac{3}{4}x+\cos\frac{1}{4}x\right]\frac{1}{2}\left[\cos\frac{3}{16}x+\cos\frac{1}{16}x\right]$

$=\frac{1}{4}\left[\cos\frac{3}{4}x+\cos\frac{1}{4}x\right]\left[\cos\frac{3}{16}x+\cos\frac{1}{16}x\right]$

$=\frac{1}{4}\left[\cos\frac{3}{4}x\cos\frac{3}{16}x+\cos\frac{3}{4}x\cos\frac{1}{16}x\right.$

$\left.+\cos\frac{1}{4}x\cos\frac{3}{16}x+\cos\frac{1}{4}x\cos\frac{1}{16}x\right]$

$=\frac{1}{4}\left[\frac{1}{2}\left(\cos\frac{15}{16}+\cos\frac{9}{16}x\right)+\frac{1}{2}\left(\cos\frac{13}{16}x+\cos\frac{11}{16}x\right)\right.$

$\left.+\frac{1}{2}\left(\cos\frac{7}{16}x+\cos\frac{1}{16}x\right)+\frac{1}{2}\left(\cos\frac{5}{16}x+\cos\frac{3}{16}x\right)\right]$

$=\frac{1}{8}\left[\cos\frac{15}{16}x+\cos\frac{13}{16}x+\cos\frac{11}{16}x+\cos\frac{9}{16}x\right.$

$\left.+\cos\frac{7}{16}x+\cos\frac{5}{16}x+\cos\frac{3}{16}x+\cos\frac{1}{16}x\right]$

47. The temperature function is

$T(t)=80+25\sin\left(\frac{2\pi}{12}\left(t-\frac{7}{2}\right)\right)$.

The normal high temperature for November 15^{th} is then $T(10.5)=67.5$ °F.

48. The water level function is

$F(t)=8.5+3.5\sin\left(\frac{2\pi}{12}(t-9)\right)$.

The water level at 5:30 P.M. is then $F(17.5)\approx 5.12$ ft .

49. As t increases, the point on the rim of the wheel will move around the circle of radius 2.

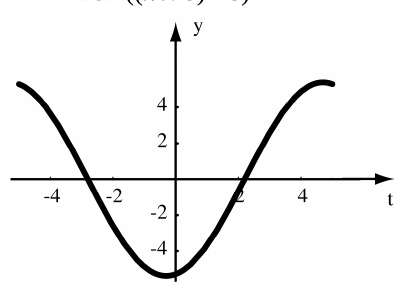
a. $x(2)\approx 1.902$
$y(2)\approx 0.618$
$x(6)\approx -1.176$
$y(6)\approx -1.618$
$x(10)=0$
$y(10)=2$
$x(0)=0$
$y(0)=2$

b. $x(t)=-2\sin\left(\frac{\pi}{5}t\right), y(t)=2\cos\left(\frac{\pi}{5}t\right)$

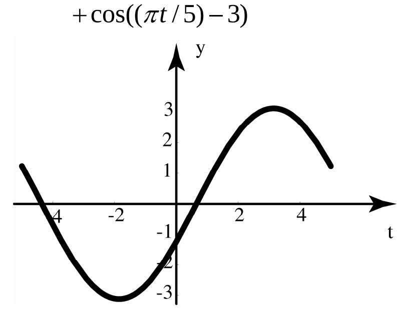
c. The point is at (2, 0) when $\frac{\pi}{5}t=\frac{\pi}{2}$; that is , when $t=\frac{5}{2}$.

50. Both functions have frequency $\frac{2\pi}{10}$. When you add functions that have the same frequency, the sum has the same frequency.

a. $y(t)=3\sin(\pi t/5)-5\cos(\pi t/5)$
$+2\sin((\pi t/5)-3)$



b. $y(t)=3\cos(\pi t/5-2)+\cos(\pi t/5)$
$+\cos((\pi t/5)-3)$



51. **a.** $C\sin(\omega t+\phi)=(C\cos\phi)\sin\omega t+(C\sin\phi)\cos\omega t$. Thus $A=C\cdot\cos\phi$ and $B=C\cdot\sin\phi$.

b. $A^2+B^2=(C\cos\phi)^2+(C\sin\phi)^2=C^2(\cos^2\phi)+C^2(\sin^2\phi)=C^2$

Also, $\dfrac{B}{A}=\dfrac{C\cdot\sin\phi}{C\cdot\cos\phi}=\tan\phi$

c. $A_1\sin(\omega t+\phi_1)+A_2\sin(\omega t+\phi_2)+A_3(\sin\omega t+\phi_3)$

$=A_1(\sin\omega t\cos\phi_1+\cos\omega t\sin\phi_1)$

$+A_2(\sin\omega t\cos\phi_2+\cos\omega t\sin\phi_2)$

$+A_3(\sin\omega t\cos\phi_3+\cos\omega t\sin\phi_3)$

$=(A_1\cos\phi_1+A_2\cos\phi_2+A_3\cos\phi_3)\sin\omega t$

$+(A_1\sin\phi_1+A_2\sin\phi_2+A_3\sin\phi_3)\cos\omega t$

$=C\sin\left(\omega t+\phi\right)$

where C and ϕ can be computed from

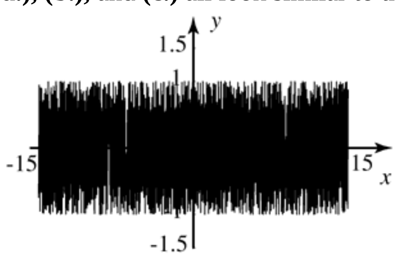
$A=A_1\cos\phi_1+A_2\cos\phi_2+A_3\cos\phi_3$

$B=A_1\sin\phi_1+A_2\sin\phi_2+A_3\sin\phi_3$

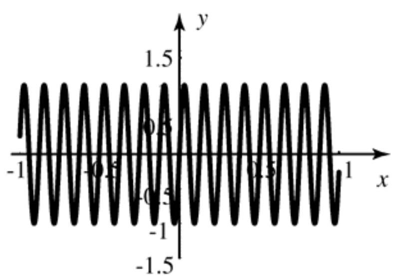
as in part (b).

d. Written response. Answers will vary.

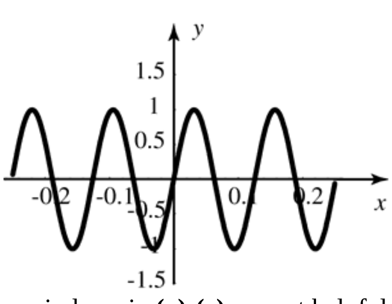
52. **(a.), (b.),** and **(c.)** all look similar to this:



d.

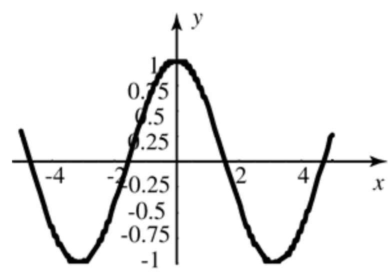


e.

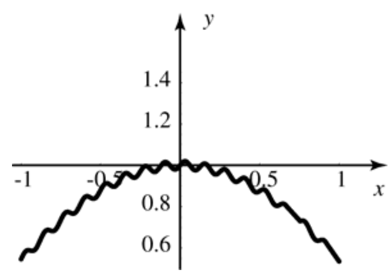


The windows in **(a)-(c)** are not helpful because the function oscillates too much over the domain plotted. Plots in **(d)** or **(e)** show the behavior of the function.

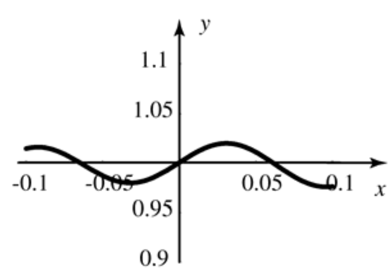
53. **a.**



b.



c.



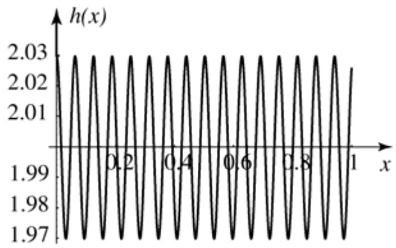
The plot in **(a)** shows the long term behavior of the function, but not the short term behavior, whereas the plot in **(c)** shows the short term behavior, but not the long term behavior. The plot in **(b)** shows a little of each.

54. a. $h(x) = (f \circ g)(x)$

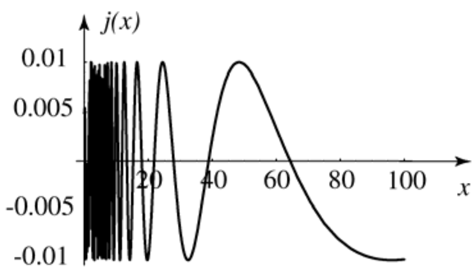
$$= \frac{\frac{3}{100}\cos(100x) + 2}{\left(\frac{1}{100}\right)^2 \cos^2(100x) + 1}$$

$$j(x) = (g \circ f)(x) = \frac{1}{100}\cos\left(100\frac{3x+2}{x^2+1}\right)$$

b.

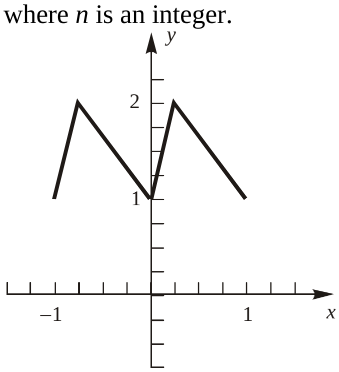


c.



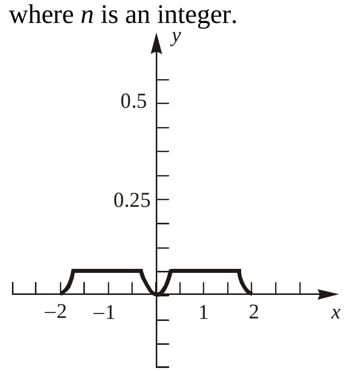
55. $$f(x) = \begin{cases} 4(x - [\![x]\!]) + 1 & : \ x \in \left[n, n + \frac{1}{4}\right) \\ -\frac{4}{3}(x - [\![x]\!]) + \frac{7}{3} & : \ x \in \left[n + \frac{1}{4}, n + 1\right) \end{cases}$$

where n is an integer.



56. $$f(x) = \begin{cases} (x - 2n)^2, & x \in \left[2n - \frac{1}{4}, 2n + \frac{1}{4}\right] \\ 0.0625, & \text{otherwise} \end{cases}$$

where n is an integer.



0.8 Chapter Review

Concepts Test

1. False: p and q must be integers.

2. True: $\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{p_1q_2 - p_2q_1}{q_1q_2}$; since $p_1, q_1, p_2,$ and q_2 are integers, so are $p_1q_2 - p_2q_1$ and q_1q_2.

3. False: If the numbers are opposites ($-\pi$ and π) then the sum is 0, which is rational.

4. True: Between any two distinct real numbers there are both a rational and an irrational number.

5. False: 0.999... is equal to 1.

6. True: $\left(a^m\right)^n = \left(a^n\right)^m = a^{mn}$

7. False: $(a*b)*c = a^{bc}; a*(b*c) = a^{b^c}$

8. True: Since $x \le y \le z$ and $x \ge z, x = y = z$

9. True: If x was not 0, then $\varepsilon = \frac{|x|}{2}$ would be a positive number less than $|x|$.

10. True: $y-x=-(x-y)$ so
$(x-y)(y-x)=(x-y)(-1)(x-y)$
$=(-1)(x-y)^2.$
$(x-y)^2 \ge 0$ for all x and y, so
$-(x-y)^2 \le 0.$

11. True: $a<b<0;\ a<b;\ \dfrac{a}{b}>1; \dfrac{1}{b}<\dfrac{1}{a}$

12. True: $[a,b]$ and $[b,c]$ share point b in common.

13. True: If (a, b) and (c, d) share a point then $c < b$ so they share the infinitely many points between b and c.

14. True: $\sqrt{x^2}=|x|=-x$ if $x<0$.

15. False: For example, if $x=-3$, then $|-x|=|-(-3)|=|3|=3$ which does not equal x.

16. False: For example, take $x=1$ and $y=-2$.

17. True: $|x|<|y| \Leftrightarrow |x|^4<|y|^4$
$|x|^4=x^4$ and $|y|^4=y^4$, so $x^4<y^4$

18. True: $|x+y|=-(x+y)$
$=-x+(-y)=|x|+|y|$

19. True: If $r = 0$, then
$$\frac{1}{1+|r|}=\frac{1}{1-r}=\frac{1}{1-|r|}=1.$$
For any r, $1+|r| \ge 1-|r|$. Since
$|r|<1, 1-|r|>0$ so $\dfrac{1}{1+|r|} \le \dfrac{1}{1-|r|}$;
also, $-1 < r < 1$.

If $-1 < r < 0$, then $|r|=-r$ and
$1-r=1+|r|$, so
$$\frac{1}{1+|r|}=\frac{1}{1-r} \le \frac{1}{1-|r|}.$$
If $0 < r < 1$, then $|r|=r$ and
$1-r=1-|r|$, so
$$\frac{1}{1+|r|} \le \frac{1}{1-r}=\frac{1}{1-|r|}.$$

20. True: If $|r|>1$, then $1-|r|<0$. Thus,
since $1+|r| \ge 1-|r|$, $\dfrac{1}{1-|r|} \le \dfrac{1}{1+|r|}$.
If $r>1$, $|r|=r$, and $1-r=1-|r|$, so
$$\frac{1}{1-|r|}=\frac{1}{1-r} \le \frac{1}{1+|r|}.$$
If $r<-1, |r|=-r$ and $1-r=1+|r|$,
so $\dfrac{1}{1-|r|} \le \dfrac{1}{1-r}=\dfrac{1}{1+|r|}$.

21. True: If x and y are the same sign, then $\big||x|-|y|\big|=|x-y|$. $|x-y| \le |x+y|$ when x and y are the same sign, so $\big||x|-|y|\big| \le |x+y|$. If x and y have opposite signs then either
$\big||x|-|y|\big|=|x-(-y)|=|x+y|$
$(x > 0, y < 0)$ or
$\big||x|-|y|\big|=|-x-y|=|x+y|$
$(x < 0, y > 0)$. In either case
$\big||x|-|y|\big|=|x+y|$.
If either $x = 0$ or $y = 0$, the inequality is easily seen to be true.

22. True: If y is positive, then $x=\sqrt{y}$ satisfies
$$x^2=\left(\sqrt{y}\right)^2=y.$$

23. True: For every real number y, whether it is positive, zero, or negative, the cube root $x=\sqrt[3]{y}$ satisfies
$$x^3=\left(\sqrt[3]{y}\right)^3=y$$

24. True: For example $x^2 \le 0$ has solution $[0]$.

25. True:
$$x^2+ax+y^2+y=0$$
$$x^2+ax+\frac{a^2}{4}+y^2+y+\frac{1}{4}=\frac{a^2}{4}+\frac{1}{4}$$
$$\left(x+\frac{a}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\frac{a^2+1}{4}$$
is a circle for all values of a.

26. False: If $a=b=0$ and $c<0$, the equation does not represent a circle.

27. True; $y-b=\frac{3}{4}(x-a)$

$y=\frac{3}{4}x-\frac{3a}{4}+b;$

If $x = a + 4$:

$y=\frac{3}{4}(a+4)-\frac{3a}{4}+b$

$=\frac{3a}{4}+3-\frac{3a}{4}+b=b+3$

28. True: If the points are on the same line, they have equal slope. Then the reciprocals of the slopes are also equal.

29. True: If $ab > 0$, a and b have the same sign, so (a, b) is in either the first or third quadrant.

30. True: Let $x=\varepsilon/2$. If $\varepsilon>0$, then $x>0$ and $x<\varepsilon$.

31. True: If $ab = 0$, a or b is 0, so (a, b) lies on the x-axis or the y-axis. If $a = b = 0$, (a, b) is the origin.

32. True: $y_1=y_2$, so (x_1, y_1) and (x_2, y_2) are on the same horizontal line.

33. True: $d=\sqrt{[(a+b)-(a-b)]^2+(a-a)^2}$

$=\sqrt{(2b)^2}=|2b|$

34. False: The equation of a vertical line cannot be written in point-slope form.

35. True: This is the general linear equation.

36. True: Two non-vertical lines are parallel if and only if they have the same slope.

37. False: The slopes of perpendicular lines are negative reciprocals.

38. True: If a and b are rational and $(a,0),(0,b)$ are the intercepts, the slope is $-\frac{b}{a}$ which is rational.

39. False: $ax+y=c\Rightarrow y=-ax+c$

$ax-y=c\Rightarrow y=ax-c$

$(a)(-a)\neq -1.$

(unless $a=\pm 1$)

40. True: The equation is $(3+2m)x+(6m-2)y+4-2m=0$ which is the equation of a straight line unless $3+2m$ and $6m-2$ are both 0, and there is no real number m such that $3+2m=0$ and $6m-2=0$.

41. True: $f(x)=\sqrt{-(x^2+4x+3)}$

$=\sqrt{-(x+3)(x+1)}$

$-(x^2+4x+3)\geq 0$ on $-3\leq x\leq -1$.

42. False: The domain does not include $n\pi+\frac{\pi}{2}$ where n is an integer.

43. True: The domain is $(-\infty, \infty)$ and the range is $[-6, \infty)$.

44. False: The range is $(-\infty, \infty)$.

45. False: The range $(-\infty, \infty)$.

46. True: If $f(x)$ and $g(x)$ are even functions, $f(x) + g(x)$ is even.

$f(-x) + g(-x) = f(x) + g(x)$

47. True: If $f(x)$ and $g(x)$ are odd functions,

$f(-x) + g(-x) = -f(x) - g(x)$

$= -[f(x) + g(x)]$, so $f(x) + g(x)$ is odd

48. False: If $f(x)$ and $g(x)$ are odd functions,

$f(-x)g(-x) = -f(x)[-g(x)] = f(x)g(x)$, so

$f(x)g(x)$ is even.

49. True: If $f(x)$ is even and $g(x)$ is odd,

$f(-x)g(-x) = f(x)[-g(x)]$

$= -f(x)g(x)$, so $f(x)g(x)$ is odd.

50. False: If $f(x)$ is even and $g(x)$ is odd,

$f(g(-x)) = f(-g(x)) = f(g(x))$; while if

$f(x)$ is odd and $g(x)$ is even,

$f(g(-x)) = f(g(x))$; so $f(g(x))$ is even.

51. False: If $f(x)$ and $g(x)$ are odd functions,

$f(g(-x)) = f(-g(x)) = -f(g(x))$, so

$f(g(x))$ is odd.

52. True: $f(-x)=\frac{2(-x)^3+(-x)}{(-x)^2+1}=\frac{-2x^3-x}{x^2+1}$

$=-\frac{2x^3+x}{x^2+1}$

53. True: $f(-t)=\dfrac{(\sin(-t))^2+\cos(-t)}{\tan(-t)\csc(-t)}$

$$=\frac{(-\sin t)^2+\cos t}{-\tan t(-\csc t)}=\frac{(\sin t)^2+\cos t}{\tan t\csc t}$$

54. False: $f(x) = c$ has domain $(-\infty, \infty)$ and the only value of the range is c.

55. False: $f(x) = c$ has domain $(-\infty, \infty)$, yet the range has only one value, c.

56. True: $g(-1.8)=\left[\!\left[\dfrac{-1.8}{2}\right]\!\right]=[\![-0.9]\!]=-1$

57. True: $(f\circ g)(x)=(x^3)^2=x^6$
$(g\circ f)(x)=(x^2)^3=x^6$

58. False: $(f\circ g)(x)=(x^3)^2=x^6$
$f(x)\cdot g(x)=x^2x^3=x^5$

59. False: The domain of $\dfrac{f}{g}$ excludes any values where $g = 0$.

60. True: $f(a) = 0$
Let $F(x) = f(x + h)$, then
$F(a - h) = f(a - h + h) = f(a) = 0$

61. True: $\cot x=\dfrac{\cos x}{\sin x}$

$$\cot(-x)=\frac{\cos(-x)}{\sin(-x)}$$

$$=\frac{\cos x}{-\sin x}=-\cot x$$

62. False: The domain of the tangent function excludes all $n\pi+\dfrac{\pi}{2}$ where n is an integer.

63. False: The cosine function is periodic, so $\cos s = \cos t$ does not necessarily imply $s = t$; e.g., $\cos 0=\cos\ 2\pi=1$, but $0\neq 2\pi$.

Sample Test Problems

1. a. $\left(n+\dfrac{1}{n}\right)^n;\left(1+\dfrac{1}{1}\right)^1=2;\left(2+\dfrac{1}{2}\right)^2=\dfrac{25}{4};$
$\left(-2+\dfrac{1}{-2}\right)^{-2}=\dfrac{4}{25}$

b. $(n^2-n+1)^2;\left[(1)^2-(1)+1\right]^2=1;$
$\left[(2)^2-(2)+1\right]^2=9;$
$\left[(-2)^2-(-2)+1\right]^2=49$

c. $4^{3/n};4^{3/1}=64;4^{3/2}=8;4^{-3/2}=\dfrac{1}{8}$

d. $\sqrt[n]{\left|\dfrac{1}{n}\right|};\sqrt[1]{\left|\dfrac{1}{1}\right|}=1;\sqrt{\left|\dfrac{1}{2}\right|}=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2};$
$\sqrt[-2]{\left|\dfrac{1}{-2}\right|}=\sqrt{2}$

2. a. $\left(1+\dfrac{1}{m}+\dfrac{1}{n}\right)\left(1-\dfrac{1}{m}+\dfrac{1}{n}\right)^{-1}=\dfrac{1+\frac{1}{m}+\frac{1}{n}}{1-\frac{1}{m}+\frac{1}{n}}$

$$=\frac{mn+n+m}{mn-n+m}$$

b. $\dfrac{\frac{2}{x+1}-\frac{x}{x^2-x-2}}{\frac{3}{x+1}-\frac{2}{x-2}}=\dfrac{\frac{2}{x+1}-\frac{x}{(x-2)(x+1)}}{\frac{3}{x+1}-\frac{2}{x-2}}$

$$=\frac{2(x-2)-x}{3(x-2)-2(x+1)}$$

$$=\frac{x-4}{x-8}$$

c. $\dfrac{(t^3-1)}{t-1}=\dfrac{(t-1)(t^2+t+1)}{t-1}=t^2+t+1$

3. Let a, b, c, and d be integers.
$\dfrac{\frac{a}{b}+\frac{c}{d}}{2}=\dfrac{a}{2b}+\dfrac{c}{2d}=\dfrac{ad+bc}{2bd}$ which is rational.

4. $x = 4.1282828\ldots$

$$\begin{aligned} 1000x &= 4128.282828\ldots \\ 10x &= 41.282828\ldots \\ \hline 990x &= 4087 \\ x &= \frac{4087}{990} \end{aligned}$$

5. Answers will vary. Possible answer:

$\sqrt{\frac{13}{50}} \approx 0.50990...$

6. $\dfrac{\left(\sqrt[3]{8.15\times10^4} - 1.32\right)^2}{3.24} \approx 545.39$

7. $\left(\pi - \sqrt{2.0}\right)^{2.5} - \sqrt[3]{2.0} \approx 2.66$

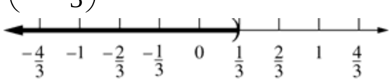
8. $\sin^2(2.45) + \cos^2(2.40) - 1.00 \approx -0.0495$

9. $1 - 3x > 0$

$3x < 1$

$x < \frac{1}{3}$

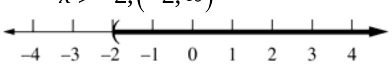
$\left(-\infty, \frac{1}{3}\right)$



10. $6x + 3 > 2x - 5$

$4x > -8$

$x > -2; (-2, \infty)$

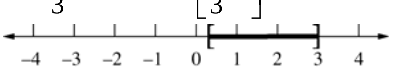


11. $3 - 2x \le 4x + 1 \le 2x + 7$

$3 - 2x \le 4x + 1$ and $4x + 1 \le 2x + 7$

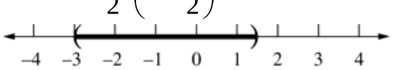
$6x \ge 2$ and $2x \ge 6$

$x \ge \frac{1}{3}$ and $x \le 3$; $\left[\frac{1}{3}, 3\right]$



12. $2x^2 + 5x - 3 < 0; (2x - 1)(x + 3) < 0;$

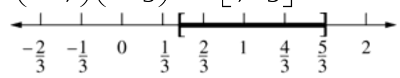
$-3 < x < \frac{1}{2}; \left(-3, \frac{1}{2}\right)$



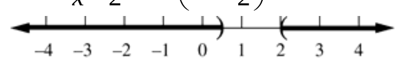
13. $21t^2 - 44t + 12 \le -3; 21t^2 - 44t + 15 \le 0;$

$t = \dfrac{44 \pm \sqrt{44^2 - 4(21)(15)}}{2(21)} = \dfrac{44 \pm 26}{42} = \dfrac{3}{7}, \dfrac{5}{3}$

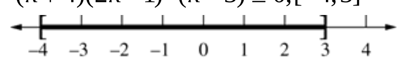
$\left(t - \frac{3}{7}\right)\left(t - \frac{5}{3}\right) \le 0; \left[\frac{3}{7}, \frac{5}{3}\right]$



14. $\dfrac{2x - 1}{x - 2} > 0; \left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$

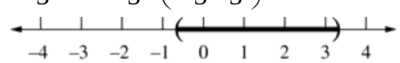


15. $(x + 4)(2x - 1)^2(x - 3) \le 0; [-4, 3]$



16. $|3x - 4| < 6; -6 < 3x - 4 < 6; -2 < 3x < 10;$

$-\frac{2}{3} < x < \frac{10}{3}; \left(-\frac{2}{3}, \frac{10}{3}\right)$



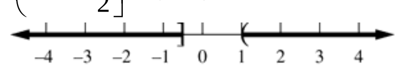
17. $\dfrac{3}{1 - x} \le 2$

$\dfrac{3}{1 - x} - 2 \le 0$

$\dfrac{3 - 2(1 - x)}{1 - x} \le 0$

$\dfrac{2x + 1}{1 - x} \le 0;$

$\left(-\infty, -\frac{1}{2}\right] \cup (1, \infty)$



18. $|12 - 3x| \ge |x|$

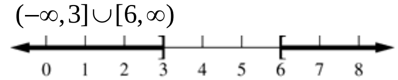
$(12 - 3x)^2 \ge x^2$

$144 - 72x + 9x^2 \ge x^2$

$8x^2 - 72x + 144 \ge 0$

$8(x - 3)(x - 6) \ge 0$

$(-\infty, 3] \cup [6, \infty)$



19. For example, if $x = -2$, $|-(-2)| = 2 \ne -2$

$|-x| \ne x$ for any $x < 0$

20. If $|-x| = x$, then $|x| = x$.

$x \ge 0$

21. $|t-5| = |-(5-t)| = |5-t|$
If $|5-t| = 5-t$, then $5-t \ge 0$.
$t \le 5$

22. $|t-a| = |-(a-t)| = |a-t|$
If $|a-t| = a-t$, then $a-t \ge 0$.
$t \le a$

23. If $|x| \le 2$, then

$$0 \le \left|2x^2+3x+2\right| \le \left|2x^2\right| + |3x| + 2 \le 8+6+2 = 16$$

also $\left|x^2+2\right| \ge 2$ so $\dfrac{1}{\left|x^2+2\right|} \le \dfrac{1}{2}$. Thus

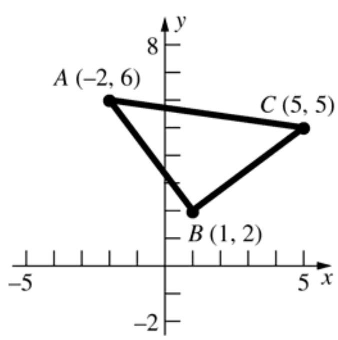
$$\left|\frac{2x^2+3x+2}{x^2+2}\right| = \left|2x^2+3x+2\right|\left|\frac{1}{x^2+2}\right| \le 16\left(\frac{1}{2}\right) = 8$$

24. **a.** The distance between x and 5 is 3.

b. The distance between x and –1 is less than or equal to 2.

c. The distance between x and a is greater than b.

25.



$$d(A, B) = \sqrt{(1+2)^2 + (2-6)^2} = \sqrt{9+16} = 5$$

$$d(B, C) = \sqrt{(5-1)^2 + (5-2)^2} = \sqrt{16+9} = 5$$

$$d(A, C) = \sqrt{(5+2)^2 + (5-6)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$

$(AB)^2 + (BC)^2 = (AC)^2$, so ΔABC is a right triangle.

26. midpoint: $\left(\dfrac{1+7}{2}, \dfrac{2+8}{2}\right) = (4, 5)$

$$d = \sqrt{(4-3)^2 + (5+6)^2} = \sqrt{1+121} = \sqrt{122}$$

27. center $= \left(\dfrac{2+10}{2}, \dfrac{0+4}{2}\right) = (6, 2)$

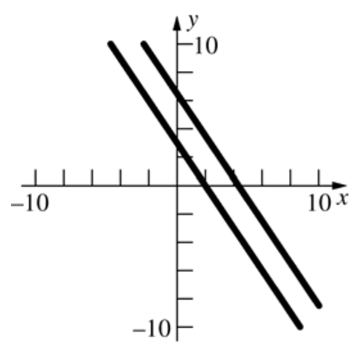
$$\text{radius} = \frac{1}{2}\sqrt{(10-2)^2 + (4-0)^2} = \frac{1}{2}\sqrt{64+16} = 2\sqrt{5}$$

circle: $(x-6)^2 + (y-2)^2 = 20$

28. $x^2 + y^2 - 8x + 6y = 0$
$x^2 - 8x + 16 + y^2 + 6y + 9 = 16 + 9$
$(x-4)^2 + (y+3)^2 = 25$;
center $= (4, -3)$, radius = 5

29. $x^2 - 2x + y^2 + 2y = 2$
$x^2 - 2x + 1 + y^2 + 2y + 1 = 2 + 1 + 1$
$(x-1)^2 + (y+1)^2 = 4$
center = (1, –1)
$x^2 + 6x + y^2 - 4y = -7$
$x^2 + 6x + 9 + y^2 - 4y + 4 = -7 + 9 + 4$
$(x+3)^2 + (y-2)^2 = 6$
center = (–3, 2)
$d = \sqrt{(-3-1)^2 + (2+1)^2} = \sqrt{16+9} = 5$

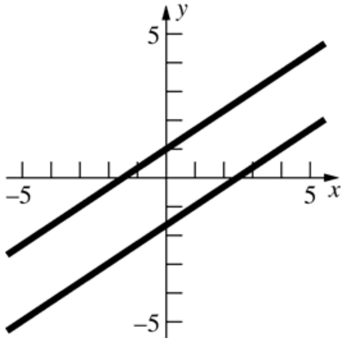
30. **a.** $3x + 2y = 6$
$2y = -3x + 6$
$y = -\dfrac{3}{2}x + 3$
$m = -\dfrac{3}{2}$
$y - 2 = -\dfrac{3}{2}(x-3)$
$y = -\dfrac{3}{2}x + \dfrac{13}{2}$



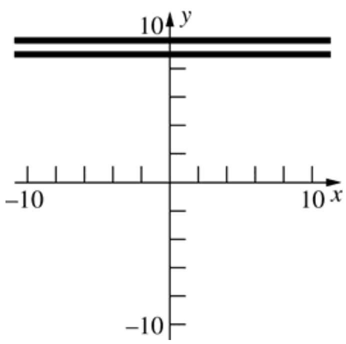
b. $m = \frac{2}{3}$;

$y + 1 = \frac{2}{3}(x - 1)$

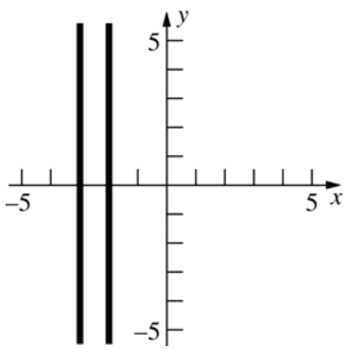
$y = \frac{2}{3}x - \frac{5}{3}$



c. $y = 9$



d. $x = -3$



31. a. $m = \frac{3-1}{7+2} = \frac{2}{9}$;

$y - 1 = \frac{2}{9}(x + 2)$

$y = \frac{2}{9}x + \frac{13}{9}$

b. $3x - 2y = 5$

$-2y = -3x + 5$

$y = \frac{3}{2}x - \frac{5}{2}$;

$m = \frac{3}{2}$

$y - 1 = \frac{3}{2}(x + 2)$

$y = \frac{3}{2}x + 4$

c. $3x + 4y = 9$

$4y = -3x + 9$;

$y = -\frac{3}{4}x + \frac{9}{4}$; $m = \frac{4}{3}$

$y - 1 = \frac{4}{3}(x + 2)$

$y = \frac{4}{3}x + \frac{11}{3}$

d. $x = -2$

e. contains (–2, 1) and (0, 3); $m = \frac{3-1}{0+2}$;

$y = x + 3$

32. $m_1 = \frac{3+1}{5-2} = \frac{4}{3}; m_2 = \frac{11-3}{11-5} = \frac{8}{6} = \frac{4}{3}$;

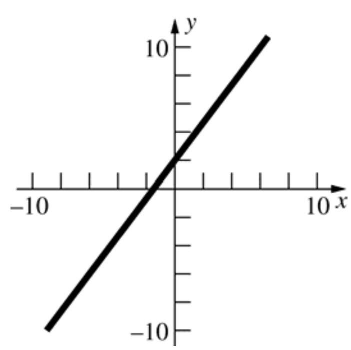
$m_3 = \frac{11+1}{11-2} = \frac{12}{9} = \frac{4}{3}$

$m_1 = m_2 = m_3$, so the points lie on the same line.

33. The figure is a cubic with respect to *y*. The equation is **(b)** $x = y^3$.

34. The figure is a quadratic, opening downward, with a negative *y*-intercept. The equation is **(c)** $y = ax^2 + bx + c$. with $a < 0$, $b > 0$, and $c < 0$.

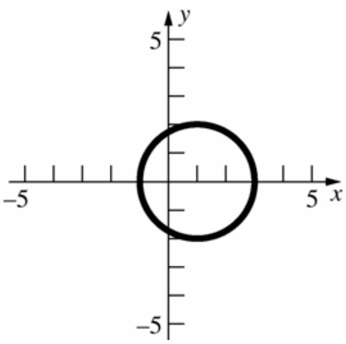
35.



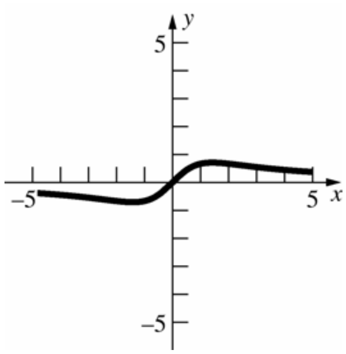
36. $x^2 - 2x + y^2 = 3$

$x^2 - 2x + 1 + y^2 = 4$

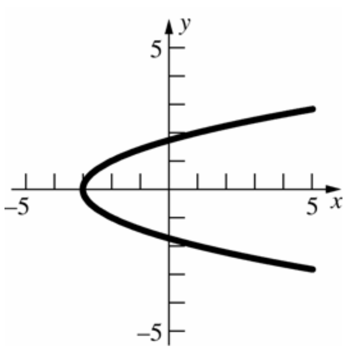
$(x-1)^2 + y^2 = 4$



37.



38.



39. $y = x^2 - 2x + 4$ and $y - x = 4$;

$x + 4 = x^2 - 2x + 4$

$x^2 - 3x = 0$

$x(x-3) = 0$

points of intersection: (0, 4) and (3, 7)

40. $4x - y = 2$

$y = 4x - 2;$

$m = -\frac{1}{4}$

contains $(a,0),(0,b)$;

$\frac{ab}{2} = 8$

$ab = 16$

$b = \frac{16}{a}$

$\frac{b-0}{0-a} = -\frac{b}{a} = -\frac{1}{4};$

$a = 4b$

$a = 4\left(\frac{16}{a}\right)$

$a^2 = 64$

$a = 8$

$b = \frac{16}{8} = 2;\ y = -\frac{1}{4}x + 2$

41. **a.** $f(1) = \frac{1}{1+1} - \frac{1}{1} = -\frac{1}{2}$

b. $f\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}+1} - \frac{1}{-\frac{1}{2}} = 4$

c. $f(-1)$ does not exist.

d. $f(t-1) = \frac{1}{t-1+1} - \frac{1}{t-1} = \frac{1}{t} - \frac{1}{t-1}$

e. $f\left(\frac{1}{t}\right) = \frac{1}{\frac{1}{t}+1} - \frac{1}{\frac{1}{t}} = \frac{t}{1+t} - t$

42. **a.** $g(2) = \frac{2+1}{2} = \frac{3}{2}$

b. $g\left(\frac{1}{2}\right) = \frac{\frac{1}{2}+1}{\frac{1}{2}} = 3$

c. $\frac{g(2+h) - g(2)}{h} = \frac{\frac{2+h+1}{2+h} - \frac{2+1}{2}}{h}$

$= \frac{\frac{2h+6-3h-6}{2(h+2)}}{h} = \frac{-\frac{h}{2(h+2)}}{h} = \frac{-1}{2(h+2)}$

43. **a.** $\{x \in \mathbb{R} : x \neq -1, 1\}$

b. $\{x \in \mathbb{R} : |x| \leq 2\}$

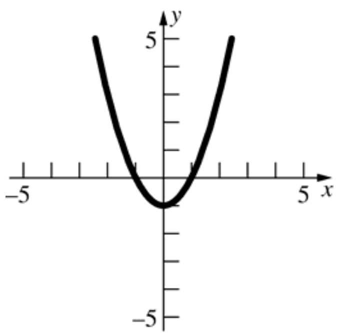
44. a. $f(-x)=\dfrac{3(-x)}{(-x)^2+1}=-\dfrac{3x}{x^2+1}$; odd

b. $g(-x)=|\sin(-x)|+\cos(-x)$
$=|-\sin x|+\cos x=|\sin x|+\cos x$; even

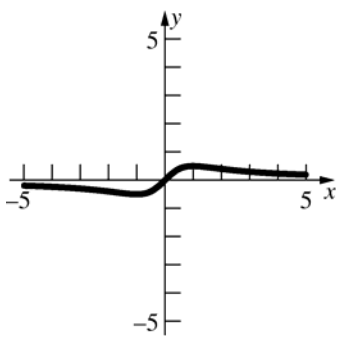
c. $h(-x)=(-x)^3+\sin(-x)=-x^3-\sin x$; odd

d. $k(-x)=\dfrac{(-x)^2+1}{|-x|+(-x)^4}=\dfrac{x^2+1}{|x|+x^4}$; even

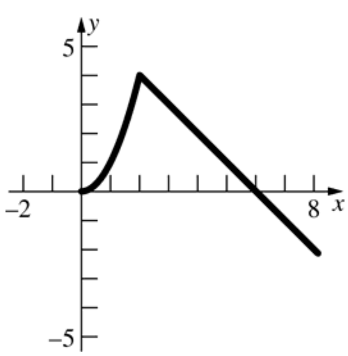
45. a. $f(x)=x^2-1$



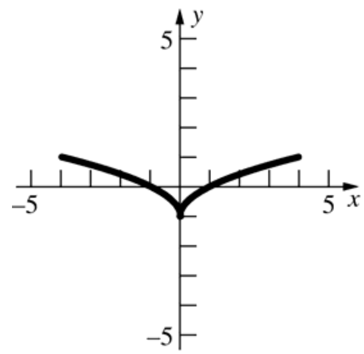
b. $g(x)=\dfrac{x}{x^2+1}$



c. $h(x)=\begin{cases} x^2 & \text{if } 0\le x\le 2 \\ 6-x & \text{if } x>2 \end{cases}$



46.



47. $V(x) = x(32 – 2x)(24 – 2x)$
Domain [0, 12]

48. a. $(f+g)(2)=\left(2-\dfrac{1}{2}\right)+(2^2+1)=\dfrac{13}{2}$

b. $(f\cdot g)(2)=\left(\dfrac{3}{2}\right)(5)=\dfrac{15}{2}$

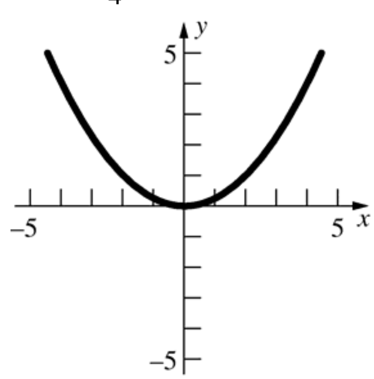
c. $(f\circ g)(2)=f(5)=5-\dfrac{1}{5}=\dfrac{24}{5}$

d. $(g\circ f)(2)=g\left(\dfrac{3}{2}\right)=\left(\dfrac{3}{2}\right)^2+1=\dfrac{13}{4}$

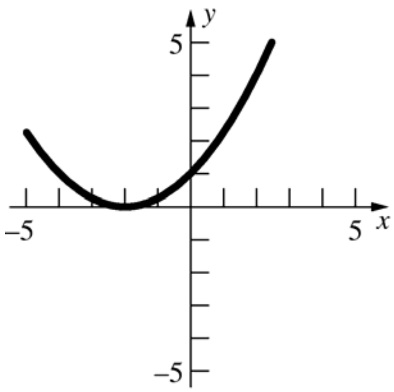
e. $f^3(-1)=\left(-1+\dfrac{1}{1}\right)^3=0$

f. $f^2(2)+g^2(2)=\left(\dfrac{3}{2}\right)^2+(5)^2$
$=\dfrac{9}{4}+25=\dfrac{109}{4}$

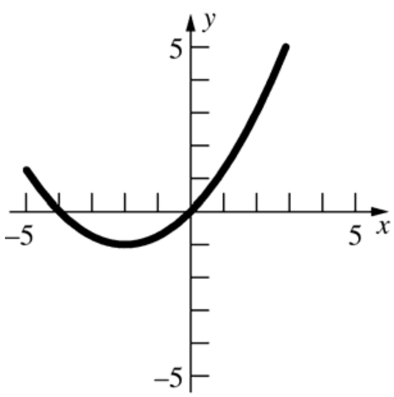
49. a. $y=\dfrac{1}{4}x^2$



b. $y = \frac{1}{4}(x+2)^2$



c. $y = -1 + \frac{1}{4}(x+2)^2$



50. a. $(-\infty, 16]$

b. $f \circ g = \sqrt{16 - x^4}$; domain $[-2, 2]$

c. $g \circ f = (\sqrt{16-x})^4 = (16-x)^2$;
domain $(-\infty, 16]$
(note: the simplification $(\sqrt{16-x})^4 = (16-x)^2$ is only true given the restricted domain)

51. $f(x) = \sqrt{x}, g(x) = 1 + x, h(x) = x^2, k(x) = \sin x,$
$F(x) = \sqrt{1 + \sin^2 x} = f \circ g \circ h \circ k$

52. a. $\sin(570°) = \sin(210°) = -\frac{1}{2}$

b. $\cos\left(\frac{9\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

c. $\cos\left(-\frac{13\pi}{6}\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

53. a. $\sin(-t) = -\sin t = -0.8$

b. $\sin^2 t + \cos^2 t = 1$
$\cos^2 t = 1 - (0.8)^2 = 0.36$
$\cos t = -0.6$

c. $\sin 2t = 2 \sin t \cos t = 2(0.8)(-0.6) = -0.96$

d. $\tan t = \frac{\sin t}{\cos t} = \frac{0.8}{-0.6} = -\frac{4}{3} \approx -1.333$

e. $\cos\left(\frac{\pi}{2} - t\right) = \sin t = 0.8$

f. $\sin(\pi + t) = -\sin t = -0.8$

54. $\sin 3t = \sin(2t + t) = \sin 2t \cos t + \cos 2t \sin t$
$= 2\sin t \cos^2 t + (1 - 2\sin^2 t)\sin t$
$= 2\sin t(1 - \sin^2 t) + \sin t - 2\sin^3 t$
$= 2\sin t - 2\sin^3 t + \sin t - 2\sin^3 t$
$= 3\sin t - 4\sin^3 t$

55. $s = rt$
$= 9\left(20\frac{\text{rev}}{\text{min}}\right)\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)(1 \text{ sec}) = 6\pi$
≈ 18.85 in.

Review and Preview Problems

1. a) $0 < 2x < 4$; $0 < x < 2$

b) $-6 < x < 16$

2. a) $13 < 2x < 14$; $6.5 < x < 7$

b) $-4 < -x/2 < 7$; $-14 < x < 8$

3. $x - 7 = 3$ or $x - 7 = -3$
$x = 10$ or $x = 4$

4. $x + 3 = 2$ or $x + 3 = -2$
$x = -1$ or $x = -5$

5. $x - 7 = 3$ or $x - 7 = -3$
$x = 10$ or $x = 4$

6. $x - 7 = d$ or $x - 7 = -d$
$x = 7 + d$ or $x = 7 - d$

7. a) $x - 7 < 3$ and $x - 7 > -3$
$x < 10$ and $x > 4$
$4 < x < 10$

b) $x-7 \le 3$ and $x-7 \ge -3$
$x \le 10$ and $x \ge 4$
$4 \le x \le 10$

c) $x-7 \le 1$ and $x-7 \ge -1$
$x \le 8$ and $x \ge 6$
$6 \le x \le 8$

d) $x-7 < 0.1$ and $x-7 > -0.1$
$x < 7.1$ and $x > 6.9$
$6.9 < x < 7.1$

8. a) $x-2 < 1$ and $x-2 > -1$
$x < 3$ and $x > 1$
$1 < x < 3$

b) $x-2 \ge 1$ or $x-2 \le -1$
$x \ge 3$ or $x \le 1$

c) $x-2 < 0.1$ and $x-2 > -0.1$
$x < 2.1$ and $x > 1.9$
$1.9 < x < 2.1$

d) $x-2 < 0.01$ and $x-2 > -0.01$
$x < 2.01$ and $x > 1.99$
$1.99 < x < 2.01$

9. a) $x-1 \ne 0;\ x \ne 1$

b) $2x^2 - x - 1 \ne 0;\ x \ne 1, -0.5$

10. a) $x \ne 0$ **b)** $x \ne 0$

11. a)

$$f(0) = \frac{0-1}{0-1} = 1$$
$$f(0.9) = \frac{0.81-1}{0.9-1} = 1.9$$
$$f(0.99) = \frac{0.9801-1}{0.99-1} = 1.99$$
$$f(0.999) = \frac{0.998001-1}{.999-1} = 1.999$$
$$f(1.001) = \frac{1.002001-1}{1.001-1} = 2.001$$
$$f(1.01) = \frac{1.0201-1}{1.01-1} = 2.01$$
$$f(1.1) = \frac{1.21-1}{1.1-1} = 2.1$$
$$f(2) = \frac{4-1}{2-1} = 3$$

b)

$$g(0) = -1$$
$$g(0.9) = -0.0357143$$
$$g(0.99) = -0.0033557$$
$$g(0.999) = -0.000333556$$
$$g(1.001) = 0.000333111$$
$$g(1.01) = 0.00331126$$
$$g(1.1) = 0.03125$$
$$g(2) = \frac{1}{5}$$

12. a)

$$F(-1) = \frac{1}{-1} = -1$$
$$F(-0.1) = \frac{0.1}{-0.1} = -1$$
$$F(-0.01) = \frac{0.01}{-0.01} = -1$$
$$F(-0.001) = \frac{0.001}{-0.001} = -1$$
$$F(0.001) = \frac{0.001}{0.001} = 1$$
$$F(0.01) = \frac{0.01}{0.01} = 1$$
$$F(0.1) = \frac{0.01}{0.01} = 1$$
$$F(1) = \frac{1}{1} = 1$$

b)

$$G(-1) = 0.841471$$
$$G(-0.1) = 0.998334$$
$$G(-0.01) = 0.999983$$
$$G(-0.001) = 0.99999983$$
$$G(0.001) = 0.99999983$$
$$G(0.01) = 0.999983$$
$$G(0.1) = 0.998334$$
$$G(1) = 0.841471$$

13. $x-5 < 0.1$ and $x-5 > -0.1$
$x < 5.1$ and $x > 4.9$
$4.9 < x < 5.1$

14. $x-5 < \varepsilon$ and $x-5 > -\varepsilon$
$x < 5+\varepsilon$ and $x > 5-\varepsilon$
$5-\varepsilon < x < 5+\varepsilon$

15. a. True. **b.** False: Choose $a = 0$.

c. True. **d.** True

16. $\sin(c+h) = \sin c \cos h + \cos c \sin h$

CHAPTER 1 Limits

1.1 Concepts Review

1. L; c

2. 6

3. L; right

4. $\lim\limits_{x \to c} f(x) = M$

Problem Set 1.1

1. $\lim\limits_{x \to 3}(x-5) = -2$

2. $\lim\limits_{t \to -1}(1-2t) = 3$

3. $\lim\limits_{x \to -2}(x^2 + 2x - 1) = (-2)^2 + 2(-2) - 1 = -1$

4. $\lim\limits_{x \to -2}(x^2 + 2t - 1) = (-2)^2 + 2t - 1 = 3 + 2t$

5. $\lim\limits_{t \to -1}\left(t^2 - 1\right) = \left((-1)^2 - 1\right) = 0$

6. $\lim\limits_{t \to -1}\left(t^2 - x^2\right) = \left((-1)^2 - x^2\right) = 1 - x^2$

7. $$\lim_{x \to 2}\frac{x^2-4}{x-2} = \lim_{x \to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2}(x+2) = 2 + 2 = 4$$

8. $$\lim_{t \to -7}\frac{t^2+4t-21}{t+7} = \lim_{t \to -7}\frac{(t+7)(t-3)}{t+7} = \lim_{t \to -7}(t-3) = -7-3 = -10$$

9. $$\lim_{x \to -1}\frac{x^3-4x^2+x+6}{x+1} = \lim_{x \to -1}\frac{(x+1)(x^2-5x+6)}{x+1} = \lim_{x \to -1}(x^2-5x+6) = (-1)^2 - 5(-1) + 6 = 12$$

10. $$\lim_{x \to 0}\frac{x^4+2x^3-x^2}{x^2} = \lim_{x \to 0}(x^2+2x-1) = -1$$

11. $$\lim_{x \to -t}\frac{x^2-t^2}{x+t} = \lim_{x \to -t}\frac{(x+t)(x-t)}{x+t} = \lim_{x \to -t}(x-t) = -t - t = -2t$$

12. $$\lim_{x \to 3}\frac{x^2-9}{x-3} = \lim_{x \to 3}\frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3}(x+3) = 3 + 3 = 6$$

13. $$\lim_{t \to 2}\frac{\sqrt{(t+4)(t-2)^4}}{(3t-6)^2} = \lim_{t \to 2}\frac{(t-2)^2\sqrt{t+4}}{9(t-2)^2} = \lim_{t \to 2}\frac{\sqrt{t+4}}{9} = \frac{\sqrt{2+4}}{9} = \frac{\sqrt{6}}{9}$$

14. $$\lim_{t \to 7^+}\frac{\sqrt{(t-7)^3}}{t-7} = \lim_{t \to 7^+}\frac{(t-7)\sqrt{t-7}}{t-7} = \lim_{t \to 7^+}\sqrt{t-7} = \sqrt{7-7} = 0$$

15. $\lim_{x\to 3}\frac{x^4-18x^2+81}{(x-3)^2}=\lim_{x\to 3}\frac{(x^2-9)^2}{(x-3)^2}$

$=\lim_{x\to 3}\frac{(x-3)^2(x+3)^2}{(x-3)^2}=\lim_{x\to 3}(x+3)^2=(3+3)^2$

$=36$

16. $\lim_{u\to 1}\frac{(3u+4)(2u-2)^3}{(u-1)^2}=\lim_{u\to 1}\frac{8(3u+4)(u-1)^3}{(u-1)^2}$

$=\lim_{u\to 1}8(3u+4)(u-1)=8[3(1)+4](1-1)=0$

17. $\lim_{h\to 0}\frac{(2+h)^2-4}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}$

$=\lim_{h\to 0}\frac{h^2+4h}{h}=\lim_{h\to 0}(h+4)=4$

18. $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$

$=\lim_{h\to 0}\frac{h^2+2xh}{h}=\lim_{h\to 0}(h+2x)=2x$

19.

x	$\frac{\sin x}{2x}$
1.	0.420735
0.1	0.499167
0.01	0.499992
0.001	0.49999992
–1.	0.420735
–0.1	0.499167
–0.01	0.499992
–0.001	0.49999992

$\lim_{x\to 0}\frac{\sin x}{2x}=0.5$

20.

t	$\frac{1-\cos t}{2t}$
1.	0.229849
0.1	0.0249792
0.01	0.00249998
0.001	0.00024999998
–1.	–0.229849
–0.1	–0.0249792
–0.01	–0.00249998
–0.001	–0.00024999998

$\lim_{t\to 0}\frac{1-\cos t}{2t}=0$

21.

x	$(x-\sin x)^2/x^2$
1.	0.0251314
0.1	2.775×10^{-6}
0.01	2.77775×10^{-10}
0.001	2.77778×10^{-14}
–1.	0.0251314
–0.1	2.775×10^{-6}
–0.01	2.77775×10^{-10}
–0.001	2.77778×10^{-14}

$\lim_{x\to 0}\frac{(x-\sin x)^2}{x^2}=0$

22.

x	$(1-\cos x)^2/x^2$
1.	0.211322
0.1	0.00249584
0.01	0.0000249996
0.001	2.5×10^{-7}
–1.	0.211322
–0.1	0.00249584
–0.01	0.0000249996
–0.001	2.5×10^{-7}

$\lim_{x\to 0}\frac{(1-\cos x)^2}{x^2}=0$

23.

t	$(t^2-1)/(\sin(t-1))$
2.	3.56519
1.1	2.1035
1.01	2.01003
1.001	2.001
0	1.1884
0.9	1.90317
0.99	1.99003
0.999	1.999

$\lim_{t\to 1}\frac{t^2-1}{\sin(t-1)}=2$

24.

x	$\frac{x-\sin(x-3)-3}{x-3}$
4.	0.158529
3.1	0.00166583
3.01	0.0000166666
3.001	1.66667×10^{-7}
2.	0.158529
2.9	0.00166583
2.99	0.0000166666
2.999	1.66667×10^{-7}

$$\lim_{x\to 3}\frac{x-\sin(x-3)-3}{x-3}=0$$

25.

x	$(1+\sin(x-3\pi/2))/(x-\pi)$
1. + π	0.4597
0.1 + π	0.0500
0.01 + π	0.0050
0.001 + π	0.0005
–1. + π	–0.4597
–0.1 + π	–0.0500
–0.01 + π	–0.0050
–0.001 + π	–0.0005

$$\lim_{x\to\pi}\frac{1+\sin\left(x-\frac{3\pi}{2}\right)}{x-\pi}=0$$

26.

t	$(1-\cot t)/(1/t)$
1.	0.357907
0.1	–0.896664
0.01	–0.989967
0.001	–0.999
–1.	–1.64209
–0.1	–1.09666
–0.01	–1.00997
–0.001	–1.001

$$\lim_{t\to 0}\frac{1-\cot t}{\frac{1}{t}}=-1$$

27.

x	$(x-\pi/4)^2/(\tan x-1)^2$
$1.+\frac{\pi}{4}$	0.0320244
$0.1+\frac{\pi}{4}$	0.201002
$0.01+\frac{\pi}{4}$	0.245009
$0.001+\frac{\pi}{4}$	0.2495
$-1.+\frac{\pi}{4}$	0.674117
$-0.1+\frac{\pi}{4}$	0.300668
$-0.01+\frac{\pi}{4}$	0.255008
$-0.001+\frac{\pi}{4}$	0.2505

$$\lim_{x\to\frac{\pi}{4}}\frac{\left(x-\frac{\pi}{4}\right)^2}{(\tan x-1)^2}=0.25$$

28.

u	$(2-2\sin u)/3u$
$1.+\frac{\pi}{2}$	0.11921
$0.1+\frac{\pi}{2}$	0.00199339
$0.01+\frac{\pi}{2}$	0.0000210862
$0.001+\frac{\pi}{2}$	2.12072×10^{-7}
$-1.+\frac{\pi}{2}$	0.536908
$-0.1+\frac{\pi}{2}$	0.00226446
$-0.01+\frac{\pi}{2}$	0.0000213564
$-0.001+\frac{\pi}{2}$	2.12342×10^{-7}

$$\lim_{u\to\frac{\pi}{2}}\frac{2-2\sin u}{3u}=0$$

29. **a.** $\lim_{x\to -3} f(x)=2$

b. $f(-3)=1$

c. $f(-1)$ does not exist.

d. $\lim_{x\to -1} f(x)=\frac{5}{2}$

e. $f(1)=2$

f. $\lim_{x\to 1} f(x)$ does not exist.

g. $\lim_{x\to 1^-} f(x)=2$

h. $\lim_{x\to 1^+} f(x)=1$

i. $\lim_{x\to -1^+} f(x)=\frac{5}{2}$

30. **a.** $\lim_{x \to -3} f(x)$ does not exist.

b. $f(-3) = 1$

c. $f(-1) = 1$

d. $\lim_{x \to -1} f(x) = 2$

e. $f(1) = 1$

f. $\lim_{x \to 1} f(x)$ does not exist.

g. $\lim_{x \to 1^-} f(x) = 1$

h. $\lim_{x \to 1^+} f(x)$ does not exist.

i. $\lim_{x \to -1^+} f(x) = 2$

31. **a.** $f(-3) = 2$

b. $f(3)$ is undefined.

c. $\lim_{x \to -3^-} f(x) = 2$

d. $\lim_{x \to -3^+} f(x) = 4$

e. $\lim_{x \to -3} f(x)$ does not exist.

f. $\lim_{x \to 3^+} f(x)$ does not exist.

32. **a.** $\lim_{x \to -1^-} f(x) = -2$

b. $\lim_{x \to -1^+} f(x) = -2$

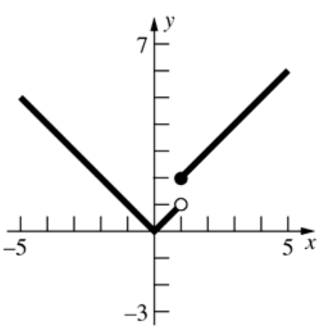
c. $\lim_{x \to -1} f(x) = -2$

d. $f(-1) = -2$

e. $\lim_{x \to 1} f(x) = 0$

f. $f(1) = 0$

33.



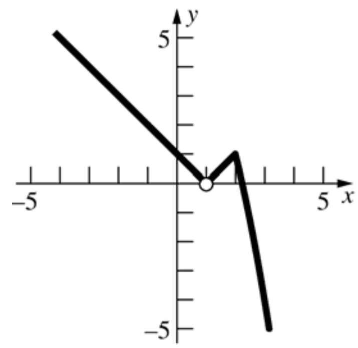
a. $\lim_{x \to 0} f(x) = 0$

b. $\lim_{x \to 1} f(x)$ does not exist.

c. $f(1) = 2$

d. $\lim_{x \to 1^+} f(x) = 2$

34.



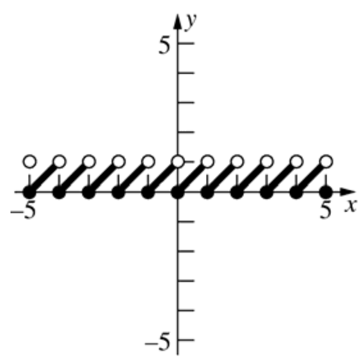
a. $\lim_{x \to 1} g(x) = 0$

b. $g(1)$ does not exist.

c. $\lim_{x \to 2} g(x) = 1$

d. $\lim_{x \to 2^+} g(x) = 1$

35. $f(x) = x - [[x]]$



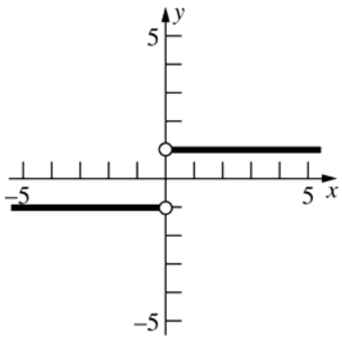
a. $f(0) = 0$

b. $\lim_{x \to 0} f(x)$ does not exist.

c. $\lim_{x \to 0^-} f(x) = 1$

d. $\lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2}$

36. $f(x) = \frac{x}{|x|}$



a. $f(0)$ does not exist.

b. $\lim_{x \to 0} f(x)$ does not exist.

c. $\lim_{x \to 0^-} f(x) = -1$

d. $\lim_{x \to \frac{1}{2}} f(x) = 1$

37. $\lim_{x \to 1} \frac{x^2 - 1}{|x-1|}$ does not exist.

$\lim_{x \to 1^-} \frac{x^2-1}{|x-1|} = -2$ and $\lim_{x \to 1^+} \frac{x^2-1}{|x-1|} = 2$

38. $\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

$$= \lim_{x \to 0} \frac{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}{x(\sqrt{x+2} + \sqrt{2})}$$

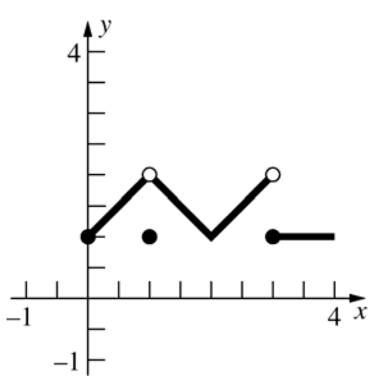
$$= \lim_{x \to 0} \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \to 0} \frac{x}{x(\sqrt{x+2} + \sqrt{2})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{\sqrt{0+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

39. a. $\lim_{x \to 1} f(x)$ does not exist.

b. $\lim_{x \to 0} f(x) = 0$

40.



41. $\lim_{x \to a} f(x)$ exists for $a = -1, 0, 1$.

42. The changed values will not change $\lim_{x \to a} f(x)$ at any a. As x approaches a, the limit is still a^2.

43. a. $\lim_{x \to 1} \frac{|x-1|}{x-1}$ does not exist.

$\lim_{x \to 1^-} \frac{|x-1|}{x-1} = -1$ and $\lim_{x \to 1^+} \frac{|x-1|}{x-1} = 1$

b. $\lim_{x \to 1^-} \frac{|x-1|}{x-1} = -1$

c. $\lim_{x \to 1^-} \frac{x^2 - |x-1| - 1}{|x-1|} = -3$

d. $\lim_{x \to 1^-} \left[\frac{1}{x-1} - \frac{1}{|x-1|} \right]$ does not exist.

44. a. $\lim_{x \to 1^+} \sqrt{x - [\![x]\!]} = 0$

b. $\lim_{x \to 0^+} \left[\!\!\left[\frac{1}{x} \right]\!\!\right]$ does not exist.

c. $\lim_{x \to 0^+} x(-1)^{[\![1/x]\!]} = 0$

d. $\lim_{x \to 0^+} [\![x]\!](-1)^{[\![1/x]\!]} = 0$

45. a) 1 **b)** 0

c) –1 **d)** –1

46. a) Does not exist **b)** 0

c) 1 **d)** 0.556

47. $\lim_{x \to 0} \sqrt{x}$ does not exist since $\sqrt{x}$ is not defined for $x < 0$.

48. $\lim_{x \to 0^+} x^x = 1$

49. $\lim_{x \to 0} \sqrt{|x|} = 0$

50. $\lim_{x \to 0} |x|^x = 1$

51. $\lim_{x \to 0} \frac{\sin 2x}{4x} = \frac{1}{2}$

52. $\lim_{x\to 0}\frac{\sin 5x}{3x}=\frac{5}{3}$

53. $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.

54. $\lim_{x\to 0}x\cos\left(\frac{1}{x}\right)=0$

55. $\lim_{x\to 1}\frac{x^3-1}{\sqrt{2x+2}-2}=6$

56. $\lim_{x\to 0}\frac{x\sin 2x}{\sin(x^2)}=2$

57. $\lim_{x\to 2^-}\frac{x^2-x-2}{|x-2|}=-3$

58. $\lim_{x\to 1^+}\frac{2}{1+2^{1/(x-1)}}=0$

59. $\lim_{x\to 0}\sqrt{x}$; The computer gives a value of 0, but $\lim_{x\to 0^-}\sqrt{x}$ does not exist.

1.2 Concepts Review

1. $L-\varepsilon$; $L+\varepsilon$

2. $0<|x-a|<\delta$; $|f(x)-L|<\varepsilon$

3. $\frac{\varepsilon}{3}$

4. $ma+b$

Problem Set 1.2

1. $0<|t-a|<\delta\Rightarrow|f(t)-M|<\varepsilon$

2. $0<|u-b|<\delta\Rightarrow|g(u)-L|<\varepsilon$

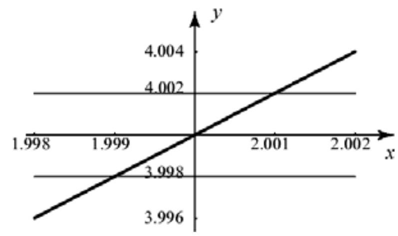
3. $0<|z-d|<\delta\Rightarrow|h(z)-P|<\varepsilon$

4. $0<|y-e|<\delta\Rightarrow|\phi(y)-B|<\varepsilon$

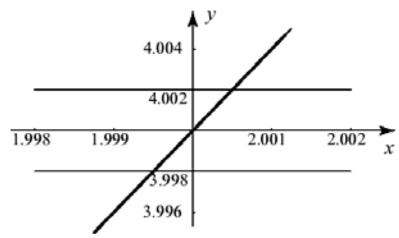
5. $0<c-x<\delta\Rightarrow|f(x)-L|<\varepsilon$

6. $0<t-a<\delta\Rightarrow|g(t)-D|<\varepsilon$

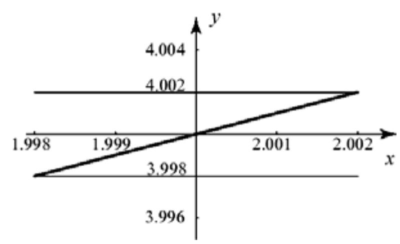
7. If x is within 0.001 of 2, then $2x$ *is* within 0.002 of 4.



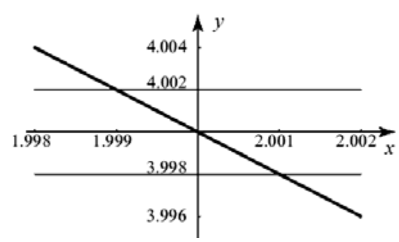
8. If x is within 0.0005 of 2, then x^2 is within 0.002 of 4.



9. If x is within 0.0019 of 2, then $\sqrt{8x}$ is within 0.002 of 4.



10. If x is within 0.001 of 2, then $\frac{8}{x}$ is within 0.002 of 4.



11. $0<|x-0|<\delta\Rightarrow|(2x-1)-(-1)|<\varepsilon$

$|2x-1+1|<\varepsilon\Leftrightarrow|2x|<\varepsilon$

$\Leftrightarrow 2|x|<\varepsilon$

$\Leftrightarrow |x|<\frac{\varepsilon}{2}$

$\delta=\frac{\varepsilon}{2}$; $0<|x-0|<\delta$

$|(2x-1)-(-1)|=|2x|=2|x|<2\delta=\varepsilon$

12. $0<|x+21|<\delta \Rightarrow |(3x-1)-(-64)|<\varepsilon$

$|3x-1+64|<\varepsilon \Leftrightarrow |3x+63|<\varepsilon$

$\Leftrightarrow |3(x+21)|<\varepsilon$

$\Leftrightarrow 3|x+21|<\varepsilon$

$\Leftrightarrow |x+21|<\dfrac{\varepsilon}{3}$

$\delta=\dfrac{\varepsilon}{3};\ 0<|x+21|<\delta$

$|(3x-1)-(-64)|=|3x+63|=3|x+21|<3\delta=\varepsilon$

13. $0<|x-5|<\delta \Rightarrow \left|\dfrac{x^2-25}{x-5}-10\right|<\varepsilon$

$\left|\dfrac{x^2-25}{x-5}-10\right|<\varepsilon \Leftrightarrow \left|\dfrac{(x-5)(x+5)}{x-5}-10\right|<\varepsilon$

$\Leftrightarrow |x+5-10|<\varepsilon$

$\Leftrightarrow |x-5|<\varepsilon$

$\delta=\varepsilon;\ 0<|x-5|<\delta$

$\left|\dfrac{x^2-25}{x-5}-10\right|=\left|\dfrac{(x-5)(x+5)}{x-5}-10\right|=|x+5-10|$

$=|x-5|<\delta=\varepsilon$

14. $0<|x-0|<\delta \Rightarrow \left|\dfrac{2x^2-x}{x}-(-1)\right|<\varepsilon$

$\left|\dfrac{2x^2-x}{x}+1\right|<\varepsilon \Leftrightarrow \left|\dfrac{x(2x-1)}{x}+1\right|<\varepsilon$

$\Leftrightarrow |2x-1+1|<\varepsilon$

$\Leftrightarrow |2x|<\varepsilon$

$\Leftrightarrow 2|x|<\varepsilon$

$\Leftrightarrow |x|<\dfrac{\varepsilon}{2}$

$\delta=\dfrac{\varepsilon}{2};\ 0<|x-0|<\delta$

$\left|\dfrac{2x^2-x}{x}-(-1)\right|=\left|\dfrac{x(2x-1)}{x}+1\right|=|2x-1+1|$

$=|2x|=2|x|<2\delta=\varepsilon$

15. $0<|x-5|<\delta \Rightarrow \left|\dfrac{2x^2-11x+5}{x-5}-9\right|<\varepsilon$

$\left|\dfrac{2x^2-11x+5}{x-5}-9\right|<\varepsilon \Leftrightarrow \left|\dfrac{(2x-1)(x-5)}{x-5}-9\right|<\varepsilon$

$\Leftrightarrow |2x-1-9|<\varepsilon$

$\Leftrightarrow |2(x-5)|<\varepsilon$

$\Leftrightarrow |x-5|<\dfrac{\varepsilon}{2}$

$\delta=\dfrac{\varepsilon}{2};\ 0<|x-5|<\delta$

$\left|\dfrac{2x^2-11x+5}{x-5}-9\right|=\left|\dfrac{(2x-1)(x-5)}{x-5}-9\right|$

$=|2x-1-9|=|2(x-5)|=2|x-5|<2\delta=\varepsilon$

16. $0<|x-1|<\delta \Rightarrow |\sqrt{2x}-\sqrt{2}|<\varepsilon$

$|\sqrt{2x}-\sqrt{2}|<\varepsilon$

$\Leftrightarrow \left|\dfrac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{2x-2}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\Leftrightarrow 2\left|\dfrac{x-1}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\delta=\dfrac{\sqrt{2}\varepsilon}{2};\ 0<|x-1|<\delta$

$|\sqrt{2x}-\sqrt{2}|=\left|\dfrac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right|$

$=\left|\dfrac{2x-2}{\sqrt{2x}+\sqrt{2}}\right|$

$\dfrac{2|x-1|}{\sqrt{2x}+\sqrt{2}}\le\dfrac{2|x-1|}{\sqrt{2}}<\dfrac{2\delta}{\sqrt{2}}=\varepsilon$

17. $0<|x-4|<\delta \Rightarrow \left|\dfrac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right|<\varepsilon$

$\left|\dfrac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right|<\varepsilon \Leftrightarrow \left|\dfrac{\sqrt{2x-1}-\sqrt{7(x-3)}}{\sqrt{x-3}}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{(\sqrt{2x-1}-\sqrt{7(x-3)})(\sqrt{2x-1}+\sqrt{7(x-3)})}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{2x-1-(7x-21)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{-5(x-4)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$$\Leftrightarrow |x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$$

To bound $\frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$, agree that $\delta \le \frac{1}{2}$. If $\delta \le \frac{1}{2}$, then $\frac{7}{2} < x < \frac{9}{2}$, so

$0.65 < \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < 1.65$ and

hence $|x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$

$$\Leftrightarrow |x-4| < \frac{\varepsilon}{1.65}$$

For whatever ε is chosen, let δ be the smaller of $\frac{1}{2}$ and $\frac{\varepsilon}{1.65}$.

$\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{1.65}\right\}$, $0 < |x-4| < \delta$

$$\left|\frac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7}\right| = |x-4| \cdot \frac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$$

$$< |x-4|(1.65) < 1.65\delta \le \varepsilon$$

since $\delta = \frac{1}{2}$ only when $\frac{1}{2} \le \frac{\varepsilon}{1.65}$ so $1.65\delta \le \varepsilon$.

18. $0 < |x-1| < \delta \Rightarrow \left|\frac{14x^2-20x+6}{x-1} - 8\right| < \varepsilon$

$$\left|\frac{14x^2-20x+6}{x-1} - 8\right| < \varepsilon \Leftrightarrow \left|\frac{2(7x-3)(x-1)}{x-1} - 8\right| < \varepsilon$$

$$\Leftrightarrow |2(7x-3)-8| < \varepsilon$$

$$\Leftrightarrow |14(x-1)| < \varepsilon$$

$$\Leftrightarrow 14|x-1| < \varepsilon$$

$$\Leftrightarrow |x-1| < \frac{\varepsilon}{14}$$

$\delta = \frac{\varepsilon}{14}$; $0 < |x-1| < \delta$

$$\left|\frac{14x^2-20x+6}{x-1} - 8\right| = \left|\frac{2(7x-3)(x-1)}{x-1} - 8\right|$$

$$= |2(7x-3)-8|$$

$$= |14(x-1)| = 14|x-1| < 14\delta = \varepsilon$$

19. $0 < |x-1| < \delta \Rightarrow \left|\frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| < \varepsilon$

$$\left|\frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| < \varepsilon$$

$$\Leftrightarrow \left|\frac{(10x-6)(x-1)^2}{(x-1)^2} - 4\right| < \varepsilon$$

$$\Leftrightarrow |10x-6-4| < \varepsilon$$

$$\Leftrightarrow |10(x-1)| < \varepsilon$$

$$\Leftrightarrow 10|x-1| < \varepsilon$$

$$\Leftrightarrow |x-1| < \frac{\varepsilon}{10}$$

$\delta = \frac{\varepsilon}{10}$; $0 < |x-1| < \delta$

$$\left|\frac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| = \left|\frac{(10x-6)(x-1)^2}{(x-1)^2} - 4\right|$$

$$= |10x-6-4| = |10(x-1)|$$

$$= 10|x-1| < 10\delta = \varepsilon$$

20. $0 < |x-1| < \delta \Rightarrow |(2x^2+1)-3| < \varepsilon$

$$|2x^2+1-3| = |2x^2-2| = 2|x+1||x-1|$$

To bound $|2x+2|$, agree that $\delta \le 1$.

$|x-1| < \delta$ implies

$$|2x+2| = |2x-2+4|$$

$$\le |2x-2| + |4|$$

$$< 2 + 4 = 6$$

$\delta \le \frac{\varepsilon}{6}$; $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$; $0 < |x-1| < \delta$

$$|(2x^2+1)-3| = |2x^2-2|$$

$$= |2x+2||x-1| < 6 \cdot \left(\frac{\varepsilon}{6}\right) = \varepsilon$$

21. $0 < |x+1| < \delta \Rightarrow |(x^2 - 2x - 1) - 2| < \varepsilon$

$|x^2 - 2x - 1 - 2| = |x^2 - 2x - 3| = |x+1||x-3|$

To bound $|x-3|$, agree that $\delta \le 1$.

$|x+1| < \delta$ implies

$|x-3| = |x+1-4| \le |x+1| + |-4| < 1 + 4 = 5$

$\delta \le \frac{\varepsilon}{5}; \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}; 0 < |x+1| < \delta$

$|(x^2 - 2x - 1) - 2| = |x^2 - 2x - 3|$

$= |x+1||x-3| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$

22. $0 < |x| < \delta \Rightarrow |x^4 - 0| = |x^4| < \varepsilon$

$|x^4| = |x||x^3|$. To bound $|x^3|$, agree that $\delta \le 1$. $|x| < \delta \le 1$ implies $|x^3| = |x|^3 \le 1$ so $\delta \le \varepsilon$.

$\delta = \min\{1, \varepsilon\}; 0 < |x| < \delta \Rightarrow |x^4| = |x||x^3| < \varepsilon \cdot 1 = \varepsilon$

23. Choose $\varepsilon > 0$. Then since $\lim_{x \to c} f(x) = L$, there is some $\delta_1 > 0$ such that

$0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$.

Since $\lim_{x \to c} f(x) = M$, there is some $\delta_2 > 0$ such that $0 < |x-c| < \delta_2 \Rightarrow |f(x) - M| < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and choose x_0 such that $0 < |x_0 - c| < \delta$.

Thus, $|f(x_0) - L| < \varepsilon \Rightarrow -\varepsilon < f(x_0) - L < \varepsilon$

$\Rightarrow -f(x_0) - \varepsilon < -L < -f(x_0) + \varepsilon$

$\Rightarrow f(x_0) - \varepsilon < L < f(x_0) + \varepsilon.$

Similarly,

$f(x_0) - \varepsilon < M < f(x_0) + \varepsilon$.

Thus,

$-2\varepsilon < L - M < 2\varepsilon$. As $\varepsilon \Rightarrow 0$, $L - M \to 0$, so $L = M$.

24. Since $\lim_{x \to c} G(x) = 0$, then given any ε > 0, we can find $\delta > 0$ such that whenever $|x-c| < \delta, |G(x)| < \varepsilon$.

Take any $\varepsilon > 0$ and the corresponding δ that works for $G(x)$, then $|x-c| < \delta$ implies

$|F(x) - 0| = |F(x)| \le |G(x)| < \varepsilon$ since $\lim_{x \to c} G(x) = 0$.

Thus, $\lim_{x \to c} F(x) = 0$.

25. For all $x \ne 0$, $0 \le \sin^2\left(\frac{1}{x}\right) \le 1$ so

$x^4 \sin^2\left(\frac{1}{x}\right) \le x^4$ for all $x \ne 0$. By Problem 18,

$\lim_{x \to 0} x^4 = 0$, so, by Problem 20,

$\lim_{x \to 0} x^4 \sin^2\left(\frac{1}{x}\right) = 0.$

26. $0 < x < \delta \Rightarrow |\sqrt{x} - 0| = |\sqrt{x}| = \sqrt{x} < \varepsilon$

For $x > 0$, $(\sqrt{x})^2 = x$.

$\sqrt{x} < \varepsilon \Leftrightarrow (\sqrt{x})^2 = x < \varepsilon^2$

$\delta = \varepsilon^2; 0 < x < \delta \Rightarrow \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$

27. $\lim_{x \to 0^+} |x| : 0 < x < \delta \Rightarrow ||x| - 0| < \varepsilon$

For $x \ge 0$, $|x| = x$.

$\delta = \varepsilon; 0 < x < \delta \Rightarrow ||x| - 0| = |x| = x < \delta = \varepsilon$

Thus, $\lim_{x \to 0^+} |x| = 0.$

$\lim_{x \to 0^-} |x| : 0 < 0 - x < \delta \Rightarrow ||x| - 0| < \varepsilon$

For $x < 0$, $|x| = -x$; note also that $||x|| = |x|$ since $|x| \ge 0$.

$\delta = \varepsilon; 0 < -x < \delta \Rightarrow ||x|| = |x| = -x < \delta = \varepsilon$

Thus, $\lim_{x \to 0^-} |x| = 0,$

since $\lim_{x \to 0^+} |x| = \lim_{x \to 0^-} |x| = 0, \lim_{x \to 0} |x| = 0.$

28. Choose $\varepsilon > 0$. Since $\lim_{x \to a} g(x) = 0$ there is some $\delta_1 > 0$ such that

$0 < |x-a| < \delta_1 \Rightarrow |g(x) - 0| < \frac{\varepsilon}{B}$.

Let $\delta = \min\{1, \delta_1\}$, then $|f(x)| < B$ for $|x-a| < \delta$ or $|x-a| < \delta \Rightarrow |f(x)| < B$. Thus,

$|x-a| < \delta \Rightarrow |f(x)g(x) - 0| = |f(x)g(x)|$

$= |f(x)||g(x)| < B \cdot \frac{\varepsilon}{B} = \varepsilon$ so $\lim_{x \to a} f(x)g(x) = 0.$

29. Choose $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there is a $\delta > 0$ such that for $0 < |x-a| < \delta$, $|f(x) - L| < \varepsilon$. That is, for
$a - \delta < x < a$ or $a < x < a + \delta$,
$L - \varepsilon < f(x) < L + \varepsilon$.
Let $f(a) = A$,
$M = \max\{|L-\varepsilon|, |L+\varepsilon|, |A|\}$, $c = a - \delta$,
$d = a + \delta$. Then for x in (c, d), $|f(x)| \le M$, since either $x = a$, in which case
$|f(x)| = |f(a)| = |A| \le M$ or $0 < |x-a| < \delta$ so
$L - \varepsilon < f(x) < L + \varepsilon$ and $|f(x)| < M$.

30. Suppose that $L > M$. Then $L - M = \alpha > 0$. Now take $\varepsilon < \frac{\alpha}{2}$ and $\delta = \min\{\delta_1, \delta_2\}$ where
$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$ and
$0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$.
Thus, for $0 < |x-a| < \delta$,
$L - \varepsilon < f(x) < L + \varepsilon$ and $M - \varepsilon < g(x) < M + \varepsilon$.
Combine the inequalities and use the fact that $f(x) \le g(x)$ to get
$L - \varepsilon < f(x) \le g(x) < M + \varepsilon$ which leads to
$L - \varepsilon < M + \varepsilon$ or $L - M < 2\varepsilon$.
However,
$L - M = \alpha > 2\varepsilon$
which is a contradiction.
Thus $L \le M$.

31. (b) and (c) are equivalent to the definition of limit.

32. For every $\varepsilon > 0$ and $\delta > 0$ there is some x with $0 < |x-c| < \delta$ such that $|f(x) - L| > \varepsilon$.

33. **a.** $g(x) = \dfrac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}$

b. No, because $\dfrac{x+6}{x^4 - 4x^3 + x^2 + x + 6} + 1$ has an asymptote at $x \approx 3.49$.

c. If $\delta \le \frac{1}{4}$, then $2.75 < x < 3$ or $3 < x < 3.25$ and by graphing

$$y = |g(x)| = \left|\frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right|$$

on the interval [2.75, 3.25], we see that

$$0 < \left|\frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right| < 3$$

so m must be at least three.

1.3 Concepts Review

1. 48

2. 4

3. -8; $-4 + 5c$

4. 0

Problem Set 1.3

1. $\lim_{x\to 1}(2x+1)$ 4
$= \lim_{x\to 1} 2x + \lim_{x\to 1} 1$ 3
$= 2\lim_{x\to 1} x + \lim_{x\to 1} 1$ 2,1
$= 2(1) + 1 = 3$

2. $\lim_{x\to -1}(3x^2 - 1)$ 5
$= \lim_{x\to -1} 3x^2 - \lim_{x\to -1} 1$ 3
$= 3\lim_{x\to -1} x^2 - \lim_{x\to -1} 1$ 8
$= 3\left(\lim_{x\to -1} x\right)^2 - \lim_{x\to -1} 1$ 2, 1
$= 3(-1)^2 - 1 = 2$

3. $\lim_{x\to 0}[(2x+1)(x-3)]$ 6
$= \lim_{x\to 0}(2x+1) \cdot \lim_{x\to 0}(x-3)$ 4, 5
$= \left(\lim_{x\to 0} 2x + \lim_{x\to 0} 1\right) \cdot \left(\lim_{x\to 0} x - \lim_{x\to 0} 3\right)$ 3
$= \left(2\lim_{x\to 0} x + \lim_{x\to 0} 1\right) \cdot \left(\lim_{x\to 0} x - \lim_{x\to 0} 3\right)$ 2, 1
$= [2(0) + 1](0 - 3) = -3$

4. $\lim_{x\to\sqrt{2}}[(2x^2+1)(7x^2+13)]$ 6
$= \lim_{x\to\sqrt{2}}(2x^2+1) \cdot \lim_{x\to\sqrt{2}}(7x^2+13)$ 4, 3
$= \left(2\lim_{x\to\sqrt{2}} x^2 + \lim_{x\to\sqrt{2}} 1\right) \cdot \left(7\lim_{x\to\sqrt{2}} x^2 + \lim_{x\to\sqrt{2}} 13\right)$ 8,1
$= \left[2\left(\lim_{x\to\sqrt{2}} x\right)^2 + 1\right]\left[7\left(\lim_{x\to\sqrt{2}} x\right)^2 + 13\right]$ 2
$= [2(\sqrt{2})^2 + 1][7(\sqrt{2})^2 + 13] = 135$

5. $\lim_{x\to 2} \frac{2x+1}{5-3x}$ 7

$= \frac{\lim_{x\to 2}(2x+1)}{\lim_{x\to 2}(5-3x)}$ 4, 5

$= \frac{\lim_{x\to 2} 2x + \lim_{x\to 2} 1}{\lim_{x\to 2} 5 - \lim_{x\to 2} 3x}$ 3, 1

$= \frac{2\lim_{x\to 2} x + 1}{5 - 3\lim_{x\to 2} x}$ 2

$= \frac{2(2)+1}{5-3(2)} = -5$

6. $\lim_{x\to -3} \frac{4x^3+1}{7-2x^2}$ 7

$= \frac{\lim_{x\to -3}(4x^3+1)}{\lim_{x\to -3}(7-2x^2)}$ 4, 5

$= \frac{\lim_{x\to -3} 4x^3 + \lim_{x\to -3} 1}{\lim_{x\to -3} 7 - \lim_{x\to -3} 2x^2}$ 3, 1

$= \frac{4\lim_{x\to -3} x^3 + 1}{7 - 2\lim_{x\to -3} x^2}$ 8

$= \frac{4\left(\lim_{x\to -3} x\right)^3 + 1}{7 - 2\left(\lim_{x\to -3} x\right)^2}$ 2

$= \frac{4(-3)^3+1}{7-2(-3)^2} = \frac{107}{11}$

7. $\lim_{x\to 3} \sqrt{3x-5}$ 9

$= \sqrt{\lim_{x\to 3}(3x-5)}$ 5, 3

$= \sqrt{3\lim_{x\to 3} x - \lim_{x\to 3} 5}$ 2, 1

$= \sqrt{3(3)-5} = 2$

8. $\lim_{x\to -3} \sqrt{5x^2+2x}$ 9

$= \sqrt{\lim_{x\to -3}(5x^2+2x)}$ 4, 3

$= \sqrt{5\lim_{x\to -3} x^2 + 2\lim_{x\to -3} x}$ 8

$= \sqrt{5\left(\lim_{x\to -3} x\right)^2 + 2\lim_{x\to -3} x}$ 2

$= \sqrt{5(-3)^2 + 2(-3)} = \sqrt{39}$

9. $\lim_{t\to -2} (2t^3+15)^{13}$ 8

$= \left[\lim_{t\to -2}(2t^3+15)\right]^{13}$ 4, 3

$= \left[2\lim_{t\to -2} t^3 + \lim_{t\to -2} 15\right]^{13}$ 8

$= \left[2\left(\lim_{t\to -2} t\right)^3 + \lim_{t\to -2} 15\right]^{13}$ 2, 1

$= [2(-2)^3+15]^{13} = -1$

10. $\lim_{w\to -2} \sqrt{-3w^3+7w^2}$ 9

$= \sqrt{\lim_{w\to -2}(-3w^3+7w^2)}$ 4, 3

$= \sqrt{-3\lim_{w\to -2} w^3 + 7\lim_{w\to -2} w^2}$ 8

$= \sqrt{-3\left(\lim_{w\to -2} w\right)^3 + 7\left(\lim_{w\to -2} w\right)^2}$ 2

$= \sqrt{-3(-2)^3+7(-2)^2} = 2\sqrt{13}$

11. $\lim_{y\to 2} \left(\frac{4y^3+8y}{y+4}\right)^{1/3}$ 9

$= \left(\lim_{y\to 2} \frac{4y^3+8y}{y+4}\right)^{1/3}$ 7

$= \left[\frac{\lim_{y\to 2}(4y^3+8y)}{\lim_{y\to 2}(y+4)}\right]^{1/3}$ 4, 3

$= \left(\frac{4\lim_{y\to 2} y^3 + 8\lim_{y\to 2} y}{\lim_{y\to 2} y + \lim_{y\to 2} 4}\right)^{1/3}$ 8, 1

$= \left[\frac{4\left(\lim_{y\to 2} y\right)^3 + 8\lim_{y\to 2} y}{\lim_{y\to 2} y + 4}\right]^{1/3}$ 2

$= \left[\frac{4(2)^3+8(2)}{2+4}\right]^{1/3} = 2$

12. $\lim_{w\to 5}(2w^4-9w^3+19)^{-1/2}$

$= \lim_{w\to 5}\frac{1}{\sqrt{2w^4-9w^3+19}}$ 7

$= \frac{\lim_{w\to 5} 1}{\lim_{w\to 5}\sqrt{2w^4-9w^3+19}}$ 1, 9

$= \frac{1}{\sqrt{\lim_{w\to 5}(2w^4-9w^3+19)}}$ 4,5

$= \frac{1}{\sqrt{\lim_{w\to 5} 2w^4 - \lim_{w\to 5} 9w^3 + \lim_{w\to 5} 19}}$ 1,3

$= \frac{1}{\sqrt{2\lim_{w\to 5} w^4 - 9\lim_{w\to 5} w^3 + 19}}$ 8

$= \frac{1}{\sqrt{2\left(\lim_{w\to 5} w\right)^4 - 9\left(\lim_{w\to 5} w\right)^3 + 19}}$ 2

$= \frac{1}{\sqrt{2(5)^4-9(5)^3+19}}$

$= \frac{1}{\sqrt{144}} = \frac{1}{12}$

13. $\lim_{x\to 2}\frac{x^2-4}{x^2+4} = \frac{\lim_{x\to 2}\left(x^2-4\right)}{\lim_{x\to 2}\left(x^2+4\right)} = \frac{4-4}{4+4} = 0$

14. $\lim_{x\to 2}\frac{x^2-5x+6}{x-2} = \lim_{x\to 2}\frac{(x-3)(x-2)}{(x-2)}$
$= \lim_{x\to 2}(x-3) = -1$

15. $\lim_{x\to -1}\frac{x^2-2x-3}{x+1} = \lim_{x\to -1}\frac{(x-3)(x+1)}{(x+1)}$
$= \lim_{x\to -1}(x-3) = -4$

16. $\lim_{x\to -1}\frac{x^2+x}{x^2+1} = \frac{\lim_{x\to -1}\left(x^2+x\right)}{\lim_{x\to -1}\left(x^2+1\right)} = \frac{0}{2} = 0$

17. $\lim_{x\to -1}\frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+7)} = \lim_{x\to -1}\frac{x-3}{x+7}$
$= \frac{-1-3}{-1+7} = -\frac{2}{3}$

18. $\lim_{x\to 2}\frac{x^2+7x+10}{x+2} = \lim_{x\to 2}\frac{(x+2)(x+5)}{x+2}$
$= \lim_{x\to 2}(x+5) = 7$

19. $\lim_{x\to 1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\frac{(x+2)(x-1)}{(x+1)(x-1)}$
$= \lim_{x\to 1}\frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}$

20. $\lim_{x\to -3}\frac{x^2-14x-51}{x^2-4x-21} = \lim_{x\to -3}\frac{(x+3)(x-17)}{(x+3)(x-7)}$
$= \lim_{x\to -3}\frac{x-17}{x-7} = \frac{-3-17}{-3-7} = 2$

21. $\lim_{u\to -2}\frac{u^2-ux+2u-2x}{u^2-u-6} = \lim_{u\to -2}\frac{(u+2)(u-x)}{(u+2)(u-3)}$
$= \lim_{u\to -2}\frac{u-x}{u-3} = \frac{x+2}{5}$

22. $\lim_{x\to 1}\frac{x^2+ux-x-u}{x^2+2x-3} = \lim_{x\to 1}\frac{(x-1)(x+u)}{(x-1)(x+3)}$
$= \lim_{x\to 1}\frac{x+u}{x+3} = \frac{1+u}{1+3} = \frac{u+1}{4}$

23. $\lim_{x\to \pi}\frac{2x^2-6x\pi+4\pi^2}{x^2-\pi^2} = \lim_{x\to \pi}\frac{2(x-\pi)(x-2\pi)}{(x-\pi)(x+\pi)}$
$= \lim_{x\to \pi}\frac{2(x-2\pi)}{x+\pi} = \frac{2(\pi-2\pi)}{\pi+\pi} = -1$

24. $\lim_{w\to -2}\frac{(w+2)(w^2-w-6)}{w^2+4w+4}$
$= \lim_{w\to -2}\frac{(w+2)^2(w-3)}{(w+2)^2} = \lim_{w\to -2}(w-3)$
$= -2-3 = -5$

25. $\lim_{x\to a}\sqrt{f^2(x)+g^2(x)}$
$= \sqrt{\lim_{x\to a} f^2(x) + \lim_{x\to a} g^2(x)}$
$= \sqrt{\left(\lim_{x\to a} f(x)\right)^2 + \left(\lim_{x\to a} g(x)\right)^2}$
$= \sqrt{(3)^2+(-1)^2} = \sqrt{10}$

26. $\lim_{x\to a}\frac{2f(x)-3g(x)}{f(x)+g(x)} = \frac{\lim_{x\to a}[2f(x)-3g(x)]}{\lim_{x\to a}[f(x)+g(x)]}$
$= \frac{2\lim_{x\to a} f(x) - 3\lim_{x\to a} g(x)}{\lim_{x\to a} f(x) + \lim_{x\to a} g(x)} = \frac{2(3)-3(-1)}{3+(-1)} = \frac{9}{2}$

27. $\lim_{x\to a} \sqrt[3]{g(x)}[f(x)+3] = \lim_{x\to a}\sqrt[3]{g(x)} \cdot \lim_{x\to a}[f(x)+3]$

$= \sqrt[3]{\lim_{x\to a} g(x)} \cdot \left[\lim_{x\to a} f(x) + \lim_{x\to a} 3\right] = \sqrt[3]{-1}\cdot(3+3)$

$= -6$

28. $\lim_{x\to a}[f(x)-3]^4 = \left[\lim_{x\to a}(f(x)-3)\right]^4$

$= \left[\lim_{x\to a} f(x) - \lim_{x\to a} 3\right]^4 = (3-3)^4 = 0$

29. $\lim_{t\to a}\left[|f(t)| + |3g(t)|\right] = \lim_{t\to a}|f(t)| + 3\lim_{t\to a}|g(t)|$

$= \left|\lim_{t\to a} f(t)\right| + 3\left|\lim_{t\to a} g(t)\right|$

$= |3| + 3|-1| = 6$

30. $\lim_{u\to a}[f(u)+3g(u)]^3 = \left(\lim_{u\to a}[f(u)+3g(u)]\right)^3$

$= \left[\lim_{u\to a} f(u) + 3\lim_{u\to a} g(u)\right]^3 = [3+3(-1)]^3 = 0$

31. $\lim_{x\to 2}\frac{3x^2-12}{x-2} = \lim_{x\to 2}\frac{3(x-2)(x+2)}{x-2}$

$= 3\lim_{x\to 2}(x+2) = 3(2+2) = 12$

32. $\lim_{x\to 2}\frac{(3x^2+2x+1)-17}{x-2} = \lim_{x\to 2}\frac{3x^2+2x-16}{x-2}$

$= \lim_{x\to 2}\frac{(3x+8)(x-2)}{x-2} = \lim_{x\to 2}(3x+8)$

$= 3\lim_{x\to 2} x + 8 = 3(2)+8 = 14$

33. $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim_{x\to 2}\frac{\frac{2-x}{2x}}{x-2} = \lim_{x\to 2}\frac{-\frac{x-2}{2x}}{x-2}$

$= \lim_{x\to 2} -\frac{1}{2x} = \frac{-1}{2\lim_{x\to 2} x} = \frac{-1}{2(2)} = -\frac{1}{4}$

34. $\lim_{x\to 2}\frac{\frac{3}{x^2}-\frac{3}{4}}{x-2} = \lim_{x\to 2}\frac{\frac{3(4-x^2)}{4x^2}}{x-2} = \lim_{x\to 2}\frac{\frac{-3(x+2)(x-2)}{4x^2}}{x-2}$

$= \lim_{x\to 2}\frac{-3(x+2)}{4x^2} = \frac{-3\left(\lim_{x\to 2} x+2\right)}{4\left(\lim_{x\to 2} x\right)^2} = \frac{-3(2+2)}{4(2)^2}$

$= -\frac{3}{4}$

35. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M.$

$|f(x)g(x) - LM| \le |g(x)||f(x)-L| + |L||g(x)-M|$

as shown in the text. Choose $\varepsilon_1 = 1.$ Since $\lim_{x\to c} g(x) = M$, there is some $\delta_1 > 0$ such that if $0 < |x-c| < \delta_1$, $|g(x)-M| < \varepsilon_1 = 1$ or

$M - 1 < g(x) < M + 1$

$|M-1| \le |M| + 1$ and $|M+1| \le |M| + 1$ so for $0 < |x-c| < \delta_1, |g(x)| < |M| + 1.$ Choose $\varepsilon > 0.$

Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, there exist δ_2 and δ_3 such that $0 < |x-c| < \delta_2 \Rightarrow$

$|f(x)-L| < \frac{\varepsilon}{|L|+|M|+1}$ and $0 < |x-c| < \delta_3 \Rightarrow$

$|g(x)-M| < \frac{\varepsilon}{|L|+|M|+1}.$ Let

$\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then $0 < |x-c| < \delta \Rightarrow$

$|f(x)g(x) - LM| \le |g(x)||f(x)-L| + |L||g(x)-M|$

$< (|M|+1)\frac{\varepsilon}{|L|+|M|+1} + |L|\frac{\varepsilon}{|L|+|M|+1} = \varepsilon$

Hence,

$\lim_{x\to c} f(x)g(x) = LM = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

36. Say $\lim_{x\to c} g(x) = M$, $M \ne 0$, and choose

$\varepsilon_1 = \frac{1}{2}|M|.$

There is some $\delta_1 > 0$ such that

$0 < |x-c| < \delta_1 \Rightarrow |g(x)-M| < \varepsilon_1 = \frac{1}{2}|M|$ or

$M - \frac{1}{2}|M| < g(x) < M + \frac{1}{2}|M|.$

$\left|M - \frac{1}{2}|M|\right| \ge \left|\frac{1}{2}|M|\right|$ and $\left|M + \frac{1}{2}|M|\right| \ge \left|\frac{1}{2}|M|\right|$

so $|g(x)| > \frac{1}{2}|M|$ and $\frac{1}{|g(x)|} < \frac{2}{|M|}$

Choose $\varepsilon > 0.$

Since $\lim_{x\to c} g(x) = M$ there is $\delta_2 > 0$ such that

$0 < |x-c| < \delta_2 \Rightarrow |g(x)-M| < \frac{1}{2}M^2.$

Let $\delta = \min\{\delta_1, \delta_2\}$, then

$0 < |x-c| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M-g(x)}{g(x)M}\right|$

$= \frac{1}{|M||g(x)|}|g(x)-M| < \frac{2}{M^2}|g(x)-M| = \frac{2}{M^2}\cdot\frac{1}{2}M^2\varepsilon$

$= \varepsilon$

Thus, $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M} = \frac{1}{\lim_{x\to c} g(x)}$.

Using statement 6 and the above result,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} f(x) \cdot \lim_{x\to c} \frac{1}{g(x)}$$

$$= \lim_{x\to c} f(x) \cdot \frac{1}{\lim_{x\to c} g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

37. $\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c} f(x) = \lim_{x\to c} L$

$\Leftrightarrow \lim_{x\to c} f(x) - \lim_{x\to c} L = 0$

$\Leftrightarrow \lim_{x\to c} [f(x) - L] = 0$

38. $\lim_{x\to c} f(x) = 0 \Leftrightarrow \left[\lim_{x\to c} f(x)\right]^2 = 0$

$\Leftrightarrow \lim_{x\to c} f^2(x) = 0$

$\Leftrightarrow \sqrt{\lim_{x\to c} f^2(x)} = 0$

$\Leftrightarrow \lim_{x\to c} \sqrt{f^2(x)} = 0$

$\Leftrightarrow \lim_{x\to c} |f(x)| = 0$

39. $\lim_{x\to c} |x| = \sqrt{\left(\lim_{x\to c} |x|\right)^2} = \sqrt{\lim_{x\to c} |x|^2} = \sqrt{\lim_{x\to c} x^2}$

$= \sqrt{\left(\lim_{x\to c} x\right)^2} = \sqrt{c^2} = |c|$

40. **a.** If $f(x) = \frac{x+1}{x-2}, g(x) = \frac{x-5}{x-2}$ and $c = 2$, then $\lim_{x\to c} [f(x) + g(x)]$ exists, but neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exists.

b. If $f(x) = \frac{2}{x}, g(x) = x,$ and $c = 0$, then $\lim_{x\to c} [f(x) \cdot g(x)]$ exists, but $\lim_{x\to c} f(x)$ does not exist.

41. $\lim_{x\to -3^+} \frac{\sqrt{3+x}}{x} = \frac{\sqrt{3-3}}{-3} = 0$

42. $\lim_{x\to -\pi^+} \frac{\sqrt{\pi^3 + x^3}}{x} = \frac{\sqrt{\pi^3 + (-\pi)^3}}{-\pi} = 0$

43. $\lim_{x\to 3^+} \frac{x-3}{\sqrt{x^2-9}} = \lim_{x\to 3^+} \frac{(x-3)\sqrt{x^2-9}}{x^2-9}$

$= \lim_{x\to 3^+} \frac{(x-3)\sqrt{x^2-9}}{(x-3)(x+3)} = \lim_{x\to 3^+} \frac{\sqrt{x^2-9}}{x+3}$

$= \frac{\sqrt{3^2-9}}{3+3} = 0$

44. $\lim_{x\to 1^-} \frac{\sqrt{1+x}}{4+4x} = \frac{\sqrt{1+1}}{4+4(1)} = \frac{\sqrt{2}}{8}$

45. $\lim_{x\to 2^+} \frac{(x^2+1)[\![x]\!]}{(3x-1)^2} = \frac{(2^2+1)[\![2]\!]}{(3\cdot 2-1)^2} = \frac{5\cdot 2}{5^2} = \frac{2}{5}$

46. $\lim_{x\to 3^-} (x - [\![x]\!]) = \lim_{x\to 3^-} x - \lim_{x\to 3^-} [\![x]\!] = 3 - 2 = 1$

47. $\lim_{x\to 0^-} \frac{x}{|x|} = -1$

48. $\lim_{x\to 3^+} [\![x^2 + 2x]\!] = [\![3^2 + 2\cdot 3]\!] = 15$

49. $f(x)g(x) = 1; g(x) = \frac{1}{f(x)}$

$\lim_{x\to a} g(x) = 0 \Leftrightarrow \lim_{x\to a} \frac{1}{f(x)} = 0$

$\Leftrightarrow \frac{1}{\lim_{x\to a} f(x)} = 0$

No value satisfies this equation, so $\lim_{x\to a} f(x)$ must not exist.

50. R has the vertices $\left(\pm\frac{x}{2}, \pm\frac{1}{2}\right)$

Each side of Q has length $\sqrt{x^2+1}$ so the perimeter of Q is $4\sqrt{x^2+1}$. R has two sides of length 1 and two sides of length $\sqrt{x^2}$ so the perimeter of R is $2 + 2\sqrt{x^2}$.

$\lim_{x\to 0^+} \frac{\text{perimeter of } R}{\text{perimeter of } Q} = \lim_{x\to 0^+} \frac{2\sqrt{x^2}+2}{4\sqrt{x^2+1}}$

$= \frac{2\sqrt{0^2}+2}{4\sqrt{0^2+1}} = \frac{2}{4} = \frac{1}{2}$

51. a. $NO = \sqrt{(0-0)^2 + (1-0)^2} = 1$

$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

$= \sqrt{x^2 + x}$

$NP = \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2 + y^2 - 2y + 1}$

$= \sqrt{x^2 + x - 2\sqrt{x} + 1}$

$MO = \sqrt{(1-0)^2 + (0-0)^2} = 1$

$MP = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{y^2 + x^2 - 2x + 1}$

$= \sqrt{x^2 - x + 1}$

$$\lim_{x\to 0^+} \frac{\text{perimeter of } \Delta NOP}{\text{perimeter of } \Delta MOP}$$

$$= \lim_{x\to 0^+} \frac{1 + \sqrt{x^2 + x} + \sqrt{x^2 + x - 2\sqrt{x} + 1}}{1 + \sqrt{x^2 + x} + \sqrt{x^2 - x + 1}}$$

$$= \frac{1 + \sqrt{1}}{1 + \sqrt{1}} = 1$$

b. Area of $\Delta NOP = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area of $\Delta MOP = \frac{1}{2}(1)(y) = \frac{\sqrt{x}}{2}$

$$\lim_{x\to 0^+} \frac{\text{area of } \Delta NOP}{\text{area of } \Delta MOP} = \lim_{x\to 0^+} \frac{\frac{x}{2}}{\frac{\sqrt{x}}{2}} = \lim_{x\to 0^+} \frac{x}{\sqrt{x}}$$

$$= \lim_{x\to 0^+} \sqrt{x} = 0$$

1.4 Concepts Review

1. 0

2. 1

3. the denominator is 0 when $t = 0$.

4. 1

Problem Set 1.4

1. $\lim_{x\to 0} \frac{\cos x}{x+1} = \frac{1}{1} = 1$

2. $\lim_{\theta\to\pi/2} \theta\cos\theta = \frac{\pi}{2}\cdot 0 = 0$

3. $\lim_{t\to 0} \frac{\cos^2 t}{1+\sin t} = \frac{\cos^2 0}{1+\sin 0} = \frac{1}{1+0} = 1$

4. $$\lim_{x\to 0} \frac{3x\tan x}{\sin x} = \lim_{x\to 0} \frac{3x(\sin x/\cos x)}{\sin x} = \lim_{x\to 0} \frac{3x}{\cos x}$$
$$= \frac{0}{1} = 0$$

5. $$\lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

6. $$\lim_{\theta\to 0} \frac{\sin 3\theta}{2\theta} = \lim_{\theta\to 0} \frac{3}{2}\cdot\frac{\sin 3\theta}{3\theta} = \frac{3}{2}\lim_{\theta\to 0} \frac{\sin 3\theta}{3\theta}$$
$$= \frac{3}{2}\cdot 1 = \frac{3}{2}$$

7. $$\lim_{\theta\to 0} \frac{\sin 3\theta}{\tan\theta} = \lim_{\theta\to 0} \frac{\sin 3\theta}{\frac{\sin\theta}{\cos\theta}} = \lim_{\theta\to 0} \frac{\cos\theta\sin 3\theta}{\sin\theta}$$
$$= \lim_{\theta\to 0}\left[\cos\theta\cdot 3\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right]$$
$$= 3\lim_{\theta\to 0}\left[\cos\theta\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right] = 3\cdot 1\cdot 1\cdot 1 = 3$$

8. $$\lim_{\theta\to 0} \frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\frac{\sin 5\theta}{\cos 5\theta}}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\sin 5\theta}{\cos 5\theta\sin 2\theta}$$
$$= \lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot 5\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{1}{2}\cdot\frac{2\theta}{\sin 2\theta}\right]$$
$$= \frac{5}{2}\lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{2\theta}{\sin 2\theta}\right]$$
$$= \frac{5}{2}\cdot 1\cdot 1\cdot 1 = \frac{5}{2}$$

9. $$\lim_{\theta\to 0} \frac{\cot\pi\theta\sin\theta}{2\sec\theta} = \lim_{\theta\to 0} \frac{\frac{\cos\pi\theta}{\sin\pi\theta}\sin\theta}{\frac{2}{\cos\theta}}$$
$$= \lim_{\theta\to 0} \frac{\cos\pi\theta\sin\theta\cos\theta}{2\sin\pi\theta}$$
$$= \lim_{\theta\to 0}\left[\frac{\cos\pi\theta\cos\theta}{2}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\pi}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$$
$$= \frac{1}{2\pi}\lim_{\theta\to 0}\left[\cos\pi\theta\cos\theta\cdot\frac{\sin\theta}{\theta}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$$
$$= \frac{1}{2\pi}\cdot 1\cdot 1\cdot 1\cdot 1 = \frac{1}{2\pi}$$

10. $$\lim_{t\to 0} \frac{\sin^2 3t}{2t} = \lim_{t\to 0} \frac{9t}{2}\cdot\frac{\sin 3t}{3t}\cdot\frac{\sin 3t}{3t} = 0\cdot 1\cdot 1 = 0$$

11. $$\lim_{t\to 0} \frac{\tan^2 3t}{2t} = \lim_{t\to 0} \frac{\sin^2 3t}{(2t)(\cos^2 3t)}$$
$$= \lim_{t\to 0} \frac{3(\sin 3t)}{2\cos^2 3t}\cdot\frac{\sin 3t}{3t} = 0\cdot 1 = 0$$

12. $$\lim_{t\to 0} \frac{\tan 2t}{\sin 2t - 1} = \frac{0}{-1} = 0$$

13. $\lim_{t\to 0}\frac{\sin(3t)+4t}{t\sec t}=\lim_{t\to 0}\left(\frac{\sin 3t}{t\sec t}+\frac{4t}{t\sec t}\right)$

$=\lim_{t\to 0}\frac{\sin 3t}{t\sec t}+\lim_{t\to 0}\frac{4t}{t\sec t}$

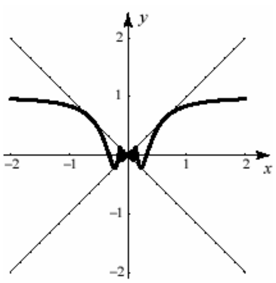
$=\lim_{t\to 0}3\cos t\cdot\frac{\sin 3t}{3t}+\lim_{t\to 0}4\cos t$

$=3\cdot 1+4=7$

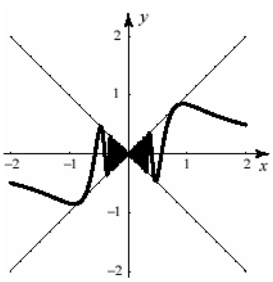
14. $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta^2}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{\sin\theta}{\theta}$

$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\times 1=1$

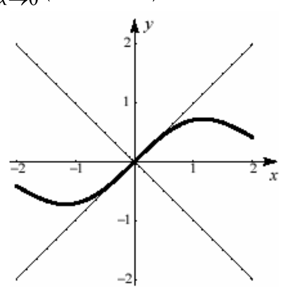
15. $\lim_{x\to 0}x\sin\left(1/x\right)=0$



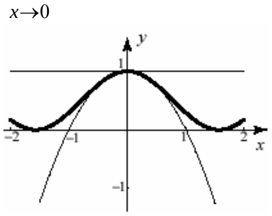
16. $\lim_{x\to 0}x\sin\left(1/x^2\right)=0$



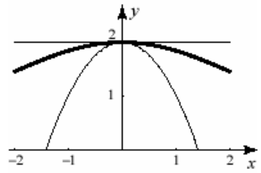
17. $\lim_{x\to 0}\left(1-\cos^2 x\right)/x=0$



18. $\lim_{x\to 0}\cos^2 x=1$



19. $\lim_{x\to 0}1+\frac{\sin x}{x}=2$



20. The result that $\lim_{t\to 0}\cos t=1$ was established in the proof of the theorem. Then

$\lim_{t\to c}\cos t=\lim_{h\to 0}\cos(c+h)$

$=\lim_{h\to 0}(\cos c\cos h-\sin c\sin h)$

$=\lim_{h\to 0}\cos c\lim_{h\to 0}\cos h-\sin c\lim_{h\to 0}\sin h$

$=\cos c$

21. $\lim_{t\to c}\tan t=\lim_{t\to c}\frac{\sin t}{\cos t}=\frac{\lim_{t\to c}\sin t}{\lim_{t\to c}\cos t}=\frac{\sin c}{\cos c}=\tan c$

$\lim_{t\to c}\cot t=\lim_{t\to c}\frac{\cos t}{\sin t}=\frac{\lim_{t\to c}\cos t}{\lim_{t\to c}\sin t}=\frac{\cos c}{\sin c}=\cot c$

22. $\lim_{t\to c}\sec t=\lim_{t\to c}\frac{1}{\cos t}=\frac{1}{\cos c}=\sec c$

$\lim_{t\to c}\csc t=\lim_{t\to c}\frac{1}{\sin t}=\frac{1}{\sin c}=\csc c$

23. $\overline{BP}=\sin t,\overline{OB}=\cos t$

$\text{area}(\Delta OBP)\le\text{area (sector }OAP)$

$\le\text{area }(\Delta OBP)+\text{area}(ABPQ)$

$\frac{1}{2}\overline{OB}\cdot\overline{BP}\le\frac{1}{2}t(1)^2\le\frac{1}{2}\overline{OB}\cdot\overline{BP}+(1-\overline{OB})\overline{BP}$

$\frac{1}{2}\sin t\cos t\le\frac{1}{2}t\le\frac{1}{2}\sin t\cos t+(1-\cos t)\sin t$

$\cos t\le\frac{t}{\sin t}\le 2-\cos t$

$\frac{1}{2-\cos t}\le\frac{\sin t}{t}\le\frac{1}{\cos t}$ for $-\frac{\pi}{2}<t<\frac{\pi}{2}$.

$\lim_{t\to 0}\frac{1}{2-\cos t}\le\lim_{t\to 0}\frac{\sin t}{t}\le\lim_{t\to 0}\frac{1}{\cos t}$

$1\le\lim_{t\to 0}\frac{\sin t}{t}\le 1$

Thus, $\lim_{t\to 0}\frac{\sin t}{t}=1.$

24. a. Written response

b. $D = \frac{1}{2}\overline{AB} \cdot \overline{BP} = \frac{1}{2}(1-\cos t)\sin t$

$= \frac{\sin t(1-\cos t)}{2}$

$E = \frac{1}{2}t(1)^2 - \frac{1}{2}\overline{OB} \cdot \overline{BP} = \frac{t}{2} - \frac{\sin t \cos t}{2}$

$\frac{D}{E} = \frac{\sin t(1-\cos t)}{t - \sin t \cos t}$

c. $\lim_{t\to 0^+}\left(\frac{D}{E}\right) = 0.75$

1.5 Concepts Review

1. x increases without bound; $f(x)$ gets close to L as x increases without bound

2. $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left

3. $y = 6$; horizontal

4. $x = 6$; vertical

Problem Set 1.5

1. $\lim_{x\to\infty} \frac{x}{x-5} = \lim_{x\to\infty} \frac{1}{1-\frac{5}{x}} = 1$

2. $\lim_{x\to\infty} \frac{x^2}{5-x^3} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{5}{x^3}-1} = 0$

3. $\lim_{t\to-\infty} \frac{t^2}{7-t^2} = \lim_{t\to-\infty} \frac{1}{\frac{7}{t^2}-1} = -1$

4. $\lim_{t\to-\infty} \frac{t}{t-5} = \lim_{t\to-\infty} \frac{1}{1-\frac{5}{t}} = 1$

5. $\lim_{x\to\infty} \frac{x^2}{(x-5)(3-x)} = \lim_{x\to\infty} \frac{x^2}{-x^2+8x-15}$

$= \lim_{x\to\infty} \frac{1}{-1+\frac{8}{x}-\frac{15}{x^2}} = -1$

6. $\lim_{x\to\infty} \frac{x^2}{x^2-8x+15} = \lim_{x\to\infty} \frac{1}{1-\frac{8}{x}+\frac{15}{x^2}} = 1$

7. $\lim_{x\to\infty} \frac{x^3}{2x^3-100x^2} = \lim_{x\to\infty} \frac{1}{2-\frac{100}{x}} = \frac{1}{2}$

8. $\lim_{\theta\to-\infty} \frac{\pi\theta^5}{\theta^5-5\theta^4} = \lim_{\theta\to-\infty} \frac{\pi}{1-\frac{5}{\theta}} = \pi$

9. $\lim_{x\to\infty} \frac{3x^3-x^2}{\pi x^3-5x^2} = \lim_{x\to\infty} \frac{3-\frac{1}{x}}{\pi-\frac{5}{x}} = \frac{3}{\pi}$

10. $\lim_{\theta\to\infty} \frac{\sin^2\theta}{\theta^2-5}$; $0 \le \sin^2\theta \le 1$ for all θ and

$\lim_{\theta\to\infty} \frac{1}{\theta^2-5} = \lim_{\theta\to\infty} \frac{\frac{1}{\theta^2}}{1-\frac{5}{\theta^2}} = 0$ so $\lim_{\theta\to\infty} \frac{\sin^2\theta}{\theta^2-5} = 0$

11. $\lim_{x\to\infty} \frac{3\sqrt{x^3}+3x}{\sqrt{2x^3}} = \lim_{x\to\infty} \frac{3x^{3/2}+3x}{\sqrt{2}x^{3/2}}$

$= \lim_{x\to\infty} \frac{3+\frac{3}{\sqrt{x}}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

12. $\lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3+3x}{\sqrt{2}x^3+7x}} = \sqrt[3]{\lim_{x\to\infty} \frac{\pi x^3+3x}{\sqrt{2}x^3+7x}}$

$= \sqrt[3]{\lim_{x\to\infty} \frac{\pi+\frac{3}{x^2}}{\sqrt{2}+\frac{7}{x^2}}} = \sqrt[3]{\frac{\pi}{\sqrt{2}}}$

13. $\lim_{x\to\infty} \sqrt[3]{\frac{1+8x^2}{x^2+4}} = \sqrt[3]{\lim_{x\to\infty} \frac{1+8x^2}{x^2+4}}$

$= \sqrt[3]{\lim_{x\to\infty} \frac{\frac{1}{x^2}+8}{1+\frac{4}{x^2}}} = \sqrt[3]{8} = 2$

14. $\lim_{x\to\infty} \sqrt{\frac{x^2+x+3}{(x-1)(x+1)}} = \sqrt{\lim_{x\to\infty} \frac{x^2+x+3}{x^2-1}}$

$= \sqrt{\lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{3}{x^2}}{1-\frac{1}{x^2}}} = \sqrt{1} = 1$

15. $\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$

16. $\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1$

17. $\lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} n}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)} = \frac{\infty}{1+0} = \infty$

18. $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1+0} = 0$

19. For $x > 0$, $x = \sqrt{x^2}$.

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+3}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\frac{\sqrt{x^2+3}}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{3}{x^2}}}$$

$$= \frac{2}{\sqrt{1}} = 2$$

20. $\lim_{x\to\infty} \frac{\sqrt{2x+1}}{x+4} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x+1}}{\sqrt{x^2}}}{1+\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{2}{x}+\frac{1}{x^2}}}{1+\frac{4}{x}} = 0$

21. $\lim_{x\to\infty}\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)\left(\sqrt{2x^2+3}+\sqrt{2x^2-5}\right)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$$

$$= \lim_{x\to\infty} \frac{2x^2+3-(2x^2-5)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$$

$$= \lim_{x\to\infty} \frac{8}{\sqrt{2x^2+3}+\sqrt{2x^2-5}} = \lim_{x\to\infty} \frac{\frac{8}{x}}{\frac{\sqrt{2x^2+3}+\sqrt{2x^2-5}}{\sqrt{x^2}}}$$

$$= \lim_{x\to\infty} \frac{\frac{8}{x}}{\sqrt{2+\frac{3}{x^2}}+\sqrt{2-\frac{5}{x^2}}} = 0$$

22.

$$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x}-x\right)\left(\sqrt{x^2+2x}+x\right)}{\sqrt{x^2+2x}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+2x}+x}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{2}{x}}+1} = \frac{2}{2} = 1$$

23. $\lim_{y\to-\infty} \frac{9y^3+1}{y^2-2y+2} = \lim_{y\to-\infty} \frac{9y+\frac{1}{y^2}}{1-\frac{2}{y}+\frac{2}{y^2}} = -\infty$

24. $\lim_{x\to\infty} \frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$

$$= \lim_{x\to\infty} \frac{a_0+\frac{a_1}{x}+\ldots+\frac{a_{n-1}}{x^{n-1}}+\frac{a_n}{x^n}}{b_0+\frac{b_1}{x}+\ldots+\frac{b_{n-1}}{x^{n-1}}+\frac{b_n}{x^n}} = \frac{a_0}{b_0}$$

25. $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = 1$

26. $\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+2n+1}} = \lim_{n\to\infty} \frac{\frac{n^2}{n^{3/2}}}{\sqrt{1+\frac{2}{n^2}+\frac{1}{n^3}}} = \frac{\infty}{1} = \infty$

27. As $x \to 4^+, x \to 4$ while $x-4 \to 0^+$.

$$\lim_{x\to 4^+} \frac{x}{x-4} = \infty$$

28. $\lim_{t\to-3^+} \frac{t^2-9}{t+3} = \lim_{t\to-3^+} \frac{(t+3)(t-3)}{t+3}$

$$= \lim_{t\to-3^+}(t-3) = -6$$

29. As $t \to 3^-, t^2 \to 9$ while $9-t^2 \to 0^+$.

$$\lim_{t\to 3^-} \frac{t^2}{9-t^2} = \infty$$

30. As $x \to \sqrt[3]{5}^+, x^2 \to 5^{2/3}$ while $5-x^3 \to 0^-$.

$$\lim_{x\to\sqrt[3]{5}^+} \frac{x^2}{5-x^3} = -\infty$$

31. As $x \to 5^-, x^2 \to 25, x-5 \to 0^-$, and $3-x \to -2$.

$$\lim_{x\to 5^-} \frac{x^2}{(x-5)(3-x)} = \infty$$

32. As $\theta \to \pi^+, \theta^2 \to \pi^2$ while $\sin\theta \to 0^-$.

$$\lim_{\theta\to\pi^+} \frac{\theta^2}{\sin\theta} = -\infty$$

33. As $x \to 3^-, x^3 \to 27$, while $x-3 \to 0^-$.

$$\lim_{x\to 3^-} \frac{x^3}{x-3} = -\infty$$

34. As $\theta \to \frac{\pi}{2}^+, \pi\theta \to \frac{\pi^2}{2}$ while $\cos\theta \to 0^-$.

$$\lim_{\theta\to\frac{\pi}{2}^+} \frac{\pi\theta}{\cos\theta} = -\infty$$

35. $\lim\limits_{x\to 3^-} \frac{x^2-x-6}{x-3} = \lim\limits_{x\to 3^-} \frac{(x+2)(x-3)}{x-3}$

$= \lim\limits_{x\to 3^-} (x+2) = 5$

36. $\lim\limits_{x\to 2^+} \frac{x^2+2x-8}{x^2-4} = \lim\limits_{x\to 2^+} \frac{(x+4)(x-2)}{(x+2)(x-2)}$

$= \lim\limits_{x\to 2^+} \frac{x+4}{x+2} = \frac{6}{4} = \frac{3}{2}$

37. For $0 \le x < 1$, $[\![x]\!] = 0$, so for $0 < x < 1$, $\frac{[\![x]\!]}{x} = 0$

thus $\lim\limits_{x\to 0^+} \frac{[\![x]\!]}{x} = 0$

38. For $-1 \le x < 0$, $[\![x]\!] = -1$, so for $-1 < x < 0$,

$\frac{[\![x]\!]}{x} = -\frac{1}{x}$ thus $\lim\limits_{x\to 0^-} \frac{[\![x]\!]}{x} = \infty$.

(Since $x < 0$, $-\frac{1}{x} > 0$.)

39. For $x < 0$, $|x| = -x$, thus

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

40. For $x > 0$, $|x| = x$, thus $\lim\limits_{x\to 0^+} \frac{|x|}{x} = \lim\limits_{x\to 0^+} \frac{x}{x} = 1$

41. As $x \to 0^-, 1+\cos x \to 2$ while $\sin x \to 0^-$.

$$\lim_{x\to 0^-} \frac{1+\cos x}{\sin x} = -\infty$$

42. $-1 \le \sin x \le 1$ for all x, and

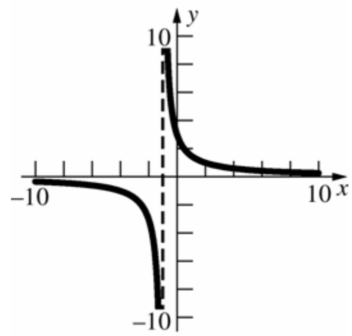
$\lim\limits_{x\to\infty} \frac{1}{x} = 0$, so $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$.

43. $\lim\limits_{x\to\infty} \frac{3}{x+1} = 0$, $\lim\limits_{x\to-\infty} \frac{3}{x+1} = 0$;

Horizontal asymptote $y = 0$.

$\lim\limits_{x\to -1^+} \frac{3}{x+1} = \infty$, $\lim\limits_{x\to -1^-} \frac{3}{x+1} = -\infty$;

Vertical asymptote $x = -1$

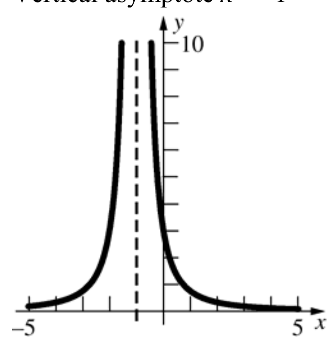


44. $\lim\limits_{x\to\infty} \frac{3}{(x+1)^2} = 0$, $\lim\limits_{x\to-\infty} \frac{3}{(x+1)^2} = 0$;

Horizontal asymptote $y = 0$.

$\lim\limits_{x\to -1^+} \frac{3}{(x+1)^2} = \infty$, $\lim\limits_{x\to -1^-} \frac{3}{(x+1)^2} = \infty$;

Vertical asymptote $x = -1$



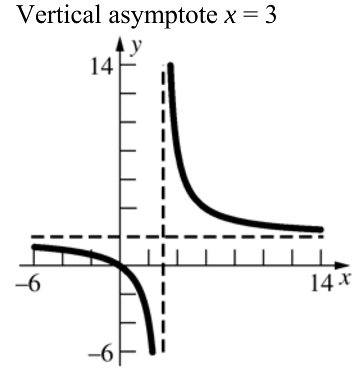
45. $\lim\limits_{x\to\infty} \frac{2x}{x-3} = \lim\limits_{x\to\infty} \frac{2}{1-\frac{3}{x}} = 2$,

$\lim\limits_{x\to-\infty} \frac{2x}{x-3} = \lim\limits_{x\to-\infty} \frac{2}{1-\frac{3}{x}} = 2$,

Horizontal asymptote $y = 2$

$\lim\limits_{x\to 3^+} \frac{2x}{x-3} = \infty$, $\lim\limits_{x\to 3^-} \frac{2x}{x-3} = -\infty$;

Vertical asymptote $x = 3$



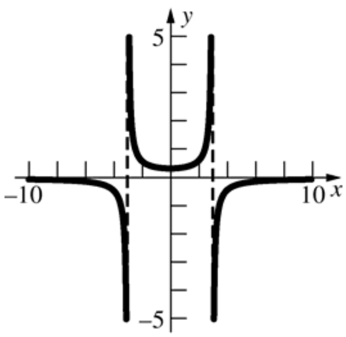
46. $\lim_{x\to\infty} \frac{3}{9-x^2} = 0$, $\lim_{x\to-\infty} \frac{3}{9-x^2} = 0$;

Horizontal asymptote $y = 0$

$\lim_{x\to 3^+} \frac{3}{9-x^2} = -\infty$, $\lim_{x\to 3^-} \frac{3}{9-x^2} = \infty$,

$\lim_{x\to -3^+} \frac{3}{9-x^2} = \infty$, $\lim_{x\to -3^-} \frac{3}{9-x^2} = -\infty$;

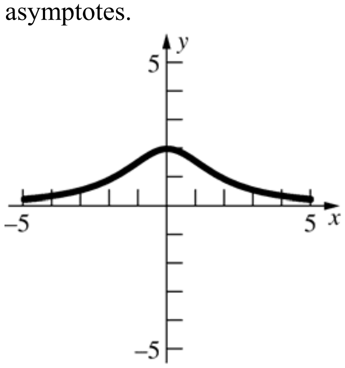
Vertical asymptotes $x = -3$, $x = 3$



47. $\lim_{x\to\infty} \frac{14}{2x^2+7} = 0$, $\lim_{x\to-\infty} \frac{14}{2x^2+7} = 0$;

Horizontal asymptote $y = 0$

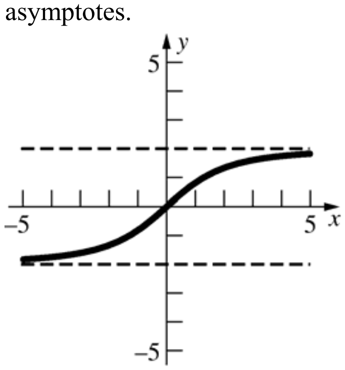
Since $2x^2 + 7 > 0$ for all x, $g(x)$ has no vertical asymptotes.



48. $\lim_{x\to\infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{5}{x^2}}} = \frac{2}{\sqrt{1}} = 2$,

$\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+5}} = \lim_{x\to-\infty} \frac{2}{-\sqrt{1+\frac{5}{x^2}}} = \frac{2}{-\sqrt{1}} = -2$

Since $\sqrt{x^2+5} > 0$ for all x, $g(x)$ has no vertical asymptotes.



49. $f(x) = 2x + 3 - \frac{1}{x^3-1}$, thus

$\lim_{x\to\infty} [f(x) - (2x+3)] = \lim_{x\to\infty} \left[-\frac{1}{x^3-1}\right] = 0$

The oblique asymptote is $y = 2x + 3$.

50. $f(x) = 3x + 4 - \frac{4x+3}{x^2+1}$, thus

$\lim_{x\to\infty} [f(x) - (3x+4)] = \lim_{x\to\infty} \left[-\frac{4x+3}{x^2+1}\right]$

$= \lim_{x\to\infty} \left[-\frac{\frac{4}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}}\right] = 0$.

The oblique asymptote is $y = 3x + 4$.

51. **a.** We say that $\lim_{x\to c^+} f(x) = -\infty$ if to each negative number M there corresponds a $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) < M$.

b. We say that $\lim_{x\to c^-} f(x) = \infty$ if to each positive number M there corresponds a $\delta > 0$ such that $0 < c - x < \delta \Rightarrow f(x) > M$.

52. **a.** We say that $\lim_{x\to\infty} f(x) = \infty$ if to each positive number M there corresponds an $N > 0$ such that $N < x \Rightarrow f(x) > M$.

b. We say that $\lim_{x\to-\infty} f(x) = \infty$ if to each positive number M there corresponds an $N < 0$ such that $x < N \Rightarrow f(x) > M$.

53. Let $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = A$, there is a corresponding number M_1 such that $x > M_1 \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}$. Similarly, there is a number M_2 such that $x > M_2 \Rightarrow |g(x) - B| < \frac{\varepsilon}{2}$.

Let $M = \max\{M_1, M_2\}$, then

$x > M \Rightarrow |f(x) + g(x) - (A+B)|$

$= |f(x) - A + g(x) - B| \le |f(x) - A| + |g(x) - B|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus, $\lim_{x\to\infty} [f(x) + g(x)] = A + B$

54. Written response

55. a. $\lim_{x\to\infty} \sin x$ does not exist as $\sin x$ oscillates between –1 and 1 as x increases.

b. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{u\to 0^+} \sin u = 0$

c. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{u\to 0^+} \frac{1}{u}\sin u = \lim_{u\to 0^+} \frac{\sin u}{u} = 1$

d. Let $u = \frac{1}{x}$, then

$\lim_{x\to\infty} x^{3/2} \sin\frac{1}{x} = \lim_{u\to 0^+} \left(\frac{1}{u}\right)^{3/2} \sin u$

$= \lim_{u\to 0^+} \left[\left(\frac{1}{\sqrt{u}}\right)\left(\frac{\sin u}{u}\right)\right] = \infty$

e. As $x \to \infty$, $\sin x$ oscillates between –1 and 1, while $x^{-1/2} = \frac{1}{\sqrt{x}} \to 0$.

$\lim_{x\to\infty} x^{-1/2} \sin x = 0$

f. Let $u = \frac{1}{x}$, then

$\lim_{x\to\infty} \sin\left(\frac{\pi}{6} + \frac{1}{x}\right) = \lim_{u\to 0^+} \sin\left(\frac{\pi}{6} + u\right)$

$= \sin\frac{\pi}{6} = \frac{1}{2}$

g. As $x \to \infty, x + \frac{1}{x} \to \infty$, so $\lim_{x\to\infty} \sin\left(x + \frac{1}{x}\right)$ does not exist. (See part a.)

h. $\sin\left(x + \frac{1}{x}\right) = \sin x\cos\frac{1}{x} + \cos x\sin\frac{1}{x}$

$\lim_{x\to\infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right]$

$= \lim_{x\to\infty} \left[\sin x\left(\cos\frac{1}{x} - 1\right) + \cos x\sin\frac{1}{x}\right]$

As $x \to \infty, \cos\frac{1}{x} \to 1$ so $\cos\frac{1}{x} - 1 \to 0$.

From part **b.**, $\lim_{x\to\infty} \sin\frac{1}{x} = 0$.

As $x \to \infty$ both $\sin x$ and $\cos x$ oscillate between –1 and 1.

$\lim_{x\to\infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right] = 0.$

56. $\lim_{v\to c^-} m(v) = \lim_{v\to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \infty$

57. $\lim_{x\to\infty} \frac{3x^2 + x + 1}{2x^2 - 1} = \frac{3}{2}$

58. $\lim_{x\to -\infty} \sqrt{\frac{2x^2 - 3x}{5x^2 + 1}} = \sqrt{\frac{2}{5}}$

59. $\lim_{x\to -\infty} \left(\sqrt{2x^2 + 3x} - \sqrt{2x^2 - 5}\right) = -\frac{3}{2\sqrt{2}}$

60. $\lim_{x\to\infty} \frac{2x + 1}{\sqrt{3x^2 + 1}} = \frac{2}{\sqrt{3}}$

61. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{10} = 1$

62. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x} = e \approx 2.718$

63. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x^2} = \infty$

64. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{\sin x} = 1$

65. $\lim_{x\to 3^-} \frac{\sin|x - 3|}{x - 3} = -1$

66. $\lim_{x\to 3^-} \frac{\sin|x - 3|}{\tan(x - 3)} = -1$

67. $\lim_{x\to 3^-} \frac{\cos(x - 3)}{x - 3} = -\infty$

68. $\lim_{x\to \frac{\pi}{2}^+} \frac{\cos x}{x - \frac{\pi}{2}} = -1$

69. $\lim_{x\to 0^+} (1 + \sqrt{x})^{\frac{1}{\sqrt{x}}} = e \approx 2.718$

70. $\lim_{x\to 0^+} (1 + \sqrt{x})^{1/x} = \infty$

71. $\lim_{x\to 0^+} (1 + \sqrt{x})^{x} = 1$

1.6 Concepts Review

1. $\lim_{x \to c} f(x)$

2. every integer

3. $\lim_{x \to a^+} f(x) = f(a);\ \lim_{x \to b^-} f(x) = f(b)$

4. a; b; $f(c) = W$

Problem Set 1.6

1. $\lim_{x \to 3}[(x-3)(x-4)] = 0 = f(3);$ continuous

2. $\lim_{x \to 3}(x^2 - 9) = 0 = g(3);$ continuous

3. $\lim_{x \to 3} \frac{3}{x-3}$ and $h(3)$ do not exist, so $h(x)$ is not continuous at 3.

4. $\lim_{t \to 3} \sqrt{t-4}$ and $g(3)$ do not exist, so $g(t)$ is not continuous at 3.

5. $\lim_{t \to 3} \frac{|t-3|}{t-3}$ and $h(3)$ do not exist, so $h(t)$ is not continuous at 3.

6. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

7. $\lim_{t \to 3} |t| = 3 = f(3);$ continuous

8. $\lim_{t \to 3} |t-2| = 1 = g(3);$ continuous

9. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

10. $f(3)$ does not exist, so $f(x)$ is not continuous at 3.

11. $\lim_{t \to 3} \frac{t^3 - 27}{t-3} = \lim_{t \to 3} \frac{(t-3)(t^2+3t+9)}{t-3}$
$= \lim_{t \to 3}(t^2 + 3t + 9) = (3)^2 + 3(3) + 9 = 27 = r(3)$
continuous

12. From Problem 11, $\lim_{t \to 3} r(t) = 27,$ so $r(t)$ is not continuous at 3 because $\lim_{t \to 3} r(t) \neq r(3).$

13. $\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+} (3-t) = 0$
$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-} (t-3) = 0$
$\lim_{t \to 3} f(t) = f(3);$ continuous

14. $\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+} (3-t)^2 = 0$
$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-} (t^2-9) = 0$
$\lim_{t \to 3} f(t) = f(3);$ continuous

15. $\lim_{t \to 3} f(x) = -2 = f(3);$ continuous

16. g is discontinuous at $x = -3, 4, 6, 8$; g is left continuous at $x = 4, 8$; g is right continuous at $x = -3, 6$

17. h is continuous on the intervals $(-\infty,-5),\ [-5,4],\ (4,6),\ [6,8],\ (8,\infty)$

18. $\lim_{x \to 7} \frac{x^2-49}{x-7} = \lim_{x \to 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x \to 7}(x+7)$
$= 7 + 7 = 14$
Define $f(7) = 14$.

19. $\lim_{x \to 3} \frac{2x^2-18}{3-x} = \lim_{x \to 3} \frac{2(x+3)(x-3)}{3-x}$
$= \lim_{x \to 3}[-2(x+3)] = -2(3+3) = -12$
Define $f(3) = -12$.

20. $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$
Define $g(0) = 1$

21. $\lim_{t \to 1} \frac{\sqrt{t}-1}{t-1} = \lim_{t \to 1} \frac{(\sqrt{t}-1)(\sqrt{t}+1)}{(t-1)(\sqrt{t}+1)}$
$= \lim_{t \to 1} \frac{t-1}{(t-1)(\sqrt{t}+1)} = \lim_{t \to 1} \frac{1}{\sqrt{t}+1} = \frac{1}{2}$
Define $H(1) = \frac{1}{2}$.

22. $\lim_{x \to -1} \frac{x^4+2x^2-3}{x+1} = \lim_{x \to -1} \frac{(x^2-1)(x^2+3)}{x+1}$
$= \lim_{x \to -1} \frac{(x+1)(x-1)(x^2+3)}{x+1}$
$= \lim_{x \to -1} [(x-1)(x^2+3)]$
$= (-1-1)[(-1)^2+3] = -8$
Define $\phi(-1) = -8$.

23. $\lim_{x\to -1} \sin\left(\frac{x^2-1}{x+1}\right) = \lim_{x\to -1} \sin\left(\frac{(x-1)(x+1)}{x+1}\right)$
$= \lim_{x\to -1} \sin(x-1) = \sin(-1-1) = \sin(-2) = -\sin 2$
Define $F(-1) = -\sin 2$.

24. Discontinuous at $x = \pi, 30$

25. $f(x) = \frac{33-x^2}{(\pi - x)(x-3)}$
Discontinuous at $x = 3, \pi$

26. Continuous at all points

27. Discontinuous at all $\theta = n\pi + \frac{\pi}{2}$ where n is any integer.

28. Discontinuous at all $u \le -5$

29. Discontinuous at $u = -1$

30. Continuous at all points

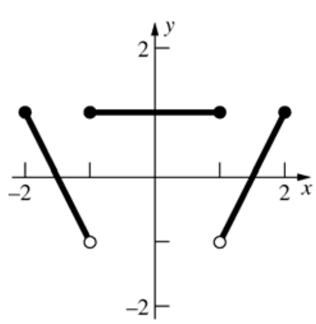
31. $G(x) = \frac{1}{\sqrt{(2-x)(2+x)}}$
Discontinuous on $(-\infty, -2] \cup [2, \infty)$

32. Continuous at all points since
$\lim_{x\to 0} f(x) = 0 = f(0)$ and $\lim_{x\to 1} f(x) = 1 = f(1)$.

33. $\lim_{x\to 0} g(x) = 0 = g(0)$
$\lim_{x\to 1^+} g(x) = 1,\ \lim_{x\to 1^-} g(x) = -1$
$\lim_{x\to 1} g(x)$ does not exist, so $g(x)$ is discontinuous at $x = 1$.

34. Discontinuous at every integer

35. Discontinuous at $t = n + \frac{1}{2}$ where n is any integer

36.


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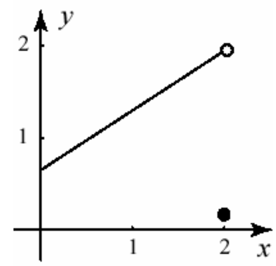

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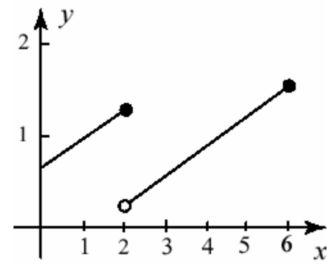

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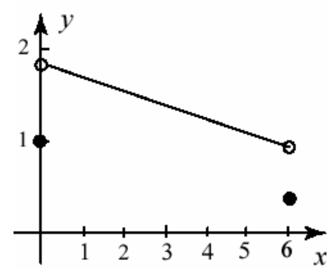

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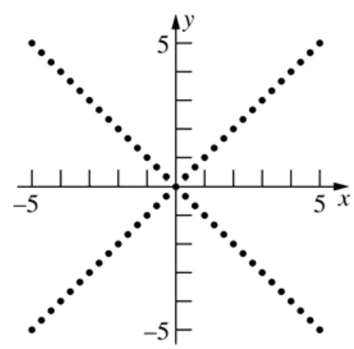


Discontinuous at all points except $x = 0$, because $\lim_{x\to c} f(x) \ne f(c)$ for $c \ne 0$. $\lim_{x\to c} f(x)$ exists only at $c = 0$ and $\lim_{x\to 0} f(x) = 0 = f(0)$.

41. Continuous.

42. Discontinuous: removable, define $f(10) = 20$

43. Discontinuous: removable, define $f(0) = 1$

44. Discontinuous: nonremovable.

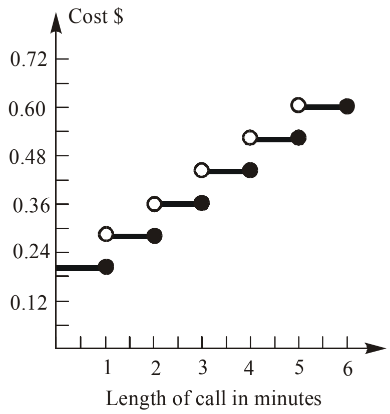
45. Discontinuous, removable, redefine $g(0) = 1$

46. Discontinuous: removable, define $F(0) = 0$

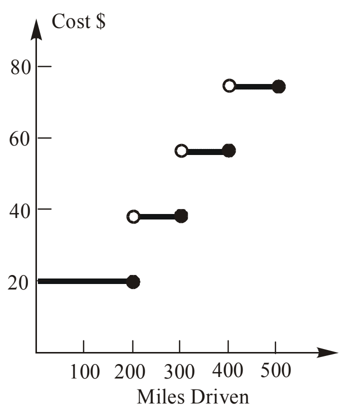
47. Discontinuous: nonremovable.

48. Discontinuous: removable, define $f(4) = 4$

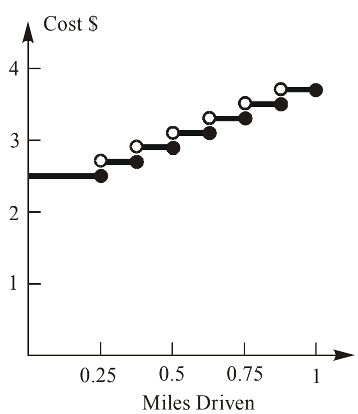
49. The function is continuous on the intervals $(0,1], (1,2], (2,3], \ldots$



50. The function is continuous on the intervals $[0, 200], (200, 300], (300, 400], \ldots$



51. The function is continuous on the intervals $(0, 0.25], (0.25, 0.375], (0.375, 0.5], \ldots$



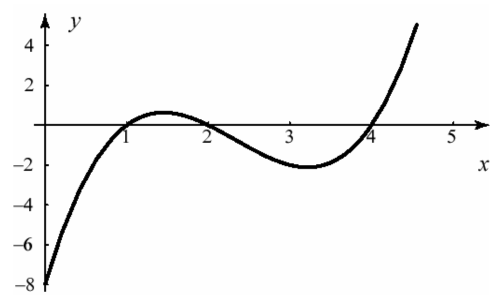
52. Let $f(x) = x^3 + 3x - 2$. f is continuous on [0, 1]. $f(0) = -2 < 0$ and $f(1) = 2 > 0$. Thus, there is at least one number c between 0 and 1 such that $x^3 + 3x - 2 = 0$.

53. Because the function is continuous on $[0, 2\pi]$ and

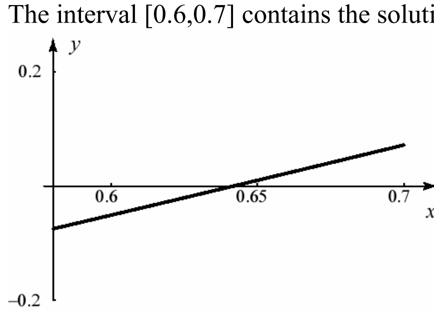
$(\cos 0)0^3 + 6\sin^5 0 - 3 = -3 < 0,$

$(\cos 2\pi)(2\pi)^3 + 6\sin^5(2\pi) - 3 = 8\pi^3 - 3 > 0,$ there is at least one number c between 0 and 2π such that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$.

54. Let $f(x) = x^3 - 7x^2 + 14x - 8$. $f(x)$ is continuous at all values of x.
$f(0) = -8$, $f(5) = 12$
Because 0 is between –8 and 12, there is at least one number c between 0 and 5 such that
$f(x) = x^3 - 7x^2 + 14x - 8 = 0$.
This equation has three solutions ($x = 1,2,4$)



55. Let $f(x) = \sqrt{x} - \cos x$.. $f(x)$ is continuous at all values of $x \geq 0$. $f(0) = -1$, $f(\pi/2) = \sqrt{\pi/2}$
Because 0 is between –1 and $\sqrt{\pi/2}$, there is at least one number c between 0 and $\pi/2$ such that $f(x) = \sqrt{x} - \cos x = 0$.
The interval [0.6,0.7] contains the solution.



56. Let $f(x) = x^5 + 4x^3 - 7x + 14$
$f(x)$ is continuous at all values of x.
$f(-2) = -36$, $f(0) = 14$
Because 0 is between –36 and 14, there is at least one number c between –2 and 0 such that
$f(x) = x^5 + 4x^3 - 7x + 14 = 0$.

57. Suppose that f is continuous at c, so $\lim_{x\to c} f(x) = f(c)$. Let $x = c + t$, so $t = x - c$, then as $x \to c$, $t \to 0$ and the statement $\lim_{x\to c} f(x) = f(c)$ becomes $\lim_{t\to 0} f(t+c) = f(c)$. Suppose that $\lim_{t\to 0} f(t+c) = f(c)$ and let $x = t + c$, so $t = x - c$. Since c is fixed, $t \to 0$ means that $x \to c$ and the statement $\lim_{t\to 0} f(t+c) = f(c)$ becomes $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c.

58. Since $f(x)$ is continuous at c, $\lim_{x\to c} f(x) = f(c) > 0$. Choose $\varepsilon = f(c)$, then there exists a $\delta > 0$ such that

$0 < |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

Thus, $f(x) - f(c) > -\varepsilon = -f(c)$, or $f(x) > 0$.

Since also $f(c) > 0$, $f(x) > 0$ for all x in $(c-\delta, c+\delta)$.

59. Let $g(x) = x - f(x)$. Then, $g(0) = 0 - f(0) = -f(0) \le 0$ and $g(1) = 1 - f(1) \ge 0$ since $0 \le f(x) \le 1$ on $[0, 1]$. If $g(0) = 0$, then $f(0) = 0$ and $c = 0$ is a fixed point of f. If $g(1) = 0$, then $f(1) = 1$ and $c = 1$ is a fixed point of f. If neither $g(0) = 0$ nor $g(1) = 0$, then $g(0) < 0$ and $g(1) > 0$ so there is some c in $[0, 1]$ such that $g(c) = 0$. If $g(c) = 0$ then $c - f(c) = 0$ or $f(c) = c$ and c is a fixed point of f.

60. For $f(x)$ to be continuous everywhere, $f(1) = a(1) + b = 2$ and $f(2) = 6 = a(2) + b$

$$\begin{aligned} a + b &= 2 \\ \underline{2a + b} &\underline{= 6} \\ -a &= -4 \\ a = 4,\ \ b &= -2 \end{aligned}$$

61. For x in $[0, 1]$, let $f(x)$ indicate where the string originally at x ends up. Thus $f(0) = a$, $f(1) = b$. $f(x)$ is continuous since the string is unbroken. Since $0 \le a,\ b \le 1$, $f(x)$ satisfies the conditions of Problem 59, so there is some c in $[0, 1]$ with $f(c) = c$, i.e., the point of string originally at c ends up at c.

62. The Intermediate Value Theorem does not imply the existence of a number c between –2 and 2 such that $f(c) = 0$. The reason is that the function $f(x)$ is not continuous on $[-2, 2]$.

63. Let $f(x)$ be the difference in times on the hiker's watch where x is a point on the path, and suppose $x = 0$ at the bottom and $x = 1$ at the top of the mountain.
So $f(x)$ = (time on watch on the way up) – (time on watch on the way down).
$f(0) = 4 - 11 = -7$, $f(1) = 12 - 5 = 7$. Since time is continuous, $f(x)$ is continuous, hence there is some c between 0 and 1 where $f(c) = 0$. This c is the point where the hiker's watch showed the same time on both days.

64. Let f be the function on $\left[0, \frac{\pi}{2}\right]$ such that $f(\theta)$ is the length of the side of the rectangle which makes angle θ with the x-axis minus the length of the sides perpendicular to it. f is continuous on $\left[0, \frac{\pi}{2}\right]$. If $f(0) = 0$ then the region is circumscribed by a square. If $f(0) \ne 0$, then observe that $f(0) = -f\left(\frac{\pi}{2}\right)$. Thus, by the Intermediate Value Theorem, there is an angle θ_0 between 0 and $\frac{\pi}{2}$ such that $f(\theta_0) = 0$. Hence, D can be circumscribed by a square.

65. Yes, g is continuous at R.

$$\lim_{r\to R^-} g(r) = \frac{GMm}{R^2} = \lim_{r\to R^+} g(r)$$

66. No. By the Intermediate Value Theorem, if f were to change signs on $[a,b]$, then f must be 0 at some c in $[a,b]$. Therefore, f cannot change sign.

67. **a.** $f(x) = f(x + 0) = f(x) + f(0)$, so $f(0) = 0$. We want to prove that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, $\lim_{x\to c} [f(x) - f(c)] = 0$. But $f(x) - f(c) = f(x - c)$, so $\lim_{x\to c} [f(x) - f(c)] = \lim_{x\to c} f(x-c)$. Let $h = x - c$ then as $x \to c$, $h \to 0$ and $\lim_{x\to c} f(x-c) = \lim_{h\to 0} f(h) = f(0) = 0$. Hence $\lim_{x\to c} f(x) = f(c)$ and f is continuous at c. Thus, f is continuous everywhere, since c was arbitrary.

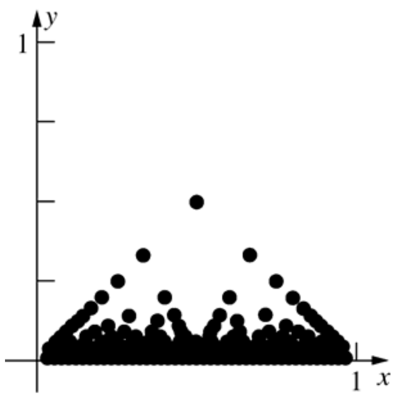
b. By Problem 43 of Section 0.5, $f(t) = mt$ for all t in Q. Since $g(t) = mt$ is a polynomial function, it is continuous for all real numbers. $f(t) = g(t)$ for all t in Q, thus $f(t) = g(t)$ for all t in R, i.e. $f(t) = mt$.

68. If $f(x)$ is continuous on an interval then $\lim_{x\to c} f(x) = f(c)$ for all points in the interval:

$$\lim_{x\to c} f(x) = f(c) \Rightarrow \lim_{x\to c} |f(x)|$$
$$= \lim_{x\to c} \sqrt{f^2(x)} = \sqrt{\left(\lim_{x\to c} f(x)\right)^2}$$
$$= \sqrt{(f(c))^2} = |f(c)|$$

69. Suppose $f(x) = \begin{cases} 1 \text{ if } x \ge 0 \\ -1 \text{ if } x < 0 \end{cases}$. $f(x)$ is discontinuous at $x = 0$, but $g(x) = |f(x)| = 1$ is continuous everywhere.

70. a.



b. If r is any rational number, then any deleted interval about r contains an irrational number. Thus, if $f(r) = \frac{1}{q}$, any deleted interval about r contains at least one point c such that $|f(r) - f(c)| = \left|\frac{1}{q} - 0\right| = \frac{1}{q}$. Hence, $\lim_{x\to r} f(x)$ does not exist.

If c is any irrational number in (0, 1), then as $x = \frac{p}{q} \to c$ (where $\frac{p}{q}$ is the reduced form of the rational number) $q \to \infty$, so $f(x) \to 0$ as $x \to c$. Thus, $\lim_{x\to c} f(x) = 0 = f(c)$ for any irrational number c.

71. a. Suppose the block rotates to the left. Using geometry, $f(x) = -\frac{3}{4}$. Suppose the block rotates to the right. Using geometry, $f(x) = \frac{3}{4}$. If $x = 0$, the block does not rotate, so $f(x) = 0$.

Domain: $\left[-\frac{3}{4}, \frac{3}{4}\right]$;

Range: $\left\{-\frac{3}{4}, 0, \frac{3}{4}\right\}$

b. At $x = 0$

c. If $x = 0$, $f(x) = 0$, if $x = -\frac{3}{4}$, $f(x) = -\frac{3}{4}$ and if $x = \frac{3}{4}$, $f(x) = \frac{3}{4}$, so $x = -\frac{3}{4}, 0, \frac{3}{4}$ are fixed points of f.

1.7 Chapter Review

Concepts Test

1. False. Consider $f(x) = [\![x]\!]$ at $x = 2$.

2. False: c may not be in the domain of $f(x)$, or it may be defined separately.

3. False: c may not be in the domain of $f(x)$, or it may be defined separately.

4. True. By definition, where $c = 0, L = 0$.

5. False: If $f(c)$ is not defined, $\lim_{x\to c} f(x)$ might exist; e.g., $f(x) = \frac{x^2 - 4}{x + 2}$.

$f(-2)$ does not exist, but $\lim_{x\to -2} \frac{x^2 - 4}{x + 2} = -4$.

6. True: $\lim_{x\to 5} \frac{x^2 - 25}{x - 5} = \lim_{x\to 5} \frac{(x-5)(x+5)}{x-5}$
$= \lim_{x\to 5} (x+5) = 5 + 5 = 10$

7. True: Substitution Theorem

8. False: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

9. False: The tangent function is not defined for all values of c.

10. True: If x is in the domain of $\tan x = \frac{\sin x}{\cos x}$, then $\cos x \ne 0$, and Theorem A.7 applies..

11. True: Since both $\sin x$ and $\cos x$ are continuous for all real numbers, by Theorem C we can conclude that $f(x) = 2\sin^2 x - \cos x$ is also continuous for all real numbers.

12. True. By definition, $\lim_{x\to c} f(x) = f(c)$.

13. True. $2 \in [1,3]$

14. False: $\lim_{x\to 0^-}$ may not exist

15. False: Consider $f(x) = \sin x$.

16. True. By the definition of continuity on an interval.

17. False: Since $-1 \le \sin x \le 1$ for all x and $\lim_{x\to\infty} \frac{1}{x} = 0$, we get $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

18. False. It could be the case where $\lim_{x\to -\infty} f(x) = 2$

19. False: The graph has many vertical asymptotes; e.g., $x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

20. True: $x = 2$; $x = -2$

21. True: As $x \to 1^+$ both the numerator and denominator are positive. Since the numerator approaches a constant and the denominator approaches zero, the limit goes to $+\infty$.

22. False: $\lim_{x\to c} f(x)$ must equal $f(c)$ for f to be continuous at $x = c$.

23. True: $\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right) = f(c)$, so f is continuous at $x = c$.

24. True: $\lim_{x\to 2.3} \left[\!\left[\frac{x}{2}\right]\!\right] = 1 = f(2.3)$

25. True: Choose $\varepsilon = 0.001 f(2)$ then since $\lim_{x\to 2} f(x) = f(2)$, there is some δ such that $0 < |x-2| < \delta \Rightarrow |f(x) - f(2)| < 0.001 f(2)$, or $-0.001 f(2) < f(x) - f(2) < 0.001 f(2)$
Thus, $0.999 f(2) < f(x) < 1.001 f(2)$ and $f(x) < 1.001 f(2)$ for $0 < |x-2| < \delta$. Since $f(2) < 1.001 f(2)$, as $f(2) > 0$, $f(x) < 1.001 f(2)$ on $(2-\delta,\ 2+\delta)$.

26. False: That $\lim_{x\to c} [f(x) + g(x)]$ exists does not imply that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist; e.g., $f(x) = \frac{x-3}{x+2}$ and $g(x) = \frac{x+7}{x+2}$ for $c = -2$.

27. True: Squeeze Theorem

28. True: A function has only one limit at a point, so if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, $L = M$

29. False: That $f(x) \ne g(x)$ for all x does not imply that $\lim_{x\to c} f(x) \ne \lim_{x\to c} g(x)$. For example, if $f(x) = \frac{x^2 + x - 6}{x-2}$ and $g(x) = \frac{5}{2}x$, then $f(x) \ne g(x)$ for all x, but $\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = 5$.

30. False: If $f(x) < 10$, $\lim_{x\to 2} f(x)$ could equal 10 if there is a discontinuity point (2, 10). For example,
$f(x) = \frac{-x^3 + 6x^2 - 2x - 12}{x-2} < 10$ for all x, but $\lim_{x\to 2} f(x) = 10$.

31. True: $\lim_{x\to a} |f(x)| = \lim_{x\to a} \sqrt{f^2(x)}$
$= \sqrt{\left[\lim_{x\to a} f(x)\right]^2} = \sqrt{(b)^2} = |b|$

32. True: If f is continuous and positive on $[a, b]$, the reciprocal is also continuous, so it will assume all values between $\frac{1}{f(a)}$ and $\frac{1}{f(b)}$.

Sample Test Problems

1. $\lim_{x\to 2}\frac{x-2}{x+2}=\frac{2-2}{2+2}=\frac{0}{4}=0$

2. $\lim_{u\to 1}\frac{u^2-1}{u+1}=\frac{1^2-1}{1+1}=0$

3. $\lim_{u\to 1}\frac{u^2-1}{u-1}=\lim_{u\to 1}\frac{(u-1)(u+1)}{u-1}=\lim_{u\to 1}(u+1)$
$= 1 + 1 = 2$

4. $\lim_{u\to 1}\frac{u+1}{u^2-1}=\lim_{u\to 1}\frac{u+1}{(u+1)(u-1)}=\lim_{u\to 1}\frac{1}{u-1}$; does not exist

5. $\lim_{x\to 2}\frac{1-\frac{2}{x}}{x^2-4}=\lim_{x\to 2}\frac{\frac{x-2}{x}}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x(x+2)}$
$=\frac{1}{2(2+2)}=\frac{1}{8}$

6. $\lim_{z\to 2}\frac{z^2-4}{z^2+z-6}=\lim_{z\to 2}\frac{(z+2)(z-2)}{(z+3)(z-2)}$
$=\lim_{z\to 2}\frac{z+2}{z+3}=\frac{2+2}{2+3}=\frac{4}{5}$

7. $\lim_{x\to 0}\frac{\tan x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{2\sin x\cos x}=\lim_{x\to 0}\frac{1}{2\cos^2 x}$
$=\frac{1}{2\cos^2 0}=\frac{1}{2}$

8. $\lim_{y\to 1}\frac{y^3-1}{y^2-1}=\lim_{y\to 1}\frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}$
$=\lim_{y\to 1}\frac{y^2+y+1}{y+1}=\frac{1^2+1+1}{1+1}=\frac{3}{2}$

9. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2}$
$=\lim_{x\to 4}(\sqrt{x}+2)=\sqrt{4}+2=4$

10. $\lim_{x\to 0}\frac{\cos x}{x}$ does not exist.

11. $\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$

12. $\lim_{x\to (1/2)^+}[\![4x]\!]=2$

13. $\lim_{t\to 2^-}\left([\![t]\!]-t\right)=\lim_{t\to 2^-}[\![t]\!]-\lim_{t\to 2^-}t=1-2=-1$

14. $\lim_{x\to 1^-}\frac{|x-1|}{x-1}=\lim_{x\to 1^-}\frac{1-x}{x-1}=-1$ since $x-1<0$ as $x\to 1^-$

15. $\lim_{x\to 0}\frac{\sin 5x}{3x}=\lim_{x\to 0}\frac{5}{3}\frac{\sin 5x}{5x}$
$=\frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{3}\times 1=\frac{5}{3}$

16. $\lim_{x\to 0}\frac{1-\cos 2x}{3x}=\lim_{x\to 0}\frac{2}{3}\frac{1-\cos 2x}{2x}$
$=\frac{2}{3}\lim_{x\to 0}\frac{1-\cos 2x}{2x}=\frac{2}{3}\times 0=0$

17. $\lim_{x\to\infty}\frac{x-1}{x+2}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{2}{x}}=\frac{1+0}{1+0}=1$

18. Since $-1\le \sin t\le 1$ for all t and $\lim_{t\to\infty}\frac{1}{t}=0$, we get $\lim_{t\to\infty}\frac{\sin t}{t}=0$.

19. $\lim_{t\to 2}\frac{t+2}{(t-2)^2}=\infty$ because as $t\to 0$, $t+2\to 4$ while the denominator goes to 0 from the right.

20. $\lim_{x\to 0^+}\frac{\cos x}{x}=\infty$**,** because as $x\to 0^+$, $\cos x\to 1$ while the denominator goes to 0 from the right.

21. $\lim_{x\to \pi/4^-}\tan 2x=\infty$ because as $x\to(\pi/4)^-$, $2x\to(\pi/2)^-$, so $\tan 2x\to\infty$.

22. $\lim_{x\to 0^+}\frac{1+\sin x}{x}=\infty$**,** because as $x\to 0^+$, $1+\sin x\to 1$ while the denominator goes to 0 from the right.

23. Preliminary analysis: Let ε > 0. We need to find a δ > 0 such that
$0<|x-3|<\delta\Rightarrow|(2x+1)-7|<\varepsilon.$
$|2x-6|<\varepsilon\Leftrightarrow 2|x-3|<\varepsilon$
$\Leftrightarrow|x-3|<\frac{\varepsilon}{2}.$ Choose $\delta=\frac{\varepsilon}{2}.$

Let ε > 0. Choose $\delta=\varepsilon/2$. Thus,

$|(2x+1)-7|=|2x-6|=2|x-3|<2(\varepsilon/2)=\varepsilon.$

24. **a.** $f(1) = 0$

b. $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (1-x) = 0$

c. $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1$

d. $\lim_{x\to-1} f(x) = -1$ because

$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} x^3 = -1$ and

$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} x = -1$

25. **a.** f is discontinuous at $x = 1$ because $f(1) = 0$, but $\lim_{x\to1} f(x)$ does not exist. f is discontinuous at $x = -1$ because $f(-1)$ does not exist.

b. Define $f(-1) = -1$

26. **a.** $0 < |u-a| < \delta \Rightarrow |g(u)-M| < \varepsilon$

b. $0 < a-x < \delta \Rightarrow |f(x)-L| < \varepsilon$

27. **a.** $\lim_{x\to3}[2f(x)-4g(x)]$

$= 2\lim_{x\to3} f(x) - 4\lim_{x\to3} g(x)$

$= 2(3) - 4(-2) = 14$

b. $\lim_{x\to3} g(x)\frac{x^2-9}{x-3} = \lim_{x\to3} g(x)(x+3)$

$= \lim_{x\to3} g(x)\cdot\lim_{x\to3}(x+3) = -2\cdot(3+3) = -12$

c. $g(3) = -2$

d. $\lim_{x\to3} g(f(x)) = g\left(\lim_{x\to3} f(x)\right) = g(3) = -2$

e. $\lim_{x\to3}\sqrt{f^2(x)-8g(x)}$

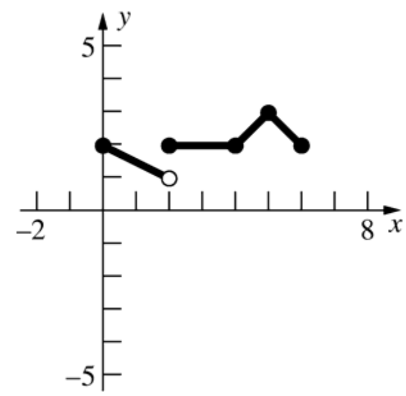
$= \sqrt{\left[\lim_{x\to3} f(x)\right]^2 - 8\lim_{x\to3} g(x)}$

$= \sqrt{(3)^2-8(-2)} = 5$

f. $\lim_{x\to3}\frac{|g(x)-g(3)|}{f(x)} = \frac{|-2-g(3)|}{3} = \frac{|-2-(-2)|}{3}$

$= 0$

28.



29. $a(0) + b = -1$ and $a(1) + b = 1$

$b = -1;\quad a + b = 1$

$a - 1 = 1$

$a = 2$

30. Let $f(x) = x^5 - 4x^3 - 3x + 1$

$f(2) = -5$, $f(3) = 127$

Because $f(x)$ is continuous on $[2, 3]$ and $f(2) < 0 < f(3)$, there exists some number c between 2 and 3 such that $f(c) = 0$.

31. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x\to\infty}\frac{x}{x^2+1} = \lim_{x\to-\infty}\frac{x}{x^2+1} = 0$, so $y = 0$ is a horizontal asymptote.

32. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x\to\infty}\frac{x^2}{x^2+1} = \lim_{x\to-\infty}\frac{x^2}{x^2+1} = 1$, so $y = 1$ is a horizontal asymptote.

33. Vertical: $x = 1, x = -1$ because $\lim_{x\to1^+}\frac{x^2}{x^2-1} = \infty$

and $\lim_{x\to-1^-}\frac{x^2}{x^2-1} = \infty$

Horizontal: $\lim_{x\to\infty}\frac{x^2}{x^2-1} = \lim_{x\to-\infty}\frac{x^2}{x^2-1} = 1$, so $y = 1$ is a horizontal asymptote.

34. Vertical: $x = 2, x = -2$ because

$$\lim_{x\to 2^+} \frac{x^3}{x^2-4} = \infty \text{ and } \lim_{x\to -2^-} \frac{x^3}{x^2-4} = \infty$$

Horizontal: $\lim_{x\to\infty} \frac{x^3}{x^2-4} = \infty$ and

$\lim_{x\to -\infty} \frac{x^3}{x^2-4} = -\infty$, so there are no horizontal asymptotes.

35. Vertical: $x = \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4, \ldots$ because $\lim_{x\to \pi/4^-} \tan 2x = \infty$ and similarly for other odd multiples of $\pi/4$.

Horizontal: None, because $\lim_{x\to\infty} \tan 2x$ and $\lim_{x\to -\infty} \tan 2x$ do not exist.

36. Vertical: $x = 0$, because

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{1}{x}\frac{\sin x}{x} = \infty .$$

Horizontal: $y = 0$, because

$$\lim_{x\to\infty} \frac{\sin x}{x^2} = \lim_{x\to -\infty} \frac{\sin x}{x^2} = 0.$$

Review and Preview Problems

1. **a.** $f(2) = 2^2 = 4$

b. $f(2.1) = 2.1^2 = 4.41$

c. $f(2.1) - f(2) = 4.41 - 4 = 0.41$

d. $\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{0.41}{0.1} = 4.1$

e. $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2$

f. $f(a+h) - f(a) = a^2 + 2ah + h^2 - a^2$
$= 2ah + h^2$

g. $\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{2ah + h^2}{h} = 2a + h$

h. $\lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h\to 0}(2a+h) = 2a$

2. **a.** $g(2) = 1/2$

b. $g(2.1) = 1/2.1 \approx 0.476$

c. $g(2.1) - g(2) = 0.476 - 0.5 = -0.024$

d. $\frac{g(2.1) - g(2)}{2.1 - 2} = \frac{-0.024}{0.1} = -0.24$

e. $g(a+h) = 1/(a+h)$

f. $g(a+h) - g(a) = 1/(a+h) - 1/a = \frac{-h}{a(a+h)}$

g. $\frac{g(a+h) - g(a)}{(a+h) - a} = \frac{\frac{-h}{a(a+h)}}{h} = \frac{-1}{a(a+h)}$

h. $\lim_{h\to 0} \frac{g(a+h) - g(a)}{(a+h) - a} = \frac{-1}{a^2}$

3. **a.** $F(2) = \sqrt{2} \approx 1.414$

b. $F(2.1) = \sqrt{2.1} \approx 1.449$

c. $F(2.1) - F(2) = 1.449 - 1.414 = 0.035$

d. $\frac{F(2.1) - F(2)}{2.1 - 2} = \frac{0.035}{0.1} = 0.35$

e. $F(a+h) = \sqrt{a+h}$

f. $F(a+h) - F(a) = \sqrt{a+h} - \sqrt{a}$

g. $\frac{F(a+h) - F(a)}{(a+h) - a} = \frac{\sqrt{a+h} - \sqrt{a}}{h}$

h. $\lim_{h\to 0} \frac{F(a+h) - F(a)}{(a+h) - a} = \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{a+h} - \sqrt{a}\right)\left(\sqrt{a+h} + \sqrt{a}\right)}{h\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim_{h\to 0} \frac{a+h-a}{h\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}} = \frac{\sqrt{a}}{2a}$$

4. a. $G(2)=(2)^3+1=8+1=9$

b. $G(2.1)=(2.1)^3+1=9.261+1=10.261$

c. $G(2.1)-G(2)=10.261-9=1.261$

d. $\dfrac{G(2.1)-G(2)}{2.1-2}=\dfrac{1.261}{0.1}=12.61$

e. $G(a+h)=(a+h)^3+1$
$=a^3+3a^2h+3ah^2+h^3+1$

f. $G(a+h)-G(a)=\left[(a+h)^3+1\right]-\left[a^3+1\right]$
$=\left(a^3+3a^2h+3ah^2+h^3+1\right)-\left(a^3+1\right)$
$=3a^2h+3ah^2+h^3$

g. $\dfrac{G(a+h)-G(a)}{(a+h)-a}=\dfrac{3a^2h+3ah^2+h^3}{h}$
$=3a^2+3ah+h^2$

h. $\lim\limits_{h\to 0}\dfrac{G(a+h)-G(a)}{(a+h)-a}=\lim\limits_{h\to 0}3a^2+3ah+h^2$
$=3a^2$

5. a. $(a+b)^3=a^3+3a^2b+\cdots$

b. $(a+b)^4=a^4+4a^3b+\cdots$

c. $(a+b)^5=a^5+5a^4b+\cdots$

6. $(a+b)^n=a^n+na^{n-1}b+\cdots$

7. $\sin(x+h)=\sin x\cos h+\cos x\sin h$

8. $\cos(x+h)=\cos x\cos h-\sin x\sin h$

9. a. The point will be at position $(10,0)$ in all three cases ($t=1,2,3$) because it will have made 4, 8, and 12 revolutions respectively.

b. Since the point is rotating at a rate of 4 revolutions per second, it will complete 1 revolution after $\frac{1}{4}$ second. Therefore, the point will first return to its starting position at time $t=\frac{1}{4}$.

10. $V_0=\dfrac{4}{3}\pi(2)^3=\dfrac{32\pi}{3}\text{cm}^3$
$V_1=\dfrac{4}{3}\pi(2.5)^3=\dfrac{62.5\pi}{3}=\dfrac{125\pi}{6}\text{cm}^3$
$\Delta V=V_1-V_0=\dfrac{125\pi}{6}\text{cm}^3-\dfrac{32\pi}{3}\text{cm}^3$
$=\dfrac{61}{6}\pi\,\text{cm}^3\approx 31.940\,\text{cm}^3$

11. a. North plane has traveled 600miles. East plane has traveled 400 miles.

b. $d=\sqrt{600^2+400^2}$
$=721$ miles

c. $d=\sqrt{675^2+500^2}$
$=840$ miles

CHAPTER 2 The Derivative

2.1 Concepts Review

1. tangent line

2. secant line

3. $\dfrac{f(c+h)-f(c)}{h}$

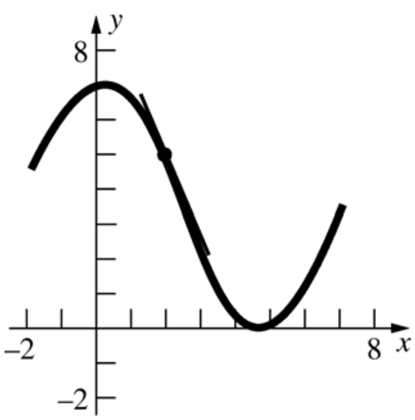
4. average velocity

Problem Set 2.1

1. Slope $= \dfrac{5-3}{2-\frac{3}{2}} = 4$

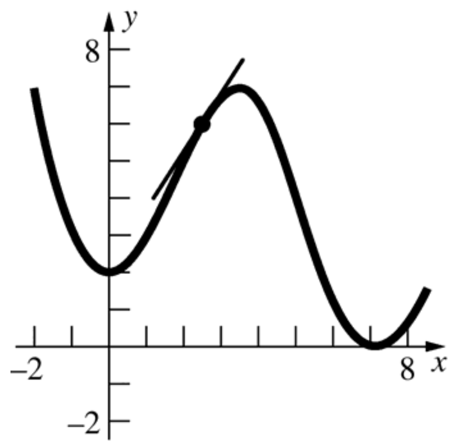
2. Slope $= \dfrac{6-4}{4-6} = -2$

3.



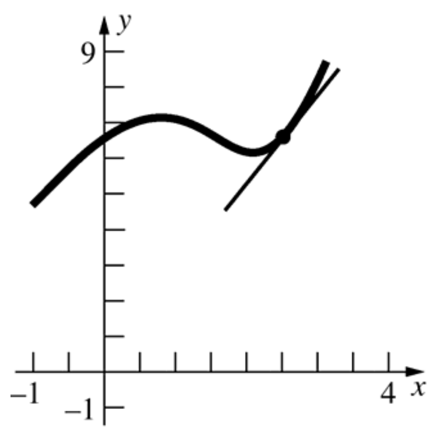
Slope ≈ -2

4.



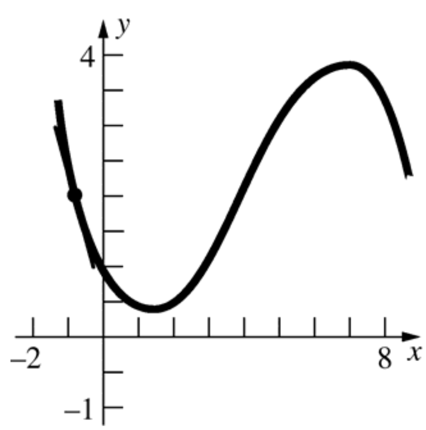
Slope ≈ 1.5

5.



Slope $\approx \dfrac{5}{2}$

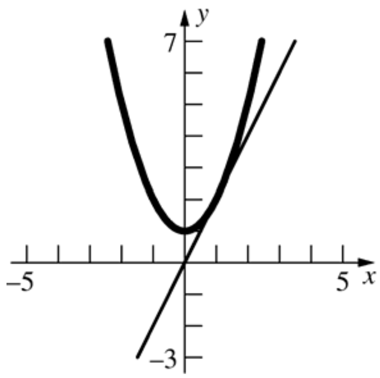
6.



Slope $\approx -\dfrac{3}{2}$

7. $y = x^2 + 1$

a., b.



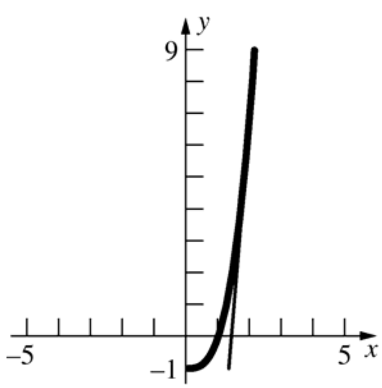
c. $m_{\tan} = 2$

d. $m_{\sec} = \dfrac{(1.01)^2 + 1.0 - 2}{1.01 - 1}$
$= \dfrac{0.0201}{.01}$
$= 2.01$

e. $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$
$= \lim\limits_{h \to 0} \dfrac{[(1+h)^2 + 1] - (1^2 + 1)}{h}$
$= \lim\limits_{h \to 0} \dfrac{2 + 2h + h^2 - 2}{h}$
$= \lim\limits_{h \to 0} \dfrac{h(2+h)}{h}$
$= \lim\limits_{h \to 0} (2+h) = 2$

8. $y = x^3 - 1$

a., b.



c. $m_{\tan} = 12$

d. $m_{\sec} = \dfrac{[(2.01)^3 - 1.0] - 7}{2.01 - 2}$
$= \dfrac{0.120601}{0.01}$
$= 12.0601$

e. $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$
$= \lim\limits_{h \to 0} \dfrac{[(2+h)^3 - 1] - (2^3 - 1)}{h}$
$= \lim\limits_{h \to 0} \dfrac{12h + 6h^2 + h^3}{h}$
$= \lim\limits_{h \to 0} \dfrac{h(12 + 6h + h^2)}{h}$
$= 12$

9. $f(x) = x^2 - 1$

$m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$
$= \lim\limits_{h \to 0} \dfrac{[(c+h)^2 - 1] - (c^2 - 1)}{h}$
$= \lim\limits_{h \to 0} \dfrac{c^2 + 2ch + h^2 - 1 - c^2 + 1}{h}$
$= \lim\limits_{h \to 0} \dfrac{h(2c+h)}{h} = 2c$

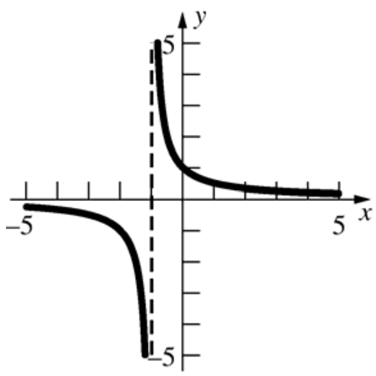
At $x = -2$, $m_{\tan} = -4$
$x = -1$, $m_{\tan} = -2$
$x = 1$, $m_{\tan} = 2$
$x = 2$, $m_{\tan} = 4$

10. $f(x) = x^3 - 3x$

$m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$
$= \lim\limits_{h \to 0} \dfrac{[(c+h)^3 - 3(c+h)] - (c^3 - 3c)}{h}$
$= \lim\limits_{h \to 0} \dfrac{c^3 + 3c^2h + 3ch^2 + h^3 - 3c - 3h - c^3 + 3c}{h}$
$= \lim\limits_{h \to 0} \dfrac{h(3c^2 + 3ch + h^2 - 3)}{h} = 3c^2 - 3$

At $x = -2$, $m_{\tan} = 9$
$x = -1$, $m_{\tan} = 0$
$x = 0$, $m_{\tan} = -3$
$x = 1$, $m_{\tan} = 0$
$x = 2$, $m_{\tan} = 9$

11.



$$f(x)=\frac{1}{x+1}$$

$$m_{\tan}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}$$

$$=\lim_{h\to 0}\frac{-\frac{h}{2(2+h)}}{h}$$

$$=\lim_{h\to 0}-\frac{1}{2(2+h)}$$

$$=-\frac{1}{4}$$

$$y-\frac{1}{2}=-\frac{1}{4}(x-1)$$

12. $f(x)=\dfrac{1}{x-1}$

$$m_{\tan}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{h-1}+1}{h}$$

$$=\lim_{h\to 0}\frac{\frac{h}{h-1}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h-1}$$

$$=-1$$

$y+1=-1(x-0);\ y=-x-1$

13. **a.** $16(1^2)-16(0^2)=16$ ft

b. $16(2^2)-16(1^2)=48$ ft

c. $V_{\text{ave}}=\dfrac{144-64}{3-2}=80$ ft/sec

d. $V_{\text{ave}}=\dfrac{16(3.01)^2-16(3)^2}{3.01-3}$

$$=\frac{0.9616}{0.01}$$

$= 96.16$ ft/s

e. $f(t)=16t^2;\ v=32c$

$v = 32(3) = 96$ ft/s

14. **a.** $V_{\text{ave}}=\dfrac{(3^2+1)-(2^2+1)}{3-2}=5$ m/sec

b. $V_{\text{ave}}=\dfrac{[(2.003)^2+1]-(2^2+1)}{2.003-2}$

$$=\frac{0.012009}{0.003}$$

$= 4.003$ m/sec

c. $V_{\text{ave}}=\dfrac{[(2+h)^2+1]-(2^2+1)}{2+h-2}$

$$=\frac{4h+h^2}{h}$$

$$=4+h$$

d. $f(t)=t^2+1$

$$v=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{[(2+h)^2+1]-(2^2+1)}{h}$$

$$=\lim_{h\to 0}\frac{4h+h^2}{h}$$

$$=\lim_{h\to 0}(4+h)$$

$$=4$$

15. a. $v = \lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h}$

$= \lim_{h\to 0} \frac{\sqrt{2(\alpha+h)+1}-\sqrt{2\alpha+1}}{h}$

$= \lim_{h\to 0} \frac{\sqrt{2\alpha+2h+1}-\sqrt{2\alpha+1}}{h}$

$= \lim_{h\to 0} \frac{(\sqrt{2\alpha+2h+1}-\sqrt{2\alpha+1})(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}{h(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}$

$= \lim_{h\to 0} \frac{2h}{h(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}$

$= \frac{2}{\sqrt{2\alpha+1}+\sqrt{2\alpha+1}} = \frac{1}{\sqrt{2\alpha+1}}$ ft/s

b. $\frac{1}{\sqrt{2\alpha+1}} = \frac{1}{2}$

$\sqrt{2\alpha+1} = 2$

$2\alpha+1 = 4;\ \alpha = \frac{3}{2}$

The object reaches a velocity of $\frac{1}{2}$ ft/s when $t = \frac{3}{2}$.

16. $f(t) = -t^2 + 4t$

$v = \lim_{h\to 0} \frac{[-(c+h)^2+4(c+h)]-(-c^2+4c)}{h}$

$= \lim_{h\to 0} \frac{-c^2-2ch-h^2+4c+4h+c^2-4c}{h}$

$= \lim_{h\to 0} \frac{h(-2c-h+4)}{h} = -2c+4$

$-2c + 4 = 0$ when $c = 2$

The particle comes to a momentary stop at $t = 2$.

17. a. $\left[\frac{1}{2}(2.01)^2+1\right]-\left[\frac{1}{2}(2)^2+1\right] = 0.02005$ g

b. $r_{\text{ave}} = \frac{0.02005}{2.01-2} = 2.005$ g/hr

c. $f(t) = \frac{1}{2}t^2+1$

$r = \lim_{h\to 0} \frac{\left[\frac{1}{2}(2+h)^2+1\right]-\left[\frac{1}{2}2^2+1\right]}{h}$

$= \lim_{h\to 0} \frac{2+2h+\frac{1}{2}h^2+1-2-1}{h}$

$= \lim_{h\to 0} \frac{h\left(2+\frac{1}{2}h\right)}{h} = 2$

At $t = 2$, $r = 2$

18. a. $1000(3)^2 - 1000(2)^2 = 5000$

b. $\frac{1000(2.5)^2-1000(2)^2}{2.5-2} = \frac{2250}{0.5} = 4500$

c. $f(t) = 1000t^2$

$r = \lim_{h\to 0} \frac{1000(2+h)^2-1000(2)^2}{h}$

$= \lim_{h\to 0} \frac{4000+4000h+1000h^2-4000}{h}$

$= \lim_{h\to 0} \frac{h(4000+1000h)}{h} = 4000$

19. a. $d_{\text{ave}} = \frac{5^3-3^3}{5-3} = \frac{98}{2} = 49$ g/cm

b. $f(x) = x^3$

$d = \lim_{h\to 0} \frac{(3+h)^3-3^3}{h}$

$= \lim_{h\to 0} \frac{27+27h+9h^2+h^3-27}{h}$

$= \lim_{h\to 0} \frac{h(27+9h+h^2)}{h} = 27$ g/cm

20. $MR = \lim_{h\to 0} \frac{R(c+h)-R(c)}{h}$

$= \lim_{h\to 0} \frac{[0.4(c+h)-0.001(c+h)^2]-(0.4c-0.001c^2)}{h}$

$= \lim_{h\to 0} \frac{0.4c+0.4h-0.001c^2-0.002ch-0.001h^2-0.4c+0.001c^2}{h}$

$= \lim_{h\to 0} \frac{h(0.4-0.002c-0.001h)}{h} = 0.4-0.002c$

When n = 10, MR = 0.38; when n = 100, MR = 0.2

21. $a = \lim_{h\to 0} \frac{2(1+h)^2-2(1)^2}{h}$

$= \lim_{h\to 0} \frac{2+4h+2h^2-2}{h}$

$= \lim_{h\to 0} \frac{h(4+2h)}{h} = 4$

22. $r = \lim_{h\to 0} \frac{p(c+h)-p(c)}{h}$

$= \lim_{h\to 0} \frac{[120(c+h)^2-2(c+h)^3]-(120c^2-2c^3)}{h}$

$= \lim_{h\to 0} \frac{h(240c-6c^2+120h-6ch-2h^2)}{h}$

$= 240c-6c^2$

When $t = 10,\ r = 240(10)-6(10)^2 = 1800$

$t = 20,\ r = 240(20)-6(20)^2 = 2400$

$t = 40,\ r = 240(40)-6(40)^2 = 0$

23. $r_{\text{ave}} = \frac{100-800}{24-0} = -\frac{175}{6} \approx -29.167$

29,167 gal/hr

At 8 o'clock, $r \approx \frac{700-400}{6-10} \approx -75$

75,000 gal/hr

24. **a.** The elevator reached the seventh floor at time $t = 80$. The average velocity is $v_{avg} = (84-0)/80 = 1.05$ feet per second

b. The slope of the line is approximately $\frac{60-12}{55-15} = 1.2$. The velocity is approximately 1.2 feet per second.

c. The building averages 84/7=12 feet from floor to floor. Since the velocity is zero for two intervals between time 0 and time 85, the elevator stopped twice. The heights are approximately 12 and 60. Thus, the elevator stopped at floors 1 and 5.

25. **a.** A tangent line at $t = 91$ has slope approximately $(63-48)/(91-61) = 0.5$. The normal high temperature increases at the rate of 0.5 degree F per day.

b. A tangent line at $t = 191$ has approximate slope $(90-88)/30 \approx 0.067$. The normal high temperature increases at the rate of 0.067 degree per day.

c. There is a time in January, about January 15, when the rate of change is zero. There is also a time in July, about July 15, when the rate of change is zero.

d. The greatest rate of increase occurs around day 61, that is, some time in March. The greatest rate of decrease occurs between day 301 and 331, that is, sometime in November.

26. The slope of the tangent line at $t = 1930$ is approximately $(8-6)/(1945-1930) \approx 0.13$. The rate of growth in 1930 is approximately 0.13 million, or 130,000, persons per year. In 1990, the tangent line has approximate slope $(24-16)/(20000-1980) \approx 0.4$. Thus, the rate of growth in 1990 is 0.4 million, or 400,000, persons per year. The approximate percentage growth in 1930 is $0.107/6 \approx 0.018$ and in 1990 it is approximately $0.4/20 \approx 0.02$.

27. In both (a) and (b), the tangent line is always positive. In (a) the tangent line becomes steeper and steeper as *t* increases; thus, the velocity is increasing. In (b) the tangent line becomes flatter and flatter as *t* increases; thus, the velocity is decreasing.

28. $f(t)=\frac{1}{3}t^3+t$

$$\text{current} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$
$$= \lim_{h\to 0}\frac{\left[\frac{1}{3}(c+h)^3+(c+h)\right]-\left(\frac{1}{3}c^3+c\right)}{h}$$
$$= \lim_{h\to 0}\frac{h\left(c^2+ch+\frac{1}{3}h^2+1\right)}{h}=c^2+1$$

When $t = 3$, the current $=10$

$c^2+1=20$

$c^2=19$

$c=\sqrt{19}\approx 4.4$

A 20-amp fuse will blow at $t = 4.4$ s.

29. $A=\pi r^2,\ r=2t$

$A=4\pi t^2$

$$\text{rate} = \lim_{h\to 0}\frac{4\pi(3+h)^2-4\pi(3)^2}{h}$$
$$= \lim_{h\to 0}\frac{h(24\pi+4\pi h)}{h}=24\pi\ \text{km}^2/\text{day}$$

30. $V=\frac{4}{3}\pi r^3, r=\frac{1}{4}t$

$V=\frac{1}{48}\pi t^3$

$$\text{rate} = \frac{1}{48}\pi\lim_{h\to 0}\frac{(3+h)^3-3^3}{h}=\frac{27}{48}\pi$$
$$=\frac{9}{16}\pi\ \text{inch}^3/\text{sec}$$

31. $y=f(x)=x^3-2x^2+1$

a. $m_{\tan}=7$ **b.** $m_{\tan}=0$

c. $m_{\tan}=-1$ **d.** $m_{\tan}=17.92$

32. $y=f(x)=\sin x\sin^2 2x$

a. $m_{\tan}=-1.125$ **b.** $m_{\tan}\approx -1.0315$

c. $m_{\tan}=0$ **d.** $m_{\tan}\approx 1.1891$

33. $s=f(t)=t+t\cos^2 t$

At $t = 3$, $v\approx 2.818$

34. $s=f(t)=\frac{(t+1)^3}{t+2}$

At $t = 1.6$, $v\approx 4.277$

2.2 Concepts Review

1. $\frac{f(c+h)-f(c)}{h};\frac{f(t)-f(c)}{t-c}$

2. $f'(c)$

3. continuous; $f(x)=|x|$

4. $f'(x);\frac{dy}{dx}$

Problem Set 2.2

1. $$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{(1+h)^2-1^2}{h}=\lim_{h\to 0}\frac{2h+h^2}{h}$$
$$=\lim_{h\to 0}(2+h)=2$$

2. $$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0}\frac{[2(2+h)]^2-[2(2)]^2}{h}$$
$$=\lim_{h\to 0}\frac{16h+4h^2}{h}=\lim_{h\to 0}(16+4h)=16$$

3. $$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$
$$=\lim_{h\to 0}\frac{[(3+h)^2-(3+h)]-(3^2-3)}{h}$$
$$=\lim_{h\to 0}\frac{5h+h^2}{h}=\lim_{h\to 0}(5+h)=5$$

4. $$f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{4-1}}{h}=\lim_{h\to 0}\frac{\frac{3-(3+h)}{3(3+h)}}{h}=\lim_{h\to 0}\frac{-1}{3(3+h)}$$
$$=-\frac{1}{9}$$

5. $$s'(x)=\lim_{h\to 0}\frac{s(x+h)-s(x)}{h}$$
$$=\lim_{h\to 0}\frac{[2(x+h)+1]-(2x+1)}{h}$$
$$=\lim_{h\to 0}\frac{2h}{h}=2$$

6. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[\alpha(x+h)+\beta] - (\alpha x + \beta)}{h}$

$= \lim_{h\to 0} \frac{\alpha h}{h} = \alpha$

7. $r'(x) = \lim_{h\to 0} \frac{r(x+h) - r(x)}{h}$

$= \lim_{h\to 0} \frac{[3(x+h)^2 + 4] - (3x^2 + 4)}{h}$

$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0} (6x + 3h) = 6x$

8. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[(x+h)^2 + (x+h) + 1] - (x^2 + x + 1)}{h}$

$= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} = \lim_{h\to 0} (2x + h + 1) = 2x + 1$

9. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[a(x+h)^2 + b(x+h) + c] - (ax^2 + bx + c)}{h}$

$= \lim_{h\to 0} \frac{2axh + ah^2 + bh}{h} = \lim_{h\to 0} (2ax + ah + b)$

$= 2ax + b$

10. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$

$= \lim_{h\to 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4}{h}$

$= \lim_{h\to 0} (4x^3 + 6hx^2 + 4h^2x + h^3) = 4x^3$

11. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[(x+h)^3 + 2(x+h)^2 + 1] - (x^3 + 2x^2 + 1)}{h}$

$= \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3 + 4hx + 2h^2}{h}$

$= \lim_{h\to 0} (3x^2 + 3hx + h^2 + 4x + 2h) = 3x^2 + 4x$

12. $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$= \lim_{h\to 0} \frac{[(x+h)^4 + (x+h)^2] - (x^4 + x^2)}{h}$

$= \lim_{h\to 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4 + 2hx + h^2}{h}$

$= \lim_{h\to 0} (4x^3 + 6hx^2 + 4h^2x + h^3 + 2x + h)$

$= 4x^3 + 2x$

13. $h'(x) = \lim_{h\to 0} \frac{h(x+h) - h(x)}{h}$

$= \lim_{h\to 0} \left[\left(\frac{2}{x+h} - \frac{2}{x} \right) \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \left[\frac{-2h}{x(x+h)} \cdot \frac{1}{h} \right] = \lim_{h\to 0} \frac{-2}{x(x+h)} = -\frac{2}{x^2}$

14. $S'(x) = \lim_{h\to 0} \frac{S(x+h) - S(x)}{h}$

$= \lim_{h\to 0} \left[\left(\frac{1}{x+h+1} - \frac{1}{x+1} \right) \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \left[\frac{-h}{(x+1)(x+h+1)} \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} = -\frac{1}{(x+1)^2}$

15. $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$

$= \lim_{h\to 0} \left[\left(\frac{6}{(x+h)^2 + 1} - \frac{6}{x^2 + 1} \right) \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \left[\frac{6(x^2+1) - 6(x^2 + 2hx + h^2 + 1)}{(x^2+1)(x^2 + 2hx + h^2 + 1)} \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \left[\frac{-12hx - 6h^2}{(x^2+1)(x^2 + 2hx + h^2 + 1)} \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \frac{-12x - 6h}{(x^2+1)(x^2 + 2hx + h^2 + 1)} = -\frac{12x}{(x^2+1)^2}$

16. $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$

$= \lim_{h\to 0} \left[\left(\frac{x+h-1}{x+h+1} - \frac{x-1}{x+1} \right) \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \left[\frac{x^2 + hx + h - 1 - (x^2 + hx - h - 1)}{(x+h+1)(x+1)} \cdot \frac{1}{h} \right]$

$= \lim_{h\to 0} \left[\frac{2h}{(x+h+1)(x+1)} \cdot \frac{1}{h} \right] = \frac{2}{(x+1)^2}$

17. $G'(x)=\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{2(x+h)-1}{x+h-4}-\frac{2x-1}{x-4}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{2x^2+2hx-9x-8h+4-(2x^2+2hx-9x-h+4)}{(x+h-4)(x-4)}\cdot\frac{1}{h}\right]=\lim_{h\to 0}\left[\frac{-7h}{(x+h-4)(x-4)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{-7}{(x+h-4)(x-4)}=-\frac{7}{(x-4)^2}$

18. $G'(x)=\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{2(x+h)}{(x+h)^2-(x+h)}-\frac{2x}{x^2-x}\right)\cdot\frac{1}{h}\right]=\lim_{h\to 0}\left[\frac{(2x+2h)(x^2-x)-2x(x^2+2xh+h^2-x-h)}{(x^2+2hx+h^2-x-h)(x^2-x)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{-2h^2x-2hx^2}{(x^2+2hx+h^2-x-h)(x^2-x)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{-2hx-2x^2}{(x^2+2hx+h^2-x-h)(x^2-x)}$

$=\frac{-2x^2}{(x^2-x)^2}=-\frac{2}{(x-1)^2}$

19. $g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$=\lim_{h\to 0}\frac{\sqrt{3(x+h)}-\sqrt{3x}}{h}$

$=\lim_{h\to 0}\frac{(\sqrt{3x+3h}-\sqrt{3x})(\sqrt{3x+3h}+\sqrt{3x})}{h(\sqrt{3x+3h}+\sqrt{3x})}$

$=\lim_{h\to 0}\frac{3h}{h(\sqrt{3x+3h}+\sqrt{3x})}=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h}+\sqrt{3x}}=\frac{3}{2\sqrt{3x}}$

20. $g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{1}{\sqrt{3(x+h)}}-\frac{1}{\sqrt{3x}}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{\sqrt{3x}-\sqrt{3x+3h}}{\sqrt{9x(x+h)}}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{(\sqrt{3x}-\sqrt{3x+3h})(\sqrt{3x}+\sqrt{3x+3h})}{\sqrt{9x(x+h)}(\sqrt{3x}+\sqrt{3x+3h})}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{-3h}{h\sqrt{9x(x+h)}(\sqrt{3x}+\sqrt{3x+3h})}=\frac{-3}{3x\cdot 2\sqrt{3x}}=-\frac{1}{2x\sqrt{3x}}$

21. $H'(x)=\lim_{h\to 0}\frac{H(x+h)-H(x)}{h}$

$=\lim_{h\to 0}\left[\left(\frac{3}{\sqrt{x+h-2}}-\frac{3}{\sqrt{x-2}}\right)\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{3\sqrt{x-2}-3\sqrt{x+h-2}}{\sqrt{(x+h-2)(x-2)}}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\frac{3(\sqrt{x-2}-\sqrt{x+h-2})(\sqrt{x-2}+\sqrt{x+h-2})}{h\sqrt{(x+h-2)(x-2)}(\sqrt{x-2}+\sqrt{x+h-2})}$

$=\lim_{h\to 0}\frac{-3h}{h[(x-2)\sqrt{x+h-2}+(x+h-2)\sqrt{x-2}]}$

$=\lim_{h\to 0}\frac{-3}{(x-2)\sqrt{x+h-2}+(x+h-2)\sqrt{x-2}}$

$=-\frac{3}{2(x-2)\sqrt{x-2}}=-\frac{3}{2(x-2)^{3/2}}$

22. $H'(x)=\lim_{h\to 0}\frac{H(x+h)-H(x)}{h}$

$=\lim_{h\to 0}\frac{\sqrt{(x+h)^2+4}-\sqrt{x^2+4}}{h}$

$=\lim_{h\to 0}\frac{\left(\sqrt{x^2+2hx+h^2+4}-\sqrt{x^2+4}\right)\left(\sqrt{x^2+2hx+h^2+4}+\sqrt{x^2+4}\right)}{h\left(\sqrt{x^2+2hx+h^2+4}+\sqrt{x^2+4}\right)}$

$=\lim_{h\to 0}\frac{2hx+h^2}{h\left(\sqrt{x^2+2hx+h^2+4}+\sqrt{x^2+4}\right)}$

$=\lim_{h\to 0}\frac{2x+h}{\sqrt{x^2+2hx+h^2+4}+\sqrt{x^2+4}}$

$=\frac{2x}{2\sqrt{x^2+4}}=\frac{x}{\sqrt{x^2+4}}$

23. $f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$

$=\lim_{t\to x}\frac{(t^2-3t)-(x^2-3x)}{t-x}$

$=\lim_{t\to x}\frac{t^2-x^2-(3t-3x)}{t-x}$

$=\lim_{t\to x}\frac{(t-x)(t+x)-3(t-x)}{t-x}$

$=\lim_{t\to x}\frac{(t-x)(t+x-3)}{t-x}=\lim_{t\to x}(t+x-3)$

$=2x-3$

24. $f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$

$=\lim_{t\to x}\frac{(t^3+5t)-(x^3+5x)}{t-x}$

$=\lim_{t\to x}\frac{t^3-x^3+5t-5x}{t-x}$

$=\lim_{t\to x}\frac{(t-x)(t^2+tx+x^2)+5(t-x)}{t-x}$

$=\lim_{t\to x}\frac{(t-x)(t^2+tx+x^2+5)}{t-x}$

$=\lim_{t\to x}(t^2+tx+x^2+5)=3x^2+5$

25. $f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$

$=\lim_{t\to x}\left[\left(\frac{t}{t-5}-\frac{x}{x-5}\right)\left(\frac{1}{t-x}\right)\right]$

$=\lim_{t\to x}\frac{tx-5t-tx+5x}{(t-5)(x-5)(t-x)}$

$=\lim_{t\to x}\frac{-5(t-x)}{(t-5)(x-5)(t-x)}=\lim_{t\to x}\frac{-5}{(t-5)(x-5)}$

$=-\frac{5}{(x-5)^2}$

26. $f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$

$=\lim_{t\to x}\left[\left(\frac{t+3}{t}-\frac{x+3}{x}\right)\left(\frac{1}{t-x}\right)\right]$

$=\lim_{t\to x}\frac{3x-3t}{xt(t-x)}=\lim_{t\to x}\frac{-3}{xt}=-\frac{3}{x^2}$

27. $f(x)=2x^3$ at $x=5$

28. $f(x)=x^2+2x$ at $x=3$

29. $f(x)=x^2$ at $x=2$

30. $f(x)=x^3+x$ at $x=3$

31. $f(x)=x^2$ at x

32. $f(x)=x^3$ at x

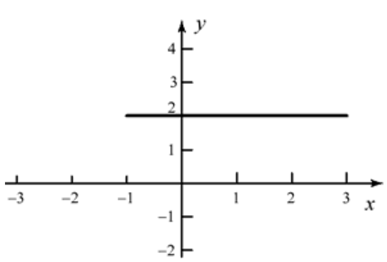
33. $f(t)=\frac{2}{t}$ at t

34. $f(y)=\sin y$ at y

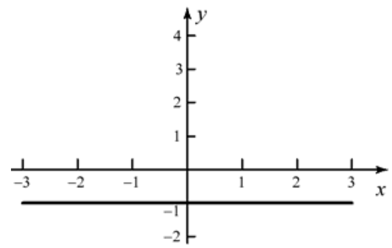
35. $f(x)=\cos x$ at x

36. $f(t)=\tan t$ at t

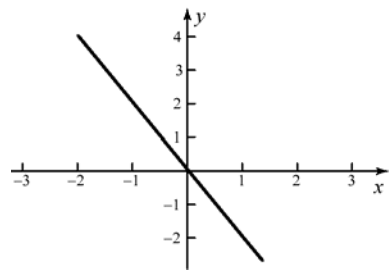
37. The slope of the tangent line is always 2.



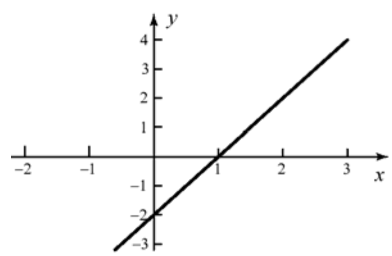
38. The slope of the tangent line is always -1.



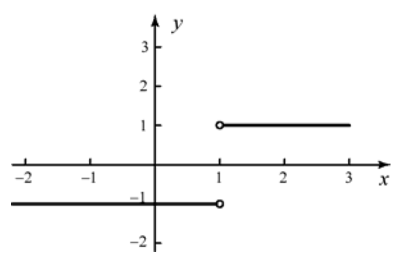
39. The derivative is positive until $x=0$, then becomes negative.



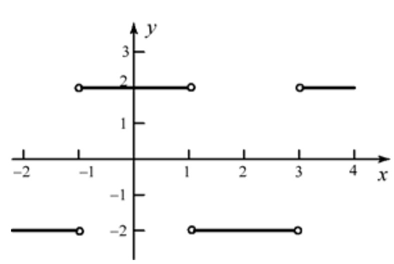
40. The derivative is negative until $x=1$, then becomes positive.



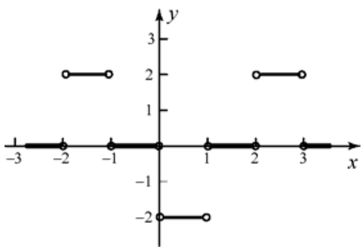
41. The derivative is -1 until $x=1$. To the right of $x=1$, the derivative is 1. The derivative is undefined at $x=1$.



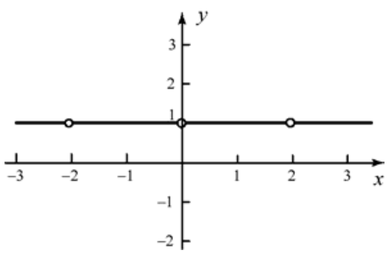
42. The derivative is -2 to the left of $x=-1$; from -1 to 1, the derivative is 2, etc. The derivative is not defined at $x=-1, 1, 3$.



43. The derivative is 0 on $(-3,-2)$, 2 on $(-2,-1)$, 0 on $(-1,0)$, -2 on $(0,1)$, 0 on $(1,2)$, 2 on $(2,3)$ and 0 on $(3,4)$. The derivative is undefined at $x=-2,\ -1,\ 0,\ 1,\ 2,\ 3$.



44. The derivative is 1 except at $x=-2,\ 0,\ 2$ where it is undefined.



45. Δy = [3(1.5) + 2] – [3(1) + 2] = 1.5

46. $\Delta y=[3(0.1)^2+2(0.1)+1]-[3(0.0)^2+2(0.0)+1]$ = 0.23

47. Δy = 1/1.2 – 1/1 = – 0.1667

48. Δy = 2/(0.1+1) – 2/(0+1) = – 0.1818

49. $\Delta y=\dfrac{3}{2.31+1}-\dfrac{3}{2.34+1}\approx 0.0081$

50. $\Delta y=\cos[2(0.573)]-\cos[2(0.571)]\approx -0.0036$

51. $\dfrac{\Delta y}{\Delta x}=\dfrac{(x+\Delta x)^2-x^2}{\Delta x}=\dfrac{2x\Delta x+(\Delta x)^2}{\Delta x}=2x+\Delta x$

$\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}(2x+\Delta x)=2x$

52. $\dfrac{\Delta y}{\Delta x}=\dfrac{[(x+\Delta x)^3-3(x+\Delta x)^2]-(x^3-3x^2)}{\Delta x}$

$=\dfrac{3x^2\Delta x+3x(\Delta x)^2-6x\Delta x-3(\Delta x)^2+\Delta x^3}{\Delta x}$

$=3x^2+3x\Delta x-6x-3\Delta x+(\Delta x)^2$

$\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}(3x^2+3x\Delta x-6x-3\Delta x+(\Delta x)^2)$

$=3x^2-6x$

53. $\dfrac{\Delta y}{\Delta x}=\dfrac{\frac{1}{x+\Delta x+1}-\frac{1}{x+1}}{\Delta x}$

$=\left(\dfrac{x+1-(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}\right)\left(\dfrac{1}{\Delta x}\right)$

$=\dfrac{-\Delta x}{(x+\Delta x+1)(x+1)\Delta x}$

$=-\dfrac{1}{(x+\Delta x+1)(x+1)}$

$\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\left[-\dfrac{1}{(x+\Delta x+1)(x+1)}\right]=-\dfrac{1}{(x+1)^2}$

54. $\dfrac{\Delta y}{\Delta x}=\dfrac{1+\frac{1}{x+\Delta x}-\left(1+\frac{1}{x}\right)}{\Delta x}$

$=\dfrac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x}=\dfrac{\frac{-\Delta x}{x(x+\Delta x)}}{\Delta x}=-\dfrac{1}{x(x+\Delta x)}$

$\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}-\dfrac{1}{x(x+\Delta x)}=-\dfrac{1}{x^2}$

55.

$\dfrac{\Delta y}{\Delta x}=\dfrac{\frac{x+\Delta x-1}{x+\Delta x+1}-\frac{x-1}{x+1}}{\Delta x}$

$=\dfrac{(x+1)(x+\Delta x-1)-(x-1)(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}\times\dfrac{1}{\Delta x}$

$=\dfrac{x^2+x\Delta x-x+x+\Delta x-1-\left[x^2+x\Delta x-x+x-\Delta x-1\right]}{x^2+x\Delta x+x+x+\Delta x+1}\times\dfrac{1}{\Delta x}$

$=\dfrac{2\Delta x}{x^2+x\Delta x+x+x+\Delta x+1}\times\dfrac{1}{\Delta x}=\dfrac{2}{x^2+x\Delta x+x+x+\Delta x+1}$

$\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\dfrac{2}{x^2+x\Delta x+x+x+\Delta x+1}=\dfrac{2}{x^2+2x+1}=\dfrac{2}{(x+1)^2}$

56. $\dfrac{\Delta y}{\Delta x}=\dfrac{\frac{(x+\Delta x)^2-1}{x+\Delta x}-\frac{x^2-1}{x}}{\Delta x}$

$=\left[\dfrac{x(x+\Delta x)^2-x-(x+\Delta x)(x^2-1)}{x(x+\Delta x)}\right]\times\dfrac{1}{\Delta x}$

$=\left[\dfrac{x\left(x^2+2x\Delta x+\left(\Delta x^2\right)\right)-x-\left(x^3+x^2\Delta x-x-\Delta x\right)}{x^2+x\Delta x}\right]\times\dfrac{1}{\Delta x}$

$=\dfrac{x^2\Delta x+x(\Delta x)^2+\Delta x}{x^2+x\Delta x}\times\dfrac{1}{\Delta x}=\dfrac{x^2+x\Delta x+1}{x^2+x\Delta x}$

$\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\dfrac{x^2+x\Delta x+1}{x^2+x\Delta x}=\dfrac{x^2+1}{x^2}$

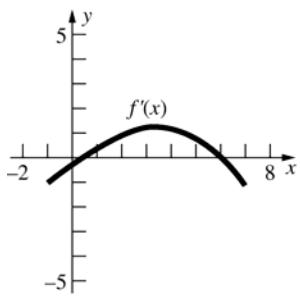
57. $f'(0) \approx -\frac{1}{2}$; $f'(2) \approx 1$

$f'(5) \approx \frac{2}{3}$; $f'(7) \approx -3$

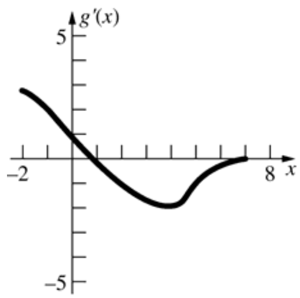
58. $g'(-1) \approx 2$; $g'(1) \approx 0$

$g'(4) \approx -2$; $g'(6) \approx -\frac{1}{3}$

59.



60.



61. **a.** $f(2) \approx \frac{5}{2}$; $f'(2) \approx \frac{3}{2}$

$f(0.5) \approx 1.8$; $f'(0.5) \approx -0.6$

b. $\frac{2.9-1.9}{2.5-0.5} = 0.5$

c. $x = 5$

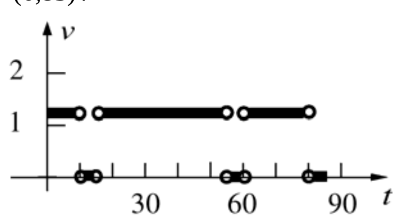
d. $x = 3, 5$

e. $x = 1, 3, 5$

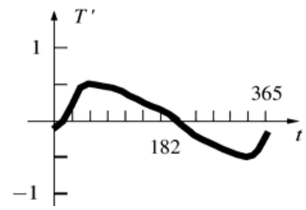
f. $x = 0$

g. $x \approx -0.7, \frac{3}{2}$ and $5 < x < 7$

62. The derivative fails to exist at the corners of the graph; that is, at $t = 10, 15, 55, 60, 80$. The derivative exists at all other points on the interval $(0,85)$.



63. The derivative is 0 at approximately $t = 15$ and $t = 201$. The greatest rate of increase occurs at about $t = 61$ and it is about 0.5 degree F per day. The greatest rate of decrease occurs at about $t = 320$ and it is about 0.5 degree F per day. The derivative is positive on (15,201) and negative on (0,15) and (201,365).



64. The slope of a tangent line for the dashed function is zero when *x* is approximately 0.3 or 1.9. The solid function is zero at both of these points. The graph indicates that the solid function is negative when the dashed function has a tangent line with a negative slope and positive when the dashed function has a tangent line with a positive slope. Thus, the solid function is the derivative of the dashed function.

65. The short-dash function has a tangent line with zero slope at about $x = 2.1$, where the solid function is zero. The solid function has a tangent line with zero slope at about $x = 0.4$, 1.2 and 3.5. The long-dash function is zero at these points. The graph shows that the solid function is positive (negative) when the slope of the tangent line of the short-dash function is positive (negative). Also, the long-dash function is positive (negative) when the slope of the tangent line of the solid function is positive (negative). Thus, the short-dash function is *f*, the solid function is $f' = g$, and the dash function is g'.

66. Note that since $x = 0 + x$, $f(x) = f(0 + x) = f(0)f(x)$, hence $f(0) = 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a)f(h) - f(a)}{h}$$

$$= f(a)\lim_{h\to 0} \frac{f(h)-1}{h} = f(a)\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= f(a)f'(0)$$

$f'(a)$ exists since $f'(0)$ exists.

67. If f is differentiable everywhere, then it is continuous everywhere, so
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(mx+b) = 2m+b = f(2) = 4$
and $b = 4 - 2m$.
For f to be differentiable everywhere,

$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$ must exist.

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+}\frac{x^2-4}{x-2} = \lim_{x\to 2^+}(x+2) = 4$$

$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-}\frac{mx+b-4}{x-2}$$

$$= \lim_{x\to 2^-}\frac{mx+4-2m-4}{x-2} = \lim_{x\to 2^-}\frac{m(x-2)}{x-2} = m$$

Thus $m = 4$ and $b = 4 - 2(4) = -4$

68. $$f_s(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)+f(x)-f(x-h)}{2h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{2h} + \frac{f(x-h)-f(x)}{-2h}\right]$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \frac{1}{2}\lim_{-h\to 0}\frac{f[x+(-h)]-f(x)}{-h}$$

$$= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x).$$

For the converse, let $f(x) = |x|$. Then

$$f_s(0) = \lim_{h\to 0}\frac{|h|-|-h|}{2h} = \lim_{h\to 0}\frac{|h|-|h|}{2h} = 0$$

but $f'(0)$ does not exist.

69. $f'(x_0) = \lim_{t\to x_0}\frac{f(t)-f(x_0)}{t-x_0}$, so

$$f'(-x_0) = \lim_{t\to -x_0}\frac{f(t)-f(-x_0)}{t-(-x_0)}$$

$$= \lim_{t\to -x_0}\frac{f(t)-f(-x_0)}{t+x_0}$$

a. If f is an odd function,

$$f'(-x_0) = \lim_{t\to -x_0}\frac{f(t)-[-f(-x_0)]}{t+x_0}$$

$$= \lim_{t\to -x_0}\frac{f(t)+f(-x_0)}{t+x_0}.$$

Let $u = -t$. As $t \to -x_0$, $u \to x_0$ and so

$$f'(-x_0) = \lim_{u\to x_0}\frac{f(-u)+f(x_0)}{-u+x_0}$$

$$= \lim_{u\to x_0}\frac{-f(u)+f(x_0)}{-(u-x_0)} = \lim_{u\to x_0}\frac{-[f(u)-f(x_0)]}{-(u-x_0)}$$

$$= \lim_{u\to x_0}\frac{f(u)-f(x_0)}{u-x_0} = f'(x_0) = m.$$

b. If f is an even function,

$f'(-x_0) = \lim_{t\to -x_0}\frac{f(t)-f(x_0)}{t+x_0}$. Let $u = -t$, as above, then $f'(-x_0) = \lim_{u\to x_0}\frac{f(-u)-f(x_0)}{-u+x_0}$

$$= \lim_{u\to x_0}\frac{f(u)-f(x_0)}{-(u-x_0)} = -\lim_{u\to x_0}\frac{f(u)-f(x_0)}{u-x_0}$$

$$= -f'(x_0) = -m.$$

70. Say $f(-x) = -f(x)$. Then

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h}$$

$$= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} = -\lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$$

$= \lim_{-h\to 0}\frac{f[x+(-h)]-f(x)}{-h} = f'(x)$ so $f'(x)$ is an even function if $f(x)$ is an odd function.

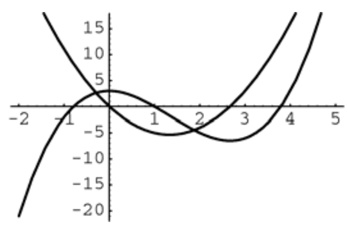
Say $f(-x) = f(x)$. Then

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$$

$= -\lim_{-h\to 0}\frac{f[x+(-h)]-f(x)}{-h} = -f'(x)$ so $f'(x)$ is an odd function if $f(x)$ is an even function.

71.

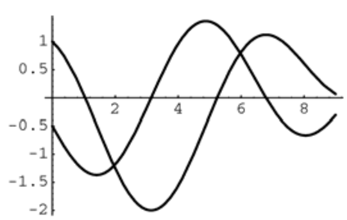


a. $0 < x < \frac{8}{3}$; $\left(0, \frac{8}{3}\right)$

b. $0 \le x \le \frac{8}{3}$; $\left[0, \frac{8}{3}\right]$

c. A function $f(x)$ decreases as x increases when $f'(x) < 0$.

72.



a. $\pi < x < 6.8$ **b.** $\pi < x < 6.8$

c. A function $f(x)$ increases as x increases when $f'(x) > 0$.

2.3 Concepts Review

1. the derivative of the second; second; $f(x)g'(x)+g(x)f'(x)$

2. denominator; denominator; square of the denominator; $\dfrac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$

3. $nx^{n-1}h$; nx^{n-1}

4. $kL(f)$; $L(f)+L(g)$; D_x

Problem Set 2.3

1. $D_x(2x^2)=2D_x(x^2)=2\cdot 2x=4x$

2. $D_x(3x^3)=3D_x(x^3)=3\cdot 3x^2=9x^2$

3. $D_x(\pi x)=\pi D_x(x)=\pi\cdot 1=\pi$

4. $D_x(\pi x^3)=\pi D_x(x^3)=\pi\cdot 3x^2=3\pi x^2$

5. $D_x(2x^{-2})=2D_x(x^{-2})=2(-2x^{-3})=-4x^{-3}$

6. $D_x(-3x^{-4})=-3D_x(x^{-4})=-3(-4x^{-5})=12x^{-5}$

7. $D_x\left(\dfrac{\pi}{x}\right)=\pi D_x(x^{-1})=\pi(-1x^{-2})=-\pi x^{-2}$
$=-\dfrac{\pi}{x^2}$

8. $D_x\left(\dfrac{\alpha}{x^3}\right)=\alpha D_x(x^{-3})=\alpha(-3x^{-4})=-3\alpha x^{-4}$
$=-\dfrac{3\alpha}{x^4}$

9. $D_x\left(\dfrac{100}{x^5}\right)=100D_x(x^{-5})=100(-5x^{-6})$
$=-500x^{-6}=-\dfrac{500}{x^6}$

10. $D_x\left(\dfrac{3\alpha}{4x^5}\right)=\dfrac{3\alpha}{4}D_x(x^{-5})=\dfrac{3\alpha}{4}(-5x^{-6})$
$=-\dfrac{15\alpha}{4}x^{-6}=-\dfrac{15\alpha}{4x^6}$

11. $D_x(x^2+2x)=D_x(x^2)+2D_x(x)=2x+2$

12. $D_x(3x^4+x^3)=3D_x(x^4)+D_x(x^3)$
$=3(4x^3)+3x^2=12x^3+3x^2$

13. $D_x(x^4+x^3+x^2+x+1)$
$=D_x(x^4)+D_x(x^3)+D_x(x^2)+D_x(x)+D_x(1)$
$=4x^3+3x^2+2x+1$

14. $D_x(3x^4-2x^3-5x^2+\pi x+\pi^2)$
$=3D_x(x^4)-2D_x(x^3)-5D_x(x^2)$
$+\pi D_x(x)+D_x(\pi^2)$
$=3(4x^3)-2(3x^2)-5(2x)+\pi(1)+0$
$=12x^3-6x^2-10x+\pi$

15. $D_x(\pi x^7-2x^5-5x^{-2})$
$=\pi D_x(x^7)-2D_x(x^5)-5D_x(x^{-2})$
$=\pi(7x^6)-2(5x^4)-5(-2x^{-3})$
$=7\pi x^6-10x^4+10x^{-3}$

16. $D_x(x^{12}+5x^{-2}-\pi x^{-10})$
$=D_x(x^{12})+5D_x(x^{-2})-\pi D_x(x^{-10})$
$=12x^{11}+5(-2x^{-3})-\pi(-10x^{-11})$
$=12x^{11}-10x^{-3}+10\pi x^{-11}$

17. $D_x\left(\dfrac{3}{x^3}+x^{-4}\right)=3D_x(x^{-3})+D_x(x^{-4})$
$=3(-3x^{-4})+(-4x^{-5})=-\dfrac{9}{x^4}-4x^{-5}$

18. $D_x(2x^{-6}+x^{-1})=2D_x(x^{-6})+D_x(x^{-1})$
$=2(-6x^{-7})+(-1x^{-2})=-12x^{-7}-x^{-2}$

19. $D_x\left(\dfrac{2}{x}-\dfrac{1}{x^2}\right)=2D_x(x^{-1})-D_x(x^{-2})$
$=2(-1x^{-2})-(-2x^{-3})=-\dfrac{2}{x^2}+\dfrac{2}{x^3}$

20. $D_x\left(\dfrac{3}{x^3}-\dfrac{1}{x^4}\right)=3D_x(x^{-3})-D_x(x^{-4})$
$=3(-3x^{-4})-(-4x^{-5})=-\dfrac{9}{x^4}+\dfrac{4}{x^5}$

21. $D_x\left(\dfrac{1}{2x}+2x\right)=\dfrac{1}{2}D_x(x^{-1})+2D_x(x)$
$=\dfrac{1}{2}(-1x^{-2})+2(1)=-\dfrac{1}{2x^2}+2$

22. $D_x\left(\frac{2}{3x}-\frac{2}{3}\right)=\frac{2}{3}D_x(x^{-1})-D_x\left(\frac{2}{3}\right)$

$=\frac{2}{3}(-1x^{-2})-0=-\frac{2}{3x^2}$

23. $D_x[x(x^2+1)]=x\,D_x(x^2+1)+(x^2+1)D_x(x)$

$=x(2x)+(x^2+1)(1)=3x^2+1$

24. $D_x[3x(x^3-1)]=3x\,D_x(x^3-1)+(x^3-1)D_x(3x)$

$=3x(3x^2)+(x^3-1)(3)=12x^3-3$

25. $D_x[(2x+1)^2]$

$=(2x+1)D_x(2x+1)+(2x+1)D_x(2x+1)$

$=(2x+1)(2)+(2x+1)(2)=8x+4$

26. $D_x[(-3x+2)^2]$

$=(-3x+2)D_x(-3x+2)+(-3x+2)D_x(-3x+2)$

$=(-3x+2)(-3)+(-3x+2)(-3)=18x-12$

27. $D_x[(x^2+2)(x^3+1)]$

$=(x^2+2)D_x(x^3+1)+(x^3+1)D_x(x^2+2)$

$=(x^2+2)(3x^2)+(x^3+1)(2x)$

$=3x^4+6x^2+2x^4+2x$

$=5x^4+6x^2+2x$

28. $D_x[(x^4-1)(x^2+1)]$

$=(x^4-1)D_x(x^2+1)+(x^2+1)D_x(x^4-1)$

$=(x^4-1)(2x)+(x^2+1)(4x^3)$

$=2x^5-2x+4x^5+4x^3=6x^5+4x^3-2x$

29. $D_x[(x^2+17)(x^3-3x+1)]$

$=(x^2+17)D_x(x^3-3x+1)+(x^3-3x+1)D_x(x^2+17)$

$=(x^2+17)(3x^2-3)+(x^3-3x+1)(2x)$

$=3x^4+48x^2-51+2x^4-6x^2+2x$

$=5x^4+42x^2+2x-51$

30. $D_x[(x^4+2x)(x^3+2x^2+1)]=(x^4+2x)D_x(x^3+2x^2+1)+(x^3+2x^2+1)D_x(x^4+2x)$

$=(x^4+2x)(3x^2+4x)+(x^3+2x^2+1)(4x^3+2)$

$=7x^6+12x^5+12x^3+12x^2+2$

31. $D_x[(5x^2-7)(3x^2-2x+1)]=(5x^2-7)D_x(3x^2-2x+1)+(3x^2-2x+1)D_x(5x^2-7)$

$=(5x^2-7)(6x-2)+(3x^2-2x+1)(10x)$

$=60x^3-30x^2-32x+14$

32. $D_x[(3x^2+2x)(x^4-3x+1)]=(3x^2+2x)D_x(x^4-3x+1)+(x^4-3x+1)D_x(3x^2+2x)$

$=(3x^2+2x)(4x^3-3)+(x^4-3x+1)(6x+2)$

$=18x^5+10x^4-27x^2-6x+2$

33. $D_x\left(\frac{1}{3x^2+1}\right)=\frac{(3x^2+1)D_x(1)-(1)D_x(3x^2+1)}{(3x^2+1)^2}$

$=\frac{(3x^2+1)(0)-(6x)}{(3x^2+1)^2}=-\frac{6x}{(3x^2+1)^2}$

34. $D_x\left(\frac{2}{5x^2-1}\right)=\frac{(5x^2-1)D_x(2)-(2)D_x(5x^2-1)}{(5x^2-1)^2}$

$=\frac{(5x^2-1)(0)-2(10x)}{(5x^2-1)^2}=-\frac{20x}{(5x^2-1)^2}$

35. $D_x\left(\dfrac{1}{4x^2-3x+9}\right)=\dfrac{(4x^2-3x+9)D_x(1)-(1)D_x(4x^2-3x+9)}{(4x^2-3x+9)^2}$

$=\dfrac{(4x^2-3x+9)(0)-(8x-3)}{(4x^2-3x+9)^2}=-\dfrac{8x-3}{(4x^2-3x+9)^2}$

$=\dfrac{-8x+3}{(4x^2-3x+9)^2}$

36. $D_x\left(\dfrac{4}{2x^3-3x}\right)=\dfrac{(2x^3-3x)D_x(4)-(4)D_x(2x^3-3x)}{(2x^3-3x)^2}$

$=\dfrac{(2x^3-3x)(0)-4(6x^2-3)}{(2x^3-3x)^2}=\dfrac{-24x^2+12}{(2x^3-3x)^2}$

37. $D_x\left(\dfrac{x-1}{x+1}\right)=\dfrac{(x+1)D_x(x-1)-(x-1)D_x(x+1)}{(x+1)^2}$

$=\dfrac{(x+1)(1)-(x-1)(1)}{(x+1)^2}=\dfrac{2}{(x+1)^2}$

38. $D_x\left(\dfrac{2x-1}{x-1}\right)=\dfrac{(x-1)D_x(2x-1)-(2x-1)D_x(x-1)}{(x-1)^2}$

$=\dfrac{(x-1)(2)-(2x-1)(1)}{(x-1)^2}=-\dfrac{1}{(x-1)^2}$

39. $D_x\left(\dfrac{2x^2-1}{3x+5}\right)=\dfrac{(3x+5)D_x(2x^2-1)-(2x^2-1)D_x(3x+5)}{(3x+5)^2}$

$=\dfrac{(3x+5)(4x)-(2x^2-1)(3)}{(3x+5)^2}$

$=\dfrac{6x^2+20x+3}{(3x+5)^2}$

40. $D_x\left(\dfrac{5x-4}{3x^2+1}\right)=\dfrac{(3x^2+1)D_x(5x-4)-(5x-4)D_x(3x^2+1)}{(3x^2+1)^2}$

$=\dfrac{(3x^2+1)(5)-(5x-4)(6x)}{(3x^2+1)^2}$

$=\dfrac{-15x^2+24x+5}{(3x^2+1)^2}$

41. $D_x\left(\dfrac{2x^2-3x+1}{2x+1}\right)=\dfrac{(2x+1)D_x(2x^2-3x+1)-(2x^2-3x+1)D_x(2x+1)}{(2x+1)^2}$

$=\dfrac{(2x+1)(4x-3)-(2x^2-3x+1)(2)}{(2x+1)^2}$

$=\dfrac{4x^2+4x-5}{(2x+1)^2}$

42. $D_x\left(\frac{5x^2+2x-6}{3x-1}\right)=\frac{(3x-1)D_x(5x^2+2x-6)-(5x^2+2x-6)D_x(3x-1)}{(3x-1)^2}$

$=\frac{(3x-1)(10x+2)-(5x^2+2x-6)(3)}{(3x-1)^2}$

$=\frac{15x^2-10x+16}{(3x-1)^2}$

43. $D_x\left(\frac{x^2-x+1}{x^2+1}\right)=\frac{(x^2+1)D_x(x^2-x+1)-(x^2-x+1)D_x(x^2+1)}{(x^2+1)^2}$

$=\frac{(x^2+1)(2x-1)-(x^2-x+1)(2x)}{(x^2+1)^2}$

$=\frac{x^2-1}{(x^2+1)^2}$

44. $D_x\left(\frac{x^2-2x+5}{x^2+2x-3}\right)=\frac{(x^2+2x-3)D_x(x^2-2x+5)-(x^2-2x+5)D_x(x^2+2x-3)}{(x^2+2x-3)^2}$

$=\frac{(x^2+2x-3)(2x-2)-(x^2-2x+5)(2x+2)}{(x^2+2x-3)^2}$

$=\frac{4x^2-16x-4}{(x^2+2x-3)^2}$

45. **a.** $(f\cdot g)'(0)=f(0)g'(0)+g(0)f'(0)$
$= 4(5) + (–3)(–1) = 23$

b. $(f+g)'(0)=f'(0)+g'(0)=-1+5=4$

c. $(f/g)'(0)=\frac{g(0)f'(0)-f(0)g'(0)}{g^2(0)}$
$=\frac{-3(-1)-4(5)}{(-3)^2}=-\frac{17}{9}$

46. **a.** $(f-g)'(3)=f'(3)-g'(3)=2-(-10)=12$

b. $(f\cdot g)'(3)=f(3)g'(3)+g(3)f'(3)=7(-10)+6(2)=-58$

c. $(g/f)'(3)=\frac{f(3)g'(3)-g(3)f'(3)}{f^2(3)}=\frac{7(-10)-6(2)}{(7)^2}=-\frac{82}{49}$

47. $D_x[f(x)]^2=D_x[f(x)f(x)]$
$=f(x)D_x[f(x)]+f(x)D_x[f(x)]$
$=2\cdot f(x)\cdot D_x f(x)$

48. $D_x[f(x)g(x)h(x)]=f(x)D_x[g(x)h(x)]+g(x)h(x)D_x f(x)$
$=f(x)[g(x)D_x h(x)+h(x)D_x g(x)]+g(x)h(x)D_x f(x)$
$=f(x)g(x)D_x h(x)+f(x)h(x)D_x g(x)+g(x)h(x)D_x f(x)$

49. $D_x(x^2 - 2x + 2) = 2x - 2$
At $x = 1$: $m_{\tan} = 2(1) - 2 = 0$
Tangent line: $y = 1$

50. $D_x\left(\frac{1}{x^2+4}\right) = \frac{(x^2+4)D_x(1) - (1)D_x(x^2+4)}{(x^2+4)^2}$

$= \frac{(x^2+4)(0) - (2x)}{(x^2+4)^2} = -\frac{2x}{(x^2+4)^2}$

At $x = 1$: $m_{\tan} = -\frac{2(1)}{(1^2+4)^2} = -\frac{2}{25}$

Tangent line: $y - \frac{1}{5} = -\frac{2}{25}(x-1)$

$y = -\frac{2}{25}x + \frac{7}{25}$

51. $D_x(x^3 - x^2) = 3x^2 - 2x$
The tangent line is horizontal when $m_{\tan} = 0$:
$m_{\tan} = 3x^2 - 2x = 0$
$x(3x-2) = 0$
$x = 0$ and $x = \frac{2}{3}$
(0, 0) and $\left(\frac{2}{3}, -\frac{4}{27}\right)$

52. $D_x\left(\frac{1}{3}x^3 + x^2 - x\right) = x^2 + 2x - 1$

$m_{\tan} = x^2 + 2x - 1 = 1$
$x^2 + 2x - 2 = 0$
$x = \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2}$
$= -1 - \sqrt{3}, -1 + \sqrt{3}$
$x = -1 \pm \sqrt{3}$
$\left(-1+\sqrt{3}, \frac{5}{3} - \sqrt{3}\right), \left(-1-\sqrt{3}, \frac{5}{3} + \sqrt{3}\right)$

53. $y = 100/x^5 = 100x^{-5}$
$y' = -500x^{-6}$

Set y' equal to -1, the negative reciprocal of the slope of the line $y = x$. Solving for x gives
$x = \pm 500^{1/6} \approx \pm 2.817$
$y = \pm 100(500)^{-5/6} \approx \pm 0.563$

The points are (2.817,0.563) and (–2.817,–0.563).

54. Proof #1:

$$D_x[f(x) - g(x)] = D_x[f(x) + (-1)g(x)]$$
$$= D_x[f(x)] + D_x[(-1)g(x)]$$
$$= D_x f(x) - D_x g(x)$$

Proof #2:

Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = \lim_{h\to 0} \frac{[f(x+h) - g(x+h)] - [f(x) - g(x)]}{h}$$
$$= \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}\right]$$
$$= f'(x) - g'(x)$$

55. **a.** $D_t(-16t^2 + 40t + 100) = -32t + 40$
$v = -32(2) + 40 = -24$ ft/s

b. $v = -32t + 40 = 0$
$t = \frac{5}{4}$ s

56. $D_t(4.5t^2 + 2t) = 9t + 2$
$9t + 2 = 30$
$t = \frac{28}{9}$ s

57. $m_{\tan} = D_x(4x - x^2) = 4 - 2x$
The line through (2,5) and (x_0, y_0) has slope $\frac{y_0 - 5}{x_0 - 2}$.

$4 - 2x_0 = \frac{4x_0 - x_0^2 - 5}{x_0 - 2}$

$-2x_0^2 + 8x_0 - 8 = -x_0^2 + 4x_0 - 5$
$x_0^2 - 4x_0 + 3 = 0$
$(x_0 - 3)(x_0 - 1) = 0$
$x_0 = 1, \ x_0 = 3$
At $x_0 = 1$: $y_0 = 4(1) - (1)^2 = 3$
$m_{\tan} = 4 - 2(1) = 2$
Tangent line: $y - 3 = 2(x - 1)$; $y = 2x + 1$
At $x_0 = 3: y_0 = 4(3) - (3)^2 = 3$
$m_{\tan} = 4 - 2(3) = -2$
Tangent line: $y - 3 = -2(x - 3)$; $y = -2x + 9$

58. $D_x(x^2) = 2x$

The line through (4, 15) and (x_0, y_0) has slope $\frac{y_0 - 15}{x_0 - 4}$. If (x_0, y_0) is on the curve $y = x^2$, then

$m_{\tan} = 2x_0 = \frac{x_0^2 - 15}{x_0 - 4}$.

$2x_0^2 - 8x_0 = x_0^2 - 15$

$x_0^2 - 8x_0 + 15 = 0$

$(x_0 - 3)(x_0 - 5) = 0$

At $x_0 = 3: y_0 = (3)^2 = 9$

She should shut off the engines at (3, 9). (At $x_0 = 5$ she would not go to (4, 15) since she is moving left to right.)

59. $D_x(7 - x^2) = -2x$

The line through (4, 0) and (x_0, y_0) has slope $\frac{y_0 - 0}{x_0 - 4}$. If the fly is at (x_0, y_0) when the spider sees it, then $m_{\tan} = -2x_0 = \frac{7 - x_0^2 - 0}{x_0 - 4}$.

$-2x_0^2 + 8x_0 = 7 - x_0^2$

$x_0^2 - 8x_0 + 7 = 0$

$(x_0 - 7)(x_0 - 1) = 0$

At $x_0 = 1: y_0 = 6$

$d = \sqrt{(4-1)^2 + (0-6)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$

≈ 6.7

They are 6.7 units apart when they see each other.

60. $P(a, b)$ is $\left(a, \frac{1}{a}\right)$. $D_x y = -\frac{1}{x^2}$ so the slope of the tangent line at P is $-\frac{1}{a^2}$. The tangent line is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$ or $y = -\frac{1}{a^2}(x - 2a)$ which has x-intercept $(2a, 0)$.

$d(O,P) = \sqrt{a^2 + \frac{1}{a^2}}, d(P,A) = \sqrt{(a-2a)^2 + \frac{1}{a^2}}$

$= \sqrt{a^2 + \frac{1}{a^2}} = d(O,P)$ so AOP is an isosceles triangle. The height of AOP is a while the base, $\overline{OA}$ has length $2a$, so the area is $\frac{1}{2}(2a)(a) = a^2$.

61. The watermelon has volume $\frac{4}{3}\pi r^3$; the volume of the rind is

$V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi\left(r - \frac{r}{10}\right)^3 = \frac{271}{750}\pi r^3$.

At the end of the fifth week $r = 10$, so

$D_r V = \frac{271}{250}\pi r^2 = \frac{271}{250}\pi(10)^2 = \frac{542\pi}{5} \approx 340 \text{ cm}^3$

per cm of radius growth. Since the radius is growing 2 cm per week, the volume of the rind is growing at the rate of $\frac{542\pi}{5}(2) \approx 681 \text{ cm}^3$ per week.

2.4 Concepts Review

1. $\frac{\sin(x+h) - \sin(x)}{h}$

2. 0; 1

3. $\cos x$; $-\sin x$

4. $\cos\frac{\pi}{3} = \frac{1}{2}; y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$

Problem Set 2.4

1. $D_x(2 \sin x + 3 \cos x) = 2\, D_x(\sin x) + 3\, D_x(\cos x)$

$= 2 \cos x - 3 \sin x$

2. $D_x(\sin^2 x) = \sin x\, D_x(\sin x) + \sin x\, D_x(\sin x)$

$= \sin x \cos x + \sin x \cos x = 2 \sin x \cos x = \sin 2x$

3. $D_x(\sin^2 x + \cos^2 x) = D_x(1) = 0$

4. $D_x(1 - \cos^2 x) = D_x(\sin^2 x)$

$= \sin x\, D_x(\sin x) + \sin x\, D_x(\sin x)$

$= \sin x \cos x + \sin x \cos x$

$= 2 \sin x \cos x = \sin 2x$

5. $D_x(\sec x) = D_x\left(\frac{1}{\cos x}\right)$

$= \frac{\cos x\, D_x(1) - (1) D_x(\cos x)}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$

6. $D_x(\csc x) = D_x\left(\frac{1}{\sin x}\right)$

$$= \frac{\sin x\, D_x(1) - (1)D_x(\sin x)}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

7. $D_x(\tan x) = D_x\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{\cos x\, D_x(\sin x) - \sin x\, D_x(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

8. $D_x(\cot x) = D_x\left(\frac{\cos x}{\sin x}\right)$

$$= \frac{\sin x\, D_x(\cos x) - \cos x\, D_x(\sin x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = -\csc^2 x$$

9. $D_x\left(\frac{\sin x + \cos x}{\cos x}\right)$

$$= \frac{\cos x\, D_x(\sin x + \cos x) - (\sin x + \cos x)D_x(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x(\cos x - \sin x) - (-\sin^2 x - \sin x \cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

10. $D_x\left(\frac{\sin x + \cos x}{\tan x}\right)$

$$= \frac{\tan x\, D_x(\sin x + \cos x) - (\sin x + \cos x)D_x(\tan x)}{\tan^2 x}$$

$$= \frac{\tan x(\cos x - \sin x) - \sec^2 x(\sin x + \cos x)}{\tan^2 x}$$

$$= \left(\sin x - \frac{\sin^2 x}{\cos x} - \frac{\sin x}{\cos^2 x} - \frac{1}{\cos x}\right) \div \left(\frac{\sin^2 x}{\cos^2 x}\right)$$

$$= \left(\sin x - \frac{\sin^2 x}{\cos x} - \frac{\sin x}{\cos^2 x} - \frac{1}{\cos x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right)$$

$$= \frac{\cos^2 x}{\sin x} - \cos x - \frac{1}{\sin x} - \frac{\cos x}{\sin^2 x}$$

11. $D_x(\sin x \cos x) = \sin x D_x[\cos x] + \cos x D_x[\sin x]$

$$= \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x$$

12. $D_x(\sin x \tan x) = \sin x D_x[\tan x] + \tan x D_x[\sin x]$

$$= \sin x\left(\sec^2 x\right) + \tan x(\cos x)$$

$$= \sin x\left(\frac{1}{\cos^2 x}\right) + \frac{\sin x}{\cos x}(\cos x)$$

$$= \tan x \sec x + \sin x$$

13. $D_x\left(\frac{\sin x}{x}\right) = \frac{x D_x(\sin x) - \sin x D_x(x)}{x^2}$

$$= \frac{x \cos x - \sin x}{x^2}$$

14. $D_x\left(\frac{1 - \cos x}{x}\right) = \frac{x D_x(1 - \cos x) - (1 - \cos x) D_x(x)}{x^2}$

$$= \frac{x \sin x + \cos x - 1}{x^2}$$

15. $D_x(x^2 \cos x) = x^2 D_x(\cos x) + \cos x\, D_x(x^2)$

$$= -x^2 \sin x + 2x \cos x$$

16. $D_x\left(\frac{x \cos x + \sin x}{x^2 + 1}\right)$

$$= \frac{(x^2+1)D_x(x \cos x + \sin x) - (x \cos x + \sin x)D_x(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(-x \sin x + \cos x + \cos x) - 2x(x \cos x + \sin x)}{(x^2+1)^2}$$

$$= \frac{-x^3 \sin x - 3x \sin x + 2\cos x}{(x^2+1)^2}$$

17. $y = \tan^2 x = (\tan x)(\tan x)$

$D_x\, y = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$

$= 2\tan x \sec^2 x$

18. $y = \sec^3 x = (\sec^2 x)(\sec x)$

$D_x\, y = (\sec^2 x)\sec x \tan x + (\sec x) D_x (\sec^2 x)$

$= \sec^3 x \tan x + \sec x(\sec x \cdot \sec x \tan x$

$+ \sec x \cdot \sec x \tan x)$

$= \sec^3 x \tan x + 2\sec^3 x \tan x$

$= 3\sec^2 x \tan x$

19. $D_x(\cos x) = -\sin x$

At $x = 1$: $m_{\tan} = -\sin 1 \approx -0.8415$

$y = \cos 1 \approx 0.5403$

Tangent line: $y - 0.5403 = -0.8415(x - 1)$

20. $D_x(\cot x) = -\csc^2 x$

At $x = \frac{\pi}{4}: m_{\tan} = -2;$

$y = 1$

Tangent line: $y - 1 = -2\left(x - \frac{\pi}{4}\right)$

21. $D_x \sin 2x = D_x(2\sin x \cos x)$

$= 2\left[\sin x\, D_x \cos x + \cos x\, D_x \sin x\right]$

$= -2\sin^2 x + 2\cos^2 x$

22. $D_x \cos 2x = D_x(2\cos^2 x - 1) = 2D_x \cos^2 x - D_x 1$

$= -2\sin x \cos x$

23. $D_t(30\sin 2t) = 30D_t(2\sin t \cos t)$

$= 30\left(-2\sin^2 t + 2\cos^2 t\right)$

$= 60\cos 2t$

$30\sin 2t = 15$

$\sin 2t = \frac{1}{2}$

$2t = \frac{\pi}{6} \quad \to \quad t = \frac{\pi}{12}$

At $t = \frac{\pi}{12}; 60\cos\left(2 \cdot \frac{\pi}{12}\right) = 30\sqrt{3}$ ft/sec

The seat is moving to the left at the rate of $30\sqrt{3}$ ft/s.

24. The coordinates of the seat at time t are $(20 \cos t, 20 \sin t)$.

a. $\left(20\cos\frac{\pi}{6}, 20\sin\frac{\pi}{6}\right) = (10\sqrt{3}, 10)$

$\approx (17.32,\ 10)$

b. $D_t(20 \sin t) = 20 \cos t$

At $t = \frac{\pi}{6}$: rate $= 20\cos\frac{\pi}{6} = 10\sqrt{3} \approx 17.32$ ft/s

c. The fastest rate $20 \cos t$ can obtain is 20 ft/s.

25. $y = \tan x$

$y' = \sec^2 x$

When $y = 0$, $y = \tan 0 = 0$ and $y' = \sec^2 0 = 1$.

The tangent line at $x = 0$ is $y = x$.

26. $y = \tan^2 x = (\tan x)(\tan x)$

$y' = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$

$= 2\tan x \sec^2 x$

Now, $\sec^2 x$ is never 0, but $\tan x = 0$ at $x = k\pi$ where k is an integer.

27. $y = 9\sin x \cos x$

$y' = 9\left[\sin x(-\sin x) + \cos x(\cos x)\right]$

$= 9\left[\sin^2 x - \cos^2 x\right]$

$= 9\left[-\cos 2x\right]$

The tangent line is horizontal when $y' = 0$ or, in this case, where $\cos 2x = 0$. This occurs when $x = \frac{\pi}{4} + k\frac{\pi}{2}$ where k is an integer.

28. $f(x) = x - \sin x$

$f'(x) = 1 - \cos x$

$f'(x) = 0$ when $\cos x = 1$; i.e. when $x = 2k\pi$ where k is an integer.

$f'(x) = 2$ when $x = (2k+1)\pi$ where k is an integer.

29. The curves intersect when $\sqrt{2}\sin x = \sqrt{2}\cos x$, $\sin x = \cos x$ at $x = \frac{\pi}{4}$ for $0 < x < \frac{\pi}{2}$.

$D_x(\sqrt{2}\sin x) = \sqrt{2}\cos x$; $\sqrt{2}\cos\frac{\pi}{4} = 1$

$D_x(\sqrt{2}\cos x) = -\sqrt{2}\sin x$; $-\sqrt{2}\sin\frac{\pi}{4} = -1$

$1(-1) = -1$ so the curves intersect at right angles.

30. $v = D_t(3\sin 2t) = 6\cos 2t$

At $t = 0$: $v = 6$ cm/s

$t = \frac{\pi}{2}$: $v = -6$ cm/s

$t = \pi$: $v = 6$ cm/s

31. $D_x(\sin x^2) = \lim_{h\to 0}\frac{\sin(x+h)^2 - \sin x^2}{h}$

$= \lim_{h\to 0}\frac{\sin(x^2+2xh+h^2)-\sin x^2}{h}$

$= \lim_{h\to 0}\frac{\sin x^2\cos(2xh+h^2)+\cos x^2\sin(2xh+h^2)-\sin x^2}{h} = \lim_{h\to 0}\frac{\sin x^2[\cos(2xh+h^2)-1]+\cos x^2\sin(2xh+h^2)}{h}$

$= \lim_{h\to 0}(2x+h)\left[\sin x^2\frac{\cos(2xh+h^2)-1}{2xh+h^2}+\cos x^2\frac{\sin(2xh+h^2)}{2xh+h^2}\right] = 2x(\sin x^2\cdot 0+\cos x^2\cdot 1) = 2x\cos x^2$

32. $D_x(\sin 5x) = \lim_{h\to 0}\frac{\sin(5(x+h))-\sin 5x}{h}$

$= \lim_{h\to 0}\frac{\sin(5x+5h)-\sin 5x}{h}$

$= \lim_{h\to 0}\frac{\sin 5x\cos 5h+\cos 5x\sin 5h-\sin 5x}{h}$

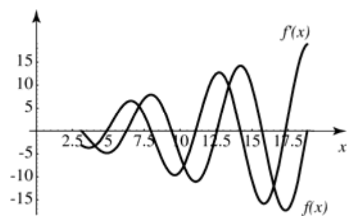
$= \lim_{h\to 0}\left[\sin 5x\frac{\cos 5h-1}{h}+\cos 5x\frac{\sin 5h}{h}\right]$

$= \lim_{h\to 0}\left[5\sin 5x\frac{\cos 5h-1}{5h}+5\cos 5x\frac{\sin 5h}{5h}\right]$

$= 0+5\cos 5x\cdot 1 = 5\cos 5x$

33. $f(x) = x\sin x$

a.



b. $f(x) = 0$ has 6 solutions on $[\pi, 6\pi]$

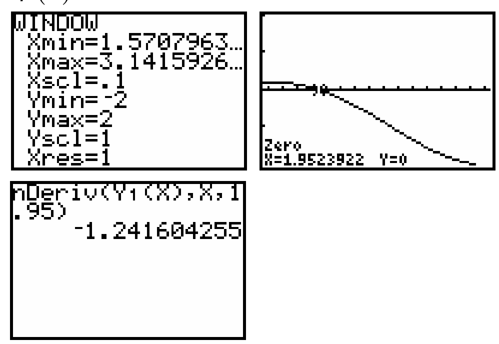
$f'(x) = 0$ has 5 solutions on $[\pi, 6\pi]$

c. $f(x) = x\sin x$ is a counterexample. Consider the interval $[0,\pi]$.

$f(-\pi) = f(\pi) = 0$ and $f(x) = 0$ has exactly two solutions in the interval (at 0 and π). However, $f'(x) = 0$ has two solutions in the interval, not 1 as the conjecture indicates it should have.

d. The maximum value of $|f(x) - f'(x)|$ on $[\pi, 6\pi]$ is about 24.93.

34. $f(x) = \cos^3 x - 1.25\cos^2 x + 0.225$



$x_0 \approx 1.95$

$f'(x_0) \approx -1.24$

2.5 Concepts Review

1. $D_t u$; $f'(g(t))g'(t)$

2. $D_v w$; $G'(H(s))H'(s)$

3. $(f(x))^2$; $(f(x))^2$

4. $2x\cos(x^2)$; $6(2x+1)^2$

Problem Set 2.5

1. $y = u^{15}$ and $u = 1 + x$

$D_x y = D_u y\cdot D_x u$

$= (15u^{14})(1)$

$= 15(1+x)^{14}$

2. $y = u^5$ and $u = 7 + x$

$D_x y = D_u y\cdot D_x u$

$= (5u^4)(1)$

$= 5(7+x)^4$

3. $y = u^5$ and $u = 3 - 2x$

$D_x y = D_u y\cdot D_x u$

$= (5u^4)(-2) = -10(3-2x)^4$

4. $y = u^7$ and $u = 4 + 2x^2$

$D_x y = D_u y \cdot D_x u$

$= (7u^6)(4x) = 28x(4 + 2x^2)^6$

5. $y = u^{11}$ and $u = x^3 - 2x^2 + 3x + 1$

$D_x y = D_u y \cdot D_x u$

$= (11u^{10})(3x^2 - 4x + 3)$

$= 11(3x^2 - 4x + 3)(x^3 - 2x^2 + 3x + 1)^{10}$

6. $y = u^{-7}$ and $u = x^2 - x + 1$

$D_x y = D_u y \cdot D_x u$

$= (-7u^{-8})(2x - 1)$

$= -7(2x - 1)(x^2 - x + 1)^{-8}$

7. $y = u^{-5}$ and $u = x + 3$

$D_x y = D_u y \cdot D_x u$

$= (-5u^{-6})(1) = -5(x + 3)^{-6} = -\dfrac{5}{(x + 3)^6}$

8. $y = u^{-9}$ and $u = 3x^2 + x - 3$

$D_x y = D_u y \cdot D_x u$

$= (-9u^{-10})(6x + 1)$

$= -9(6x + 1)(3x^2 + x - 3)^{-10}$

$= -\dfrac{9(6x + 1)}{(3x^2 + x - 3)^{10}}$

9. $y = \sin u$ and $u = x^2 + x$

$D_x y = D_u y \cdot D_x u$

$= (\cos u)(2x + 1)$

$= (2x + 1)\cos(x^2 + x)$

10. $y = \cos u$ and $u = 3x^2 - 2x$

$D_x y = D_u y \cdot D_x u$

$= (-\sin u)(6x - 2)$

$= -(6x - 2)\sin(3x^2 - 2x)$

11. $y = u^3$ and $u = \cos x$

$D_x y = D_u y \cdot D_x u$

$= (3u^2)(-\sin x)$

$= -3\sin x\cos^2 x$

12. $y = u^4$, $u = \sin v$, and $v = 3x^2$

$D_x y = D_u y \cdot D_v u \cdot D_x v$

$= (4u^3)(\cos v)(6x)$

$= 24x\sin^3(3x^2)\cos(3x^2)$

13. $y = u^3$ and $u = \dfrac{x + 1}{x - 1}$

$D_x y = D_u y \cdot D_x u$

$= (3u^2)\dfrac{(x - 1)D_x(x + 1) - (x + 1)D_x(x - 1)}{(x - 1)^2}$

$= 3\left(\dfrac{x + 1}{x - 1}\right)^2\left(\dfrac{-2}{(x - 1)^2}\right) = -\dfrac{6(x + 1)^2}{(x - 1)^4}$

14. $y = u^{-3}$ and $u = \dfrac{x - 2}{x - \pi}$

$D_x y = D_u y \cdot D_x u$

$= (-3u^{-4}) \cdot \dfrac{(x - \pi)D_x(x - 2) - (x - 2)D_x(x - \pi)}{(x - \pi)^2}$

$= -3\left(\dfrac{x - 2}{x - \pi}\right)^{-4}\dfrac{(2 - \pi)}{(x - \pi)^2} = -3\dfrac{(x - \pi)^2}{(x - 2)^4}(2 - \pi)$

15. $y = \cos u$ and $u = \dfrac{3x^2}{x + 2}$

$D_x y = D_u y \cdot D_x u \quad = (-\sin u)\dfrac{(x + 2)D_x(3x^2) - (3x^2)D_x(x + 2)}{(x + 2)^2}$

$= -\sin\left(\dfrac{3x^2}{x + 2}\right)\dfrac{(x + 2)(6x) - (3x^2)(1)}{(x + 2)^2} \quad = -\dfrac{3x^2 + 12x}{(x + 2)^2}\sin\left(\dfrac{3x^2}{x + 2}\right)$

16. $y = u^3$, $u = \cos v$, and $v = \dfrac{x^2}{1 - x}$

$D_x y = D_u y \cdot D_v u \cdot D_x v \quad = (3u^2)(-\sin v)\dfrac{(1 - x)D_x(x^2) - (x^2)D_x(1 - x)}{(1 - x)^2}$

$= -3\cos^2\left(\dfrac{x^2}{1 - x}\right)\sin\left(\dfrac{x^2}{1 - x}\right)\dfrac{(1 - x)(2x) - (x^2)(-1)}{(1 - x)^2} \quad = \dfrac{-3(2x - x^2)}{(1 - x)^2}\cos^2\left(\dfrac{x^2}{1 - x}\right)\sin\left(\dfrac{x^2}{1 - x}\right)$

17. $D_x[(3x-2)^2(3-x^2)^2] = (3x-2)^2 D_x(3-x^2)^2 + (3-x^2)^2 D_x(3x-2)^2$

$= (3x-2)^2(2)(3-x^2)(-2x) + (3-x^2)^2(2)(3x-2)(3)$

$= 2(3x-2)(3-x^2)[(3x-2)(-2x)+(3-x^2)(3)] = 2(3x-2)(3-x^2)(9+4x-9x^2)$

18. $D_x[(2-3x^2)^4(x^7+3)^3] = (2-3x^2)^4 D_x(x^7+3)^3 + (x^7+3)^3 D_x(2-3x^2)^4$

$= (2-3x^2)^4(3)(x^7+3)^2(7x^6) + (x^7+3)^3(4)(2-3x^2)^3(-6x) = 3x(3x^2-2)^3(x^7+3)^2(29x^7-14x^5+24)$

19. $D_x\left[\dfrac{(x+1)^2}{3x-4}\right] = \dfrac{(3x-4)D_x(x+1)^2 - (x+1)^2 D_x(3x-4)}{(3x-4)^2} = \dfrac{(3x-4)(2)(x+1)(1)-(x+1)^2(3)}{(3x-4)^2} = \dfrac{3x^2-8x-11}{(3x-4)^2}$

$= \dfrac{(x+1)(3x-11)}{(3x-4)^2}$

20. $D_x\left[\dfrac{2x-3}{(x^2+4)^2}\right] = \dfrac{(x^2+4)^2 D_x(2x-3)-(2x-3)D_x(x^2+4)^2}{(x^2+4)^4}$

$= \dfrac{(x^2+4)^2(2)-(2x-3)(2)(x^2+4)(2x)}{(x^2+4)^4} = \dfrac{-6x^2+12x+8}{(x^2+4)^3}$

21. $y' = 2(x^2+4)(x^2+4)' = 2(x^2+4)(2x) = 4x(x^2+4)$

22. $y' = 2(x+\sin x)(x+\sin x)' = 2(x+\sin x)(1+\cos x)$

23. $D_t\left(\dfrac{3t-2}{t+5}\right)^3 = 3\left(\dfrac{3t-2}{t+5}\right)^2 \dfrac{(t+5)D_t(3t-2)-(3t-2)D_t(t+5)}{(t+5)^2}$

$= 3\left(\dfrac{3t-2}{t+5}\right)^2 \dfrac{(t+5)(3)-(3t-2)(1)}{(t+5)^2} = \dfrac{51(3t-2)^2}{(t+5)^4}$

24. $D_s\left(\dfrac{s^2-9}{s+4}\right) = \dfrac{(s+4)D_s(s^2-9)-(s^2-9)D_s(s+4)}{(s+4)^2} = \dfrac{(s+4)(2s)-(s^2-9)(1)}{(s+4)^2} = \dfrac{s^2+8s+9}{(s+4)^2}$

25. $\dfrac{d}{dt}\left(\dfrac{(3t-2)^3}{t+5}\right) = \dfrac{(t+5)\dfrac{d}{dt}(3t-2)^3-(3t-2)^3\dfrac{d}{dt}(t+5)}{(t+5)^2} = \dfrac{(t+5)(3)(3t-2)^2(3)-(3t-2)^3(1)}{(t+5)^2}$

$= \dfrac{(6t+47)(3t-2)^2}{(t+5)^2}$

26. $\dfrac{d}{d\theta}(\sin^3\theta) = 3\sin^2\theta\cos\theta$

27. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos 2x}\right)^3 = 3\left(\dfrac{\sin x}{\cos 2x}\right)^2 \cdot \dfrac{d}{dx}\dfrac{\sin x}{\cos 2x} = 3\left(\dfrac{\sin x}{\cos 2x}\right)^2 \cdot \dfrac{(\cos 2x)\dfrac{d}{dx}(\sin x)-(\sin x)\dfrac{d}{dx}(\cos 2x)}{\cos^2 2x}$

$= 3\left(\dfrac{\sin x}{\cos 2x}\right)^2 \dfrac{\cos x\cos 2x + 2\sin x\sin 2x}{\cos^2 2x} = \dfrac{3\sin^2 x\cos x\cos 2x + 6\sin^3 x\sin 2x}{\cos^4 2x}$

$= \dfrac{3(\sin^2 x)(\cos x\cos 2x + 2\sin x\sin 2x)}{\cos^4 2x}$

28. $\frac{dy}{dt} = \frac{d}{dt}[\sin t \tan(t^2+1)] = \sin t \cdot \frac{d}{dt}[\tan(t^2+1)] + \tan(t^2+1) \cdot \frac{d}{dt}(\sin t)$

$= (\sin t)[\sec^2(t^2+1)](2t) + \tan(t^2+1)\cos t = 2t \sin t \sec^2(t^2+1) + \cos t \tan(t^2+1)$

29. $f'(x) = 3\left(\frac{x^2+1}{x+2}\right)^2 \frac{(x+2)D_x(x^2+1) - (x^2+1)D_x(x+2)}{(x+2)^2} = 3\left(\frac{x^2+1}{x+2}\right)^2 \frac{2x^2+4x-x^2-1}{(x+2)^2} = \frac{3(x^2+1)^2(x^2+4x-1)}{(x+2)^4}$

$f'(3) = 9.6$

30. $G'(t) = (t^2+9)^3 D_t(t^2-2)^4 + (t^2-2)^4 D_t(t^2+9)^3 = (t^2+9)^3(4)(t^2-2)^3(2t) + (t^2-2)^4(3)(t^2+9)^2(2t)$

$= 2t(7t^2+30)(t^2+9)^2(t^2-2)^3$

$G'(1) = -7400$

31. $F'(t) = [\cos(t^2+3t+1)](2t+3) = (2t+3)\cos(t^2+3t+1)$; $\quad F'(1) = 5\cos 5 \approx 1.4183$

32. $g'(s) = (\cos \pi s)D_s(\sin^2 \pi s) + (\sin^2 \pi s)D_s(\cos \pi s) = (\cos \pi s)(2\sin \pi s)(\cos \pi s)(\pi) + (\sin^2 \pi s)(-\sin \pi s)(\pi)$

$= \pi \sin \pi s[2\cos^2 \pi s - \sin^2 \pi s]$

$g'\left(\frac{1}{2}\right) = -\pi$

33. $D_x[\sin^4(x^2+3x)] = 4\sin^3(x^2+3x)D_x \sin(x^2+3x) = 4\sin^3(x^2+3x)\cos(x^2+3x)D_x(x^2+3x)$

$= 4\sin^3(x^2+3x)\cos(x^2+3x)(2x+3) = 4(2x+3)\sin^3(x^2+3x)\cos(x^2+3x)$

34. $D_t[\cos^5(4t-19)] = 5\cos^4(4t-19)D_t \cos(4t-19) = 5\cos^4(4t-19)[-\sin(4t-19)]D_t(4t-19)$

$= -5\cos^4(4t-19)\sin(4t-19)(4) = -20\cos^4(4t-19)\sin(4t-19)$

35. $D_t[\sin^3(\cos t)] = 3\sin^2(\cos t)D_t \sin(\cos t) = 3\sin^2(\cos t)\cos(\cos t)D_t(\cos t)$

$= 3\sin^2(\cos t)\cos(\cos t)(-\sin t) = -3\sin t \sin^2(\cos t)\cos(\cos t)$

36 . $D_u\left[\cos^4\left(\frac{u+1}{u-1}\right)\right] = 4\cos^3\left(\frac{u+1}{u-1}\right)D_u \cos\left(\frac{u+1}{u-1}\right) = 4\cos^3\left(\frac{u+1}{u-1}\right)\left[-\sin\left(\frac{u+1}{u-1}\right)\right]D_u\left(\frac{u+1}{u-1}\right)$

$= -4\cos^3\left(\frac{u+1}{u-1}\right)\sin\left(\frac{u+1}{u-1}\right)\frac{(u-1)D_u(u+1) - (u+1)D_u(u-1)}{(u-1)^2} = \frac{8}{(u-1)^2}\cos^3\left(\frac{u+1}{u-1}\right)\sin\left(\frac{u+1}{u-1}\right)$

37. $D_\theta[\cos^4(\sin \theta^2)] = 4\cos^3(\sin \theta^2)D_\theta \cos(\sin \theta^2) = 4\cos^3(\sin \theta^2)[-\sin(\sin \theta^2)]D_\theta(\sin \theta^2)$

$= -4\cos^3(\sin \theta^2)\sin(\sin \theta^2)(\cos \theta^2)D_\theta(\theta^2) = -8\theta \cos^3(\sin \theta^2)\sin(\sin \theta^2)(\cos \theta^2)$

38. $D_x[x\sin^2(2x)] = x D_x \sin^2(2x) + \sin^2(2x)D_x x = x[2\sin(2x)D_x \sin(2x)] + \sin^2(2x)(1)$

$= x[2\sin(2x)\cos(2x)D_x(2x)] + \sin^2(2x) = x[4\sin(2x)\cos(2x)] + \sin^2(2x) = 2x\sin(4x) + \sin^2(2x)$

39. $D_x\{\sin[\cos(\sin 2x)]\} = \cos[\cos(\sin 2x)]D_x \cos(\sin 2x) = \cos[\cos(\sin 2x)][-\sin(\sin 2x)]D_x(\sin 2x)$

$= -\cos[\cos(\sin 2x)]\sin(\sin 2x)(\cos 2x)D_x(2x) = -2\cos[\cos(\sin 2x)]\sin(\sin 2x)(\cos 2x)$

40. $D_t\{\cos^2[\cos(\cos t)]\} = 2\cos[\cos(\cos t)]D_t \cos[\cos(\cos t)] = 2\cos[\cos(\cos t)]\{-\sin[\cos(\cos t)]\}D_t \cos(\cos t)$

$= -2\cos[\cos(\cos t)]\sin[\cos(\cos t)][-\sin(\cos t)]D_t(\cos t) = 2\cos[\cos(\cos t)]\sin[\cos(\cos t)]\sin(\cos t)(-\sin t)$

$= -2\sin t \cos[\cos(\cos t)]\sin[\cos(\cos t)]\sin(\cos t)$

41. $(f+g)'(4) = f'(4) + g'(4)$
$\approx \frac{1}{2} + \frac{3}{2} \approx 2$

42. $(f-2g)'(2) = f'(2) - (2g)'(2)$
$= f'(2) - 2g'(2)$
$= 1 - 2(0) = 1$

43. $(fg)'(2) = (fg' + gf')(2) = 2(0) + 1(1) = 1$

44. $(f/g)'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{g^2(2)}$
$\approx \dfrac{(1)(1) - (3)(0)}{(1)^2} = 1$

45. $(f \circ g)'(6) = f'(g(6))g'(6)$
$= f'(2)g'(6) \approx (1)(-1) = -1$

46. $(g \circ f)'(3) = g'(f(3))f'(3)$
$= g'(4)f'(3) \approx \left(\frac{3}{2}\right)(1) = \frac{3}{2}$

47. $D_x F(2x) = F'(2x)D_x(2x) = 2F'(2x)$

48. $D_x F(x^2+1) = F'(x^2+1)D_x(x^2+1)$
$= 2xF'(x^2+1)$

49. $D_t\left[(F(t))^{-2}\right] = -2(F(t))^{-3}F'(t)$

50. $\dfrac{d}{dz}\left[\dfrac{1}{(F(z))^2}\right] = -2(F(z))^{-3}F'(z)$

51. $\dfrac{d}{dz}\left[(1+F(2z))^2\right] = 2(1+F(2z))\dfrac{d}{dz}(1+F(2z))$
$= 2(1+F(2z))(2F'(2z)) = 4(1+F(2z))F'(2z)$

52. $\dfrac{d}{dy}\left[y^2 + \dfrac{1}{F(y^2)}\right] = 2y + \dfrac{d}{dy}\left[\left(F(y^2)\right)^{-1}\right]$

$= 2y - F'(y^2)\dfrac{d}{dy}y^2 = 2y - \dfrac{2yF'(y^2)}{\left(F(y^2)\right)^2}$

$= 2y\left(1 - \dfrac{F'(y^2)}{\left(F(y^2)\right)^2}\right)$

53. $\dfrac{d}{dx}F(\cos x) = F'(\cos x)\dfrac{d}{dx}(\cos x)$
$= -\sin x F'(\cos x)$

54. $\dfrac{d}{dx}\cos(F(x)) = -\sin(F(x))\dfrac{d}{dx}F(x)$
$= -F'(x)\sin(F(x))$

55. $D_x\left[\tan(F(2x))\right] = \sec^2(F(2x))D_x\left[F(2x)\right]$
$= \sec^2(F(2x)) \times F'(2x) \times D_x[2x]$
$= 2F'(2x)\sec^2(F(2x))$

56. $\dfrac{d}{dx}\left[g(\tan 2x)\right] = g'(\tan 2x)\cdot\dfrac{d}{dx}\tan 2x$
$= g'(\tan 2x)(\sec^2 2x)\cdot 2$
$= 2g'(\tan 2x)\sec^2 2x$

57. $D_x\left[F(x)\sin^2 F(x)\right]$
$= F(x) \times D_x\left[\sin^2 F(x)\right] + \sin^2 F(x) \times D_x F(x)$
$= F(x) \times 2\sin F(x) \times D_x\left[\sin F(x)\right]$
$\quad + F'(x)\sin^2 F(x)$
$= F(x) \times 2\sin F(x) \times \cos(F(x)) \times D_x\left[F(x)\right]$
$\quad + F'(x)\sin^2 F(x)$
$= 2F(x)F'(x)\sin F(x)\cos F(x)$
$\quad + F'(x)\sin^2 F(x)$

58. $D_x\left[\sec^3 F(x)\right] = 3\sec^2\left[F(x)\right]D_x\left[\sec F(x)\right]$
$= 3\sec^2\left[F(x)\right]\sec F(x)\tan F(x)D_x[x]$
$= 3F'(x)\sec^3 F(x)\tan F(x)$

59. $g'(x) = -\sin f(x)D_x f(x) = -f'(x)\sin f(x)$
$g'(0) = -f'(0)\sin f(0) = -2\sin 1 \approx -1.683$

60. $G'(x) = \dfrac{(1+\sec F(2x))\frac{d}{dx}x - x\frac{d}{dx}(1+\sec F(2x))}{(1+\sec F(2x))^2}$
$= \dfrac{(1+\sec F(2x)) - 2xF'(2x)\sec F(2x)\tan F(2x)}{(1+\sec F(2x))^2}$
$G'(0) = \dfrac{1+\sec F(0) - 0}{(1+\sec F(0))^2} = \dfrac{1+\sec F(0)}{(1+\sec F(0))^2}$
$= \dfrac{1}{1+\sec F(0)} = \dfrac{1}{1+\sec 2} \approx -0.713$

61. $F'(x) = -f(x)g'(x)\sin g(x) + f'(x)\cos g(x)$

$F'(1) = -f(1)g'(1)\sin g(1) + f'(1)\cos g(1)$

$= -2(1)\sin 0 + -1\cos 0 = -1$

62. $y = 1 + x\sin 3x;\quad y' = 3x\cos 3x + \sin 3x$

$y'(\pi/3) = 3\frac{\pi}{3}\cos 3\frac{\pi}{3} + \sin\frac{\pi}{3} = -\pi + 0 = -\pi$

$y - 1 = -\pi x - \pi/3$

$y = -\pi x - \pi/3 + 1$

The line crosses the x-axis at $x = \frac{3-\pi}{3}$.

63. $y = \sin^2 x;\quad y' = 2\sin x\cos x = \sin 2x = 1$

$x = \pi/4 + k\pi,\ k = 0, \pm 1, \pm 2, \ldots$

64. $y' = \left(x^2+1\right)^3 2\left(x^4+1\right)x^3 + 3\left(x^2+1\right)^2 x\left(x^4+1\right)^2$

$= 2x^3\left(x^4+1\right)\left(x^2+1\right)^3 + 3x\left(x^4+1\right)^2\left(x^2+1\right)^2$

$y'(1) = 2(2)(2)^3 + 3(1)(2)^2(2)^2 = 32 + 48 = 80$

$y - 32 = 80x - 1,\quad y = 80x + 31$

65. $y' = -2\left(x^2+1\right)^{-3}(2x) = -4x\left(x^2+1\right)^{-3}$

$y'(1) = -4(1)(1+1)^{-3} = -1/2$

$y - \frac{1}{4} = -\frac{1}{2}x + \frac{1}{2},\quad y = -\frac{1}{2}x + \frac{3}{4}$

66. $y' = 3(2x+1)^2(2) = 6(2x+1)^2$

$y'(0) = 6(1)^2 = 6$

$y - 1 = 6x - 0,\quad y = 6x + 1$

The line crosses the x-axis at $x = -1/6$.

67. $y' = -2\left(x^2+1\right)^{-3}(2x) = -4x\left(x^2+1\right)^{-3}$

$y'(1) = -4(2)^{-3} = -1/2$

$y - \frac{1}{4} = -\frac{1}{2}x + \frac{1}{2},\quad y = -\frac{1}{2}x + \frac{3}{4}$

Set $y = 0$ and solve for x. The line crosses the x-axis at $x = 3/2$.

68. a. $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{7}\right)^2 = \left(\frac{4\cos 2t}{4}\right)^2 + \left(\frac{7\sin 2t}{7}\right)^2$

$= \cos^2 2t + \sin^2 2t = 1$

b. $L = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2}$

$= \sqrt{(4\cos 2t)^2 + (7\sin 2t)^2}$

$= \sqrt{16\cos^2 2t + 49\sin^2 2t}$

c. $D_t L = \frac{1}{2\sqrt{16\cos^2 2t + 49\sin^2 2t}} D_t(16\cos^2 2t + 49\sin^2 2t)$

$= \frac{32\cos 2t\, D_t(\cos 2t) + 98\sin 2t\, D_t(\sin 2t)}{2\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

$= \frac{-64\cos 2t\sin 2t + 196\sin 2t\cos 2t}{2\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

$= \frac{-16\sin 4t + 49\sin 4t}{\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

$= \frac{33\sin 4t}{\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

At $t = \frac{\pi}{8}$: rate $= \frac{33}{\sqrt{16\cdot\frac{1}{2} + 49\cdot\frac{1}{2}}} \approx 5.8$ ft/sec.

69. a. $(10\cos 8\pi t, 10\sin 8\pi t)$

b. $D_t(10\sin 8\pi t) = 10\cos(8\pi t) D_t(8\pi t)$

$= 80\pi\cos(8\pi t)$

At $t = 1$: rate $= 80\pi \approx 251$ cm/s

P is rising at the rate of 251 cm/s.

70. a. $(\cos 2t, \sin 2t)$

b. $(0 - \cos 2t)^2 + (y - \sin 2t)^2 = 5^2$, so

$y = \sin 2t + \sqrt{25 - \cos^2 2t}$

c. $D_t\left(\sin 2t + \sqrt{25 - \cos^2 2t}\right)$

$= 2\cos 2t + \frac{1}{2\sqrt{25-\cos^2 2t}} \cdot 4\cos 2t\sin 2t$

$= 2\cos 2t\left(1 + \frac{\sin 2t}{\sqrt{25-\cos^2 2t}}\right)$

71. 60 revolutions per minute is 120π radians per minute or 2π radians per second.

a. $(\cos 2\pi t, \sin 2\pi t)$

b. $(0 - \cos 2\pi t)^2 + (y - \sin 2\pi t)^2 = 5^2$, so

$y = \sin 2\pi t + \sqrt{25 - \cos^2 2\pi t}$

c. $D_t\left(\sin 2\pi t + \sqrt{25 - \cos^2 2\pi t}\right)$

$= 2\pi\cos 2\pi t$

$+ \frac{1}{2\sqrt{25-\cos^2 2\pi t}} \cdot 4\pi\cos 2\pi t\sin 2\pi t$

$= 2\pi\cos 2\pi t\left(1 + \frac{\sin 2\pi t}{\sqrt{25-\cos^2 2\pi t}}\right)$

72. The minute hand makes 1 revolution every hour, so at t minutes after the hour, it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. By the Law of Cosines, the length of the elastic string is

$$s = \sqrt{10^2 + 10^2 - 2(10)(10)\cos\frac{\pi t}{30}}$$
$$= 10\sqrt{2 - 2\cos\frac{\pi t}{30}}$$

$$\frac{ds}{dt} = 10 \cdot \frac{1}{2\sqrt{2 - 2\cos\frac{\pi t}{30}}} \cdot \frac{\pi}{15}\sin\frac{\pi t}{30}$$
$$= \frac{\pi \sin\frac{\pi t}{30}}{3\sqrt{2 - 2\cos\frac{\pi t}{30}}}$$

At 12:15, the string is stretching at the rate of

$$\frac{\pi \sin\frac{\pi}{2}}{3\sqrt{2 - 2\cos\frac{\pi}{2}}} = \frac{\pi}{3\sqrt{2}} \approx 0.74 \text{ cm/min}$$

73. The minute hand makes 1 revolution every hour, so at t minutes after noon it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. Similarly, at t minutes after noon the hour hand makes an angle of $\frac{\pi t}{360}$ with the vertical. Thus, by the Law of Cosines, the distance between the tips of the hands is

$$s = \sqrt{6^2 + 8^2 - 2\cdot 6\cdot 8\cos\left(\frac{\pi t}{30} - \frac{\pi t}{360}\right)}$$
$$= \sqrt{100 - 96\cos\frac{11\pi t}{360}}$$

$$\frac{ds}{dt} = \frac{1}{2\sqrt{100 - 96\cos\frac{11\pi t}{360}}} \cdot \frac{44\pi}{15}\sin\frac{11\pi t}{360}$$
$$= \frac{22\pi \sin\frac{11\pi t}{360}}{15\sqrt{100 - 96\cos\frac{11\pi t}{360}}}$$

At 12:20,

$$\frac{ds}{dt} = \frac{22\pi \sin\frac{11\pi}{18}}{15\sqrt{100 - 96\cos\frac{11\pi}{18}}} \approx 0.38 \text{ in./min}$$

74. From Problem 73, $\frac{ds}{dt} = \frac{22\pi \sin\frac{11\pi t}{360}}{15\sqrt{100 - 96\cos\frac{11\pi t}{360}}}$.

Using a computer algebra system or graphing utility to view $\frac{ds}{dt}$ for $0 \le t \le 60$, $\frac{ds}{dt}$ is largest when $t \approx 7.5$. Thus, the distance between the tips of the hands is increasing most rapidly at about 12:08.

75.

$$\sin x_0 = \sin 2x_0$$
$$\sin x_0 = 2\sin x_0 \cos x_0$$
$$\cos x_0 = \frac{1}{2} \text{ [if } \sin x_0 \ne 0]$$
$$x_0 = \frac{\pi}{3}$$

$D_x(\sin x) = \cos x$, $D_x(\sin 2x) = 2\cos 2x$, so at x_0, the tangent lines to $y = \sin x$ and $y = \sin 2x$ have slopes of $m_1 = \frac{1}{2}$ and $m_2 = 2\left(-\frac{1}{2}\right) = -1$, respectively. From Problem 40 of Section 0.7, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$ where θ is the angle between the tangent lines. $\tan\theta = \frac{-1 - \frac{1}{2}}{1 + \left(\frac{1}{2}\right)(-1)} = \frac{-\frac{3}{2}}{\frac{1}{2}} = -3$, so $\theta \approx -1.25$. The curves intersect at an angle of 1.25 radians.

76.

$$\frac{1}{2}\overline{AB} = \overline{OA}\sin\frac{t}{2}$$
$$D = \frac{1}{2}\overline{OA}\cos\frac{t}{2} \cdot \overline{AB} = \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2}$$

$E = D +$ area (semi-circle)

$$= \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\overline{AB}\right)^2$$
$$= \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\pi\overline{OA}^2 \sin^2\frac{t}{2}$$
$$= \overline{OA}^2 \sin\frac{t}{2}\left(\cos\frac{t}{2} + \frac{1}{2}\pi\sin\frac{t}{2}\right)$$

$$\frac{D}{E} = \frac{\cos\frac{t}{2}}{\cos\frac{t}{2} + \frac{1}{2}\pi\sin\frac{t}{2}}$$

$$\lim_{t\to 0^+}\frac{D}{E} = \frac{1}{1+0} = 1$$

$$\lim_{t\to\pi^-}\frac{D}{E} = \lim_{t\to\pi^-}\frac{\cos(t/2)}{\cos(t/2) + \frac{\pi}{2}\sin(t/2)}$$
$$= \frac{0}{0 + \frac{\pi}{2}} = 0$$

77. $y = \sqrt{u}$ and $u = x^2$

$D_x y = D_u y \cdot D_x u$

$$= \frac{1}{2\sqrt{u}} \cdot 2x = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|} = \frac{|x|}{x}$$

78. $D_x \left|x^2 - 1\right| = \frac{\left|x^2 - 1\right|}{x^2 - 1} D_x(x^2 - 1)$

$$= \frac{\left|x^2 - 1\right|}{x^2 - 1}(2x) = \frac{2x\left|x^2 - 1\right|}{x^2 - 1}$$

79. $D_x |\sin x| = \frac{|\sin x|}{\sin x} D_x(\sin x)$

$$= \frac{|\sin x|}{\sin x} \cos x = \cot x |\sin x|$$

80. **a.** $D_x L\left(x^2\right) = L'\left(x^2\right) D_x\left(x^2\right) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$

b. $D_x L(\cos^4 x) = \sec^4 x\, D_x(\cos^4 x)$

$= \sec^4 x(4\cos^3 x) D_x(\cos x)$

$= 4\sec^4 x \cos^3 x(-\sin x)$

$= 4 \cdot \frac{1}{\cos^4 x} \cdot \cos^3 x \cdot (-\sin x)$

$= -4\sec x \sin x = -4\tan x$

81. $[f(f(f(f(0))))]'$

$= f'(f(f(f(0)))) \cdot f'(f(f(0))) \cdot f'(f(0)) \cdot f'(0)$

$= 2 \cdot 2 \cdot 2 \cdot 2 = 16$

82. **a.** $\frac{d}{dx} f^{[2]} = f'(f(x)) \cdot f'(x)$

$$= f'(f^{[1]}) \cdot \frac{d}{dx} f^{[1]}(x)$$

b. $\frac{d}{dx} f^{[3]} = f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$

$$= f'(f^{[2]}(x)) \cdot f'(f^{[1]}(x)) \cdot \frac{d}{dx} f^{[1]}(x)$$

$$= f'(f^{[2]}(x)) \cdot \frac{d}{dx} f^{[2]}(x)$$

c. Conjecture:

$$\frac{d}{dx} f^{[n]}(x) = f'(f^{[n-1]}(x)) \cdot \frac{d}{dx} f^{[n-1]}(x)$$

83. $D_x\left(\frac{f(x)}{g(x)}\right) = D_x\left(f(x) \cdot \frac{1}{g(x)}\right) = D_x\left(f(x) \cdot (g(x))^{-1}\right) = f(x) D_x\left((g(x))^{-1}\right) + (g(x))^{-1} D_x f(x)$

$$= f(x) \cdot (-1)(g(x))^{-2} D_x g(x) + (g(x))^{-1} D_x f(x) = -f(x)(g(x))^{-2} D_x g(x) + (g(x))^{-1} D_x f(x)$$

$$= \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{D_x f(x)}{g(x)} = \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{g(x)}{g(x)} \cdot \frac{D_x f(x)}{g(x)} = \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{g(x) D_x f(x)}{g^2(x)}$$

$$= \frac{g(x) D_x f(x) - f(x) D_x g(x)}{g^2(x)}$$

84. $g'(x) = f'\left(f\left(f\left(f(x)\right)\right)\right) f'\left(f\left(f(x)\right)\right) f'\left(f(x)\right) f'(x)$

$g'(x_1) = f'\left(f\left(f\left(f(x_1)\right)\right)\right) f'\left(f\left(f(x_1)\right)\right) f'\left(f(x_1)\right) f'(x_1)$

$= f'\left(f\left(f(x_2)\right)\right) f'\left(f(x_2)\right) f'(x_2) f'(x_1) = f'\left(f(x_1)\right) f'(x_1) f'(x_2) f'(x_1)$

$= \left[f'(x_1)\right]^2 \left[f'(x_2)\right]^2$

$g'(x_2) = f'\left(f\left(f\left(f(x_2)\right)\right)\right) f'\left(f\left(f(x_2)\right)\right) f'\left(f(x_2)\right) f'(x_2)$

$= f'\left(f\left(f(x_1)\right)\right) f'\left(f(x_1)\right) f'(x_1) f'(x_2) = f'\left(f(x_2)\right) f'(x_2) f'(x_1) f'(x_2)$

$= \left[f'(x_1)\right]^2 \left[f'(x_2)\right]^2 = g'(x_1)$

2.6 Concepts Review

1. $f'''(x), D_x^3 y, \dfrac{d^3 y}{dx^3}$, y'''

2. $\dfrac{ds}{dt}; \left|\dfrac{ds}{dt}\right|; \dfrac{d^2 s}{dt^2}$

3. $f'(t) > 0$

4. 0; < 0

Problem Set 2.6

1. $\dfrac{dy}{dx} = 3x^2 + 6x + 6$

$\dfrac{d^2 y}{dx^2} = 6x + 6$

$\dfrac{d^3 y}{dx^3} = 6$

2. $\dfrac{dy}{dx} = 5x^4 + 4x^3$

$\dfrac{d^2 y}{dx^2} = 20x^3 + 12x^2$

$\dfrac{d^3 y}{dx^3} = 60x^2 + 24x$

3. $\dfrac{dy}{dx} = 3(3x+5)^2(3) = 9(3x+5)^2$

$\dfrac{d^2 y}{dx^2} = 18(3x+5)(3) = 162x + 270$

$\dfrac{d^3 y}{dx^3} = 162$

4. $\dfrac{dy}{dx} = 5(3-5x)^4(-5) = -25(3-5x)^4$

$\dfrac{d^2 y}{dx^2} = -100(3-5x)^3(-5) = 500(3-5x)^3$

$\dfrac{d^3 y}{dx^3} = 1500(3-5x)^2(-5) = -7500(3-5x)^2$

5. $\dfrac{dy}{dx} = 7\cos(7x)$

$\dfrac{d^2 y}{dx^2} = -7^2 \sin(7x)$

$\dfrac{d^3 y}{dx^3} = -7^3 \cos(7x) = -343\cos(7x)$

6. $\dfrac{dy}{dx} = 3x^2 \cos(x^3)$

$\dfrac{d^2 y}{dx^2} = 3x^2[-3x^2 \sin(x^3)] + 6x\cos(x^3) \quad = -9x^4 \sin(x^3) + 6x\cos(x^3)$

$\dfrac{d^3 y}{dx^3} = -9x^4 \cos(x^3)(3x^2) + \sin(x^3)(-36x^3) + 6x[-\sin(x^3)(3x^2)] + 6\cos(x^3)$

$= -27x^6 \cos(x^3) - 36x^3 \sin(x^3) - 18x^3 \sin(x^3) + 6\cos(x^3) \quad = (6 - 27x^6)\cos(x^3) - 54x^3 \sin(x^3)$

7. $\dfrac{dy}{dx} = \dfrac{(x-1)(0) - (1)(1)}{(x-1)^2} = -\dfrac{1}{(x-1)^2}$

$\dfrac{d^2 y}{dx^2} = -\dfrac{(x-1)^2(0) - 2(x-1)}{(x-1)^4} = \dfrac{2}{(x-1)^3}$

$\dfrac{d^3 y}{dx^3} = \dfrac{(x-1)^3(0) - 2[3(x-1)^2]}{(x-1)^6}$

$= -\dfrac{6}{(x-1)^4}$

8. $\dfrac{dy}{dx} = \dfrac{(1-x)(3) - (3x)(-1)}{(1-x)^2} = \dfrac{3}{(x-1)^2}$

$\dfrac{d^2 y}{dx^2} = \dfrac{(x-1)^2(0) - 3[2(x-1)]}{(x-1)^4} = -\dfrac{6}{(x-1)^3}$

$\dfrac{d^3 y}{dx^3} = -\dfrac{(x-1)^3(0) - 6(3)(x-1)^2}{(x-1)^6}$

$= \dfrac{18}{(x-1)^4}$

9. $f'(x) = 2x;\ f''(x) = 2;\ f''(2) = 2$

10. $f'(x) = 15x^2 + 4x + 1$
$f''(x) = 30x + 4$
$f''(2) = 64$

11. $f'(t) = -\dfrac{2}{t^2}$
$f''(t) = \dfrac{4}{t^3}$
$f''(2) = \dfrac{4}{8} = \dfrac{1}{2}$

12. $f'(u) = \dfrac{(5-u)(4u) - (2u^2)(-1)}{(5-u)^2} = \dfrac{20u - 2u^2}{(5-u)^2}$
$f''(u) = \dfrac{(5-u)^2(20-4u) - (20u - 2u^2)2(5-u)(-1)}{(5-u)^4}$
$= \dfrac{100}{(5-u)^3}$
$f''(2) = \dfrac{100}{3^3} = \dfrac{100}{27}$

13. $f'(\theta) = -2(\cos\theta\pi)^{-3}(-\sin\theta\pi)\pi \ = 2\pi(\cos\theta\pi)^{-3}(\sin\theta\pi)$
$f''(\theta) = 2\pi[(\cos\theta\pi)^{-3}(\pi)(\cos\theta\pi) + (\sin\theta\pi)(-3)(\cos\theta\pi)^{-4}(-\sin\theta\pi)(\pi)] \ = 2\pi^2[(\cos\theta\pi)^{-2} + 3\sin^2\theta\pi(\cos\theta\pi)^{-4}]$
$f''(2) = 2\pi^2[1 + 3(0)(1)] = 2\pi^2$

14. $f'(t) = t\cos\left(\dfrac{\pi}{t}\right)\left(-\dfrac{\pi}{t^2}\right) + \sin\left(\dfrac{\pi}{t}\right) = \left(-\dfrac{\pi}{t}\right)\cos\left(\dfrac{\pi}{t}\right) + \sin\left(\dfrac{\pi}{t}\right)$
$f''(t) = \left(-\dfrac{\pi}{t}\right)\left[-\sin\left(\dfrac{\pi}{t}\right)\left(-\dfrac{\pi}{t^2}\right)\right] + \left(\dfrac{\pi}{t^2}\right)\cos\left(\dfrac{\pi}{t}\right) + \left(-\dfrac{\pi}{t^2}\right)\cos\left(\dfrac{\pi}{t}\right) = -\dfrac{\pi^2}{t^3}\sin\left(\dfrac{\pi}{t}\right)$
$f''(2) = -\dfrac{\pi^2}{8}\sin\left(\dfrac{\pi}{2}\right) = -\dfrac{\pi^2}{8} \approx -1.23$

15. $f'(s) = s(3)(1-s^2)^2(-2s) + (1-s^2)^3 \ = -6s^2(1-s^2)^2 + (1-s^2)^3 \ = -7s^6 + 15s^4 - 9s^2 + 1$
$f''(s) = -42s^5 + 60s^3 - 18s$
$f''(2) = -900$

16. $f'(x) = \dfrac{(x-1)2(x+1) - (x+1)^2}{(x-1)^2} = \dfrac{x^2 - 2x - 3}{(x-1)^2}$
$f''(x) = \dfrac{(x-1)^2(2x-2) - (x^2 - 2x - 3)2(x-1)}{(x-1)^4} = \dfrac{(x-1)(2x-2) - (x^2 - 2x - 3)(2)}{(x-1)^3} = \dfrac{8}{(x-1)^3}$
$f''(2) = \dfrac{8}{1^3} = 8$

17. $D_x(x^n) = nx^{n-1}$
$D_x^2(x^n) = n(n-1)x^{n-2}$
$D_x^3(x^n) = n(n-1)(n-2)x^{n-3}$
$D_x^4(x^n) = n(n-1)(n-2)(n-3)x^{n-4}$
$\vdots$
$D_x^{n-1}(x^n) = n(n-1)(n-2)(n-3)\ldots(2)x$
$D_x^n(x^n) = n(n-1)(n-2)(n-3)\ldots2(1)x^0 = n!$

18. Let $k < n$.
$D_x^n(x^k) = D_x^{n-k}[D_x^k(x^k)] = D_x(k!) = 0$
so $D_x^n[a_n x^{n-1} + \ldots + a_1x + a_0] = 0$

19. **a.** $D_x^4(3x^3 + 2x - 19) = 0$

b. $D_x^{12}(100x^{11} - 79x^{10}) = 0$

c. $D_x^{11}(x^2 - 3)^5 = 0$

20. $D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$D_x^2\left(\frac{1}{x}\right) = D_x(-x^{-2}) = 2x^{-3} = \frac{2}{x^3}$

$D_x^3\left(\frac{1}{x}\right) = D_x(2x^{-3}) = -\frac{3(2)}{x^4}$

$D_x^4\left(\frac{1}{x}\right) = \frac{4(3)(2)}{x^5}$

$D_x^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$

21. $f'(x) = 3x^2 + 6x - 45 = 3(x+5)(x-3)$

$3(x + 5)(x - 3) = 0$

$x = -5, x = 3$

$f''(x) = 6x + 6$

$f''(-5) = -24$

$f''(3) = 24$

22. $g'(t) = 2at + b$

$g''(t) = 2a$

$g''(1) = 2a = -4$

$a = -2$

$g'(1) = 2a + b = 3$

$2(-2) + b = 3$

$b = 7$

$g(1) = a + b + c = 5$

$(-2) + (7) + c = 5$

$c = 0$

23. **a.** $v(t) = \frac{ds}{dt} = 12 - 4t$

$a(t) = \frac{d^2s}{dt^2} = -4$

b. $12 - 4t > 0$

$4t < 12$

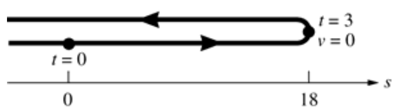
$t < 3;\ (-\infty, 3)$

c. $12 - 4t < 0$

$t > 3;\ (3, \infty)$

d. $a(t) = -4 < 0$ for all t

e.



24. **a.** $v(t) = \frac{ds}{dt} = 3t^2 - 12t$

$a(t) = \frac{d^2s}{dt^2} = 6t - 12$

b. $3t^2 - 12t > 0$

$3t(t - 4) > 0;\ (-\infty, 0) \cup (4, \infty)$

c. $3t^2 - 12t < 0$

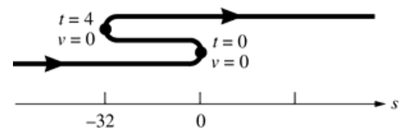
$(0, 4)$

d. $6t - 12 < 0$

$6t < 12$

$t < 2;\ (-\infty, 2)$

e.



25. **a.** $v(t) = \frac{ds}{dt} = 3t^2 - 18t + 24$

$a(t) = \frac{d^2s}{dt^2} = 6t - 18$

b. $3t^2 - 18t + 24 > 0$

$3(t - 2)(t - 4) > 0$

$(-\infty, 2) \cup (4, \infty)$

c. $3t^2 - 18t + 24 < 0$

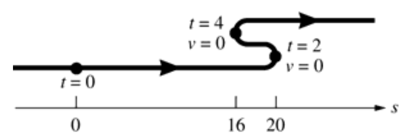
$(2, 4)$

d. $6t - 18 < 0$

$6t < 18$

$t < 3;\ (-\infty, 3)$

e.



26. **a.** $v(t) = \frac{ds}{dt} = 6t^2 - 6$

$a(t) = \frac{d^2s}{dt^2} = 12t$

b. $6t^2 - 6 > 0$

$6(t + 1)(t - 1) > 0$

$(-\infty, -1) \cup (1, \infty)$

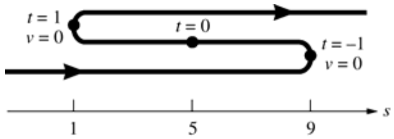
c. $6t^2 - 6 < 0$

$(-1, 1)$

d. $12t < 0$

$t < 0$

The acceleration is negative for negative t.

e. 

27. a. $v(t) = \frac{ds}{dt} = 2t - \frac{16}{t^2}$

$a(t) = \frac{d^2s}{dt^2} = 2 + \frac{32}{t^3}$

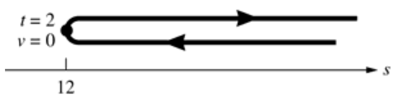
b. $2t - \frac{16}{t^2} > 0$

$\frac{2t^3 - 16}{t^2} > 0;\ (2, \infty)$

c. $2t - \frac{16}{t^2} < 0;\ (0, 2)$

d. $2 + \frac{32}{t^3} < 0$

$\frac{2t^3 + 32}{t^3} < 0$; The acceleration is not negative for any positive t.

e. 

28. a. $v(t) = \frac{ds}{dt} = 1 - \frac{4}{t^2}$

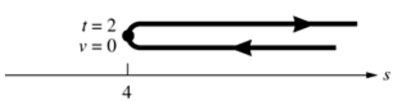
$a(t) = \frac{d^2s}{dt^2} = \frac{8}{t^3}$

b. $1 - \frac{4}{t^2} > 0$

$\frac{t^2 - 4}{t^2} > 0;\ (2, \infty)$

c. $1 - \frac{4}{t^2} < 0;\ (0, 2)$

d. $\frac{8}{t^3} < 0$; The acceleration is not negative for any positive t.

e. 

29. $v(t) = \frac{ds}{dt} = 2t^3 - 15t^2 + 24t$

$a(t) = \frac{d^2s}{dt^2} = 6t^2 - 30t + 24$

$6t^2 - 30t + 24 = 0$
$6(t - 4)(t - 1) = 0$
$t = 4, 1$
$v(4) = -16,\ v(1) = 11$

30. $v(t) = \frac{ds}{dt} = \frac{1}{10}(4t^3 - 42t^2 + 120t)$

$a(t) = \frac{d^2s}{dt^2} = \frac{1}{10}(12t^2 - 84t + 120)$

$\frac{1}{10}(12t^2 - 84t + 120) = 0$

$\frac{12}{10}(t - 2)(t - 5) = 0$

$t = 2,\ t = 5$
$v(2) = 10.4,\ v(5) = 5$

31. $v_1(t) = \frac{ds_1}{dt} = 4 - 6t$

$v_2(t) = \frac{ds_2}{dt} = 2t - 2$

a. $4 - 6t = 2t - 2$
$8t = 6$
$t = \frac{3}{4}$ sec

b. $|4 - 6t| = |2t - 2|;\ 4 - 6t = -2t + 2$

$t = \frac{1}{2}$ sec and $t = \frac{3}{4}$ sec

c. $4t - 3t^2 = t^2 - 2t$
$4t^2 - 6t = 0$
$2t(2t - 3) = 0$
$t = 0$ sec and $t = \frac{3}{2}$ sec

32. $v_1(t) = \frac{ds_1}{dt} = 9t^2 - 24t + 18$

$v_2(t) = \frac{ds_2}{dt} = -3t^2 + 18t - 12$

$9t^2 - 24t + 18 = -3t^2 + 18t - 12$
$12t^2 - 42t + 30 = 0$
$2t^2 - 7t + 5 = 0$
$(2t - 5)(t - 1) = 0$
$t = 1,\ \frac{5}{2}$

33. a. $v(t) = -32t + 48$
initial velocity $= v_0 = 48$ ft/sec

b. $-32t + 48 = 0$
$t = \frac{3}{2}$ sec

c. $s = -16(1.5)^2 + 48(1.5) + 256 = 292$ ft

d. $-16t^2 + 48t + 256 = 0$
$$t = \frac{-48 \pm \sqrt{48^2 - 4(-16)(256)}}{-32} \approx -2.77, 5.77$$
The object hits the ground at $t = 5.77$ sec.

e. $v(5.77) \approx -137$ ft/sec;
speed $= |-137| = 137$ ft/sec.

34. $v(t) = 48 - 32t$

a. $48 - 32t = 0$
$t = 1.5$
$s = 48(1.5) - 16(1.5)^2 = 36$ ft

b. $v(1) = 16$ ft/sec upward

c. $48t - 16t^2 = 0$
$-16t(-3 + t) = 0$
$t = 3$ sec

35. $v(t) = v_0 - 32t$
$v_0 - 32t = 0$
$$t = \frac{v_0}{32}$$
$$v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = 5280$$
$$\frac{{v_0}^2}{32} - \frac{{v_0}^2}{64} = 5280$$
$$\frac{{v_0}^2}{64} = 5280$$
$v_0 = \sqrt{337,920} \approx 581$ ft/sec

36. $v(t) = v_0 + 32t$
$v_0 + 32t = 140$
$v_0 + 32(3) = 140$
$v_0 = 44$
$s = 44(3) + 16(3)^2 = 276$ ft

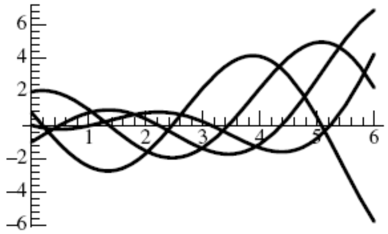
37. $v(t) = 3t^2 - 6t - 24$
$$\frac{d}{dt}\left|3t^2 - 6t - 24\right| = \frac{\left|3t^2 - 6t - 24\right|}{3t^2 - 6t - 24}(6t - 6)$$
$$= \frac{|(t-4)(t+2)|}{(t-4)(t+2)}(6t-6)$$
$$\frac{|(t-4)(t+2)|(6t-6)}{(t-4)(t+2)} < 0$$
$t < -2$, $1 < t < 4$; $(-\infty, -2) \cup (1, 4)$

38. Point slowing down when
$$\frac{d}{dt}|v(t)| < 0$$
$$\frac{d}{dt}|v(t)| = \frac{|v(t)|a(t)}{v(t)}$$
$\frac{|v(t)|a(t)}{v(t)} < 0$ when $a(t)$ and $v(t)$ have opposite signs.

39. $D_x(uv) = uv' + u'v$
$$D_x^2(uv) = uv'' + u'v' + u'v' + u''v$$
$$= uv'' + 2u'v' + u''v$$
$$D_x^3(uv) = uv''' + u'v'' + 2(u'v'' + u''v') + u''v' + u'''v$$
$$= uv''' + 3u'v'' + 3u''v' + u'''v$$
$$D_x^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_x^{n-k}(u) D_x^k(v)$$
where $\binom{n}{k}$ is the binomial coefficient
$$\frac{n!}{(n-k)!k!}.$$

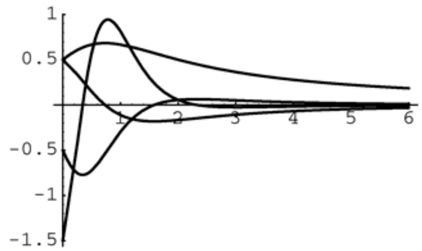
40. $D_x^4(x^4 \sin x) = \binom{4}{0} D_x^4(x^4) D_x^0(\sin x)$
$+ \binom{4}{1} D_x^3(x^4) D_x^1(\sin x) + \binom{4}{2} D_x^2(x^4) D_x^2(\sin x)$
$+ \binom{4}{3} D_x^1(x^4) D_x^3(\sin x) + \binom{4}{4} D_x^0(x^4) D_x^4(\sin x)$
$= 24 \sin x + 96x \cos x - 72x^2 \sin x$
$-16x^3 \cos x + x^4 \sin x$

41. a.



b. $f'''(2.13) \approx -1.2826$

42. a.



b. $f'''(2.13) \approx 0.0271$

2.7 Concepts Review

1. $\dfrac{9}{x^3-3}$

2. $3y^2 \dfrac{dy}{dx}$

3. $x(2y)\dfrac{dy}{dx} + y^2 + 3y^2 \dfrac{dy}{dx} - \dfrac{dy}{dx} = 3x^2$

4. $\dfrac{p}{q} x^{p/q-1}; \dfrac{5}{3}(x^2-5x)^{2/3}(2x-5)$

Problem Set 2.7

1. $2y\,D_x y - 2x = 0$

$D_x y = \dfrac{2x}{2y} = \dfrac{x}{y}$

2. $18x + 8y\,D_x y = 0$

$D_x y = \dfrac{-18x}{8y} = -\dfrac{9x}{4y}$

3. $x\,D_x y + y = 0$

$D_x y = -\dfrac{y}{x}$

4. $2x + 2\alpha^2 y D_x y = 0$

$D_x y = -\dfrac{2x}{2\alpha^2 y} = -\dfrac{x}{\alpha^2 y}$

5. $x(2y)D_x y + y^2 = 1$

$D_x y = \dfrac{1-y^2}{2xy}$

6. $2x + 2x^2 D_x y + 4xy + 3x\,D_x y + 3y = 0$

$D_x y(2x^2+3x) = -2x - 4xy - 3y$

$D_x y = \dfrac{-2x-4xy-3y}{2x^2+3x}$

7. $12x^2 + 7x(2y)D_x y + 7y^2 = 6y^2 D_x y$

$12x^2 + 7y^2 = 6y^2 D_x y - 14xy D_x y$

$D_x y = \dfrac{12x^2+7y^2}{6y^2-14xy}$

8. $x^2 D_x y + 2xy = y^2 + x(2y)D_x y$

$x^2 D_x y - 2xy D_x y = y^2 - 2xy$

$D_x y = \dfrac{y^2-2xy}{x^2-2xy}$

9. $\dfrac{1}{2\sqrt{5xy}} \cdot (5x\,D_x y + 5y) + 2D_x y$

$= 2y\,D_x y + x(3y^2)D_x y + y^3$

$\dfrac{5x}{2\sqrt{5xy}} D_x y + 2D_x y - 2y\,D_x y - 3xy^2 D_x y$

$= y^3 - \dfrac{5y}{2\sqrt{5xy}}$

$D_x y = \dfrac{y^3 - \frac{5y}{2\sqrt{5xy}}}{\frac{5x}{2\sqrt{5xy}} + 2 - 2y - 3xy^2}$

10. $x\dfrac{1}{2\sqrt{y+1}} D_x y + \sqrt{y+1} = x\,D_x y + y$

$\dfrac{x}{2\sqrt{y+1}} D_x y - x\,D_x y = y - \sqrt{y+1}$

$D_x y = \dfrac{y-\sqrt{y+1}}{\frac{x}{2\sqrt{y+1}} - x}$

11. $x D_x y + y + \cos(xy)(x D_x y + y) = 0$
$x D_x y + x\cos(xy) D_x y = -y - y\cos(xy)$
$D_x y = \frac{-y - y\cos(xy)}{x + x\cos(xy)} = -\frac{y}{x}$

12. $-\sin(xy^2)(2xy D_x y + y^2) = 2y D_x y + 1$
$-2xy\sin(xy^2) D_x y - 2y D_x y = 1 + y^2\sin(xy^2)$
$D_x y = \frac{1 + y^2\sin(xy^2)}{-2xy\sin(xy^2) - 2y}$

13. $x^3 y' + 3x^2 y + y^3 + 3xy^2 y' = 0$
$y'(x^3 + 3xy^2) = -3x^2 y - y^3$
$y' = \frac{-3x^2 y - y^3}{x^3 + 3xy^2}$
At (1, 3), $y' = -\frac{36}{28} = -\frac{9}{7}$
Tangent line: $y - 3 = -\frac{9}{7}(x - 1)$

14. $x^2(2y)y' + 2xy^2 + 4xy' + 4y = 12y'$
$y'(2x^2 y + 4x - 12) = -2xy^2 - 4y$
$y' = \frac{-2xy^2 - 4y}{2x^2 y + 4x - 12} = \frac{-xy^2 - 2y}{x^2 y + 2x - 6}$
At (2, 1), $y' = -2$
Tangent line: $y - 1 = -2(x - 2)$

15. $\cos(xy)(xy' + y) = y'$
$y'[x\cos(xy) - 1] = -y\cos(xy)$
$y' = \frac{-y\cos(xy)}{x\cos(xy) - 1} = \frac{y\cos(xy)}{1 - x\cos(xy)}$
At $\left(\frac{\pi}{2}, 1\right)$, $y' = 0$
Tangent line: $y - 1 = 0\left(x - \frac{\pi}{2}\right)$
$y = 1$

16. $y' + [-\sin(xy^2)][2xyy' + y^2] + 6x = 0$
$y'[1 - 2xy\sin(xy^2)] = y^2\sin(xy^2) - 6x$
$y' = \frac{y^2\sin(xy^2) - 6x}{1 - 2xy\sin(xy^2)}$
At (1, 0), $y' = -\frac{6}{1} = -6$
Tangent line: $y - 0 = -6(x - 1)$

17. $\frac{2}{3}x^{-1/3} - \frac{2}{3}y^{-1/3}y' - 2y' = 0$
$\frac{2}{3}x^{-1/3} = y'\left(\frac{2}{3}y^{-1/3} + 2\right)$
$y' = \frac{\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3} + 2}$
At (1, –1), $y' = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2}$
Tangent line: $y + 1 = \frac{1}{2}(x - 1)$

18. $\frac{1}{2\sqrt{y}}y' + 2xyy' + y^2 = 0$
$y'\left(\frac{1}{2\sqrt{y}} + 2xy\right) = -y^2$
$y' = \frac{-y^2}{\frac{1}{2\sqrt{y}} + 2xy}$
At (4, 1), $y' = \frac{-1}{\frac{17}{2}} = -\frac{2}{17}$
Tangent line: $y - 1 = -\frac{2}{17}(x - 4)$

19. $\frac{dy}{dx} = 5x^{2/3} + \frac{1}{2\sqrt{x}}$

20. $\frac{dy}{dx} = \frac{1}{3}x^{-2/3} - 7x^{5/2} = \frac{1}{3\sqrt[3]{x^2}} - 7x^{5/2}$

21. $\frac{dy}{dx} = \frac{1}{3}x^{-2/3} - \frac{1}{3}x^{-4/3} = \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{3\sqrt[3]{x^4}}$

22. $\frac{dy}{dx} = \frac{1}{4}(2x+1)^{-3/4}(2) = \frac{1}{2\sqrt[4]{(2x+1)^3}}$

23. $\frac{dy}{dx} = \frac{1}{4}(3x^2 - 4x)^{-3/4}(6x - 4)$
$= \frac{6x - 4}{4\sqrt[4]{(3x^2 - 4x)^3}} = \frac{3x - 2}{2\sqrt[4]{(3x^2 - 4x)^3}}$

24. $\frac{dy}{dx} = \frac{1}{3}(x^3 - 2x)^{-2/3}(3x^2 - 2)$

25. $\frac{dy}{dx} = \frac{d}{dx}[(x^3 + 2x)^{-2/3}]$
$= -\frac{2}{3}(x^3 + 2x)^{-5/3}(3x^2 + 2) = -\frac{6x^2 + 4}{3\sqrt[3]{(x^3 + 2x)^5}}$

26. $\frac{dy}{dx} = -\frac{5}{3}(3x-9)^{-8/3}(3) = -5(3x-9)^{-8/3}$

27. $\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+\sin x}}(2x+\cos x)$

$= \frac{2x+\cos x}{2\sqrt{x^2+\sin x}}$

28. $\frac{dy}{dx} = \frac{1}{2\sqrt{x^2\cos x}}[x^2(-\sin x)+2x\cos x]$

$= \frac{2x\cos x - x^2\sin x}{2\sqrt{x^2\cos x}}$

29. $\frac{dy}{dx} = \frac{d}{dx}[(x^2\sin x)^{-1/3}]$

$= -\frac{1}{3}(x^2\sin x)^{-4/3}(x^2\cos x + 2x\sin x)$

$= -\frac{x^2\cos x + 2x\sin x}{3\sqrt[3]{(x^2\sin x)^4}}$

30. $\frac{dy}{dx} = \frac{1}{4}(1+\sin 5x)^{-3/4}(\cos 5x)(5)$

$= \frac{5\cos 5x}{4\sqrt[4]{(1+\sin 5x)^3}}$

31. $\frac{dy}{dx} = \frac{[1+\cos(x^2+2x)]^{-3/4}[-\sin(x^2+2x)(2x+2)]}{4}$

$= -\frac{(x+1)\sin(x^2+2x)}{2\sqrt[4]{[1+\cos(x^2+2x)]^3}}$

32. $\frac{dy}{dx} = \frac{(\tan^2 x+\sin^2 x)^{-1/2}(2\tan x\sec^2 x + 2\sin x\cos x)}{2}$

$= \frac{\tan x\sec^2 x + \sin x\cos x}{\sqrt{\tan^2 x + \sin^2 x}}$

33. $s^2 + 2st\frac{ds}{dt} + 3t^2 = 0$

$\frac{ds}{dt} = \frac{-s^2-3t^2}{2st} = -\frac{s^2+3t^2}{2st}$

$s^2\frac{dt}{ds} + 2st + 3t^2\frac{dt}{ds} = 0$

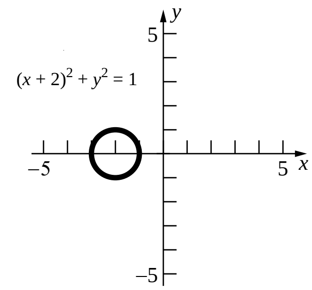
$\frac{dt}{ds}(s^2+3t^2) = -2st$

$\frac{dt}{ds} = -\frac{2st}{s^2+3t^2}$

34. $1 = \cos(x^2)(2x)\frac{dx}{dy} + 6x^2\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{1}{2x\cos(x^2)+6x^2}$

35.



$2x + 4 + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x+4}{2y} = -\frac{x+2}{y}$

The tangent line at (x_0, y_0) has equation

$y - y_0 = -\frac{x_0+2}{y_0}(x - x_0)$ which simplifies to

$2x_0 - yy_0 - 2x - xx_0 + y_0^2 + x_0^2 = 0.$ Since

(x_0, y_0) is on the circle, $x_0^2 + y_0^2 = -3 - 4x_0$,

so the equation of the tangent line is

$-yy_0 - 2x_0 - 2x - xx_0 = 3.$

If (0, 0) is on the tangent line, then $x_0 = -\frac{3}{2}.$

Solve for y_0 in the equation of the circle to get

$y_0 = \pm\frac{\sqrt{3}}{2}.$ Put these values into the equation of the tangent line to get that the tangent lines are $\sqrt{3}y + x = 0$ and $\sqrt{3}y - x = 0.$

36. $16(x^2+y^2)(2x+2yy') = 100(2x-2yy')$

$32x^3 + 32x^2yy' + 32xy^2 + 32y^3y' = 200x - 200yy'$

$y'(4x^2y + 4y^3 + 25y) = 25x - 4x^3 - 4xy^2$

$y' = \frac{25x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 25y}$

The slope of the normal line $= -\frac{1}{y'}$

$= \frac{4x^2y + 4y^3 + 25y}{4x^3 + 4xy^2 - 25x}$

At (3, 1), slope $= \frac{65}{45} = \frac{13}{9}$

Normal line: $y - 1 = \frac{13}{9}(x-3)$

37. a. $xy' + y + 3y^2 y' = 0$

$y'(x + 3y^2) = -y$

$y' = -\dfrac{y}{x + 3y^2}$

b. $xy'' + \left(\dfrac{-y}{x+3y^2}\right) + \left(\dfrac{-y}{x+3y^2}\right) + 3y^2 y''$

$+6y\left(\dfrac{-y}{x+3y^2}\right)^2 = 0$

$xy'' + 3y^2 y'' - \dfrac{2y}{x+3y^2} + \dfrac{6y^3}{(x+3y^2)^2} = 0$

$y''(x+3y^2) = \dfrac{2y}{x+3y^2} - \dfrac{6y^3}{(x+3y^2)^2}$

$y''(x+3y^2) = \dfrac{2xy}{(x+3y^2)^2}$

$y'' = \dfrac{2xy}{(x+3y^2)^3}$

38. $3x^2 - 8yy' = 0$

$y' = \dfrac{3x^2}{8y}$

$6x - 8(yy'' + (y')^2) = 0$

$6x - 8yy'' - 8\left(\dfrac{3x^2}{8y}\right)^2 = 0$

$6x - 8yy'' - \dfrac{9x^4}{8y^2} = 0$

$\dfrac{48xy^2 - 9x^4}{8y^2} = 8yy''$

$y'' = \dfrac{48xy^2 - 9x^4}{64y^3}$

39. $2(x^2 y' + 2xy) - 12y^2 y' = 0$

$2x^2 y' - 12y^2 y' = -4xy$

$y' = \dfrac{2xy}{6y^2 - x^2}$

$2(x^2 y'' + 2xy' + 2xy' + 2y) - 12[y^2 y'' + 2y(y')^2] = 0$

$2x^2 y'' - 12y^2 y'' = -8xy' - 4y + 24y(y')^2$

$y''(2x^2 - 12y^2) = -\dfrac{16x^2 y}{6y^2 - x^2} - 4y + \dfrac{96x^2 y^3}{(6y^2 - x^2)^2}$

$y''(2x^2 - 12y^2) = \dfrac{12x^4 y + 48x^2 y^3 - 144y^5}{(6y^2 - x^2)^2}$

$y''(6y^2 - x^2) = \dfrac{72y^5 - 6x^4 y - 24x^2 y^3}{(6y^2 - x^2)^2}$

$y'' = \dfrac{72y^5 - 6x^4 y - 24x^2 y^3}{(6y^2 - x^2)^3}$

At (2, 1), $y'' = \dfrac{-120}{8} = -15$

40. $2x + 2yy' = 0$

$y' = -\dfrac{2x}{2y} = -\dfrac{x}{y}$

$2 + 2[yy'' + (y')^2] = 0$

$2 + 2yy'' + 2\left(-\dfrac{x}{y}\right)^2 = 0$

$2yy'' = -2 - \dfrac{2x^2}{y^2}$

$y'' = -\dfrac{1}{y} - \dfrac{x^2}{y^3} = -\dfrac{y^2 + x^2}{y^3}$

At (3, 4), $y'' = -\dfrac{25}{64}$

41. $3x^2 + 3y^2 y' = 3(xy' + y)$

$y'(3y^2 - 3x) = 3y - 3x^2$

$y' = \dfrac{y - x^2}{y^2 - x}$

At $\left(\dfrac{3}{2}, \dfrac{3}{2}\right)$, $y' = -1$

Slope of the normal line is 1.

Normal line: $y - \dfrac{3}{2} = 1\left(x - \dfrac{3}{2}\right)$; $y = x$

This line includes the point (0, 0).

42. $xy' + y = 0$

$y' = -\dfrac{y}{x}$

$2x - 2yy' = 0$

$y' = \dfrac{x}{y}$

The slopes of the tangents are negative reciprocals, so the hyperbolas intersect at right angles.

43. Implicitly differentiate the first equation.

$4x + 2yy' = 0$

$y' = -\dfrac{2x}{y}$

Implicitly differentiate the second equation.

$2yy' = 4$

$y' = \dfrac{2}{y}$

Solve for the points of intersection.

$2x^2 + 4x = 6$

$2(x^2 + 2x - 3) = 0$

$(x + 3)(x - 1) = 0$

$x = -3, x = 1$

$x = -3$ is extraneous, and $y = -2, 2$ when $x = 1$.

The graphs intersect at (1, –2) and (1, 2).

At (1, –2): $m_1 = 1, m_2 = -1$

At (1, 2): $m_1 = -1, m_2 = 1$

44. Find the intersection points:

$x^2 + y^2 = 1 \quad \rightarrow \quad y^2 = 1 - x^2$

$(x-1)^2 + y^2 = 1$

$(x-1)^2 + (1-x^2) = 1$

$x^2 - 2x + 1 + 1 - x^2 = 1 \quad \Rightarrow \quad x = \dfrac{1}{2}$

Points of intersection: $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$

Implicitly differentiate the first equation.

$2x + 2yy' = 0$

$y' = -\dfrac{x}{y}$

Implicitly differentiate the second equation.

$2(x-1) + 2yy' = 0$

$y' = \dfrac{1-x}{y}$

At $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$: $m_1 = -\dfrac{1}{\sqrt{3}}, m_2 = \dfrac{1}{\sqrt{3}}$

$$\tan\theta = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3} \quad \rightarrow \quad \theta = \frac{\pi}{3}$$

At $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$: $m_1 = \dfrac{1}{\sqrt{3}}, m_2 = -\dfrac{1}{\sqrt{3}}$

$$\tan\theta = \frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} = -\sqrt{3}$$

$\theta = \dfrac{2\pi}{3}$

45. $x^2 - x(2x) + 2(2x)^2 = 28$

$7x^2 = 28$

$x^2 = 4$

$x = -2, 2$

Intersection point in first quadrant: (2, 4)

$y_1' = 2$

$2x - xy_2' - y + 4yy_2' = 0$

$y_2'(4y - x) = y - 2x$

$y_2' = \dfrac{y - 2x}{4y - x}$

At (2, 4): $m_1 = 2, m_2 = 0$

$\tan\theta = \dfrac{0 - 2}{1 + (0)(2)} = -2; \theta = \pi + \tan^{-1}(-2) \approx 2.034$

46. The equation is $mv^2 - mv_0^2 = kx_0^2 - kx^2$.

Differentiate implicitly with respect to t to get $2mv\dfrac{dv}{dt} = -2kx\dfrac{dx}{dt}$. Since $v = \dfrac{dx}{dt}$ this simplifies to $2mv\dfrac{dv}{dt} = -2kxv$ or $m\dfrac{dv}{dt} = -kx$.

47. $x^2 - xy + y^2 = 16$, when $y = 0$,

$x^2 = 16$

$x = -4, 4$

The ellipse intersects the x-axis at (–4, 0) and (4, 0).

$2x - xy' - y + 2yy' = 0$

$y'(2y - x) = y - 2x$

$y' = \dfrac{y - 2x}{2y - x}$

At (–4, 0), $y' = 2$

At (4, 0), $y' = 2$

Tangent lines: $y = 2(x + 4)$ and $y = 2(x - 4)$

48. $x^2 + 2xy\frac{dx}{dy} - 2xy - y^2\frac{dx}{dy} = 0$

$\frac{dx}{dy}(2xy - y^2) = 2xy - x^2;$

$\frac{dx}{dy} = \frac{2xy - x^2}{2xy - y^2}$

$\frac{2xy - x^2}{2xy - y^2} = 0$ if $x(2y - x) = 0$, which occurs when $x = 0$ or $y = \frac{x}{2}$. There are no points on $x^2y - xy^2 = 2$ where $x = 0$. If $y = \frac{x}{2}$, then

$2 = x^2\left(\frac{x}{2}\right) - x\left(\frac{x}{2}\right)^2 = \frac{x^3}{2} - \frac{x^3}{4} = \frac{x^3}{4}$ so $x = 2$,

$y = \frac{2}{2} = 1.$

The tangent line is vertical at (2, 1).

49. $2x + 2y\frac{dy}{dx} = 0; \frac{dy}{dx} = -\frac{x}{y}$

The tangent line at (x_0, y_0) has slope $-\frac{x_0}{y_0}$, hence the equation of the tangent line is $y - y_0 = -\frac{x_0}{y_0}(x - x_0)$ which simplifies to $yy_0 + xx_0 - (x_0^2 + y_0^2) = 0$ or $yy_0 + xx_0 = 1$ since (x_0, y_0) is on $x^2 + y^2 = 1$. If (1.25, 0) is on the tangent line through (x_0, y_0), $x_0 = 0.8$. Put this into $x^2 + y^2 = 1$ to get $y_0 = 0.6$, since $y_0 > 0$. The line is $6y + 8x = 10$. When $x = -2$, $y = \frac{13}{3}$, so the light bulb must be $\frac{13}{3}$ units high.

2.8 Concepts Review

1. $\frac{du}{dt}; t = 2$

2. 400 mi/hr

3. negative

4. negative; positive

Problem Set 2.8

1. $V = x^3; \frac{dx}{dt} = 3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

When $x = 12$, $\frac{dV}{dt} = 3(12)^2(3) = 1296$ in.^{3}/s.

2. $V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 3$

$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

When $r = 3$, $3 = 4\pi(3)^2\frac{dr}{dt}$.

$\frac{dr}{dt} = \frac{1}{12\pi} \approx 0.027$ in./s

3. $y^2 = x^2 + 1^2; \frac{dx}{dt} = 400$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$ mi/hr

When $x = 5, y = \sqrt{26}, \frac{dy}{dt} = \frac{5}{\sqrt{26}}(400)$

$\approx$ 392 mi/h.

4. $V = \frac{1}{3}\pi r^2 h; \frac{r}{h} = \frac{3}{10}; r = \frac{3h}{10}$

$V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h = \frac{3\pi h^3}{100}; \frac{dV}{dt} = 3, h = 5$

$\frac{dV}{dt} = \frac{9\pi h^2}{100}\frac{dh}{dt}$

When $h = 5$, $3 = \frac{9\pi(5)^2}{100}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{3\pi} \approx 0.42$ cm/s

5. $s^2 = (x + 300)^2 + y^2; \frac{dx}{dt} = 300, \frac{dy}{dt} = 400,$

$2s\frac{ds}{dt} = 2(x + 300)\frac{dx}{dt} + 2y\frac{dy}{dt}$

$s\frac{ds}{dt} = (x + 300)\frac{dx}{dt} + y\frac{dy}{dt}$

When $x = 300$, $y = 400$, $s = 200\sqrt{13}$, so

$200\sqrt{13}\frac{ds}{dt} = (300 + 300)(300) + 400(400)$

$\frac{ds}{dt} \approx 471$ mi/h

6. $y^2 = x^2 + (10)^2; \frac{dy}{dt} = 2$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

When $y = 25$, $x \approx 22.9$, so

$\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} \approx \frac{25}{22.9}(2) \approx 2.18$ ft/s

7. $20^2 = x^2 + y^2; \frac{dx}{dt} = 1$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

When $x = 5$, $y = \sqrt{375} = 5\sqrt{15}$, so

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{5}{5\sqrt{15}}(1) \approx -0.258$ ft/s

The top of the ladder is moving down at 0.258 ft/s.

8. $\frac{dV}{dt} = -4$ ft^3/h; $V = \pi hr^2; \frac{dh}{dt} = -0.0005$ ft/h

$A = \pi r^2 = \frac{V}{h} = Vh^{-1}$, so $\frac{dA}{dt} = h^{-1}\frac{dV}{dt} - \frac{V}{h^2}\frac{dh}{dt}$.

When $h = 0.001$ ft, $V = \pi(0.001)(250)^2 = 62.5\pi$

and $\frac{dA}{dt} = 1000(-4) - 1{,}000{,}000(62.5\pi)(-0.0005)$

$= -4000 + 31{,}250\pi \approx 94{,}175$ ft^2/h.

(The height is decreasing due to the spreading of the oil rather than the bacteria.)

9. $V = \frac{1}{3}\pi r^2 h; h = \frac{d}{4} = \frac{r}{2}, r = 2h$

$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3; \frac{dV}{dt} = 16$

$\frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}$

When $h = 4$, $16 = 4\pi(4)^2\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{4\pi} \approx 0.0796$ ft/s

10. $y^2 = x^2 + (90)^2; \frac{dx}{dt} = 5$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

When $y = 150$, $x = 120$, so

$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{120}{150}(5) = 4$ ft/s

11. $V = \frac{hx}{2}(20); \frac{40}{5} = \frac{x}{h}, x = 8h$

$V = 10h(8h) = 80h^2; \frac{dV}{dt} = 40$

$\frac{dV}{dt} = 160h\frac{dh}{dt}$

When $h = 3$, $40 = 160(3)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{12}$ ft/min

12. $y = \sqrt{x^2 - 4}; \frac{dx}{dt} = 5$

$\frac{dy}{dt} = \frac{1}{2\sqrt{x^2-4}}(2x)\frac{dx}{dt} = \frac{x}{\sqrt{x^2-4}}\frac{dx}{dt}$

When $x = 3$, $\frac{dy}{dt} = \frac{3}{\sqrt{3^2-4}}(5) = \frac{15}{\sqrt{5}} \approx 6.7$ units/s

13. $A = \pi r^2; \frac{dr}{dt} = 0.02$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $r = 8.1$, $\frac{dA}{dt} = 2\pi(0.02)(8.1) = 0.324\pi$

≈ 1.018 in.^{2}/s

14. $s^2 = x^2 + (y+48)^2; \frac{dx}{dt} = 30, \frac{dy}{dt} = 24$

$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2(y+48)\frac{dy}{dt}$

$s\frac{ds}{dt} = x\frac{dx}{dt} + (y+48)\frac{dy}{dt}$

At 2:00 p.m., $x = 3(30) = 90$, $y = 3(24) = 72$, so $s = 150$.

$(150)\frac{ds}{dt} = 90(30) + (72+48)(24)$

$\frac{ds}{dt} = \frac{5580}{150} = 37.2$ knots/h

15. Let x be the distance from the beam to the point opposite the lighthouse and θ be the angle between the beam and the line from the lighthouse to the point opposite.

$\tan\theta = \frac{x}{1}; \frac{d\theta}{dt} = 2(2\pi) = 4\pi$ rad/min,

$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$

At $x = \frac{1}{2}, \theta = \tan^{-1}\frac{1}{2}$ and $\sec^2\theta = \frac{5}{4}$.

$\frac{dx}{dt} = \frac{5}{4}(4\pi) \approx 15.71$ km/min

16. $\tan\theta = \frac{4000}{x}$

$\sec^2\theta \frac{d\theta}{dt} = -\frac{4000}{x^2}\frac{dx}{dt}$

When $\theta = \frac{1}{2}, \frac{d\theta}{dt} = \frac{1}{10}$ and $x = \frac{4000}{\tan\frac{1}{2}} \approx 7322$.

$\frac{dx}{dt} \approx \sec^2\frac{1}{2}\left(\frac{1}{10}\right)\left[-\frac{(7322)^2}{4000}\right]$

≈ -1740 ft/s or -1186 mi/h

The plane's ground speed is 1186 mi/h.

17. a. Let x be the distance along the ground from the light pole to Chris, and let s be the distance from Chris to the tip of his shadow. By similar triangles, $\frac{6}{s} = \frac{30}{x+s}$, so $s = \frac{x}{4}$ and $\frac{ds}{dt} = \frac{1}{4}\frac{dx}{dt}$. $\frac{dx}{dt} = 2$ ft/s, hence $\frac{ds}{dt} = \frac{1}{2}$ ft/s no matter how far from the light pole Chris is.

b. Let $l = x + s$, then

$\frac{dl}{dt} = \frac{dx}{dt} + \frac{ds}{dt} = 2 + \frac{1}{2} = \frac{5}{2}$ ft/s.

c. The angular rate at which Chris must lift his head to follow his shadow is the same as the rate at which the angle that the light makes with the ground is decreasing. Let θ be the angle that the light makes with the ground at the tip of Chris' shadow.

$\tan\theta = \frac{6}{s}$ so $\sec^2\theta \frac{d\theta}{dt} = -\frac{6}{s^2}\frac{ds}{dt}$ and

$\frac{d\theta}{dt} = -\frac{6\cos^2\theta}{s^2}\frac{ds}{dt}$. $\frac{ds}{dt} = \frac{1}{2}$ ft/s

When $s = 6$, $\theta = \frac{\pi}{4}$, so

$$\frac{d\theta}{dt} = -\frac{6\left(\frac{1}{\sqrt{2}}\right)^2}{6^2}\left(\frac{1}{2}\right) = -\frac{1}{24}.$$

Chris must lift his head at the rate of $\frac{1}{24}$ rad/s.

18. Let θ be the measure of the vertex angle, a be the measure of the equal sides, and b be the measure of the base. Observe that $b = 2a\sin\frac{\theta}{2}$ and the height of the triangle is $a\cos\frac{\theta}{2}$.

$A = \frac{1}{2}\left(2a\sin\frac{\theta}{2}\right)\left(a\cos\frac{\theta}{2}\right) = \frac{1}{2}a^2\sin\theta$

$A = \frac{1}{2}(100)^2\sin\theta = 5000\sin\theta; \frac{d\theta}{dt} = \frac{1}{10}$

$\frac{dA}{dt} = 5000\cos\theta\frac{d\theta}{dt}$

When $\theta = \frac{\pi}{6}, \frac{dA}{dt} = 5000\left(\cos\frac{\pi}{6}\right)\left(\frac{1}{10}\right) = 250\sqrt{3}$

$\approx 433\ \text{cm}^2/\text{min}$.

19. Let p be the point on the bridge directly above the railroad tracks. If a is the distance between p and the automobile, then $\frac{da}{dt} = 66$ ft/s. If l is the distance between the train and the point directly below p, then $\frac{dl}{dt} = 88$ ft/s. The distance from the train to p is $\sqrt{100^2 + l^2}$, while the distance from p to the automobile is a. The distance between the train and automobile is

$D = \sqrt{a^2 + \left(\sqrt{100^2 + l^2}\right)^2} = \sqrt{a^2 + l^2 + 100^2}$.

$\frac{dD}{dt} = \frac{1}{2\sqrt{a^2 + l^2 + 100^2}}\cdot\left(2a\frac{da}{dt} + 2l\frac{dl}{dt}\right)$

$= \frac{a\frac{da}{dt} + l\frac{dl}{dt}}{\sqrt{a^2 + l^2 + 100^2}}$. After 10 seconds, $a = 660$ and $l = 880$, so

$\frac{dD}{dt} = \frac{660(66) + 880(88)}{\sqrt{660^2 + 880^2 + 100^2}} \approx 110$ ft/s.

20. $V = \frac{1}{3}\pi h \cdot (a^2 + ab + b^2)$; $a = 20, b = \frac{h}{4} + 20$,

$$V = \frac{1}{3}\pi h\left(400 + 5h + 400 + \frac{h^2}{16} + 10h + 400\right)$$

$$= \frac{1}{3}\pi\left(1200h + 15h^2 + \frac{h^3}{16}\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(1200 + 30h + \frac{3h^2}{16}\right)\frac{dh}{dt}$$

When $h = 30$ and $\frac{dV}{dt} = 2000$,

$$2000 = \frac{1}{3}\pi\left(1200 + 900 + \frac{675}{4}\right)\frac{dh}{dt} = \frac{3025\pi}{4}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{320}{121\pi} \approx 0.84 \text{ cm/min.}$$

21. $V = \pi h^2\left[r - \frac{h}{3}\right]; \frac{dV}{dt} = -2, r = 8$

$$V = \pi r h^2 - \frac{\pi h^3}{3} = 8\pi h^2 - \frac{\pi h^3}{3}$$

$$\frac{dV}{dt} = 16\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt}$$

When $h = 3$, $-2 = \frac{dh}{dt}[16\pi(3) - \pi(3)^2]$

$$\frac{dh}{dt} = \frac{-2}{39\pi} \approx -0.016 \text{ ft/hr}$$

22. $s^2 = a^2 + b^2 - 2ab\cos\theta$;

$a = 5$, $b = 4$, $\frac{d\theta}{dt} = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ rad/h

$$s^2 = 41 - 40\cos\theta$$

$$2s\frac{ds}{dt} = 40\sin\theta\frac{d\theta}{dt}$$

At 3:00, $\theta = \frac{\pi}{2}$ and $s = \sqrt{41}$, so

$$2\sqrt{41}\frac{ds}{dt} = 40\sin\left(\frac{\pi}{2}\right)\left(\frac{11\pi}{6}\right) = \frac{220\pi}{3}$$

$$\frac{ds}{dt} \approx 18 \text{ in./hr}$$

23. Let P be the point on the ground where the ball hits. Then the distance from P to the bottom of the light pole is 10 ft. Let s be the distance between P and the shadow of the ball. The height of the ball t seconds after it is dropped is $64 - 16t^2$.

By similar triangles, $\frac{48}{64 - 16t^2} = \frac{10 + s}{s}$

(for $t > 1$), so $s = \frac{10t^2 - 40}{1 - t^2}$.

$$\frac{ds}{dt} = \frac{20t(1 - t^2) - (10t^2 - 40)(-2t)}{(1 - t^2)^2} = -\frac{60t}{(1 - t^2)^2}$$

The ball hits the ground when $t = 2$, $\frac{ds}{dt} = -\frac{120}{9}$.

The shadow is moving $\frac{120}{9} \approx 13.33$ ft/s.

24. $V = \pi h^2\left(r - \frac{h}{3}\right)$; $r = 20$

$$V = \pi h^2\left(20 - \frac{h}{3}\right) = 20\pi h^2 - \frac{\pi}{3}h^3$$

$$\frac{dV}{dt} = (40\pi h - \pi h^2)\frac{dh}{dt}$$

At 7:00 a.m., $h = 15$, $\frac{dh}{dt} \approx -3$, so

$$\frac{dV}{dt} = (40\pi(15) - \pi(15)^2)(-3) \approx -1125\pi \approx -3534.$$

Webster City residents used water at the rate of $2400 + 3534 = 5934$ ft^3/h.

25. Assuming that the tank is now in the shape of an upper hemisphere with radius r, we again let t be the number of hours past midnight and h be the height of the water at time t. The volume, V, of water in the tank at that time is given by

$$V = \frac{2}{3}\pi r^3 - \frac{\pi}{3}(r - h)^2(2r + h)$$

and so $V = \frac{16000}{3}\pi - \frac{\pi}{3}(20 - h)^2(40 + h)$

from which

$$\frac{dV}{dt} = -\frac{\pi}{3}(20 - h)^2\frac{dh}{dt} + \frac{2\pi}{3}(20 - h)(40 + h)\frac{dh}{dt}$$

At $t = 7$, $\frac{dV}{dt} \approx -525\pi \approx -1649$

Thus Webster City residents were using water at the rate of $2400 + 1649 = 4049$ cubic feet per hour at 7:00 A.M.

26. The amount of water used by Webster City can be found by:

usage = beginning amount + added amount
− remaining amount

Thus the usage is

$$\approx \pi(20)^2(9) + 2400(12) - \pi(20)^2(10.5) \approx 26{,}915 \text{ ft}^3$$

over the 12 hour period.

27. a. Let x be the distance from the bottom of the wall to the end of the ladder on the ground, so $\frac{dx}{dt}=2$ ft/s. Let y be the height of the opposite end of the ladder. By similar triangles, $\frac{y}{12}=\frac{18}{\sqrt{144+x^2}}$, so $y=\frac{216}{\sqrt{144+x^2}}$.

$$\frac{dy}{dt}=-\frac{216}{2(144+x^2)^{3/2}}2x\frac{dx}{dt}=-\frac{216x}{(144+x^2)^{3/2}}\frac{dx}{dt}$$

When the ladder makes an angle of 60° with the ground, $x=4\sqrt{3}$ and $\frac{dy}{dt}=-\frac{216(4\sqrt{3})}{(144+48)^{3/2}}\cdot 2=-1.125$ ft/s.

b. $$\frac{d^2y}{dt^2}=\frac{d}{dt}\left(-\frac{216x}{(144+x^2)^{3/2}}\frac{dx}{dt}\right)=\frac{d}{dt}\left(-\frac{216x}{(144+x^2)^{3/2}}\right)\frac{dx}{dt}-\frac{216x}{(144+x^2)^{3/2}}\cdot\frac{d^2x}{dt^2}$$

Since $\frac{dx}{dt}=2,\ \frac{d^2x}{dt^2}=0,$ thus

$$\frac{d^2y}{dt^2}=\left[\frac{-216(144+x^2)^{3/2}\frac{dx}{dt}+216x\left(\frac{3}{2}\right)\sqrt{144+x^2}(2x)\frac{dx}{dt}}{(144+x^2)^3}\right]\frac{dx}{dt}$$

$$=\frac{-216(144+x^2)+648x^2}{(144+x^2)^{5/2}}\left(\frac{dx}{dt}\right)^2=\frac{432x^2-31{,}104}{(144+x^2)^{5/2}}\left(\frac{dx}{dt}\right)^2$$

When the ladder makes an angle of 60° with the ground,

$$\frac{d^2y}{dt^2}=\frac{432\cdot 48-31{,}104}{(144+48)^{5/2}}(2)^2\approx -0.08\ \text{ft/s}^2$$

28. a. If the ball has radius 6 in., the volume of the water in the tank is

$$V=8\pi h^2-\frac{\pi h^3}{3}-\frac{4}{3}\pi\left(\frac{1}{2}\right)^3$$

$$=8\pi h^2-\frac{\pi h^3}{3}-\frac{\pi}{6}$$

$$\frac{dV}{dt}=16\pi h\frac{dh}{dt}-\pi h^2\frac{dh}{dt}$$

This is the same as in Problem 21, so $\frac{dh}{dt}$ is again –0.016 ft/hr.

b. If the ball has radius 2 ft, and the height of the water in the tank is h feet with $2\le h\le 3$, the part of the ball in the water has volume

$$\frac{4}{3}\pi(2)^3-\pi(4-h)^2\left[2-\frac{4-h}{3}\right]=\frac{(6-h)h^2\pi}{3}.$$

The volume of water in the tank is

$$V=8\pi h^2-\frac{\pi h^3}{3}-\frac{(6-h)h^2\pi}{3}=6h^2\pi$$

$$\frac{dV}{dt}=12h\pi\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{1}{12h\pi}\frac{dV}{dt}$$

When h = 3, $\frac{dh}{dt}=\frac{1}{36\pi}(-2)\approx -0.018$ ft/hr.

29. $\frac{dV}{dt}=k(4\pi r^2)$

a. $$V=\frac{4}{3}\pi r^3$$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

$$k(4\pi r^2)=4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt}=k$$

b. If the original volume was V_0, the volume after 1 hour is $\frac{8}{27}V_0$. The original radius was $r_0=\sqrt[3]{\frac{3}{4\pi}V_0}$ while the radius after 1 hour is $r_1=\sqrt[3]{\frac{8}{27}V_0\cdot\frac{3}{4\pi}}=\frac{2}{3}r_0$. Since $\frac{dr}{dt}$ is constant, $\frac{dr}{dt}=-\frac{1}{3}r_0$ unit/hr. The snowball will take 3 hours to melt completely.

30. $PV = k$

$$P\frac{dV}{dt}+V\frac{dP}{dt}=0$$

At $t = 6.5$, $P \approx 67$, $\frac{dP}{dt}\approx -30$, $V = 300$

$$\frac{dV}{dt}=-\frac{V}{P}\frac{dP}{dt}=-\frac{300}{67}(-30)\approx 134 \text{ in.}^3\text{/min}$$

31. Let l be the distance along the ground from the brother to the tip of the shadow. The shadow is controlled by both siblings when $\frac{3}{l}=\frac{5}{l+4}$ or $l = 6$. Again using similar triangles, this occurs when $\frac{y}{20}=\frac{6}{3}$, so $y = 40$. Thus, the girl controls the tip of the shadow when $y \geq 40$ and the boy controls it when $y < 40$.
Let x be the distance along the ground from the light pole to the girl. $\frac{dx}{dt}=-4$

When $y \geq 40$, $\frac{20}{y}=\frac{5}{y-x}$ or $y=\frac{4}{3}x$.

When $y < 40$, $\frac{20}{y}=\frac{3}{y-(x+4)}$ or $y=\frac{20}{17}(x+4)$.

$x = 30$ when $y = 40$. Thus,

$$y=\begin{cases}\frac{4}{3}x & \text{if } x\geq 30\\ \frac{20}{17}(x+4) & \text{if } x<30\end{cases}$$

and

$$\frac{dy}{dt}=\begin{cases}\frac{4}{3}\frac{dx}{dt} & \text{if } x\geq 30\\ \frac{20}{17}\frac{dx}{dt} & \text{if } x<30\end{cases}$$

Hence, the tip of the shadow is moving at the rate of $\frac{4}{3}(4)=\frac{16}{3}$ ft/s when the girl is at least 30 feet from the light pole, and it is moving $\frac{20}{17}(4)=\frac{80}{17}$ ft/s when the girl is less than 30 ft from the light pole.

2.9 Concepts Review

1. $f'(x)dx$

2. Δy; dy

3. Δx is small.

4. larger ; smaller

Problem Set 2.9

1. $dy = (2x + 1)dx$

2. $dy=(21x^2+6x)dx$

3. $dy=-4(2x+3)^{-5}(2)dx=-8(2x+3)^{-5}dx$

4. $dy=-2(3x^2+x+1)^{-3}(6x+1)dx$
$=-2(6x+1)(3x^2+x+1)^{-3}dx$

5. $dy=3(\sin x+\cos x)^2(\cos x-\sin x)dx$

6. $dy=3(\tan x+1)^2(\sec^2 x)dx$
$=3\sec^2 x(\tan x+1)^2 dx$

7. $dy=-\frac{3}{2}(7x^2+3x-1)^{-5/2}(14x+3)dx$
$=-\frac{3}{2}(14x+3)(7x^2+3x-1)^{-5/2}dx$

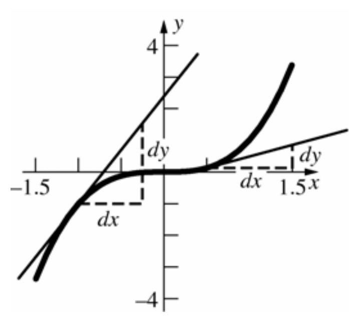
8. $dy=2(x^{10}+\sqrt{\sin 2x})[10x^9+\frac{1}{2\sqrt{\sin 2x}}\cdot(\cos 2x)(2)]dx$
$=2\left(10x^9+\frac{\cos 2x}{\sqrt{\sin 2x}}\right)(x^{10}+\sqrt{\sin 2x})dx$

9. $ds=\frac{3}{2}(t^2-\cot t+2)^{1/2}(2t+\csc^2 t)dt$
$=\frac{3}{2}(2t+\csc^2 t)\sqrt{t^2-\cot t+2}dt$

10. a. $dy=3x^2dx=3(0.5)^2(1)=0.75$

b. $dy=3x^2dx=3(-1)^2(0.75)=2.25$

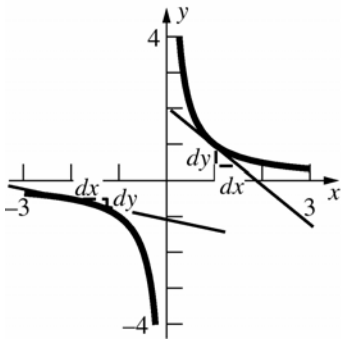
11.



12. a. $dy=-\frac{dx}{x^2}=-\frac{0.5}{(1)^2}=-0.5$

b. $dy=-\frac{dx}{x^2}=-\frac{0.75}{(-2)^2}=-0.1875$

13.



14. a. $\Delta y = (1.5)^3 - (0.5)^3 = 3.25$

b. $\Delta y = (-0.25)^3 - (-1)^3 = 0.984375$

15. a. $\Delta y = \frac{1}{1.5} - \frac{1}{1} = -\frac{1}{3}$

b. $\Delta y = \frac{1}{-1.25} + \frac{1}{2} = -0.3$

16. a. $\Delta y = [(2.5)^2 - 3] - [(2)^2 - 3] = 2.25$

$dy = 2x\,dx = 2(2)(0.5) = 2$

b. $\Delta y = [(2.88)^2 - 3] - [(3)^2 - 3] = -0.7056$

$dy = 2x\,dx = 2(3)(-0.12) = -0.72$

17. a. $\Delta y = [(3)^4 + 2(3)] - [(2)^4 + 2(2)] = 67$

$dy = (4x^3 + 2)dx = [4(2)^3 + 2](1) = 34$

b. $\Delta y = [(2.005)^4 + 2(2.005)] - [(2)^4 + 2(2)]$

≈ 0.1706

$dy = (4x^3 + 2)dx = [4(2)^3 + 2](0.005) = 0.17$

18. $y = \sqrt{x};\ dy = \frac{1}{2\sqrt{x}}dx;\ x = 400,\ dx = 2$

$dy = \frac{1}{2\sqrt{400}}(2) = 0.05$

$\sqrt{402} \approx \sqrt{400} + dy = 20 + 0.05 = 20.05$

19. $y = \sqrt{x};\ dy = \frac{1}{2\sqrt{x}}dx;\ x = 36,\ dx = -0.1$

$dy = \frac{1}{2\sqrt{36}}(-0.1) \approx -0.0083$

$\sqrt{35.9} \approx \sqrt{36} + dy = 6 - 0.0083 = 5.9917$

20. $y = \sqrt[3]{x};\ dy = \frac{1}{3}x^{-2/3}dx = \frac{1}{3\sqrt[3]{x^2}}dx;$

$x = 27,\ dx = -0.09$

$dy = \frac{1}{3\sqrt[3]{(27)^2}}(-0.09) \approx -0.0033$

$\sqrt[3]{26.91} \approx \sqrt[3]{27} + dy = 3 - 0.0033 = 2.9967$

21. $V = \frac{4}{3}\pi r^3;\ r = 5,\ dr = 0.125$

$dV = 4\pi r^2 dr = 4\pi(5)^2(0.125) \approx 39.27\ \text{cm}^3$

22. $V = x^3;\ x = \sqrt[3]{40},\ dx = 0.5$

$dV = 3x^2 dx = 3(\sqrt[3]{40})^2(0.5) \approx 17.54\ \text{in.}^3$

23. $V = \frac{4}{3}\pi r^3;\ r = 6\,\text{ft} = 72\,\text{in.},\ dr = -0.3$

$dV = 4\pi r^2 dr = 4\pi(72)^2(-0.3) \approx -19{,}543$

$V \approx \frac{4}{3}\pi(72)^3 - 19{,}543$

$\approx 1{,}543{,}915\ \text{in}^3 \approx 893\ \text{ft}^3$

24. $V = \pi r^2 h;\ r = 6\,\text{ft} = 72\,\text{in.},\ dr = -0.05,$

$h = 8\,\text{ft} = 96\,\text{in.}$

$dV = 2\pi rh\,dr = 2\pi(72)(96)(-0.05) \approx -2171\,\text{in.}^3$

About 9.4 gal of paint are needed.

25. $C = 2\pi r$; $r = 4000$ mi = 21,120,000 ft, $dr = 2$

$dC = 2\pi dr = 2\pi(2) = 4\pi \approx 12.6\ ft$

26. $T = 2\pi\sqrt{\frac{L}{32}};\ L = 4,\ dL = -0.03$

$dT = \frac{2\pi}{2\sqrt{\frac{L}{32}}} \cdot \frac{1}{32} \cdot dL = \frac{\pi}{\sqrt{32L}}dL$

$dT = \frac{\pi}{\sqrt{32(4)}}(-0.03) \approx -0.0083$

The time change in 24 hours is
$(0.0083)(60)(60)(24) \approx 717$ sec

27. $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(10)^3 \approx 4189$

$dV = 4\pi r^2 dr = 4\pi(10)^2(0.05) \approx 62.8$ The volume is $4189 \pm 62.8\ \text{cm}^3$.

The absolute error is ≈ 62.8 while the relative error is $62.8/4189 \approx 0.015$ or 1.5% .

28. $V = \pi r^2 h = \pi(3)^2(12) \approx 339$

$dV = 24\pi r dr = 24\pi(3)(0.0025) \approx 0.565$

The volume is 339 ± 0.565 in.^{3}

The absolute error is ≈ 0.565 while the relative error is $0.565/339 \approx 0.0017$ or 0.17%.

29. $s = \sqrt{a^2 + b^2 - 2ab\cos\theta}$

$= \sqrt{151^2 + 151^2 - 2(151)(151)\cos 0.53} \approx 79.097$

$s = \sqrt{45,602 - 45,602\cos\theta}$

$$ds = \frac{1}{2\sqrt{45,602 - 45,602\cos\theta}} \cdot 45,602\sin\theta\, d\theta$$

$$= \frac{22,801\sin\theta}{\sqrt{45,602 - 45,602\cos\theta}} d\theta$$

$$= \frac{22,801\sin 0.53}{\sqrt{45,602 - 45,602\cos 0.53}}(0.005) \approx 0.729$$

$s \approx 79.097 \pm 0.729$ cm

The absolute error is ≈ 0.729 while the relative error is $0.729/79.097 \approx 0.0092$ or 0.92%.

30. $A = \frac{1}{2}ab\sin\theta = \frac{1}{2}(151)(151)\sin 0.53 \approx 5763.33$

$A = \frac{22,801}{2}\sin\theta; \theta = 0.53, d\theta = 0.005$

$dA = \frac{22,801}{2}(\cos\theta)d\theta$

$= \frac{22,801}{2}(\cos 0.53)(0.005) \approx 49.18$

$A \approx 5763.33 \pm 49.18$ cm^2

The absolute error is ≈ 49.18 while the relative error is $49.18/5763.33 \approx 0.0085$ or 0.85%.

31. $y = 3x^2 - 2x + 11; x = 2, dx = 0.001$

$dy = (6x - 2)dx = [6(2) - 2](0.001) = 0.01$

$\frac{d^2y}{dx^2} = 6,$ so with $\Delta x = 0.001$,

$|\Delta y - dy| \le \frac{1}{2}(6)(0.001)^2 = 0.000003$

32. Using the approximation

$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

we let $x = 1.02$ and $\Delta x = -0.02$. We can rewrite the above form as

$f(x) \approx f(x + \Delta x) - f'(x)\Delta x$

which gives

$f(1.02) \approx f(1) - f'(1.02)(-0.02)$

$= 10 + 12(0.02) = 10.24$

33. Using the approximation

$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

we let $x = 3.05$ and $\Delta x = -0.05$. We can rewrite the above form as

$f(x) \approx f(x + \Delta x) - f'(x)\Delta x$

which gives

$f(3.05) \approx f(3) - f'(3.05)(-0.05)$

$= 8 + \frac{1}{4}(0.05) = 8.0125$

34. From similar triangles, the radius at height h is $\frac{2}{5}h$. Thus, $V = \frac{1}{3}\pi r^2 h = \frac{4}{75}\pi h^3$, so

$dV = \frac{4}{25}\pi h^2 dh.$ $h = 10$, $dh = -1$:

$dV = \frac{4}{25}\pi(100)(-1) \approx -50\,\text{cm}^3$

The ice cube has volume $3^3 = 27\,\text{cm}^3$, so there is room for the ice cube without the cup overflowing.

35. $V = \pi r^2 h + \frac{4}{3}\pi r^3$

$V = 100\pi r^2 + \frac{4}{3}\pi r^3; r = 10, dr = 0.1$

$dV = (200\pi r + 4\pi r^2)dr$

$= (2000\pi + 400\pi)(0.1) = 240\pi \approx 754$ cm^3

36. The percent increase in mass is $\frac{dm}{m}$.

$$dm = -\frac{m_0}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)dv$$

$$= \frac{m_0 v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} dv$$

$$\frac{dm}{m} = \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} dv = \frac{v}{c^2}\left(\frac{c^2}{c^2 - v^2}\right)dv$$

$$= \frac{v}{c^2 - v^2}dv$$

$v = 0.9c$, $dv = 0.02c$

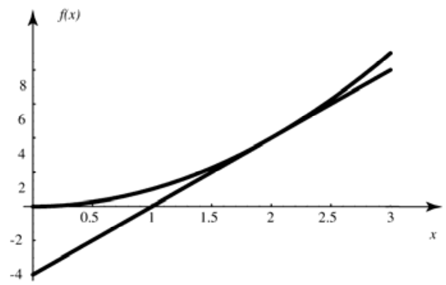
$$\frac{dm}{m} = \frac{0.9c}{c^2 - 0.81c^2}(0.02c) = \frac{0.018}{0.19} \approx 0.095$$

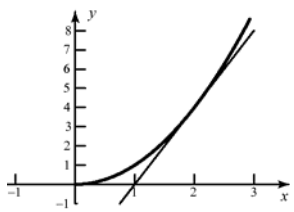
The percent increase in mass is about 9.5.

37. $f(x)=x^2;\ f'(x)=2x;\ a=2$

The linear approximation is then

$$L(x)=f(2)+f'(2)(x-2)$$
$$=4+4(x-2)=4x-4$$



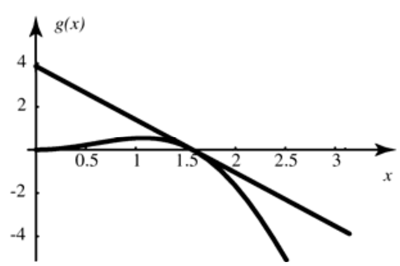


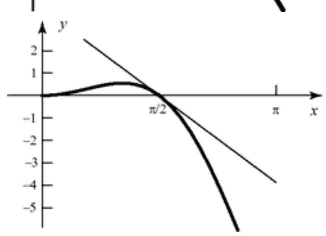
38. $g(x)=x^2\cos x;\ g'(x)=-x^2\sin x+2x\cos x$

$a=\pi/2$

The linear approximation is then

$$L(x)=0+-\left(\frac{\pi}{2}\right)^2\left(x-\frac{\pi}{2}\right)$$
$$=-\frac{\pi^2}{4}x+\frac{\pi^3}{8}$$
$$L(x)=0+-\frac{\pi^2}{4}\left(x-\frac{\pi}{2}\right)$$
$$=-\frac{\pi^2}{4}x+\frac{\pi^3}{8}$$

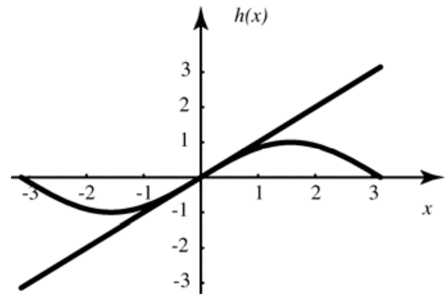


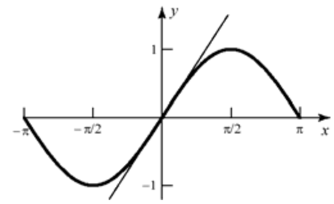


39. $h(x)=\sin x;\ h'(x)=\cos x;\ a=0$

The linear approximation is then

$L(x)=0+1(x-0)=x$

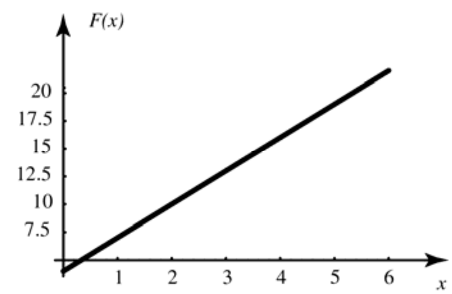


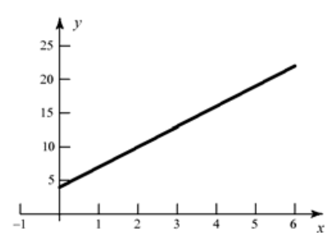


40. $F(x)=3x+4;\ F'(x)=3;\ a=3$

The linear approximation is then

$$L(x)=13+3(x-3)=13+3x-9$$
$$=3x+4$$





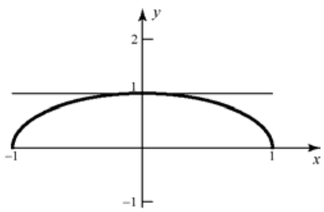
41. $f(x)=\sqrt{1-x^2}$;

$$f'(x)=\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)$$

$$=\frac{-x}{\sqrt{1-x^2}},\ \ a=0$$

The linear approximation is then

$L(x)=1+0(x-0)=1$

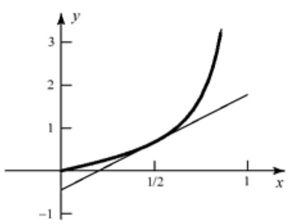


42. $g(x)=\dfrac{x}{1-x^2}$;

$$g'(x)=\frac{\left(1-x^2\right)-x(-2x)}{\left(1-x^2\right)^2}=\frac{1+x^2}{\left(1-x^2\right)^2},a=\frac{1}{2}$$

The linear approximation is then

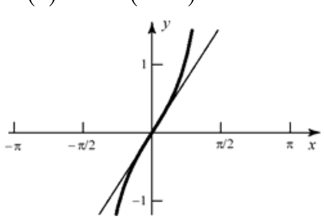
$$L(x)=\frac{2}{3}+\frac{20}{9}\left(x-\frac{1}{2}\right)=\frac{20}{9}x-\frac{4}{9}$$



43. $h(x)=x\sec x; h'(x)=\sec x+x\sec x\tan x, a=0$

The linear approximation is then

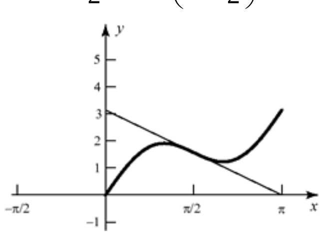
$L(x)=0+1(x-0)=x$



44. $G(x)=x+\sin 2x;\ \ G'(x)=1+2\cos 2x, a=\pi/2$

The linear approximation is then

$$L(x)=\frac{\pi}{2}+(-1)\left(x-\frac{\pi}{2}\right)=-x+\pi$$



45. $f(x)=mx+b;\ \ f'(x)=m$

The linear approximation is then

$L(x)=ma+b+m(x-a)=am+b+mx-ma$

$=mx+b\qquad\qquad f(x)=L(x)$

46. $L(x)-f(x)=\sqrt{a}+\dfrac{1}{2\sqrt{a}}(x-a)-\sqrt{x}$

$$=\frac{x}{2\sqrt{a}}-\sqrt{x}+\frac{\sqrt{a}}{2}=\frac{x-2\sqrt{a}\sqrt{x}+a}{2\sqrt{a}}$$

$$=\frac{\left(\sqrt{x}-\sqrt{a}\right)^2}{2\sqrt{a}}\ge 0$$

47. The linear approximation to $f(x)$ at a is

$L(x)=f(a)+f'(a)(x-a)$

$=a^2+2a(x-a)$

$=2ax-a^2$

Thus,

$f(x)-L(x)=x^2-\left(2ax-a^2\right)$

$=x^2-2ax+a^2$

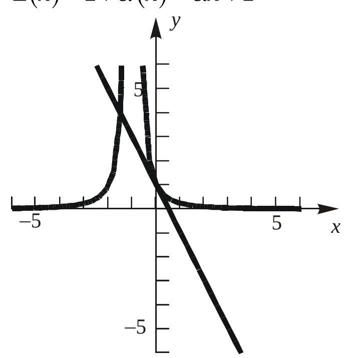
$=(x-a)^2$

≥ 0

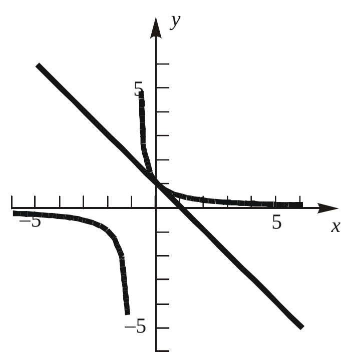
48. $f(x)=(1+x)^{\alpha},\ \ f'(x)=\alpha(1+x)^{\alpha-1},\ a=0$

The linear approximation is then

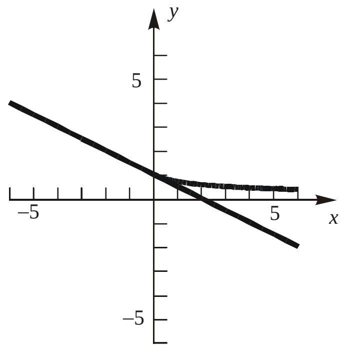
$L(x)=1+\alpha(x)=\alpha x+1$



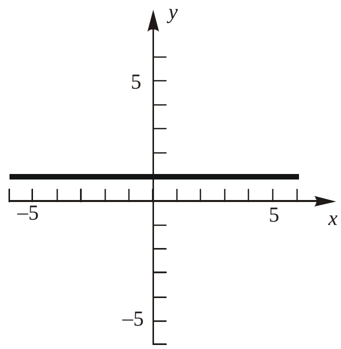
$\alpha=-2$



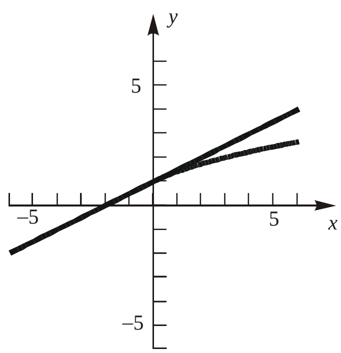
$\alpha=-1$



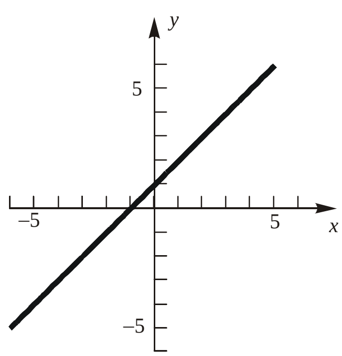
$\alpha = -0.5$



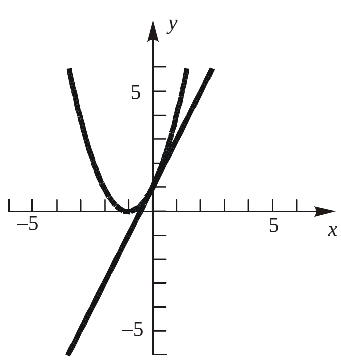
$\alpha = 0$



$\alpha = 0.5$



$\alpha = 1$



$\alpha = 2$

49. a. $\lim_{h\to 0} \varepsilon(h) = \lim_{h\to 0}\left(f(x+h) - f(x) - f'(x)h \right)$
$= f(x) - f(x) - f'(x)0 = 0$

b. $\lim_{h\to 0} \frac{\varepsilon(h)}{h} = \lim\left[\frac{f(x+h)-f(x)}{h} - f'(x) \right]$
$= f'(x) - f'(x) = 0$

2.10 Chapter Review

Concepts Test

1. False: If $f(x) = x^3$, $f'(x) = 3x^2$ and the tangent line $y = 0$ at $x = 0$ crosses the curve at the point of tangency.

2. False: The tangent line can touch the curve at infinitely many points.

3. True: $m_{\tan} = 4x^3$, which is unique for each value of x.

4. False: $m_{\tan} = -\sin x$, which is periodic.

5. True: If the velocity is negative and increasing, the speed is decreasing.

6. True: If the velocity is negative and decreasing, the speed is increasing.

7. True: If the tangent line is horizontal, the slope must be 0.

8. False: $f(x) = ax^2 + b$, $g(x) = ax^2 + c$, $b \neq c$. Then $f'(x) = 2ax = g'(x)$, but $f(x) \neq g(x)$.

9. True: $D_x f(g(x)) = f'(g(x))g'(x)$; since $g(x) = x$, $g'(x) = 1$, so $D_x f(g(x)) = f'(g(x))$.

10. False: $D_x y = 0$ because π is a constant, not a variable.

11. True: Theorem 3.2.A

12. True: The derivative does not exist when the tangent line is vertical.

13. False: $(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$

14. True: Negative acceleration indicates decreasing velocity.

15. True: If $f(x) = x^3 g(x)$, then

$$D_x f(x) = x^3 g'(x) + 3x^2 g(x)$$
$$= x^2[xg'(x) + 3g(x)].$$

16. False: $D_x y = 3x^2$; At (1, 1):

$m_{\tan} = 3(1)^2 = 3$

Tangent line: $y – 1 = 3(x – 1)$

17. False:

$$D_x y = f(x)g'(x) + g(x)f'(x)$$
$$D_x^2 y = f(x)g''(x) + g'(x)f'(x) + g(x)f''(x) + f'(x)g'(x)$$
$$= f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$$

18. True: The degree of $y = (x^3 + x)^8$ is 24, so $D_x^{25} y = 0.$

19. True: $f(x) = ax^n$; $f'(x) = anx^{n-1}$

20. True: $D_x \dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

21. True:

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$
$$h'(c) = f(c)g'(c) + g(c)f'(c)$$
$$= f(c)(0) + g(c)(0) = 0$$

22. True:

$$f'\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} \frac{\sin x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}}$$
$$= \lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{x - \frac{\pi}{2}}$$

23. True: $D^2(kf) = kD^2 f$ and $D^2(f + g) = D^2 f + D^2 g$

24. True:

$$h'(x) = f'(g(x)) \cdot g'(x)$$
$$h'(c) = f'(g(c)) \cdot g'(c) = 0$$

25. True:

$$(f \circ g)'(2) = f'(g(2)) \cdot g'(2)$$
$$= f'(2) \cdot g'(2) = 2 \cdot 2 = 4$$

26. False: Consider $f(x) = \sqrt{x}$. The curve always lies below the tangent.

27. False: The rate of volume change depends on the radius of the sphere.

28. True: $c = 2\pi r$; $\dfrac{dr}{dt} = 4$

$$\frac{dc}{dt} = 2\pi \frac{dr}{dt} = 2\pi(4) = 8\pi$$

29. True:

$$D_x(\sin x) = \cos x;$$
$$D_x^2(\sin x) = -\sin x;$$
$$D_x^3(\sin x) = -\cos x;$$
$$D_x^4(\sin x) = \sin x;$$
$$D_x^5(\sin x) = \cos x$$

30. False:

$$D_x(\cos x) = -\sin x;$$
$$D_x^2(\cos x) = -\cos x;$$
$$D_x^3(\cos x) = \sin x;$$
$$D_x^4(\cos x) = D_x[D_x^3(\cos x)] = D_x(\sin x)$$

Since $D_x^{1+3}(\cos x) = D_x^1(\sin x)$,

$D_x^{n+3}(\cos x) = D_x^n(\sin x).$

31. True:

$$\lim_{x\to 0} \frac{\tan x}{3x} = \frac{1}{3} \lim_{x\to 0} \frac{\sin x}{x\cos x}$$
$$= \frac{1}{3} \cdot 1 = \frac{1}{3}$$

32. True: $v = \dfrac{ds}{dt} = 15t^2 + 6$ which is greater than 0 for all t.

33. True:

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

If $\dfrac{dV}{dt} = 3$, then $\dfrac{dr}{dt} = \dfrac{3}{4\pi r^2}$ so $\dfrac{dr}{dt} > 0.$

$$\frac{d^2r}{dt^2} = -\frac{3}{2\pi r^3}\frac{dr}{dt} \text{ so } \frac{d^2r}{dt^2} < 0$$

34. True: When $h > r$, then $\dfrac{d^2h}{dt^2} > 0$

35. True:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$
$$dV = 4\pi r^2 dr = S \cdot dr$$

If $\Delta r = dr$, then $dV = S \cdot \Delta r$

36. False: $dy = 5x^4 dx$, so $dy > 0$ when $dx > 0$, but $dy < 0$ when $dx < 0$.

37. False: The slope of the linear approximation is equal to $f'(a) = f'(0) = -\sin(0) = 0$.

Sample Test Problems

1. **a.** $f'(x)=\lim_{h\to 0}\frac{3(x+h)^3-3x^3}{h}=\lim_{h\to 0}\frac{9x^2h+9xh^2+3h^3}{h}=\lim_{h\to 0}(9x^2+9xh+3h^2)=9x^2$

b. $f'(x)=\lim_{h\to 0}\frac{[2(x+h)^5+3(x+h)]-(2x^5+3x)}{h}=\lim_{h\to 0}\frac{10x^4h+20x^3h^2+20x^2h^3+10xh^4+2h^5+3h}{h}$

$=\lim_{h\to 0}(10x^4+20x^3h+20x^2h^2+10xh^3+2h^4+3)=10x^4+3$

c. $f'(x)=\lim_{h\to 0}\frac{\frac{1}{3(x+h)}-\frac{1}{3x}}{h}=\lim_{h\to 0}\left[-\frac{h}{3(x+h)x}\right]\frac{1}{h}=\lim_{h\to 0}-\left(\frac{1}{3x(x+h)}\right)=-\frac{1}{3x^2}$

d. $f'(x)=\lim_{h\to 0}\left[\left(\frac{1}{3(x+h)^2+2}-\frac{1}{3x^2+2}\right)\frac{1}{h}\right]=\lim_{h\to 0}\left[\frac{3x^2+2-3(x+h)^2-2}{(3(x+h)^2+2)(3x^2+2)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{-6xh-3h^2}{(3(x+h)^2+2)(3x^2+2)}\cdot\frac{1}{h}\right]=\lim_{h\to 0}\frac{-6x-3h}{(3(x+h)^2+2)(3x^2+2)}=-\frac{6x}{(3x^2+2)^2}$

e. $f'(x)=\lim_{h\to 0}\frac{\sqrt{3(x+h)}-\sqrt{3x}}{h}=\lim_{h\to 0}\frac{(\sqrt{3x+3h}-\sqrt{3x})(\sqrt{3x+3h}+\sqrt{3x})}{h(\sqrt{3x+3h}+\sqrt{3x})}$

$=\lim_{h\to 0}\frac{3h}{h(\sqrt{3x+3h}+\sqrt{3x})}=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h}+\sqrt{3x}}=\frac{3}{2\sqrt{3x}}$

f. $f'(x)=\lim_{h\to 0}\frac{\sin[3(x+h)]-\sin 3x}{h}=\lim_{h\to 0}\frac{\sin(3x+3h)-\sin 3x}{h}$

$=\lim_{h\to 0}\frac{\sin 3x\cos 3h+\sin 3h\cos 3x-\sin 3x}{h}=\lim_{h\to 0}\frac{\sin 3x(\cos 3h-1)}{h}+\lim_{h\to 0}\frac{\sin 3h\cos 3x}{h}$

$=3\sin 3x\lim_{h\to 0}\frac{\cos 3h-1}{3h}+\cos 3x\lim_{h\to 0}\frac{\sin 3h}{h}=(3\sin 3x)(0)+(\cos 3x)3\lim_{h\to 0}\frac{\sin 3h}{3h}=(\cos 3x)(3)(1)=3\cos 3x$

g. $f'(x)=\lim_{h\to 0}\frac{\sqrt{(x+h)^2+5}-\sqrt{x^2+5}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{(x+h)^2+5}-\sqrt{x^2+5}\right)\left(\sqrt{(x+h)^2+5}+\sqrt{x^2+5}\right)}{h\left(\sqrt{(x+h)^2+5}+\sqrt{x^2+5}\right)}$

$=\lim_{h\to 0}\frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+5}+\sqrt{x^2+5}\right)}=\lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+5}+\sqrt{x^2+5}}=\frac{2x}{2\sqrt{x^2+5}}=\frac{x}{\sqrt{x^2+5}}$

h. $f'(x)=\lim_{h\to 0}\frac{\cos[\pi(x+h)]-\cos\pi x}{h}=\lim_{h\to 0}\frac{\cos(\pi x+\pi h)-\cos\pi x}{h}=\lim_{h\to 0}\frac{\cos\pi x\cos\pi h-\sin\pi x\sin\pi h-\cos\pi x}{h}$

$=\lim_{h\to 0}\left(-\pi\cos\pi x\frac{1-\cos\pi h}{\pi h}\right)-\lim_{h\to 0}\left(\pi\sin\pi x\frac{\sin\pi h}{\pi h}\right)=(-\pi\cos\pi x)(0)-(\pi\sin\pi x)=-\pi\sin\pi x$

2. **a.** $g'(x)=\lim_{t\to x}\frac{2t^2-2x^2}{t-x}=\lim_{t\to x}\frac{2(t-x)(t+x)}{t-x}$

$=2\lim_{t\to x}(t+x)=2(2x)=4x$

b. $g'(x) = \lim_{t\to x} \frac{(t^3+t)-(x^3+x)}{t-x}$

$= \lim_{t\to x} \frac{(t-x)(t^2+tx+x^2)+(t-x)}{t-x}$

$= \lim_{t\to x}(t^2+tx+x^2+1) = 3x^2+1$

c. $g'(x) = \lim_{t\to x} \frac{\frac{1}{t}-\frac{1}{x}}{t-x} = \lim_{t\to x} \frac{x-t}{tx(t-x)}$

$= \lim_{t\to x} \frac{-1}{tx} = -\frac{1}{x^2}$

d. $g'(x) = \lim_{t\to x}\left[\left(\frac{1}{t^2+1}-\frac{1}{x^2+1}\right)\left(\frac{1}{t-x}\right)\right]$

$= \lim_{t\to x} \frac{x^2-t^2}{(t^2+1)(x^2+1)(t-x)}$

$= \lim_{t\to x} \frac{-(x+t)(t-x)}{(t^2+1)(x^2+1)(t-x)}$

$= \lim_{t\to x} \frac{-(x+t)}{(t^2+1)(x^2+1)} = -\frac{2x}{(x^2+1)^2}$

e. $g'(x) = \lim_{t\to x} \frac{\sqrt{t}-\sqrt{x}}{t-x}$

$= \lim_{t\to x} \frac{(\sqrt{t}-\sqrt{x})(\sqrt{t}+\sqrt{x})}{(t-x)(\sqrt{t}+\sqrt{x})}$

$= \lim_{t\to x} \frac{t-x}{(t-x)(\sqrt{t}+\sqrt{x})} = \lim_{t\to x} \frac{1}{\sqrt{t}+\sqrt{x}}$

$= \frac{1}{2\sqrt{x}}$

f. $g'(x) = \lim_{t\to x} \frac{\sin \pi t - \sin \pi x}{t-x}$

Let $v = t - x$, then $t = v + x$ and as $t \to x, v \to 0$.

$\lim_{t\to x} \frac{\sin \pi t - \sin \pi x}{t-x} = \lim_{v\to 0} \frac{\sin \pi(v+x) - \sin \pi x}{v}$

$= \lim_{v\to 0} \frac{\sin \pi v \cos \pi x + \sin \pi x \cos \pi v - \sin \pi x}{v}$

$= \lim_{v\to 0}\left[\pi \cos \pi x \frac{\sin \pi v}{\pi v} + \pi \sin \pi x \frac{\cos \pi v - 1}{\pi v}\right]$

$= \pi \cos \pi x \cdot 1 + \pi \sin \pi x \cdot 0 = \pi \cos \pi x$

Other method:

Use the subtraction formula

$\sin \pi t - \sin \pi x = 2\cos\frac{\pi(t+x)}{2}\sin\frac{\pi(t-x)}{2}$

g. $g'(x) = \lim_{t\to x} \frac{\sqrt{t^3+C}-\sqrt{x^3+C}}{t-x}$

$= \lim_{t\to x} \frac{\left(\sqrt{t^3+C}-\sqrt{x^3+C}\right)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}{(t-x)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}$

$= \lim_{t\to x} \frac{t^3-x^3}{(t-x)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}$

$= \lim_{t\to x} \frac{t^2+tx+x^2}{\sqrt{t^3+C}+\sqrt{x^3+C}} = \frac{3x^2}{2\sqrt{x^3+C}}$

h. $g'(x) = \lim_{t\to x} \frac{\cos 2t - \cos 2x}{t-x}$

Let $v = t - x$, then $t = v + x$ and as $t \to x, v \to 0$.

$\lim_{t\to x} \frac{\cos 2t - \cos 2x}{t-x} = \lim_{v\to 0} \frac{\cos 2(v+x) - \cos 2x}{v}$

$= \lim_{v\to 0} \frac{\cos 2v \cos 2x - \sin 2v \sin 2x - \cos 2x}{v}$

$= \lim_{v\to 0}\left[2\cos 2x \frac{\cos 2v - 1}{2v} - 2\sin 2x \frac{\sin 2v}{2v}\right]$

$= 2\cos 2x \cdot 0 - 2\sin 2x \cdot 1 = -2\sin 2x$

Other method:

Use the subtraction formula

$\cos 2t - \cos 2x = -2\sin(t+x)\sin(t-x)$.

3. a. $f(x) = 3x$ at $x = 1$

b. $f(x) = 4x^3$ at $x = 2$

c. $f(x) = \sqrt{x^3}$ at $x = 1$

d. $f(x) = \sin x$ at $x = \pi$

e. $f(x) = \frac{4}{x}$ at x

f. $f(x) = -\sin 3x$ at x

g. $f(x) = \tan x$ at $x = \frac{\pi}{4}$

h. $f(x) = \frac{1}{\sqrt{x}}$ at $x = 5$

4. a. $f'(2) \approx -\frac{3}{4}$

b. $f'(6) \approx \frac{3}{2}$

c. $V_{\text{avg}} = \dfrac{6-\frac{3}{2}}{7-3} = \dfrac{9}{8}$

d. $\dfrac{d}{dt} f(t^2) = f'(t^2)(2t)$

At $t = 2$, $4f'(4) \approx 4\left(\dfrac{2}{3}\right) = \dfrac{8}{3}$

e. $\dfrac{d}{dt}[f^2(t)] = 2f(t)f'(t)$

At $t = 2$,

$2f(2)f'(2) \approx 2(2)\left(-\dfrac{3}{4}\right) = -3$

f. $\dfrac{d}{dt}(f(f(t))) = f'(f(t))f'(t)$

At $t = 2$, $f'(f(2))f'(2) = f'(2)f'(2)$

$\approx \left(-\dfrac{3}{4}\right)\left(-\dfrac{3}{4}\right) = \dfrac{9}{16}$

5. $D_x(3x^5) = 15x^4$

6. $D_x(x^3 - 3x^2 + x^{-2}) = 3x^2 - 6x + (-2)x^{-3}$

$= 3x^2 - 6x - 2x^{-3}$

7. $D_z(z^3 + 4z^2 + 2z) = 3z^2 + 8z + 2$

8. $D_x\left(\dfrac{3x-5}{x^2+1}\right) = \dfrac{(x^2+1)(3)-(3x-5)(2x)}{(x^2+1)^2}$

$= \dfrac{-3x^2+10x+3}{(x^2+1)^2}$

9. $D_t\left(\dfrac{4t-5}{6t^2+2t}\right) = \dfrac{(6t^2+2t)(4)-(4t-5)(12t+2)}{(6t^2+2t)^2}$

$= \dfrac{-24t^2+60t+10}{(6t^2+2t)^2}$

10. $D_x(3x+2)^{2/3} = \dfrac{2}{3}(3x+2)^{-1/3}(3)$

$= 2(3x+2)^{-1/3}$

$D_x^2(3x+2)^{2/3} = -\dfrac{2}{3}(3x+2)^{-4/3}(3)$

$= -2(3x+2)^{-4/3}$

11. $\dfrac{d}{dx}\left(\dfrac{4x^2-2}{x^3+x}\right) = \dfrac{(x^3+x)(8x)-(4x^2-2)(3x^2+1)}{(x^3+x)^2}$

$= \dfrac{-4x^4+10x^2+2}{(x^3+x)^2}$

12. $D_t(t\sqrt{2t+6}) = t\dfrac{1}{2\sqrt{2t+6}}(2) + \sqrt{2t+6}$

$= \dfrac{t}{\sqrt{2t+6}} + \sqrt{2t+6}$

13. $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x^2+4}}\right) = \dfrac{d}{dx}(x^2+4)^{-1/2}$

$= -\dfrac{1}{2}(x^2+4)^{-3/2}(2x)$

$= -\dfrac{x}{\sqrt{(x^2+4)^3}}$

14. $\dfrac{d}{dx}\sqrt{\dfrac{x^2-1}{x^3-x}} = \dfrac{d}{dx}\dfrac{1}{\sqrt{x}} = \dfrac{d}{dx}x^{-1/2} = -\dfrac{1}{2x^{3/2}}$

15. $D_\theta(\sin\theta + \cos^3\theta) = \cos\theta + 3\cos^2\theta(-\sin\theta)$

$= \cos\theta - 3\sin\theta\cos^2\theta$

$D_\theta^2(\sin\theta + \cos^3\theta)$

$= -\sin\theta - 3[\sin\theta(2)(\cos\theta)(-\sin\theta) + \cos^3\theta]$

$= -\sin\theta + 6\sin^2\theta\cos\theta - 3\cos^3\theta$

16. $\dfrac{d}{dt}[\sin(t^2) - \sin^2(t)] = \cos(t^2)(2t) - (2\sin t)(\cos t)$

$= 2t\cos(t^2) - \sin(2t)$

17. $D_\theta[\sin(\theta^2)] = \cos(\theta^2)(2\theta) = 2\theta\cos(\theta^2)$

18. $\dfrac{d}{dx}(\cos^3 5x) = (3\cos^2 5x)(-\sin 5x)(5)$

$= -15\cos^2 5x\sin 5x$

19. $\frac{d}{d\theta}[\sin^2(\sin(\pi\theta))] = 2\sin(\sin(\pi\theta))\cos(\sin(\pi\theta))(\cos(\pi\theta))(\pi) = 2\pi\sin(\sin(\pi\theta))\cos(\sin(\pi\theta))\cos(\pi\theta)$

20. $\frac{d}{dt}[\sin^2(\cos 4t)] = 2\sin(\cos 4t)\big(\cos(\cos 4t)\big)(-\sin 4t)(4) = -8\sin(\cos 4t)\cos(\cos 4t)\sin 4t$

21. $D_\theta \tan 3\theta = (\sec^2 3\theta)(3) = 3\sec^2 3\theta$

22. $\frac{d}{dx}\left(\frac{\sin 3x}{\cos 5x^2}\right) = \frac{(\cos 5x^2)(\cos 3x)(3) - (\sin 3x)(-\sin 5x^2)(10x)}{\cos^2 5x^2} = \frac{3\cos 5x^2 \cos 3x + 10x\sin 3x \sin 5x^2}{\cos^2 5x^2}$

23. $f'(x) = (x^2-1)^2(9x^2-4) + (3x^3-4x)(2)(x^2-1)(2x) = (x^2-1)^2(9x^2-4) + 4x(x^2-1)(3x^3-4x)$

$f'(2) = 672$

24. $g'(x) = 3\cos 3x + 2(\sin 3x)(\cos 3x)(3) = 3\cos 3x + 3\sin 6x$

$g''(x) = -9\sin 3x + 18\cos 6x$

$g''(0) = 18$

25. $\frac{d}{dx}\left(\frac{\cot x}{\sec x^2}\right) = \frac{(\sec x^2)(-\csc^2 x) - (\cot x)(\sec x^2)(\tan x^2)(2x)}{\sec^2 x^2} = \frac{-\csc^2 x - 2x\cot x\tan x^2}{\sec x^2}$

26. $D_t\left(\frac{4t\sin t}{\cos t - \sin t}\right) = \frac{(\cos t - \sin t)(4t\cos t + 4\sin t) - (4t\sin t)(-\sin t - \cos t)}{(\cos t - \sin t)^2}$

$= \frac{4t\cos^2 t + 2\sin 2t - 4\sin^2 t + 4t\sin^2 t}{(\cos t - \sin t)^2} = \frac{4t + 2\sin 2t - 4\sin^2 t}{(\cos t - \sin t)^2}$

27. $f'(x) = (x-1)^3 2(\sin \pi x - x)(\pi\cos \pi x - 1) + (\sin \pi x - x)^2 3(x-1)^2$

$= 2(x-1)^3(\sin \pi x - x)(\pi\cos \pi x - 1) + 3(\sin \pi x - x)^2(x-1)^2$

$f'(2) = 16 - 4\pi \approx 3.43$

28. $h'(t) = 5(\sin(2t) + \cos(3t))^4(2\cos(2t) - 3\sin(3t))$

$h''(t) = 5(\sin(2t) + \cos(3t))^4(-4\sin(2t) - 9\cos(3t)) + 20(\sin(2t) + \cos(3t))^3(2\cos(2t) - 3\sin(3t))^2$

$h''(0) = 5\cdot 1^4 \cdot (-9) + 20\cdot 1^3 \cdot 2^2 = 35$

29. $g'(r) = 3(\cos^2 5r)(-\sin 5r)(5) = -15\cos^2 5r\sin 5r$

$g''(r) = -15[(\cos^2 5r)(\cos 5r)(5) + (\sin 5r)2(\cos 5r)(-\sin 5r)(5)] = -15[5\cos^3 5r - 10(\sin^2 5r)(\cos 5r)]$

$g'''(r) = -15[5(3)(\cos^2 5r)(-\sin 5r)(5) - (10\sin^2 5r)(-\sin 5r)(5) - (\cos 5r)(20\sin 5r)(\cos 5r)(5)]$

$= -15[-175(\cos^2 5r)(\sin 5r) + 50\sin^3 5r]$

$g'''(1) \approx 458.8$

30. $f'(t) = h'(g(t))g'(t) + 2g(t)g'(t)$

31. $G'(x) = F'(r(x) + s(x))(r'(x) + s'(x)) + s'(x)$

$G''(x) = F'(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))F''(r(x) + s(x))(r'(x) + s'(x)) + s''(x)$

$= F'(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))^2 F''(r(x) + s(x)) + s''(x)$

32. $F'(x) = Q'(R(x))R'(x) = 3[R(x)]^2(-\sin x)$
$= -3\cos^2 x \sin x$

33. $F'(z) = r'(s(z))s'(z) = [3\cos(3s(z))](9z^2)$
$= 27z^2 \cos(9z^3)$

34. $\frac{dy}{dx} = 2(x-2)$
$2x - y + 2 = 0$; $y = 2x + 2$; $m = 2$
$2(x-2) = -\frac{1}{2}$
$x = \frac{7}{4}$
$y = \left(\frac{7}{4} - 2\right)^2 = \frac{1}{16}; \left(\frac{7}{4}, \frac{1}{16}\right)$

35. $V = \frac{4}{3}\pi r^3$
$\frac{dV}{dr} = 4\pi r^2$
When $r = 5$, $\frac{dV}{dr} = 4\pi(5)^2 = 100\pi \approx 314$ m^3 per meter of increase in the radius.

36. $V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 10$
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
When $r = 5$, $10 = 4\pi(5)^2 \frac{dr}{dt}$
$\frac{dr}{dt} = \frac{1}{10\pi} \approx 0.0318$ m/h

37. $V = \frac{1}{2}bh(12); \frac{6}{4} = \frac{b}{h}; b = \frac{3h}{2}$
$V = 6\left(\frac{3h}{2}\right)h = 9h^2; \frac{dV}{dt} = 9$
$\frac{dV}{dt} = 18h\frac{dh}{dt}$
When $h = 3$, $9 = 18(3)\frac{dh}{dt}$
$\frac{dh}{dt} = \frac{1}{6} \approx 0.167$ ft/min

38. a. $v = 128 - 32t$
$v = 0$, when $t = 4s$
$s = 128(4) - 16(4)^2 = 256$ ft

b. $128t - 16t^2 = 0$
$-16t(t-8) = 0$
The object hits the ground when $t = 8$s
$v = 128 - 32(8) = -128$ ft/s

39. $s = t^3 - 6t^2 + 9t$
$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$
$a(t) = \frac{d^2s}{dt^2} = 6t - 12$

a. $3t^2 - 12t + 9 < 0$
$3(t-3)(t-1) < 0$
$1 < t < 3$; $(1, 3)$

b. $3t^2 - 12t + 9 = 0$
$3(t-3)(t-1) = 0$
$t = 1, 3$
$a(1) = -6$, $a(3) = 6$

c. $6t - 12 > 0$
$t > 2$; $(2, \infty)$

40. a. $D_x^{20}(x^{19} + x^{12} + x^5 + 100) = 0$

b. $D_x^{20}(x^{20} + x^{19} + x^{18}) = 20!$

c. $D_x^{20}(7x^{21} + 3x^{20}) = (7 \cdot 21!)x + (3 \cdot 20!)$

d. $D_x^{20}(\sin x + \cos x) = D_x^4(\sin x + \cos x)$
$= \sin x + \cos x$

e. $D_x^{20}(\sin 2x) = 2^{20}\sin 2x$
$= 1{,}048{,}576 \sin 2x$

f. $D_x^{20}\left(\frac{1}{x}\right) = \frac{(-1)^{20}(20!)}{x^{21}} = \frac{20!}{x^{21}}$

41. a. $2(x-1) + 2y\frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{-(x-1)}{y} = \frac{1-x}{y}$

b. $x(2y)\frac{dy}{dx} + y^2 + y(2x) + x^2\frac{dy}{dx} = 0$
$\frac{dy}{dx}(2xy + x^2) = -(y^2 + 2xy)$
$\frac{dy}{dx} = -\frac{y^2 + 2xy}{x^2 + 2xy}$

c. $3x^2 + 3y^2 \frac{dy}{dx} = x^3(3y^2)\frac{dy}{dx} + 3x^2y^3$

$\frac{dy}{dx}(3y^2 - 3x^3y^2) = 3x^2y^3 - 3x^2$

$\frac{dy}{dx} = \frac{3x^2y^3 - 3x^2}{3y^2 - 3x^3y^2} = \frac{x^2y^3 - x^2}{y^2 - x^3y^2}$

d. $x\cos(xy)\left[x\frac{dy}{dx} + y\right] + \sin(xy) = 2x$

$x^2\cos(xy)\frac{dy}{dx} = 2x - \sin(xy) - xy\cos(xy)$

$\frac{dy}{dx} = \frac{2x - \sin(xy) - xy\cos(xy)}{x^2\cos(xy)}$

e. $x\sec^2(xy)\left(x\frac{dy}{dx} + y\right) + \tan(xy) = 0$

$x^2\sec^2(xy)\frac{dy}{dx} = -[\tan(xy) + xy\sec^2(xy)]$

$\frac{dy}{dx} = -\frac{\tan(xy) + xy\sec^2(xy)}{x^2\sec^2(xy)}$

42. $2yy_1' = 12x^2$

$y_1' = \frac{6x^2}{y}$

At (1, 2): $y_1' = 3$

$4x + 6yy_2' = 0$

$y_2' = -\frac{2x}{3y}$

At (1, 2): $y_2' = -\frac{1}{3}$

Since $(y_1')(y_2') = -1$ at (1, 2), the tangents are perpendicular.

43. $dy = [\pi\cos(\pi x) + 2x]dx$; $x = 2$, $dx = 0.01$

$dy = [\pi\cos(2\pi) + 2(2)](0.01) = (4 + \pi)(0.01)$

≈ 0.0714

44. $x(2y)\frac{dy}{dx} + y^2 + 2y[2(x+2)] + (x+2)^2(2)\frac{dy}{dx} = 0$

$\frac{dy}{dx}[2xy + 2(x+2)^2] = -[y^2 + 2y(2x+4)]$

$\frac{dy}{dx} = \frac{-(y^2 + 4xy + 8y)}{2xy + 2(x+2)^2}$

$dy = -\frac{y^2 + 4xy + 8y}{2xy + 2(x+2)^2}dx$

When $x = -2$, $y = \pm 1$

a. $dy = -\frac{(1)^2 + 4(-2)(1) + 8(1)}{2(-2)(1) + 2(-2+2)^2}(-0.01)$

$= -0.0025$

b. $dy = -\frac{(-1)^2 + 4(-2)(-1) + 8(-1)}{2(-2)(-1) + 2(-2+2)^2}(-0.01)$

$= 0.0025$

45. a. $\frac{d}{dx}[f^2(x) + g^3(x)]$

$= 2f(x)f'(x) + 3g^2(x)g'(x)$

$2f(2)f'(2) + 3g^2(2)g'(2)$

$= 2(3)(4) + 3(2)^2(5) = 84$

b. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

$f(2)g'(2) + g(2)f'(2) = (3)(5) + (2)(4) = 23$

c. $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

$f'(g(2))g'(2) = f'(2)g'(2) = (4)(5) = 20$

d. $D_x[f^2(x)] = 2f(x)f'(x)$

$D_x^2[f^2(x)] = 2[f(x)f''(x) + f'(x)f'(x)]$

$= 2f(2)f''(2) + 2[f'(2)]^2$

$= 2(3)(-1) + 2(4)^2 = 26$

46. $(13)^2 = x^2 + y^2$; $\frac{dx}{dt} = 2$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

When $y = 5$, $x = 12$, so

$\frac{dy}{dt} = -\frac{12}{5}(2) = -\frac{24}{5} = -4.8$ ft/s

47. $\sin 15° = \frac{y}{x}$, $\frac{dx}{dt} = 400$

$y = x\sin 15°$

$\frac{dy}{dt} = \sin 15° \frac{dx}{dt}$

$\frac{dy}{dt} = 400\sin 15° \approx 104$ mi/hr

48. a. $D_x(|x|^2) = 2|x| \cdot \frac{|x|}{x} = \frac{2(|x|^2)}{x} = \frac{2x^2}{x} = 2x$

b. $D_x^2|x| = D_x\left(\frac{|x|}{x}\right) = \frac{x\left(\frac{|x|}{x}\right) - |x|}{x^2} = \frac{|x| - |x|}{x^2} = 0$

c. $D_x^3|x| = D_x(D_x^2|x|) = D_x(0) = 0$

d. $D_x^2(|x|^2) = D_x(2x) = 2$

49. a. $D_\theta|\sin\theta| = \dfrac{|\sin\theta|}{\sin\theta}\cos\theta = \cot\theta|\sin\theta|$

b. $D_\theta|\cos\theta| = \dfrac{|\cos\theta|}{\cos\theta}(-\sin\theta) = -\tan\theta|\cos\theta|$

50. a. $f(x) = \sqrt{x+1};\ f'(x) = -\dfrac{1}{2}(x+1)^{-1/2};\ a = 3$

$$
\begin{aligned}
L(x) &= f(3) + f'(3)(x-3) \\
&= \sqrt{4} + -\frac{1}{2}(4)^{-1/2}(x-3) \\
&= 2 - \frac{1}{4}x + \frac{3}{4} = -\frac{1}{4}x + \frac{11}{4}
\end{aligned}
$$

b. $f(x) = x\cos x;\ f'(x) = -x\sin x + \cos x;\ a = 1$

$$
\begin{aligned}
L(x) &= f(1) + f'(1)(x-1) \\
&= \cos 1 + (-\sin 1 + \cos 1)(x-1) \\
&= \cos 1 - (\sin 1)x + \sin 1 + (\cos 1)x - \cos 1 \\
&= (\cos 1 - \sin 1)x + \sin 1 \\
&\approx -0.3012x + 0.8415
\end{aligned}
$$

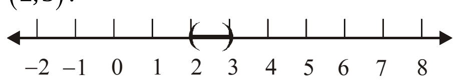
Review and Preview Problems

1. $(x-2)(x-3) < 0$

$(x-2)(x-3) = 0$

$x = 2$ or $x = 3$

The split points are 2 and 3. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty, 2)$, $(2,3)$, and $(3,\infty)$. The solution set is $\{x \mid 2 < x < 3\}$ or $(2,3)$.



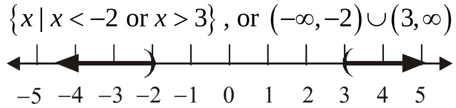
2. $x^2 - x - 6 > 0$

$(x-3)(x+2) > 0$

$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$

The split points are 3 and -2. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty,-2)$, $(-2,3)$, and $(3,\infty)$. The solution set is $\{x \mid x < -2 \text{ or } x > 3\}$, or $(-\infty,-2)\cup(3,\infty)$.

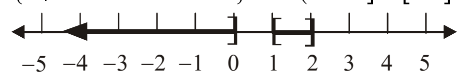


3. $x(x-1)(x-2) \le 0$

$x(x-1)(x-2) = 0$

$x = 0, x = 1$ or $x = 2$

The split points are 0, 1, and 2. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty,0)$, $(0,1)$, $(1,2)$, and $(2,\infty)$. The solution set is $\{x \mid x \le 0 \text{ or } 1 \le x \le 2\}$, or $(-\infty,0]\cup[1,2]$.



4. $x^3 + 3x^2 + 2x \ge 0$

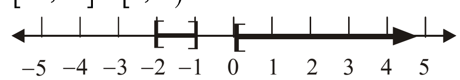
$x(x^2 + 3x + 2) \ge 0$

$x(x+1)(x+2) \ge 0$

$x(x+1)(x+2) = 0$

$x = 0, x = -1, x = -2$

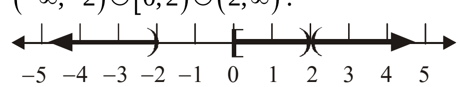
The split points are 0, -1, and -2. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty,-2)$, $(-2,-1)$, $(-1,0)$, and $(0,\infty)$. The solution set is $\{x \mid -2 \le x \le -1 \text{ or } x \ge 0\}$, or $[-2,-1]\cup[0,\infty)$.



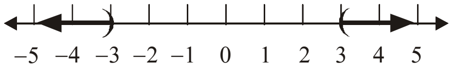
5. $\dfrac{x(x-2)}{x^2-4} \ge 0$

$\dfrac{x(x-2)}{(x-2)(x+2)} \ge 0$

The expression on the left is equal to 0 or undefined at $x = 0$, $x = 2$, and $x = -2$. These are the split points. The expression on the left can only change signs at the split points. Check a point in the intervals: $(-\infty,-2)$, $(-2,0)$, $(0,2)$, and $(2,\infty)$. The solution set is $\{x \mid x < -2 \text{ or } 0 \le x < 2 \text{ or } x > 2\}$, or $(-\infty,-2)\cup[0,2)\cup(2,\infty)$.



6. $$\frac{x^2-9}{x^2+2}>0$$
$$\frac{(x-3)(x+3)}{x^2+2}>0$$
The expression on the left is equal to 0 at $x=3$, and $x=-3$. These are the split points. The expression on the left can only change signs at the split points. Check a point in the intervals: $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$. The solution set is $\{x \mid x<-3 \text{ or } x>3\}$, or $(-\infty,-3)\cup(3,\infty)$.



7. $f'(x)=4(2x+1)^3(2)=8(2x+1)^3$

8. $f'(x)=\cos(\pi x)\cdot\pi=\pi\cos(\pi x)$

9. $f'(x)=\left(x^2-1\right)\cdot -\sin(2x)\cdot 2+\cos(2x)\cdot(2x)$
$=-2\left(x^2-1\right)\sin(2x)+2x\cos(2x)$

10. $f'(x)=\dfrac{x\cdot\sec x\tan x-\sec x\cdot 1}{x^2}$
$=\dfrac{\sec x(x\tan x-1)}{x^2}$

11. $f'(x)=2(\tan 3x)\cdot\sec^2 3x\cdot 3$
$=6\left(\sec^2 3x\right)(\tan 3x)$

12. $f'(x)=\dfrac{1}{2}\left(1+\sin^2 x\right)^{-1/2}(2\sin x)(\cos x)$
$=\dfrac{\sin x\cos x}{\sqrt{1+\sin^2 x}}$

13. $f'(x)=\cos\left(\sqrt{x}\right)\cdot\dfrac{1}{2}x^{-1/2}=\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$

(note: you cannot cancel the $\sqrt{x}$ here because it is not a factor of both the numerator and denominator. It is the argument for the cosine in the numerator.)

14. $f'(x)=\dfrac{1}{2}(\sin 2x)^{-1/2}\cdot\cos 2x\cdot 2=\dfrac{\cos 2x}{\sqrt{\sin 2x}}$

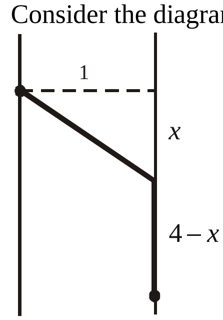
15. The tangent line is horizontal when the derivative is 0.
$y'=2\tan x\cdot\sec^2 x$
$2\tan x\sec x=0$
$$\frac{2\sin x}{\cos^2 x}=0$$
The tangent line is horizontal whenever $\sin x=0$. That is, for $x=k\pi$ where k is an integer.

16. The tangent line is horizontal when the derivative is 0.
$y'=1+\cos x$
The tangent line is horizontal whenever $\cos x=-1$. That is, for $x=(2k+1)\pi$ where k is an integer.

17. The line $y=2+x$ has slope 1, so any line parallel to this line will also have a slope of 1.
For the tangent line to $y=x+\sin x$ to be parallel to the given line, we need its derivative to equal 1.
$y'=1+\cos x=1$
$\cos x=0$
The tangent line will be parallel to $y=2+x$ whenever $x=(2k+1)\dfrac{\pi}{2}$.

18. Length: $24-2x$
Width: $9-2x$
Height: x
Volume: $l\cdot w\cdot h=(24-2x)(9-2x)x$
$=x(9-2x)(24-2x)$

19. Consider the diagram:



His distance swimming will be $\sqrt{1^2+x^2}=\sqrt{x^2+1}$ kilometers. His distance running will be $4-x$ kilometers.
Using the distance traveled formula, $d=r\cdot t$, we solve for t to get $t=\dfrac{d}{r}$. Andy can swim at 4 kilometers per hour and run 10 kilometers per hour. Therefore, the time to get from A to D will be $\dfrac{\sqrt{x^2+1}}{4}+\dfrac{4-x}{10}$ hours.

20. **a.** $f(0) = 0 - \cos(0) = 0 - 1 = -1$

$f(\pi) = \pi - \cos(\pi) = \pi - (-1) = \pi + 1$

Since $x - \cos x$ is continuous, $f(0) < 0$, and $f(\pi) > 0$, there is at least one point c .in the interval $(0, \pi)$ where $f(c) = 0$. (Intermediate Value Theorem)

b. $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

$f'(x) = 1 + \sin x$

$f'\left(\frac{\pi}{2}\right) = 1 + \sin\left(\frac{\pi}{2}\right) = 1 + 1 = 2$

The slope of the tangent line is $m = 2$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore,

$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{2}\right)$ or $y = 2x - \frac{\pi}{2}$.

c. $2x - \frac{\pi}{2} = 0$.

$2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

The tangent line will intersect the x-axis at $x = \frac{\pi}{4}$.

21. **a.** The derivative of x^2 is $2x$ and the derivative of a constant is 0. Therefore, one possible function is $f(x) = x^2 + 3$.

b. The derivative of $-\cos x$ is $\sin x$ and the derivative of a constant is 0. Therefore, one possible function is $f(x) = -(\cos x) + 8$.

c. The derivative of x^3 is $3x^2$, so the derivative of $\frac{1}{3}x^3$ is x^2. The derivative of x^2 is $2x$, so the derivative of $\frac{1}{2}x^2$ is x. The derivative of x is 1, and the derivative of a constant is 0. Therefore, one possible function is $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 2$.

22. Yes. Adding 1 only changes the constant term in the function and the derivative of a constant is 0. Therefore, we would get the same derivative regardless of the value of the constant.

CHAPTER 3 Applications of the Derivative

3.1 Concepts Review

1. continuous; closed and bounded

2. extreme

3. endpoints; stationary points; singular points

4. $f'(c) = 0$; $f'(c)$ does not exist

Problem Set 3.1

1. Endpoints: -2, 4
Singular points: none
Stationary points: 0, 2
Critical points: $-2, 0, 2, 4$

2. Endpoints: -2, 4
Singular points: 2
Stationary points: 0
Critical points: $-2, 0, 2, 4$

3. Endpoints: -2, 4
Singular points: none
Stationary points: $-1, 0, 1, 2, 3$
Critical points: $-2, -1, 0, 1, 2, 3, 4$

4. Endpoints: -2, 4
Singular points: none
Stationary points: none
Critical points: $-2, 4$

5. $f'(x) = 2x + 4; 2x + 4 = 0$ when $x = -2$.
Critical points: $-4, -2, 0$
$f(-4) = 4, f(-2) = 0, f(0) = 4$
Maximum value = 4, minimum value = 0

6. $h'(x) = 2x + 1; 2x + 1 = 0$ when $x = -\frac{1}{2}$.

Critical points: $-2, -\frac{1}{2}, 2$

$h(-2) = 2,\ h\left(-\frac{1}{2}\right) = -\frac{1}{4},\ h(2) = 6$

Maximum value = 6, minimum value = $-\frac{1}{4}$

7. $\Psi'(x) = 2x + 3;\ 2x + 3 = 0$ when $x = -\frac{3}{2}$.

Critical points: $-2, -\frac{3}{2}, 1$

$\Psi(-2) = -2, \Psi\left(-\frac{3}{2}\right) = -\frac{9}{4}, \Psi(1) = 4$

Maximum value = 4, minimum value = $-\frac{9}{4}$

8. $G'(x) = \frac{1}{5}(6x^2 + 6x - 12) = \frac{6}{5}(x^2 + x - 2);$

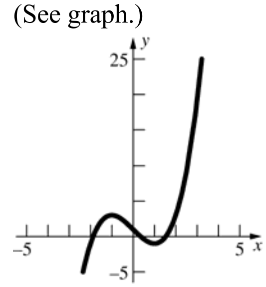
$x^2 + x - 2 = 0$ when $x = -2, 1$
Critical points: $-3, -2, 1, 3$

$G(-3) = \frac{9}{5}, G(-2) = 4, G(1) = -\frac{7}{5}, G(3) = 9$

Maximum value = 9,
minimum value $= -\frac{7}{5}$

9. $f'(x) = 3x^2 - 3; 3x^2 - 3 = 0$ when $x = -1, 1$.
Critical points: $-1, 1$
$f(-1) = 3, f(1) = -1$
No maximum value, minimum value = -1

(See graph.)



10. $f'(x) = 3x^2 - 3; 3x^2 - 3 = 0$ when $x = -1, 1$.

Critical points: $-\frac{3}{2}, -1, 1, 3$

$f\left(-\frac{3}{2}\right) = \frac{17}{8},\ f(-1) = 3,\ f(1) = -1,\ f(3) = 19$

Maximum value = 19, minimum value = -1

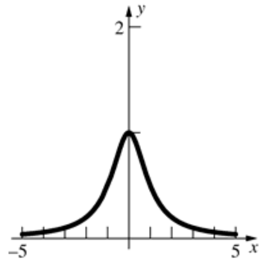
11. $h'(r) = -\frac{1}{r^2}$; $h'(r)$ is never 0; $h'(r)$ is not defined when $r = 0$, but $r = 0$ is not in the domain on $[-1, 3]$ since $h(0)$ is not defined.
Critical points: –1, 3
Note that $\lim_{x\to 0^-} h(r) = -\infty$ and $\lim_{x\to 0^+} h(x) = \infty$.
No maximum value, no minimum value.

12. $g'(x) = -\frac{2x}{(1+x^2)^2}$; $-\frac{2x}{(1+x^2)^2} = 0$ when $x = 0$
Critical points: –3, 0, 1
$g(-3) = \frac{1}{10}$, $g(0) = 1$, $g(1) = \frac{1}{2}$
Maximum value = 1, minimum value = $\frac{1}{10}$

13.
$$\begin{aligned} f'(x) &= 4x^3 - 4x \\ &= 4x\left(x^2 - 1\right) \\ &= 4x(x-1)(x+1) \end{aligned}$$
$4x(x-1)(x+1) = 0$ when $x = 0, 1, -1$.
Critical points: $-2, -1, 0, 1, 2$
$f(-2) = 10$; $f(-1) = 1$; $f(0) = 2$; $f(1) = 1$; $f(2) = 10$
Maximum value: 10
Minimum value: 1

14.
$$\begin{aligned} f'(x) &= 5x^4 - 25x^2 + 20 \\ &= 5\left(x^4 - 5x^2 + 4\right) \\ &= 5\left(x^2 - 4\right)\left(x^2 - 1\right) \\ &= 5(x-2)(x+2)(x-1)(x+1) \end{aligned}$$
$5(x-2)(x+2)(x-1)(x+1) = 0$ when $x = -2, -1, 1, 2$
Critical points: $-3, -2, -1, 1, 2$
$f(-3) = -79$; $f(-2) = -\frac{19}{3}$; $f(-1) = -\frac{41}{3}$;
$f(1) = \frac{35}{3}$; $f(2) = \frac{13}{3}$
Maximum value: $\frac{35}{3}$
Minimum value: -79

15. $g'(x) = -\frac{2x}{(1+x^2)^2}$; $-\frac{2x}{(1+x^2)^2} = 0$ when $x = 0$.
Critical point: 0
$g(0) = 1$
As $x \to \infty, g(x) \to 0^+$; as $x \to -\infty, g(x) \to 0^+$.
Maximum value = 1, no minimum value
(See graph.)



16. $f'(x) = \frac{1-x^2}{(1+x^2)^2}$;
$\frac{1-x^2}{(1+x^2)^2} = 0$ when $x = -1, 1$
Critical points: –1, 1, 4
$f(-1) = -\frac{1}{2}$, $f(1) = \frac{1}{2}$, $f(4) = \frac{4}{17}$
Maximum value = $\frac{1}{2}$,
minimum value $= -\frac{1}{2}$

17. $r'(\theta) = \cos\theta$; $\cos\theta = 0$ when $\theta = \frac{\pi}{2} + k\pi$
Critical points: $-\frac{\pi}{4}, \frac{\pi}{6}$
$r\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $r\left(\frac{\pi}{6}\right) = \frac{1}{2}$
Maximum value $= \frac{1}{2}$, minimum value $= -\frac{1}{\sqrt{2}}$

18. $s'(t) = \cos t + \sin t$; $\cos t + \sin t = 0$ when $\tan t = -1$ or $t = -\frac{\pi}{4} + k\pi$.
Critical points: 0, $\frac{3\pi}{4}$, π
$s(0) = -1$, $s\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $s(\pi) = 1$.
Maximum value $= \sqrt{2}$,
minimum value = –1

19. $a'(x) = \frac{x-1}{|x-1|}$; $a'(x)$ does not exist when $x = 1$.

Critical points: 0, 1, 3
$a(0) = 1$, $a(1) = 0$, $a(3) = 2$
Maximum value = 2, minimum value = 0

20. $f'(s) = \frac{3(3s-2)}{|3s-2|}$; $f'(s)$ does not exist when $s = \frac{2}{3}$.

Critical points: $-1, \frac{2}{3}, 4$

$f(-1) = 5$, $f\left(\frac{2}{3}\right) = 0$, $f(4) = 10$

Maximum value = 10, minimum value = 0

21. $g'(x) = \frac{1}{3x^{2/3}}$; $f'(x)$ does not exist when $x = 0$.

Critical points: –1, 0, 27
$g(-1) = -1$, $g(0) = 0$, $g(27) = 3$
Maximum value = 3, minimum value = –1

22. $s'(t) = \frac{2}{5t^{3/5}}$; $s'(t)$ does not exist when $t = 0$.

Critical points: –1, 0, 32
$s(-1) = 1$, $s(0) = 0$, $s(32) = 4$
Maximum value = 4, minimum value = 0

23. $H'(t) = -\sin t$

$-\sin t = 0$ when
$t = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi$
Critical points: $0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi$

$H(0) = 1$; $H(\pi) = -1$; $H(2\pi) = 1$;

$H(3\pi) = -1$; $H(4\pi) = 1$; $H(5\pi) = -1$;

$H(6\pi) = 1$; $H(7\pi) = -1$; $H(8\pi) = 1$

Maximum value: 1
Minimum value: –1

24. $g'(x) = 1 - 2\cos x$

$1 - 2\cos x = 0 \to \cos x = \frac{1}{2}$ when

$x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

Critical points: $-2\pi, -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$

$g(-2\pi) = -2\pi$; $g\left(-\frac{5\pi}{3}\right) = \frac{-5\pi}{3} - \sqrt{3}$;

$g\left(-\frac{\pi}{3}\right) = -\frac{\pi}{3} + \sqrt{3}$; $g\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$;

$g\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$; $g(2\pi) = 2\pi$

Maximum value: $\frac{5\pi}{3} + \sqrt{3}$

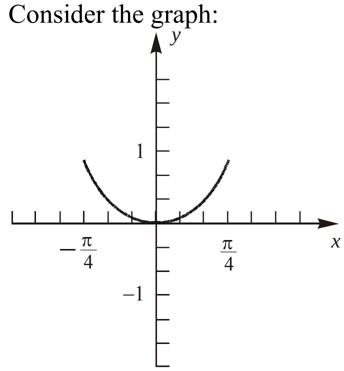
Minimum value: $-\frac{5\pi}{3} - \sqrt{3}$

25. $g'(\theta) = \theta^2(\sec\theta\tan\theta) + 2\theta\sec\theta$

$= \theta\sec\theta(\theta\tan\theta + 2)$

$\theta\sec\theta(\theta\tan\theta + 2) = 0$ when $\theta = 0$.

Consider the graph:



Critical points: $-\frac{\pi}{4}, 0, \frac{\pi}{4}$

$g\left(-\frac{\pi}{4}\right) = \frac{\pi^2\sqrt{2}}{16}$; $g(0) = 0$; $g\left(\frac{\pi}{4}\right) = \frac{\pi^2\sqrt{2}}{16}$

Maximum value: $\frac{\pi^2\sqrt{2}}{16}$; Minimum value: 0

26. $h'(t) = \frac{(2+t)\left(\frac{5}{3}t^{2/3}\right) - t^{5/3}(1)}{(2+t)^2}$

$= \frac{t^{2/3}\left(\frac{5}{3}(2+t) - t\right)}{(2+t)^2} = \frac{t^{2/3}\left(\frac{10}{3} + \frac{2}{3}t\right)}{(2+t)^2}$

$= \frac{2t^{2/3}(t+5)}{3(2+t)^2}$

$h'(t)$ is undefined when $t = -2$ and $h'(t) = 0$ when $t = 0$ or $t = -5$. Since -5 is not in the interval of interest, it is not a critical point.
Critical points: $-1, 0, 8$

$h(-1) = -1$; $h(0) = 0$; $h(8) = \frac{16}{5}$

Maximum value: $\frac{16}{5}$; Minimum value: –1

27. a. $f'(x) = 3x^2 - 12x + 1; 3x^2 - 12x + 1 = 0$ when $x = 2 - \frac{\sqrt{33}}{3}$ and $x = 2 + \frac{\sqrt{33}}{3}$.

Critical points: $-1, 2 - \frac{\sqrt{33}}{3}, 2 + \frac{\sqrt{33}}{3}, 5$

$f(-1) = -6$, $f\left(2 - \frac{\sqrt{33}}{3}\right) \approx 2.04$,

$f\left(2 + \frac{\sqrt{33}}{3}\right) \approx -26.04$, $f(5) = -18$

Maximum value ≈ 2.04;
minimum value ≈ –26.04

b. $g'(x) = \dfrac{(x^3 - 6x^2 + x + 2)(3x^2 - 12x + 1)}{|x^3 - 6x^2 + x + 2|}$;

$g'(x) = 0$ when $x = 2 - \dfrac{\sqrt{33}}{3}$ and $x = 2 + \dfrac{\sqrt{33}}{3}$. $g'(x)$ does not exist when $f(x) = 0$; on $[-1, 5]$, $f(x) = 0$ when $x \approx -0.4836$ and $x \approx 0.7172$

Critical points: $-1, -0.4836, 2 - \dfrac{\sqrt{33}}{3}, 0.7172, 2 + \dfrac{\sqrt{33}}{3}, 5$

$g(-1) = 6$, $g(-0.4836) = 0$, $g\left(2 - \dfrac{\sqrt{33}}{3}\right) \approx 2.04$, $g(0.7172) = 0$, $g\left(2 + \dfrac{\sqrt{33}}{3}\right) \approx 26.04$, $g(5) = 18$

Maximum value ≈ 26.04,
minimum value = 0

28. a. $f'(x) = x\cos x$; on $[-1, 5]$, $x\cos x = 0$ when $x = 0$, $x = \dfrac{\pi}{2}$, $x = \dfrac{3\pi}{2}$

Critical points: $-1, 0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, 5$

$f(-1) \approx 3.38$, $f(0) = 3$, $f\left(\dfrac{\pi}{2}\right) \approx 3.57$, $f\left(\dfrac{3\pi}{2}\right) \approx -2.71$, $f(5) \approx -2.51$

Maximum value ≈ 3.57,
minimum value ≈ -2.71

b. $g'(x) = \dfrac{(\cos x + x\sin x + 2)(x\cos x)}{|\cos x + x\sin x + 2|}$;

$g'(x) = 0$ when $x = 0$, $x = \dfrac{\pi}{2}$, $x = \dfrac{3\pi}{2}$

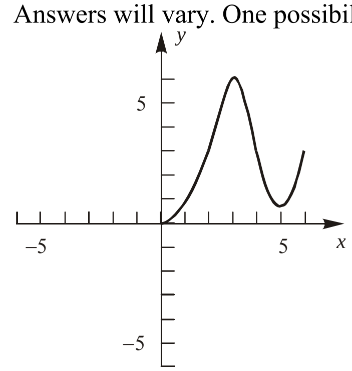
$g'(x)$ does not exist when $f(x) = 0$; on $[-1, 5]$, $f(x) = 0$ when $x \approx 3.45$

Critical points: $-1, 0, \dfrac{\pi}{2}, 3.45, \dfrac{3\pi}{2}, 5$

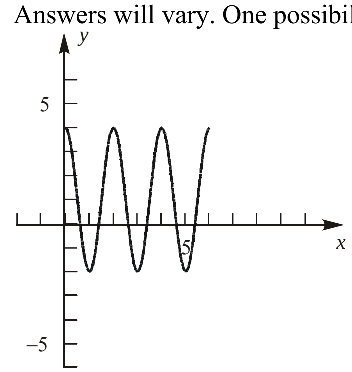
$g(-1) \approx 3.38$, $g(0) = 3$, $g\left(\dfrac{\pi}{2}\right) \approx 3.57$, $g(3.45) = 0$, $g\left(\dfrac{3\pi}{2}\right) \approx 2.71$, $g(5) \approx 2.51$

Maximum value ≈ 3.57;
minimum value = 0

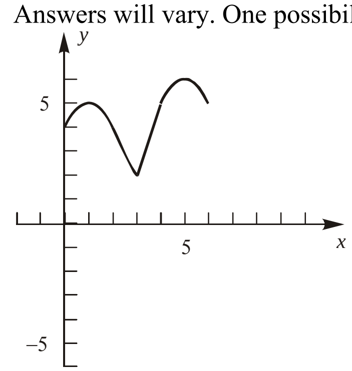
29. Answers will vary. One possibility:



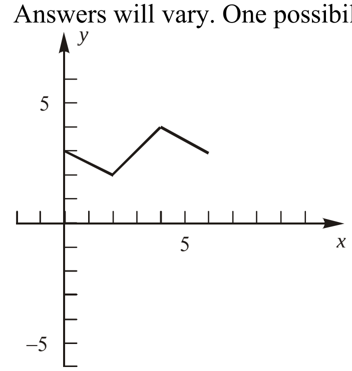
30. Answers will vary. One possibility:



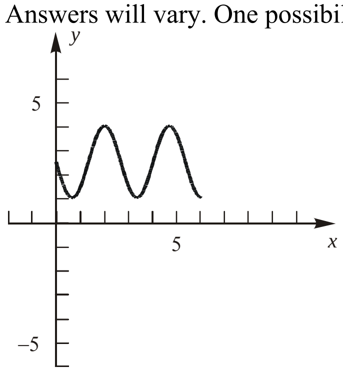
31. Answers will vary. One possibility:



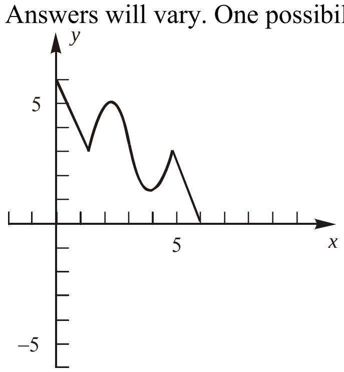
32. Answers will vary. One possibility:



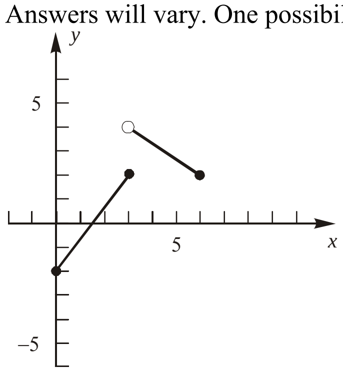
33. Answers will vary. One possibility:



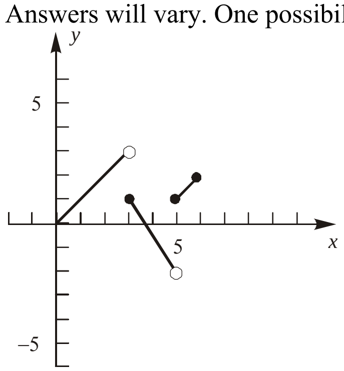
34. Answers will vary. One possibility:



35. Answers will vary. One possibility:



36. Answers will vary. One possibility:



3.2 Concepts Review

1. Increasing; concave up

2. $f'(x) > 0;\ f''(x) < 0$

3. An inflection point

4. $f''(c) = 0;\ f''(c)$ does not exist.

Problem Set 3.2

1. $f'(x) = 3;\ 3 > 0$ for all x. $f(x)$ is increasing for all x.

2. $g'(x) = 2x - 1;\ 2x - 1 > 0$ when $x > \frac{1}{2}$. $g(x)$ is increasing on $\left[\frac{1}{2}, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{2}\right]$.

3. $h'(t) = 2t + 2;\ 2t + 2 > 0$ when $t > -1$. $h(t)$ is increasing on $[-1, \infty)$ and decreasing on $(-\infty, -1]$.

4. $f'(x) = 3x^2; 3x^2 > 0$ for $x \neq 0$. $f(x)$ is increasing for all x.

5. $G'(x) = 6x^2 - 18x + 12 = 6(x-2)(x-1)$
Split the x-axis into the intervals $(-\infty, 1)$, $(1, 2)$, $(2, \infty)$.
Test points: $x = 0, \frac{3}{2}, 3; G'(0) = 12,\ G'\left(\frac{3}{2}\right) = -\frac{3}{2},$
$G'(3) = 12$
$G(x)$ is increasing on $(-\infty, 1] \cup [2, \infty)$ and decreasing on $[1, 2]$.

6. $f'(t) = 3t^2 + 6t = 3t(t+2)$
Split the x-axis into the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, \infty)$.
Test points: $t = -3, -1, 1;\ f'(-3) = 9,$
$f'(-1) = -3,\ f'(1) = 9$
$f(t)$ is increasing on $(-\infty, -2] \cup [0, \infty)$ and decreasing on $[-2, 0]$.

7. $h'(z) = z^3 - 2z^2 = z^2(z-2)$
Split the x-axis into the intervals $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$.
Test points: $z = -1, 1, 3;\ h'(-1) = -3,\ h'(1) = -1,$
$h'(3) = 9$
$h(z)$ is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$.

8. $f'(x) = \dfrac{2-x}{x^3}$

Split the x-axis into the intervals $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$.

Test points: $-1, 1, 3$; $f'(-1) = -3$, $f'(1) = 1$, $f'(3) = -\dfrac{1}{27}$

$f(x)$ is increasing on $(0, 2]$ and decreasing on $(-\infty, 0) \cup [2, \infty)$.

9. $H'(t) = \cos t; H'(t) > 0$ when $0 \le t < \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} < t \le 2\pi.$

$H(t)$ is increasing on $\left[0, \dfrac{\pi}{2}\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right]$ and decreasing on $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.

10. $R'(\theta) = -2\cos\theta \sin\theta; R'(\theta) > 0$ when $\dfrac{\pi}{2} < \theta < \pi$ and $\dfrac{3\pi}{2} < \theta < 2\pi.$

$R(\theta)$ is increasing on $\left[\dfrac{\pi}{2}, \pi\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right]$ and decreasing on $\left[0, \dfrac{\pi}{2}\right] \cup \left[\pi, \dfrac{3\pi}{2}\right]$.

11. $f''(x) = 2$; $2 > 0$ for all x. $f(x)$ is concave up for all x; no inflection points.

12. $G''(w) = 2$; $2 > 0$ for all w. $G(w)$ is concave up for all w; no inflection points.

13. $T''(t) = 18t$; $18t > 0$ when $t > 0$. $T(t)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; $(0, 0)$ is the only inflection point.

14. $f''(z) = 2 - \dfrac{6}{z^4} = \dfrac{2}{z^4}(z^4 - 3)$; $z^4 - 3 > 0$ for $z < -\sqrt[4]{3}$ and $z > \sqrt[4]{3}$.

$f(z)$ is concave up on $(-\infty, -\sqrt[4]{3}) \cup (\sqrt[4]{3}, \infty)$ and concave down on $(-\sqrt[4]{3}, 0) \cup (0, \sqrt[4]{3})$; inflection points are $\left(-\sqrt[4]{3}, \sqrt{3} - \dfrac{1}{\sqrt{3}}\right)$ and $\left(\sqrt[4]{3}, \sqrt{3} - \dfrac{1}{\sqrt{3}}\right)$.

15. $q''(x) = 12x^2 - 36x - 48; q''(x) > 0$ when $x < -1$ and $x > 4$.

$q(x)$ is concave up on $(-\infty, -1) \cup (4, \infty)$ and concave down on $(-1, 4)$; inflection points are $(-1, -19)$ and $(4, -499)$.

16. $f''(x) = 12x^2 + 48x = 12x(x+4)$; $f''(x) > 0$ when $x < -4$ and $x > 0$.

$f(x)$ is concave up on $(-\infty, -4) \cup (0, \infty)$ and concave down on $(-4, 0)$; inflection points are $(-4, -258)$ and $(0, -2)$.

17. $F''(x) = 2\sin^2 x - 2\cos^2 x + 4 = 6 - 4\cos^2 x$;

$6 - 4\cos^2 x > 0$ for all x since $0 \le \cos^2 x \le 1$.

$F(x)$ is concave up for all x; no inflection points.

18. $G''(x) = 48 + 24\cos^2 x - 24\sin^2 x$

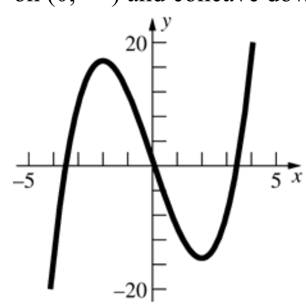
$= 24 + 48\cos^2 x$; $24 + 48\cos^2 x > 0$ for all x.

$G(x)$ is concave up for all x; no inflection points.

19. $f'(x) = 3x^2 - 12; 3x^2 - 12 > 0$ when $x < -2$ or $x > 2$.

$f(x)$ is increasing on $(-\infty, -2] \cup [2, \infty)$ and decreasing on $[-2, 2]$.

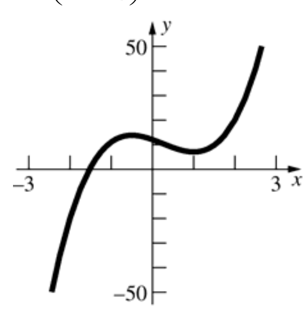
$f''(x) = 6x$; $6x > 0$ when $x > 0$. $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.



20. $g'(x) = 12x^2 - 6x - 6 = 6(2x+1)(x-1)$; $g'(x) > 0$ when $x < -\dfrac{1}{2}$ or $x > 1$. $g(x)$ is increasing on $\left(-\infty, -\dfrac{1}{2}\right] \cup [1, \infty)$ and decreasing on $\left[-\dfrac{1}{2}, 1\right]$.

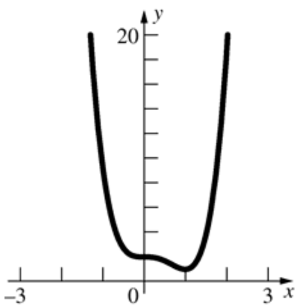
$g''(x) = 24x - 6 = 6(4x-1)$; $g''(x) > 0$ when $x > \dfrac{1}{4}$.

$g(x)$ is concave up on $\left(\dfrac{1}{4}, \infty\right)$ and concave down on $\left(-\infty, \dfrac{1}{4}\right)$.



21. $g'(x)=12x^3-12x^2=12x^2(x-1)$; $g'(x)>0$ when $x>1$. $g(x)$ is increasing on $[1,\infty)$ and decreasing on $(-\infty,1]$.

$g''(x)=36x^2-24x=12x(3x-2)$; $g''(x)>0$ when $x<0$ or $x>\frac{2}{3}$. $g(x)$ is concave up on $(-\infty,0)\cup\left(\frac{2}{3},\infty\right)$ and concave down on $\left(0,\frac{2}{3}\right)$.



22. $F'(x)=6x^5-12x^3=6x^3(x^2-2)$

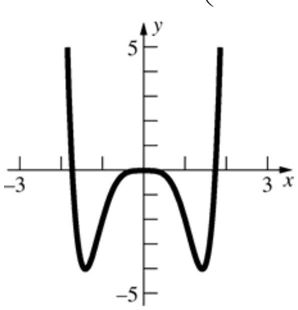
Split the x-axis into the intervals $(-\infty,-\sqrt{2})$, $(-\sqrt{2},0),(0,\sqrt{2}),(\sqrt{2},\infty)$.

Test points: $x=-2,-1,1,2$; $F'(-2)=-96$, $F'(-1)=6, F'(1)=-6, F'(2)=96$

$F(x)$ is increasing on $[-\sqrt{2},0]\cup[\sqrt{2},\infty)$ and decreasing on $(-\infty,-\sqrt{2}]\cup[0,\sqrt{2}]$

$F''(x)=30x^4-36x^2=6x^2(5x^2-6)$; $5x^2-6>0$ when $x<-\sqrt{\frac{6}{5}}$ or $x>\sqrt{\frac{6}{5}}$.

$F(x)$ is concave up on $\left(-\infty,-\sqrt{\frac{6}{5}}\right)\cup\left(\sqrt{\frac{6}{5}},\infty\right)$ and concave down on $\left(-\sqrt{\frac{6}{5}},\sqrt{\frac{6}{5}}\right)$.



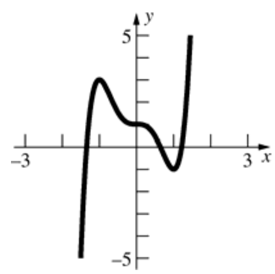
23. $G'(x)=15x^4-15x^2=15x^2(x^2-1)$; $G'(x)>0$ when $x<-1$ or $x>1$. $G(x)$ is increasing on $(-\infty,-1]\cup[1,\infty)$ and decreasing on $[-1,1]$.

$G''(x)=60x^3-30x=30x(2x^2-1)$;

Split the x-axis into the intervals $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}},0\right),\left(0,\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},\infty\right)$.

Test points: $x=-1,-\frac{1}{2},\frac{1}{2},1$; $G''(-1)=-30$, $G''\left(-\frac{1}{2}\right)=\frac{15}{2},G''\left(\frac{1}{2}\right)=-\frac{15}{2},G''(1)=30.$

$G(x)$ is concave up on $\left(-\frac{1}{\sqrt{2}},0\right)\cup\left(\frac{1}{\sqrt{2}},\infty\right)$ and concave down on $\left(-\infty,-\frac{1}{\sqrt{2}}\right)\cup\left(0,\frac{1}{\sqrt{2}}\right)$.

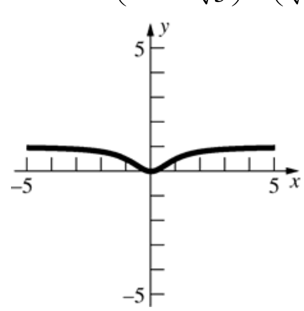


24. $H'(x)=\frac{2x}{(x^2+1)^2}$; $H'(x)>0$ when $x>0$.

$H(x)$ is increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$.

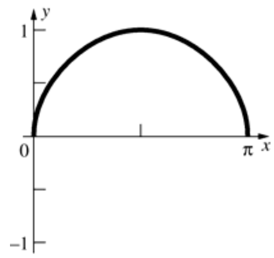
$H''(x)=\frac{2(1-3x^2)}{(x^2+1)^3}$; $H''(x)>0$ when $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$.

$H(x)$ is concave up on $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ and concave down on $\left(-\infty,-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}},\infty\right)$.



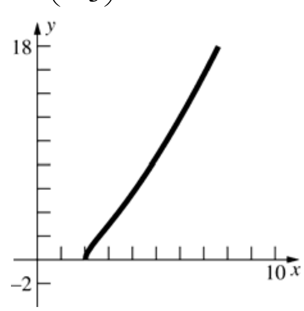
25. $f'(x) = \dfrac{\cos x}{2\sqrt{\sin x}}$; $f'(x) > 0$ when $0 < x < \dfrac{\pi}{2}$. $f(x)$ is increasing on $\left[0, \dfrac{\pi}{2}\right]$ and decreasing on $\left[\dfrac{\pi}{2}, \pi\right]$.

$f''(x) = \dfrac{-\cos^2 x - 2\sin^2 x}{4\sin^{3/2} x}$; $f''(x) < 0$ for all x in $(0, \infty)$. $f(x)$ is concave down on $(0, \pi)$.



26. $g'(x) = \dfrac{3x-4}{2\sqrt{x-2}}$; $3x - 4 > 0$ when $x > \dfrac{4}{3}$.
$g(x)$ is increasing on $[2, \infty)$.
$g''(x) = \dfrac{3x-8}{4(x-2)^{3/2}}$; $3x - 8 > 0$ when $x > \dfrac{8}{3}$.

$g(x)$ is concave up on $\left(\dfrac{8}{3}, \infty\right)$ and concave down on $\left(2, \dfrac{8}{3}\right)$.



27. $f'(x) = \dfrac{2-5x}{3x^{1/3}}$; $2 - 5x > 0$ when $x < \dfrac{2}{5}$, $f'(x)$ does not exist at $x = 0$.
Split the x-axis into the intervals $(-\infty, 0)$, $\left(0, \dfrac{2}{5}\right), \left(\dfrac{2}{5}, \infty\right)$.

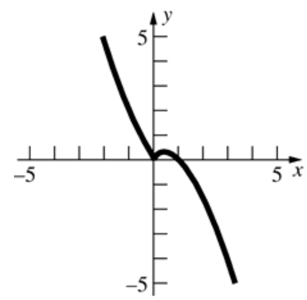
Test points: $-1, \dfrac{1}{5}, 1$; $f'(-1) = -\dfrac{7}{3}$, $f'\left(\dfrac{1}{5}\right) = \dfrac{\sqrt[3]{5}}{3}$, $f'(1) = -1$.

$f(x)$ is increasing on $\left[0, \dfrac{2}{5}\right]$ and decreasing on $(-\infty, 0] \cup \left[\dfrac{2}{5}, \infty\right)$.

$f''(x) = \dfrac{-2(5x+1)}{9x^{4/3}}$; $-2(5x + 1) > 0$ when $x < -\dfrac{1}{5}$, $f''(x)$ does not exist at $x = 0$.

Test points: $-1, -\dfrac{1}{10}, 1$; $f''(-1) = \dfrac{8}{9}$, $f''\left(-\dfrac{1}{10}\right) = -\dfrac{10^{4/3}}{9}$, $f(1) = -\dfrac{4}{3}$.

$f(x)$ is concave up on $\left(-\infty, -\dfrac{1}{5}\right)$ and concave down on $\left(-\dfrac{1}{5}, 0\right) \cup (0, \infty)$.



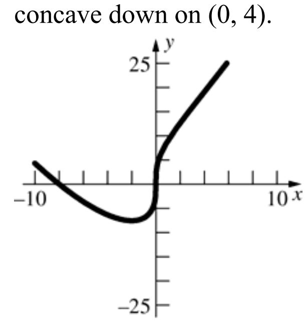
28. $g'(x) = \dfrac{4(x+2)}{3x^{2/3}}$; $x + 2 > 0$ when $x > -2$, $g'(x)$ does not exist at $x = 0$.
Split the x-axis into the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, \infty)$.

Test points: $-3, -1, 1$; $g'(-3) = -\dfrac{4}{3^{5/3}}$, $g'(-1) = \dfrac{4}{3}$, $g'(1) = 4$.

$g(x)$ is increasing on $[-2, \infty)$ and decreasing on $(-\infty, -2]$.
$g''(x) = \dfrac{4(x-4)}{9x^{5/3}}$; $x - 4 > 0$ when $x > 4$, $g''(x)$ does not exist at $x = 0$.

Test points: $-1, 1, 5$; $g''(-1) = \dfrac{20}{9}$, $g''(1) = -\dfrac{4}{3}$, $g''(5) = \dfrac{4}{9(5)^{5/3}}$.

$g(x)$ is concave up on $(-\infty, 0) \cup (4, \infty)$ and concave down on $(0, 4)$.



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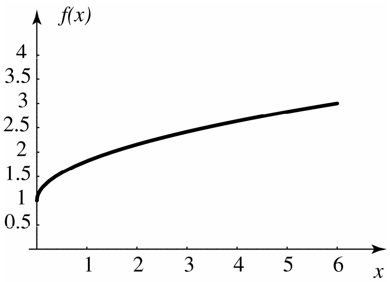

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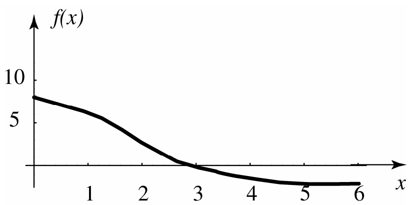

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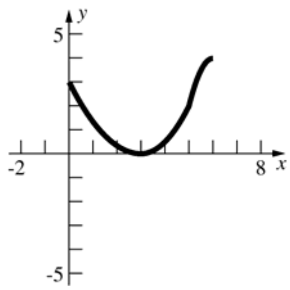

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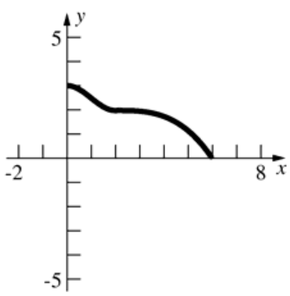

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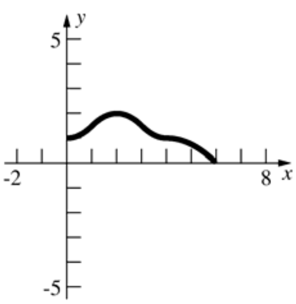
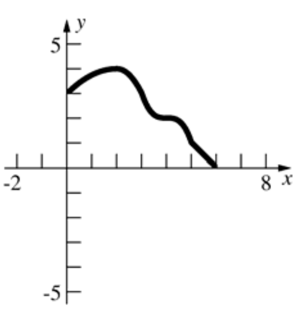


34.



35. $f(x) = ax^2 + bx + c$; $f'(x) = 2ax + b$;
$f''(x) = 2a$
An inflection point would occur where $f''(x) = 0$, or $2a = 0$. This would only occur when $a = 0$, but if $a = 0$, the equation is not quadratic. Thus, quadratic functions have no points of inflection.

36. $f(x) = ax^3 + bx^2 + cx + d$;
$f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$
An inflection point occurs where $f''(x) = 0$, or $6ax + 2b = 0$.
The function will have an inflection point at
$x = -\frac{b}{3a}$, $a \neq 0$.

37. Suppose that there are points x_1 and x_2 in I where $f'(x_1) > 0$ and $f'(x_2) < 0$. Since f' is continuous on *I*, the Intermediate Value Theorem says that there is some number *c* between x_1 and x_2 such that $f'(c) = 0$, which is a contradiction. Thus, either $f'(x) > 0$ for all *x* in *I* and *f* is increasing throughout *I* or $f'(x) < 0$ for all *x* in *I* and *f* is decreasing throughout *I*.

38. Since $x^2 + 1 = 0$ has no real solutions, $f'(x)$ exists and is continuous everywhere.
$x^2 - x + 1 = 0$ has no real solutions. $x^2 - x + 1 > 0$ and $x^2 + 1 > 0$ for all *x*, so $f'(x) > 0$ for all *x*. Thus *f* is increasing everywhere.

39. **a.** Let $f(x) = x^2$ and let $I = [0, a], a > y$.
$f'(x) = 2x > 0$ on *I*. Therefore, *f*(*x*) is increasing on *I*, so $f(x) < f(y)$ for $x < y$.

b. Let $f(x) = \sqrt{x}$ and let $I = [0, a], a > y$.
$f'(x) = \frac{1}{2\sqrt{x}} > 0$ on *I*. Therefore, *f*(*x*) is increasing on *I*, so $f(x) < f(y)$ for $x < y$.

c. Let $f(x) = \frac{1}{x}$ and let $I = [0, a]$, $a > y$.
$f'(x) = -\frac{1}{x^2} < 0$ on *I*. Therefore *f*(*x*) is decreasing on *I*, so $f(x) > f(y)$ for $x < y$.

40. $f'(x) = 3ax^2 + 2bx + c$
In order for *f*(*x*) to always be increasing, *a*, *b*, and *c* must meet the condition $3ax^2 + 2bx + c > 0$ for all *x*. More specifically, $a > 0$ and $b^2 - 3ac < 0$.

41. $f''(x) = \frac{3b - ax}{4x^{5/2}}$. If (4, 13) is an inflection point then $13 = 2a + \frac{b}{2}$ and $\frac{3b - 4a}{4 \cdot 32} = 0$. Solving these equations simultaneously, $a = \frac{39}{8}$ and $b = \frac{13}{2}$.

42.
$$f(x) = a(x - r_1)(x - r_2)(x - r_3)$$
$$f'(x) = a[(x - r_1)(2x - r_2 - r_3) + (x - r_2)(x - r_3)]$$
$$f'(x) = a[3x^2 - 2x(r_1 + r_2 + r_3) + r_1r_2 + r_2r_3 + r_1r_3]$$
$$f''(x) = a[6x - 2(r_1 + r_2 + r_3)]$$
$$a[6x - 2(r_1 + r_2 + r_3)] = 0$$
$$6x = 2(r_1 + r_2 + r_3);\ x = \frac{r_1 + r_2 + r_3}{3}$$

43. **a.** $[f(x) + g(x)]' = f'(x) + g'(x)$.
Since $f'(x) > 0$ and $g'(x) > 0$ for all x, $f'(x) + g'(x) > 0$ for all x. No additional conditions are needed.

b. $[f(x) \cdot g(x)]' = f(x)g'(x) + f'(x)g(x)$.
$f(x)g'(x) + f'(x)g(x) > 0$ if
$f(x) > -\frac{f'(x)}{g'(x)} g(x)$ for all x.

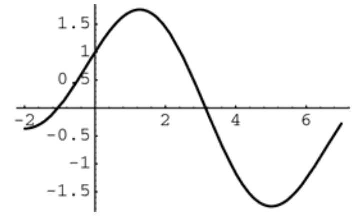
c. $[f(g(x))]' = f'(g(x))g'(x)$.
Since $f'(x) > 0$ and $g'(x) > 0$ for all x, $f'(g(x))g'(x) > 0$ for all x. No additional conditions are needed.

44. **a.** $[f(x) + g(x)]'' = f''(x) + g''(x)$.
Since $f''(x) > 0$ and $g'' > 0$ for all x, $f''(x) + g''(x) > 0$ for all x. No additional conditions are needed.

b. $[f(x) \cdot g(x)]'' = [f(x)g'(x) + f'(x)g(x)]'$
$= f(x)g''(x) + f''(x)g(x) + 2f'(x)g'(x)$.
The additional condition is that $f(x)g''(x) + f''(x)g(x) + 2f'(x)g'(x) > 0$ for all x is needed.

c. $[f(g(x))]'' = [f'(g(x))g'(x)]'$
$= f'(g(x))g''(x) + f''(g(x))[g'(x)]^2$.
The additional condition is that
$f'(g(x)) > -\frac{f''(g(x))[g'(x)]^2}{g''(x)}$ for all x.

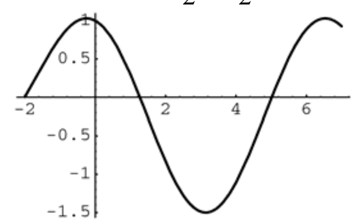
45. **a.**



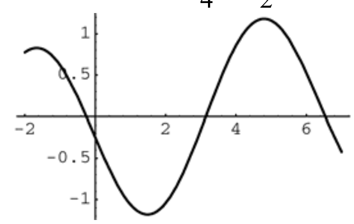
b. $f'(x) < 0 : (1.3, 5.0)$

c. $f''(x) < 0 : (-0.25, 3.1) \cup (6.5, 7]$

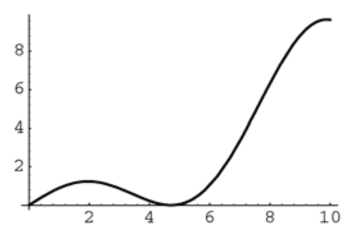
d. $f'(x) = \cos x - \frac{1}{2}\sin\frac{x}{2}$



e. $f''(x) = -\sin x - \frac{1}{4}\cos\frac{x}{2}$



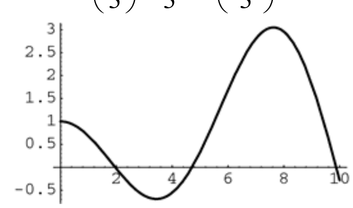
46. **a.**



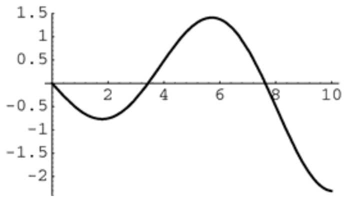
b. $f'(x) < 0 : (2.0, 4.7) \cup (9.9, 10]$

c. $f''(x) < 0 : [0, 3.4) \cup (7.6, 10]$

d. $f'(x) = x\left[-\frac{2}{3}\cos\left(\frac{x}{3}\right)\sin\left(\frac{x}{3}\right)\right] + \cos^2\left(\frac{x}{3}\right)$
$= \cos^2\left(\frac{x}{3}\right) - \frac{x}{3}\sin\left(\frac{2x}{3}\right)$



e. $f''(x) = -\frac{2x}{9}\cos\left(\frac{2x}{3}\right) - \frac{2}{3}\sin\left(\frac{2x}{3}\right)$

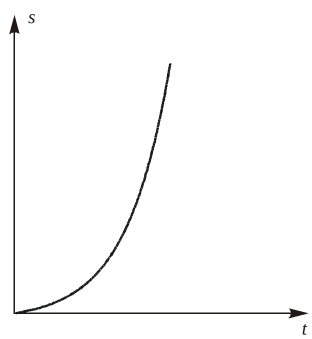


47. $f'(x) > 0$ on (–0.598, 0.680)
f is increasing on [–0.598, 0.680].

48. $f''(x) < 0$ when $x > 1.63$ in [–2, 3]
f is concave down on (1.63, 3).

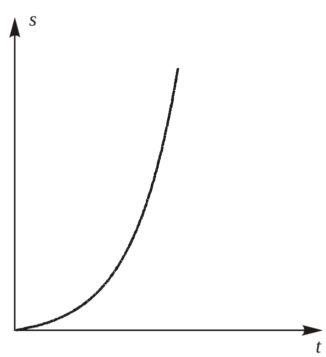
49. Let s be the distance traveled. Then $\frac{ds}{dt}$ is the speed of the car.

a. $\frac{ds}{dt} = ks$, k a constant



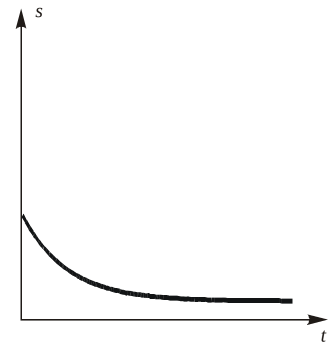
Concave up.

b. $\frac{d^2s}{dt^2} > 0$



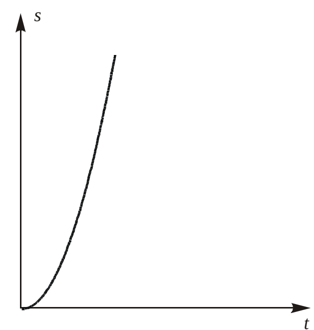
Concave up.

c. $\frac{d^3s}{dt^3} < 0, \frac{d^2s}{dt^2} > 0$



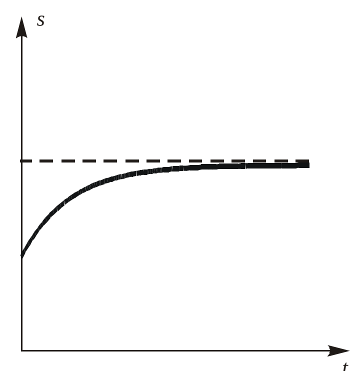
Concave up.

d. $\frac{d^2s}{dt^2} = 10$ mph/min



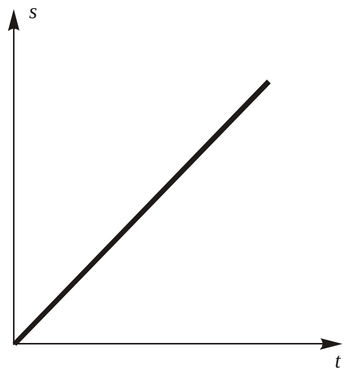
Concave up.

e. $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ are approaching zero.



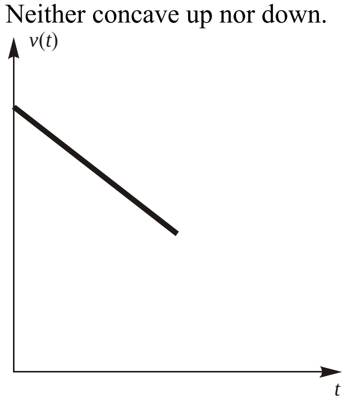
Concave down.

f. $\frac{ds}{dt}$ is constant.

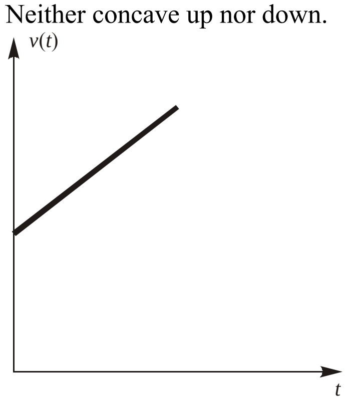


Neither concave up nor down.

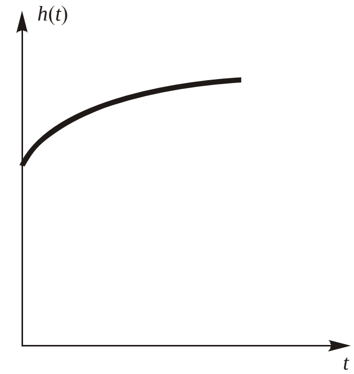
50. a. $\frac{dV}{dt} = k < 0$, V is the volume of water in the tank, k is a constant.
Neither concave up nor down.



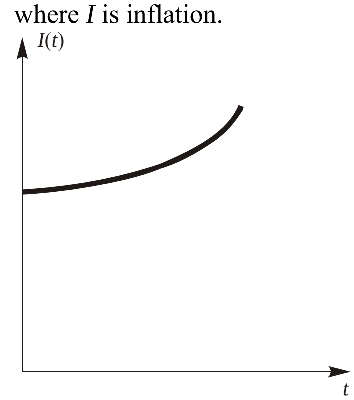
b. $\frac{dV}{dt} = 3 - \frac{1}{2} = 2\frac{1}{2}$ gal/min
Neither concave up nor down.



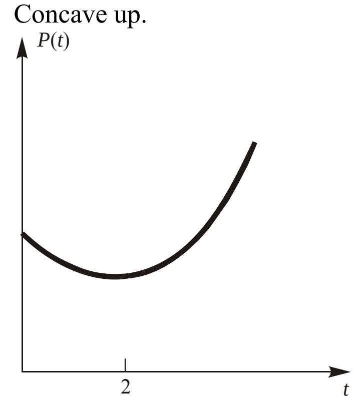
c. $\frac{dV}{dt} = k, \frac{dh}{dt} > 0, \frac{d^2h}{dt^2} < 0$
Concave down.



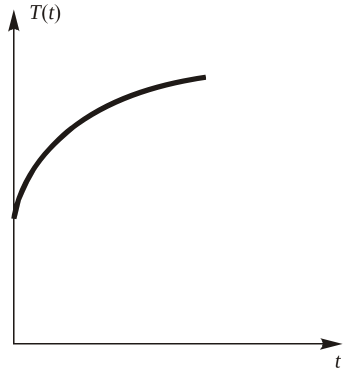
d. $I(t) = k$ now, but $\frac{dI}{dt}, \frac{d^2I}{dt^2} > 0$ in the future where I is inflation.



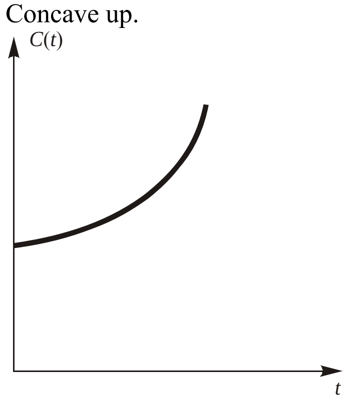
e. $\frac{dp}{dt} < 0$, but $\frac{d^2p}{dt^2} > 0$ and at $t = 2$: $\frac{dp}{dt} > 0$.
where p is the price of oil.
Concave up.



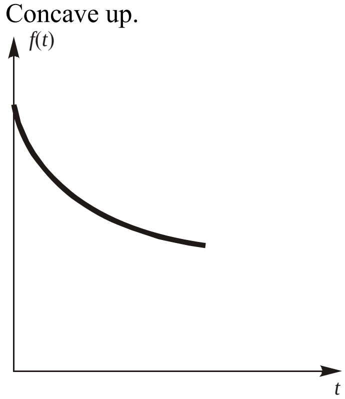
f. $\frac{dT}{dt} > 0, \frac{d^2T}{dt^2} < 0$, where T is David's temperature.
Concave down.



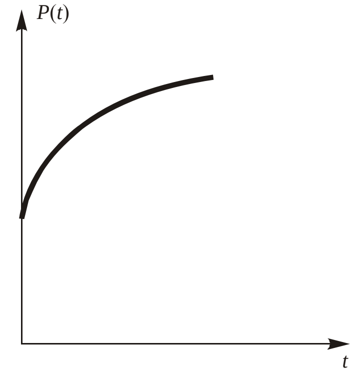
51. a. $\frac{dC}{dt} > 0, \frac{d^2C}{dt^2} > 0$, where C is the car's cost.
Concave up.



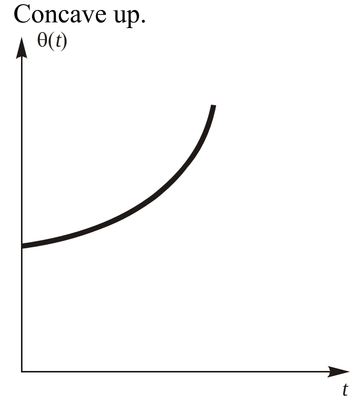
b. $f(t)$ is oil consumption at time t.
$\frac{df}{dt} < 0, \frac{d^2f}{dt^2} > 0$
Concave up.



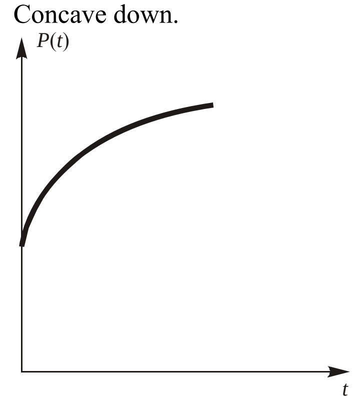
c. $\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$, where P is world population.
Concave down.



d. $\frac{d\theta}{dt} > 0, \frac{d^2\theta}{dt^2} > 0$, where θ is the angle that the tower makes with the vertical.
Concave up.



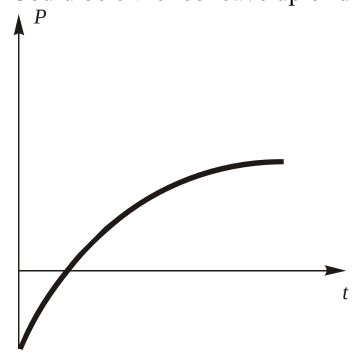
e. $P = f(t)$ is profit at time t.
$\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$
Concave down.

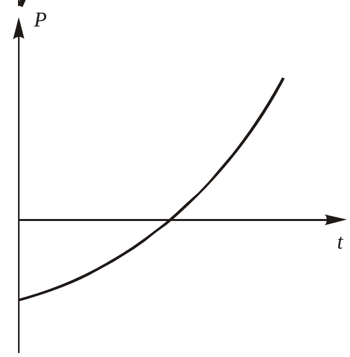


f. R is revenue at time t.

$P < 0,\ \frac{dP}{dt} > 0$

Could be either concave up or down.





52. a. $R(t) \approx 0.28,\ t < 1981$

b. On [1981, 1983], $\frac{dR}{dt} > 0, \frac{d^2R}{dt^2} > 0$,

$R(1983) \approx 0.36$

53. $\frac{dV}{dt} = 2 \text{ in}^3/\text{sec}$

The cup is a portion of a cone with the bottom cut off. If we let x represent the height of the missing cone, we can use similar triangles to show that

$$\frac{x}{3} = \frac{x+5}{3.5}$$

$$3.5x = 3x + 15$$

$$0.5x = 15$$

$$x = 30$$

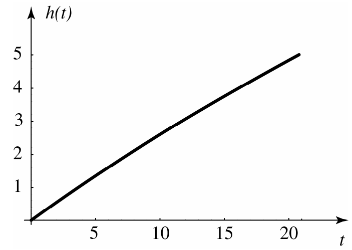
Similar triangles can be used again to show that, at any given time, the radius of the cone at water level is

$$r = \frac{h+30}{20}$$

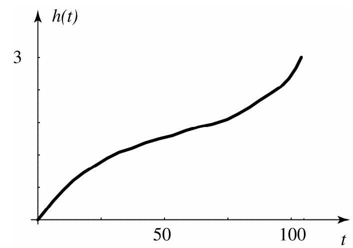
Therefore, the volume of water can be expressed as

$$V = \frac{\pi(h+30)^3}{1200} - \frac{45\pi}{2}.$$

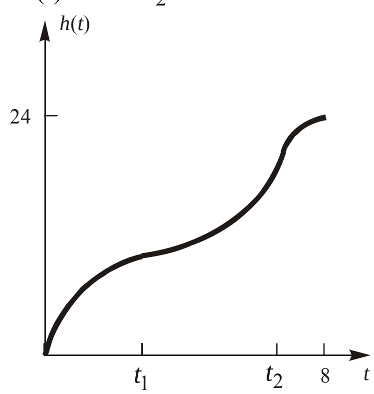
We also know that $V = 2t$ from above. Setting the two volume equations equal to each other and solving for h gives $h = \sqrt[3]{\frac{2400}{\pi}t + 27000} - 30$.



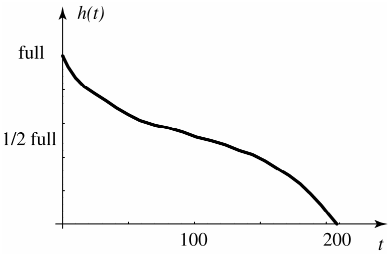
54. The height is always increasing so $h'(t) > 0$. The rate of change of the height decreases for the first 50 minutes and then increases over the next 50 minutes. Thus $h''(t) < 0$ for $0 \le t \le 50$ and $h''(t) > 0$ for $50 < t \le 100$.



55. $V = 3t,\ 0 \le t \le 8$. The height is always increasing, so $h'(t) > 0$. The rate of change of the height decreases from time $t = 0$ until time t_1 when the water reaches the middle of the rounded bottom part. The rate of change then increases until time t_2 when the water reaches the middle of the neck. Then the rate of change decreases until $t = 8$ and the vase is full. Thus, $h''(t) > 0$ for $t_1 < t < t_2$ and $h''(t) < 0$ for $t_2 < t < 8$.



56. $V = 20 - .1t$, $0 \le t \le 200$. The height of the water is always decreasing so $h'(t) < 0$. The rate of change in the height increases (the rate is negative, and its absolute value decreases) for the first 100 days and then decreases for the remaining time. Therefore we have $h''(t) > 0$ for $0 < t < 100$, and $h''(t) < 0$ for $100 < t < 200$.



57. a. The cross-sectional area of the vase is approximately equal to ΔV and the corresponding radius is $r = \sqrt{\Delta V / \pi}$. The table below gives the approximate values for r. The vase becomes slightly narrower as you move above the base, and then gets wider as you near the top.

Depth	V	$A \approx \Delta V$	$r = \sqrt{\Delta V / \pi}$
1	4	4	1.13
2	8	4	1.13
3	11	3	0.98
4	14	3	0.98
5	20	6	1.38
6	28	8	1.60

b. Near the base, this vase is like the one in part (a), but just above the base it becomes larger. Near the middle of the vase it becomes very narrow. The top of the vase is similar to the one in part (a).

Depth	V	$A \approx \Delta V$	$r = \sqrt{\Delta V / \pi}$
1	4	4	1.13
2	9	5	1.26
3	12	3	0.98
4	14	2	0.80
5	20	6	1.38
6	28	8	1.60

3.3 Concepts Review

1. maximum

2. maximum; minimum

3. maximum

4. local maximum, local minimum, 0

Problem Set 3.3

1. $f'(x) = 3x^2 - 12x = 3x(x-4)$
Critical points: 0, 4
$f'(x) > 0$ on $(-\infty, 0)$, $f'(x) < 0$ on $(0, 4)$,
$f'(x) > 0$ on $(4, \infty)$
$f''(x) = 6x - 12$; $f''(0) = -12$, $f''(4) = 12$.
Local minimum at $x = 4$;
local maximum at $x = 0$

2. $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$
Critical points: –2, 2
$f'(x) > 0$ on $(-\infty, -2)$, $f'(x) < 0$ on $(-2, 2)$,
$f'(x) > 0$ on $(2, \infty)$
$f''(x) = 6x$; $f''(-2) = -12$, $f''(2) = 12$
Local minimum at $x = 2$;
local maximum at $x = -2$

3. $f'(\theta) = 2\cos 2\theta$; $2\cos 2\theta \ne 0$ on $\left(0, \frac{\pi}{4}\right)$
No critical points; no local maxima or minima on $\left(0, \frac{\pi}{4}\right)$.

4. $f'(x) = \frac{1}{2} + \cos x$; $\frac{1}{2} + \cos x = 0$ when $\cos x = -\frac{1}{2}$.
Critical points: $\frac{2\pi}{3}, \frac{4\pi}{3}$
$f'(x) > 0$ on $\left(0, \frac{2\pi}{3}\right)$, $f'(x) < 0$ on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$,
$f'(x) > 0$ on $\left(\frac{4\pi}{3}, 2\pi\right)$
$f''(x) = -\sin x$; $f''\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $f''\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$
Local minimum at $x = \frac{4\pi}{3}$; local maximum at $x = \frac{2\pi}{3}$.

5. $\Psi'(\theta) = 2\sin\theta\cos\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Critical point: 0

$\Psi'(\theta) < 0$ on $\left(-\frac{\pi}{2}, 0\right)$, $\Psi'(\theta) > 0$ on $\left(0, \frac{\pi}{2}\right)$,

$\Psi''(\theta) = 2\cos^2\theta - 2\sin^2\theta$; $\Psi''(0) = 2$

Local minimum at $x = 0$

6. $r'(z) = 4z^3$

Critical point: 0

$r'(z) < 0$ on $(-\infty, 0)$;

$r'(z) > 0$ on $(0, \infty)$

$r''(x) = 12x^2$; $r''(0) = 0$; the Second Derivative Test fails.

Local minimum at $z = 0$; no local maxima

7. $f'(x) = \frac{(x^2+4)\cdot 1 - x(2x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$

Critical points: $-2, 2$

$f'(x) < 0$ on $(-\infty, -2)$ and $(2, \infty)$;

$f'(x) > 0$ on $(-2, 2)$

$f''(x) = \frac{2x(x^2-12)}{(x^2+4)^3}$

$f''(-2) = \frac{1}{16}$; $f''(2) = -\frac{1}{16}$

Local minima at $x = -2$; Local maxima at $x = 2$

8. $g'(z) = \frac{(1+z^2)(2z) - z^2(2z)}{(1+z^2)^2} = \frac{2z}{(1+z^2)^2}$

Critical point: $z = 0$

$g'(z) < 0$ on $(-\infty, 0)$

$g'(z) > 0$ on $(0, \infty)$

$g''(z) = \frac{-2(3z^2-1)}{(z^2+1)^3}$

$g''(0) = 2$

Local minima at $z = 0$.

9. $h'(y) = 2y + \frac{1}{y^2}$

Critical point: $-\frac{\sqrt[3]{4}}{2}$

$h'(y) < 0$ on $\left(-\infty, -\frac{\sqrt[3]{4}}{2}\right)$

$h'(y) > 0$ on $\left(-\frac{\sqrt[3]{4}}{2}, 0\right)$ and $(0, \infty)$

$h''(y) = 2 - \frac{2}{y^3}$

$h\left(-\frac{\sqrt[3]{4}}{2}\right) = 2 - \frac{2}{\left(-\frac{\sqrt[3]{4}}{2}\right)^3} = 2 + \frac{16}{4} = 6$

Local minima at $-\frac{\sqrt[3]{4}}{2}$

10. $f'(x) = \frac{(x^2+1)(3) - (3x+1)(2x)}{(x^2+1)^2} = \frac{3-2x-3x^2}{(x^2+1)^2}$

The only critical points are stationary points. Find these by setting the numerator equal to 0 and solving.

$3 - 2x - 3x^2 = 0$

$a = -3, b = -2, c = 3$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)(3)}}{2(-3)} = \frac{2 \pm \sqrt{40}}{-6} = \frac{-1 \pm \sqrt{10}}{3}$

Critical points: $\frac{-1-\sqrt{10}}{3}$ and $\frac{-1+\sqrt{10}}{3}$

$f'(x) < 0$ on $\left(-\infty, \frac{-1-\sqrt{10}}{3}\right)$ and $\left(\frac{-1+\sqrt{10}}{3}, \infty\right)$.

$f'(0) > 0$ on $\left(\frac{-1-\sqrt{10}}{3}, \frac{-1+\sqrt{10}}{3}\right)$

$f''(x) = \frac{2(3x^3+3x^2-9x-1)}{(x^2+1)^3}$

$f''\left(\frac{-1-\sqrt{10}}{3}\right) \approx 0.739$

$f''\left(\frac{-1+\sqrt{10}}{3}\right) \approx -2.739$

Local minima at $x = \frac{-1-\sqrt{10}}{3}$;

Local maxima at $x = \frac{-1+\sqrt{10}}{3}$

11. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$
Critical points: –1, 1
$f''(x) = 6x$; $f''(-1) = -6$, $f''(1) = 6$
Local minimum value $f(1) = -2$;
local maximum value $f(-1) = 2$

12. $g'(x) = 4x^3 + 2x = 2x(2x^2 + 1)$
Critical point: 0
$g''(x) = 12x^2 + 2$; $g''(0) = 2$
Local minimum value $g(0) = 3$; no local maximum values

13. $H'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$
Critical points: $0, \frac{3}{2}$
$H''(x) = 12x^2 - 12x = 12x(x-1)$; $H''(0) = 0$,
$H''\left(\frac{3}{2}\right) = 9$
$H'(x) < 0$ on $(-\infty, 0)$, $H'(x) < 0$ on $\left(0, \frac{3}{2}\right)$
Local minimum value $H\left(\frac{3}{2}\right) = -\frac{27}{16}$; no local maximum values ($x = 0$ is neither a local minimum nor maximum)

14. $f'(x) = 5(x-2)^4$
Critical point: 2
$f''(x) = 20(x-2)^3$; $f''(2) = 0$
$f'(x) > 0$ on $(-\infty, 2)$, $f'(x) > 0$ on $(2, \infty)$
No local minimum or maximum values

15. $g'(t) = -\frac{2}{3(t-2)^{1/3}}$; $g'(t)$ does not exist at $t = 2$.
Critical point: 2
$g'(1) = \frac{2}{3}$, $g'(3) = -\frac{2}{3}$
No local minimum values; local maximum value $g(2) = \pi$.

16. $r'(s) = 3 + \frac{2}{5s^{3/5}} = \frac{15s^{3/5} + 2}{5s^{3/5}}$; $r'(s) = 0$ when
$s = -\left(\frac{2}{15}\right)^{5/3}$, $r'(s)$ does not exist at $s = 0$.
Critical points: $-\left(\frac{2}{15}\right)^{5/3}, 0$

$r''(s) = -\frac{6}{25s^{8/5}}$; $r''\left(-\left(\frac{2}{15}\right)^{5/3}\right) = -\frac{6}{25}\left(\frac{15}{2}\right)^{8/3}$
$r'(s) < 0$ on $\left(-\left(\frac{2}{15}\right)^{5/3}, 0\right)$, $r'(s) > 0$ on $(0, \infty)$

Local minimum value $r(0) = 0$; local maximum value
$r\left(-\left(\frac{2}{15}\right)^{5/3}\right) = -3\left(\frac{2}{15}\right)^{5/3} + \left(\frac{2}{15}\right)^{2/3} = \frac{3}{5}\left(\frac{2}{15}\right)^{2/3}$

17. $f'(t) = 1 + \frac{1}{t^2}$
No critical points
No local minimum or maximum values

18. $f'(x) = \frac{x(x^2 + 8)}{(x^2 + 4)^{3/2}}$
Critical point: 0
$f'(x) < 0$ on $(-\infty, 0)$, $f'(x) > 0$ on $(0, \infty)$
Local minimum value $f(0) = 0$, no local maximum values

19. $\Lambda'(\theta) = -\frac{1}{1 + \sin\theta}$; $\Lambda'(\theta)$ does not exist at
$\theta = \frac{3\pi}{2}$, but $\Lambda(\theta)$ does not exist at that point either.
No critical points
No local minimum or maximum values

20. $g'(\theta) = \frac{\sin\theta\cos\theta}{|\sin\theta|}$; $g'(\theta) = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$;
$g'(\theta)$ does not exist at $x = \pi$.
Split the x-axis into the intervals $\left(0, \frac{\pi}{2}\right)$,
$\left(\frac{\pi}{2}, \pi\right), \left(\pi, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$.
Test points: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; $g'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$,
$g'\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $g'\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $g'\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
Local minimum value $g(\pi) = 0$; local maximum values $g\left(\frac{\pi}{2}\right) = 1$ and $g\left(\frac{3\pi}{2}\right) = 1$

21. $f'(x)=4(\sin 2x)(\cos 2x)$

$4(\sin 2x)(\cos 2x)=0$ when $x=\dfrac{(2k-1)\pi}{4}$ or

$x=\dfrac{k\pi}{2}$ where k is an integer.

Critical points: 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, 2

$f(0)=0$; $f\left(\dfrac{\pi}{4}\right)=1$; $f\left(\dfrac{\pi}{2}\right)=0$;

$f(2)\approx 0.5728$

Minimum value: $f(0)=f\left(\dfrac{\pi}{2}\right)=0$

Maximum value: $f\left(\dfrac{\pi}{4}\right)=1$

22. $f'(x)=\dfrac{-2\left(x^2-4\right)}{\left(x^2+4\right)^2}$

$f'(x)=0$ when $x=2$ or $x=-2$. (there are no singular points)

Critical points: 0, 2 (note: -2 is not in the given domain)

$f(0)=0$; $f(2)=\dfrac{1}{2}$; $f(x)\to 0$ as $x\to\infty$.

Minimum value: $f(0)=0$

Maximum value: $f(2)=\dfrac{1}{2}$

23. $g'(x)=\dfrac{-x\left(x^3-64\right)}{\left(x^3+32\right)^2}$

$g'(x)=0$ when $x=0$ or $x=4$.

Critical points: 0, 4

$g(0)=0$; $g(4)=\dfrac{1}{6}$

As x approaches ∞, the value of g approaches 0 but never actually gets there.

Maximum value: $g(4)=\dfrac{1}{6}$

Minimum value: $g(0)=0$

24. $h'(x)=\dfrac{-2x}{\left(x^2+4\right)^2}$

$h'(x)=0$ when $x=0$. (there are no singular points)

Critical points: 0

Since $h'(x)<0$ for $x>0$, the function is always decreasing. Thus, there is no minimum value.

Maximum value: $h(0)=\dfrac{1}{4}$

25. $F'(x)=\dfrac{3}{\sqrt{x}}-4$; $\dfrac{3}{\sqrt{x}}-4=0$ when $x=\dfrac{9}{16}$

Critical points: $0, \dfrac{9}{16}, 4$

$F(0)=0$, $F\left(\dfrac{9}{16}\right)=\dfrac{9}{4}$, $F(4)=-4$

Minimum value $F(4)=-4$; maximum value $F\left(\dfrac{9}{16}\right)=\dfrac{9}{4}$

26. From Problem 25, the critical points are 0 and $\dfrac{9}{16}$.

$F'(x)>0$ on $\left(0,\dfrac{9}{16}\right)$, $F'(x)<0$ on $\left(\dfrac{9}{16},\infty\right)$

F decreases without bound on $\left(\dfrac{9}{16},\infty\right)$. No minimum values; maximum value $F\left(\dfrac{9}{16}\right)=\dfrac{9}{4}$

27. $f'(x)=64(-1)(\sin x)^{-2}\cos x$

$+27(-1)(\cos x)^{-2}(-\sin x)$

$=-\dfrac{64\cos x}{\sin^2 x}+\dfrac{27\sin x}{\cos^2 x}$

$=\dfrac{(3\sin x-4\cos x)(9\sin^2 x+12\cos x\sin x+16\cos^2 x)}{\sin^2 x\cos^2 x}$

On $\left(0,\dfrac{\pi}{2}\right)$, $f'(x)=0$ only where $3\sin x=4\cos x$;

$\tan x=\dfrac{4}{3}$;

$x=\tan^{-1}\dfrac{4}{3}\approx 0.9273$

Critical point: 0.9273

For $0<x<0.9273$, $f'(x)<0$, while for

$0.9273<x<\dfrac{\pi}{2}$, $f'(x)>0$

Minimum value $f\left(\tan^{-1}\dfrac{4}{3}\right)=\dfrac{64}{\frac{4}{5}}+\dfrac{27}{\frac{3}{5}}=125$;

no maximum value

28. $g'(x) = 2x + \dfrac{(8-x)^2(32x) - (16x^2)2(8-x)(-1)}{(8-x)^4}$

$= 2x + \dfrac{256x}{(8-x)^3} = \dfrac{2x[(8-x)^3 + 128]}{(8-x)^3}$

For $x > 8$, $g'(x) = 0$ when $(8-x)^3 + 128 = 0$;

$(8-x)^3 = -128; 8 - x = -\sqrt[3]{128}$;

$x = 8 + 4\sqrt[3]{2} \approx 13.04$

$g'(x) < 0$ on $(8, 8+4\sqrt[3]{2})$,

$g'(x) > 0$ on $(8+4\sqrt[3]{2}, \infty)$

$g(13.04) \approx 277$ is the minimum value

29. $H'(x) = \dfrac{2x\left(x^2-1\right)}{\left|x^2-1\right|}$

$H'(x) = 0$ when $x = 0$.

$H'(x)$ is undefined when $x = -1$ or $x = 1$

Critical points: -2, -1, 0, 1, 2

$H(-2) = 3$; $H(-1) = 0$; $H(0) = 1$; $H(1) = 0$; $H(2) = 3$

Minimum value: $H(-1) = H(1) = 0$

Maximum value: $H(-2) = H(2) = 3$

30. $h'(t) = 2t\cos t^2$

$h'(t) = 0$ when $t = 0$, $t = \dfrac{\sqrt{2\pi}}{2}$, $t = \dfrac{\sqrt{6\pi}}{2}$, and $t = \dfrac{\sqrt{10\pi}}{2}$

(Consider $t^2 = \dfrac{\pi}{2}$, $t^2 = \dfrac{3\pi}{2}$, and $t^2 = \dfrac{5\pi}{2}$)

Critical points: 0, $\dfrac{\sqrt{2\pi}}{2}$, $\dfrac{\sqrt{6\pi}}{2}$, $\dfrac{\sqrt{10\pi}}{2}$, π

$h(0) = 0$; $h\left(\dfrac{\sqrt{2\pi}}{2}\right) = 1$; $h\left(\dfrac{\sqrt{6\pi}}{2}\right) = -1$;

$h\left(\dfrac{\sqrt{10\pi}}{2}\right) = 1$; $h(\pi) \approx -0.4303$

Minimum value: $h\left(\dfrac{\sqrt{6\pi}}{2}\right) = -1$

Maximum value: $h\left(\dfrac{\sqrt{2\pi}}{2}\right) = h\left(\dfrac{\sqrt{10\pi}}{2}\right) = 1$

31. $f'(x) = 0$ when $x = 0$ and $x = 1$. On the interval $(-\infty, 0)$ we get $f'(x) < 0$. On $(0, \infty)$, we get $f'(x) > 0$. Thus there is a local min at $x = 0$ but no local max.

32. $f'(x) = 0$ at $x = 1, 2, 3, 4$; $f'(x)$ is negative on $(-\infty, 1) \cup (2, 3) \cup (4, \infty)$ and positive on $(1, 2) \cup (3, 4)$. Thus, the function has a local minimum at $x = 1, 3$ and a local maximum at $x = 2, 4$.

33. $f'(x) = 0$ at $x = 1, 2, 3, 4$; $f'(x)$ is negative on $(3, 4)$ and positive on $(-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (4, \infty)$ Thus, the function has a local minimum at $x = 4$ and a local maximum at $x = 3$.

34. Since $f'(x) \geq 0$ for all x, the function is always increasing. Therefore, there are no local extrema.

35. Since $f'(x) \geq 0$ for all x, the function is always increasing. Therefore, there are no local extrema.

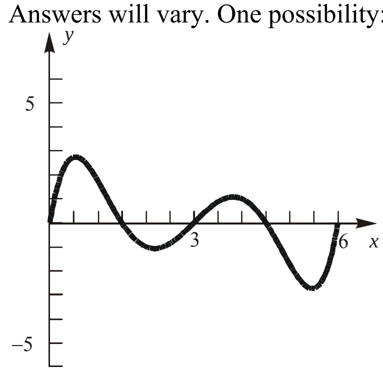
36. $f'(x) = 0$ at $x = 0, A,$ and B.

$f'(x)$ is negative on $(-\infty, 0)$ and (A, B)

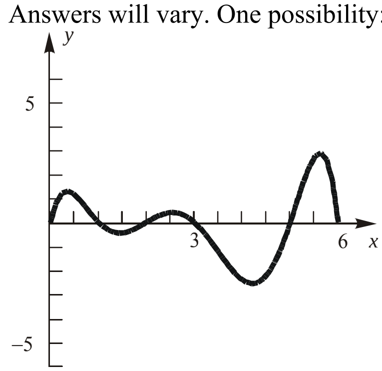
$f'(x)$ is positive on $(0, A)$ and (B, ∞)

Therefore, the function has a local minimum at $x = 0$ and $x = B$, and a local maximum at $x = A$.

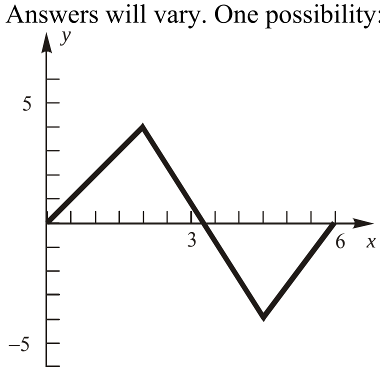
37. Answers will vary. One possibility:



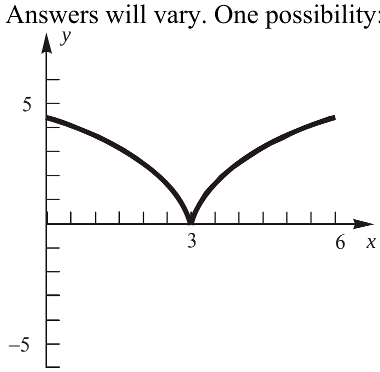
38. Answers will vary. One possibility:



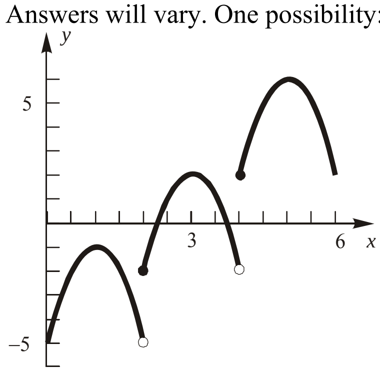
39. Answers will vary. One possibility:



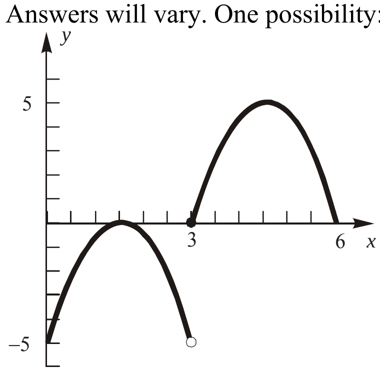
40. Answers will vary. One possibility:



41. Answers will vary. One possibility:



42. Answers will vary. One possibility:



43. The graph of f is a parabola which opens up.

$$f'(x) = 2Ax + B = 0 \quad \rightarrow \quad x = -\frac{B}{2A}$$

$$f''(x) = 2A$$

Since $A > 0$, the graph of f is always concave up. There is exactly one critical point which yields the minimum of the graph.

$$f\left(-\frac{B}{2A}\right) = A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C$$

$$= \frac{B^2}{4A} - \frac{B^2}{2A} + C$$

$$= \frac{B^2 - 2B^2 + 4AC}{4A}$$

$$= \frac{4AC - B^2}{4A} = -\frac{B^2 - 4AC}{4A}$$

If $f(x) \ge 0$ with $A > 0$, then $-\left(B^2 - 4AC\right) \ge 0$, or $B^2 - 4AC \le 0$.

If $B^2 - 4AC \le 0$, then we get $f\left(-\frac{B}{2A}\right) \ge 0$

Since $0 \le f\left(-\frac{B}{2A}\right) \le f(x)$ for all x, we get $f(x) \ge 0$ for all x.

44. A third degree polynomial will have at most two extrema.

$$f'(x) = 3Ax^2 + 2Bx + C$$

$$f''(x) = 6Ax + 2B$$

Critical points are obtained by solving $f'(x) = 0$.

$$3Ax^2 + 2Bx + C = 0$$

$$x = \frac{-2B \pm \sqrt{4B^2 - 12AC}}{6A}$$

$$= \frac{-2B \pm 2\sqrt{B^2 - 3AC}}{6A}$$

$$= \frac{-B \pm \sqrt{B^2 - 3AC}}{3A}$$

To have a relative maximum *and* a relative minimum, we must have two solutions to the above quadratic equation. That is, we must have $B^2 - 3AC > 0$.

The two solutions would be $\frac{-B - \sqrt{B^2 - 3AC}}{3A}$ and $\frac{-B + \sqrt{B^2 - 3AC}}{3A}$. Evaluating the second derivative at each of these values gives:

$$f''\left(\frac{-B-\sqrt{B^2-3AC}}{3A}\right)$$
$$= 6A\left(\frac{-B-\sqrt{B^2-3AC}}{3A}\right)+2B$$
$$= -2B-2\sqrt{B^2-3AC}+2B$$
$$= -2\sqrt{B^2-3AC}$$
and
$$f''\left(\frac{-B+\sqrt{B^2-3AC}}{3A}\right)$$
$$= 6A\left(\frac{-B+\sqrt{B^2-3AC}}{3A}\right)+2B$$
$$= -2B+2\sqrt{B^2-3AC}+2B$$
$$= 2\sqrt{B^2-3AC}$$
If $B^2-3AC>0$, then $-2\sqrt{B^2-3AC}$ exists and is negative, and $2\sqrt{B^2-3AC}$ exists and is positive.
Thus, from the Second Derivative Test, $\frac{-B-\sqrt{B^2-3AC}}{3A}$ would yield a local maximum and $\frac{-B+\sqrt{B^2-3AC}}{3A}$ would yield a local minimum.

45. $f'''(c)>0$ implies that f'' is increasing at *c*, so *f* is concave up to the right of *c* (since $f''(x)>0$ to the right of *c*) and concave down to the left of *c* (since $f''(x)<0$ to the left of *c*). Therefore *f* has a point of inflection at *c*.

3.4 Concepts Review

1. $0<x<\infty$

2. $2x+\frac{200}{x}$

3. $S=\sum_{i=1}^{n}\left(y_i-bx_i\right)^2$

4. marginal revenue; marginal cost

Problem Set 3.4

1. Let *x* be one number, *y* be the other, and *Q* be the sum of the squares.
$xy=-16$
$y=-\frac{16}{x}$
The possible values for *x* are in $(-\infty, 0)$ or $(0,\infty)$.
$$Q=x^2+y^2=x^2+\frac{256}{x^2}$$
$$\frac{dQ}{dx}=2x-\frac{512}{x^3}$$
$$2x-\frac{512}{x^3}=0$$
$x^4=256$
$x=\pm4$
The critical points are –4, 4.
$\frac{dQ}{dx}<0$ on $(-\infty, -4)$ and $(0, 4)$. $\frac{dQ}{dx}>0$ on $(-4, 0)$ and $(4, \infty)$.
When $x=-4$, $y=4$ and when $x=4$, $y=-4$. The two numbers are –4 and 4.

2. Let *x* be the number.
$Q=\sqrt{x}-8x$
x will be in the interval $(0, \infty)$.
$$\frac{dQ}{dx}=\frac{1}{2}x^{-1/2}-8$$
$$\frac{1}{2}x^{-1/2}-8=0$$
$x^{-1/2}=16$
$x=\frac{1}{256}$
$\frac{dQ}{dx}>0$ on $\left(0,\frac{1}{256}\right)$ and $\frac{dQ}{dx}<0$ on $\left(\frac{1}{256},\infty\right)$.
Q attains its maximum value at $x=\frac{1}{256}$.

3. Let *x* be the number.
$Q=\sqrt[4]{x}-2x$
x will be in the interval $(0, \infty)$.
$$\frac{dQ}{dx}=\frac{1}{4}x^{-3/4}-2$$
$$\frac{1}{4}x^{-3/4}-2=0$$
$x^{-3/4}=8$
$x=\frac{1}{16}$
$\frac{dQ}{dx}>0$ on $\left(0,\frac{1}{16}\right)$ and $\frac{dQ}{dx}<0$ on $\left(\frac{1}{16},\infty\right)$
Q attains its maximum value at $x=\frac{1}{16}$.

4. Let x be one number, y be the other, and Q be the sum of the squares.

$xy = -12$

$y = -\frac{12}{x}$

The possible values for x are in $(-\infty, 0)$ or $(0, \infty)$.

$Q = x^2 + y^2 = x^2 + \frac{144}{x^2}$

$\frac{dQ}{dx} = 2x - \frac{288}{x^3}$

$2x - \frac{288}{x^3} = 0$

$x^4 = 144$

$x = \pm 2\sqrt{3}$

The critical points are $-2\sqrt{3}, 2\sqrt{3}$

$\frac{dQ}{dx} < 0$ on $(-\infty, -2\sqrt{3})$ and $(0, 2\sqrt{3})$.

$\frac{dQ}{dx} > 0$ on $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, \infty)$.

When $x = -2\sqrt{3}, y = 2\sqrt{3}$ and when $x = 2\sqrt{3}, y = -2\sqrt{3}$.

The two numbers are $-2\sqrt{3}$ and $2\sqrt{3}$.

5. Let Q be the square of the distance between (x, y) and $(0, 5)$.

$Q = (x-0)^2 + (y-5)^2 = x^2 + (x^2 - 5)^2$

$= x^4 - 9x^2 + 25$

$\frac{dQ}{dx} = 4x^3 - 18x$

$4x^3 - 18x = 0$

$2x(2x^2 - 9) = 0$

$x = 0, \pm\frac{3}{\sqrt{2}}$

$\frac{dQ}{dx} < 0$ on $\left(-\infty, -\frac{3}{\sqrt{2}}\right)$ and $\left(0, \frac{3}{\sqrt{2}}\right)$.

$\frac{dQ}{dx} > 0$ on $\left(-\frac{3}{\sqrt{2}}, 0\right)$ and $\left(\frac{3}{\sqrt{2}}, \infty\right)$.

When $x = -\frac{3}{\sqrt{2}}, y = \frac{9}{2}$ and when $x = \frac{3}{\sqrt{2}}$, $y = \frac{9}{2}$.

The points are $\left(-\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$ and $\left(\frac{3}{\sqrt{2}}, \frac{9}{2}\right)$.

6. Let Q be the square of the distance between (x, y) and $(10, 0)$.

$Q = (x-10)^2 + (y-0)^2 = (2y^2 - 10)^2 + y^2$

$= 4y^4 - 39y^2 + 100$

$\frac{dQ}{dy} = 16y^3 - 78y$

$16y^3 - 78y = 0$

$2y(8y^2 - 39) = 0$

$y = 0, \pm\frac{\sqrt{39}}{2\sqrt{2}}$

$\frac{dQ}{dy} < 0$ on $\left(-\infty, -\frac{\sqrt{39}}{2\sqrt{2}}\right)$ and $\left(0, \frac{\sqrt{39}}{2\sqrt{2}}\right)$.

$\frac{dQ}{dy} > 0$ on $\left(-\frac{\sqrt{39}}{2\sqrt{2}}, 0\right)$ and $\left(\frac{\sqrt{39}}{2\sqrt{2}}, \infty\right)$.

When $y = -\frac{\sqrt{39}}{2\sqrt{2}}, x = \frac{39}{4}$ and when $y = \frac{\sqrt{39}}{2\sqrt{2}}, x = \frac{39}{4}$.

The points are $\left(\frac{39}{4}, -\frac{\sqrt{39}}{2\sqrt{2}}\right)$ and $\left(\frac{39}{4}, \frac{\sqrt{39}}{2\sqrt{2}}\right)$.

7. $x \ge x^2$ if $0 \le x \le 1$

$f(x) = x - x^2$; $f'(x) = 1 - 2x$;

$f'(x) = 0$ when $x = \frac{1}{2}$

Critical points: $0, \frac{1}{2}, 1$

$f(0) = 0$, $f(1) = 0$, $f\left(\frac{1}{2}\right) = \frac{1}{4}$; therefore, $\frac{1}{2}$ exceeds its square by the maximum amount.

8. For a rectangle with perimeter K and width x, the length is $\frac{K}{2} - x$. Then the area is

$A = x\left(\frac{K}{2} - x\right) = \frac{Kx}{2} - x^2$.

$\frac{dA}{dx} = \frac{K}{2} - 2x$; $\frac{dA}{dx} = 0$ when $x = \frac{K}{4}$

Critical points: $0, \frac{K}{4}, \frac{K}{2}$

At $x = 0$ or $\frac{K}{2}$, $A = 0$; at $x = \frac{K}{4}$, $A = \frac{K^2}{16}$.

The area is maximized when the width is one fourth of the perimeter, so the rectangle is a square.

9. Let x be the width of the square to be cut out and V the volume of the resulting open box.

$V = x(24-2x)^2 = 4x^3 - 96x^2 + 576x$

$\frac{dV}{dx} = 12x^2 - 192x + 576 = 12(x-12)(x-4);$

$12(x-12)(x-4) = 0$; $x = 12$ or $x = 4$.

Critical points: 0, 4, 12

At $x = 0$ or 12, $V = 0$; at $x = 4$, $V = 1024$.

The volume of the largest box is 1024 in.^3

10. Let A be the area of the pen.

$A = x(80-2x) = 80x - 2x^2; \frac{dA}{dx} = 80 - 4x;$

$80 - 4x = 0; x = 20$

Critical points: 0, 20, 40.

At $x = 0$ or 40, $A = 0$; at $x = 20$, $A = 800$.

The dimensions are 20 ft by $80 - 2(20) = 40$ ft, with the length along the barn being 40 ft.

11. Let x be the width of each pen, then the length along the barn is $80 - 4x$.

$A = x(80-4x) = 80x - 4x^2; \frac{dA}{dx} = 80 - 8x;$

$\frac{dA}{dx} = 0$ when $x = 10$.

Critical points: 0, 10, 20

At $x = 0$ or 20, $A = 0$; at $x = 10$, $A = 400$.

The area is largest with width 10 ft and length 40 ft.

12. Let A be the area of the pen. The perimeter is $100 + 180 = 280$ ft.

$y + y - 100 + 2x = 180$; $y = 140 - x$

$A = x(140-x) = 140x - x^2; \frac{dA}{dx} = 140 - 2x;$

$140 - 2x = 0; x = 70$

Since $0 \le x \le 40$, the critical points are 0 and 40. When $x = 0$, $A = 0$. When $x = 40$, $A = 4000$. The dimensions are 40 ft by 100 ft.

13. $xy = 900; y = \frac{900}{x}$

The possible values for x are in $(0, \infty)$.

$Q = 4x + 3y = 4x + 3\left(\frac{900}{x}\right) = 4x + \frac{2700}{x}$

$\frac{dQ}{dx} = 4 - \frac{2700}{x^2}$

$4 - \frac{2700}{x^2} = 0$

$x^2 = 675$

$x = \pm 15\sqrt{3}$

$x = 15\sqrt{3}$ is the only critical point in $(0, \infty)$.

$\frac{dQ}{dx} < 0$ on $(0, 15\sqrt{3})$ and

$\frac{dQ}{dx} > 0$ on $(15\sqrt{3}, \infty)$.

When $x = 15\sqrt{3}$, $y = \frac{900}{15\sqrt{3}} = 20\sqrt{3}$.

Q has a minimum when $x = 15\sqrt{3} \approx 25.98$ ft and $y = 20\sqrt{3} \approx 34.64$ ft.

14. $xy = 300; \ y = \frac{300}{x}$

The possible values for x are in $(0, \infty)$.

$Q = 6x + 4y = 6x + \frac{1200}{x}$

$\frac{dQ}{dx} = 6 - \frac{1200}{x^2}$

$6 - \frac{1200}{x^2} = 0$

$x^2 = 200$

$x = \pm 10\sqrt{2}$

$x = 10\sqrt{2}$ is the only critical point in $(0, \infty)$.

$\frac{dQ}{dx} < 0$ on $(0, 10\sqrt{2})$ and $\frac{dQ}{dx} > 0$ on $(10\sqrt{2}, \infty)$

When $x = 10\sqrt{2}$, $y = \frac{300}{10\sqrt{2}} = 15\sqrt{2}$.

Q has a minimum when $x = 10\sqrt{2} \approx 14.14$ ft and $y = 15\sqrt{2} \approx 21.21$ ft.

15. $xy = 300; \ y = \frac{300}{x}$

The possible values for x are in $(0, \infty)$.

$Q = 3(6x + 2y) + 2(2y) = 18x + 10y = 18x + \frac{3000}{x}$

$\frac{dQ}{dx} = 18 - \frac{3000}{x^2}$

$18 - \frac{3000}{x^2} = 0$

$x^2 = \frac{500}{3}$

$x = \pm \frac{10\sqrt{5}}{\sqrt{3}}$

$x = \frac{10\sqrt{5}}{\sqrt{3}}$ is the only critical point in $(0, \infty)$.

$\frac{dQ}{dx} < 0$ on $\left(0, \frac{10\sqrt{5}}{\sqrt{3}}\right)$ and

$\frac{dQ}{dx} > 0$ on $\left(\frac{10\sqrt{5}}{\sqrt{3}}, \infty\right)$.

When $x=\frac{10\sqrt{5}}{\sqrt{3}}, y=\frac{300}{\frac{10\sqrt{5}}{\sqrt{3}}}=6\sqrt{15}$

Q has a minimum when $x=\frac{10\sqrt{5}}{\sqrt{3}}\approx 12.91$ ft and $y=6\sqrt{15}\approx 23.24$ ft.

16. $xy = 900;\ y=\frac{900}{x}$

The possible values for x are in $(0, \infty)$.

$Q=6x+4y=6x+\frac{3600}{x}$

$\frac{dQ}{dx}=6-\frac{3600}{x^2}$

$6-\frac{3600}{x^2}=0$

$x^2=600$

$x=\pm 10\sqrt{6}$

$x=10\sqrt{6}$ is the only critical point in $(0, \infty)$.

$\frac{dQ}{dx}<0$ on $(0,10\sqrt{6})$ and $\frac{dQ}{dx}>0$ on $(10\sqrt{6},\infty)$.

When $x=10\sqrt{6}, y=\frac{900}{10\sqrt{6}}=15\sqrt{6}$

Q has a minimum when $x=10\sqrt{6}\approx 24.49$ ft and $y=15\sqrt{6}\approx 36.74$.

It appears that $\frac{x}{y}=\frac{2}{3}$.

Suppose that each pen has area A.

$xy=A;\ y=\frac{A}{x}$

The possible values for x are in $(0, \infty)$.

$Q=6x+4y=6x+\frac{4A}{x}$

$\frac{dQ}{dx}=6-\frac{4A}{x^2}$

$6-\frac{4A}{x^2}=0$

$x^2=\frac{2A}{3}$

$x=\pm\sqrt{\frac{2A}{3}}$

$x=\sqrt{\frac{2A}{3}}$ is the only critical point on $(0, \infty)$.

$\frac{dQ}{dx}<0$ on $\left(0,\sqrt{\frac{2\text{A}}{3}}\right)$ and

$\frac{dQ}{dx}>0$ on $\left(\sqrt{\frac{2A}{3}},\infty\right)$.

When $x=\sqrt{\frac{2A}{3}}, y=\frac{A}{\sqrt{\frac{2A}{3}}}=\sqrt{\frac{3A}{2}}$

$\frac{x}{y}=\frac{\sqrt{\frac{2A}{3}}}{\sqrt{\frac{3A}{2}}}=\frac{2}{3}$

17. Let D be the square of the distance.

$D=(x-0)^2+(y-4)^2=x^2+\left(\frac{x^2}{4}-4\right)^2$

$=\frac{x^4}{16}-x^2+16$

$\frac{dD}{dx}=\frac{x^3}{4}-2x;\ \frac{x^3}{4}-2x=0;\ x(x^2-8)=0$

$x=0, x=\pm 2\sqrt{2}$

Critical points: $0, 2\sqrt{2}, 2\sqrt{3}$

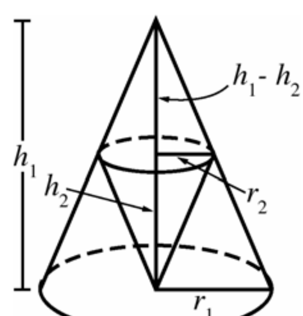
Since D is continuous and we are considering a closed interval for x, there is a maximum and minimum value of D on the interval. These extrema must occur at one of the critical points. At $x = 0$, $y = 0$, and $D = 16$. At $x=2\sqrt{2}$, $y = 2$, and $D = 12$. At $x=2\sqrt{3}$, $y = 3$, and $D = 13$.

Therefore, the point on $y=\frac{x^2}{4}$ closest to $(0,4)$ is $P\left(2\sqrt{2},2\right)$ and the point farthest from $(0,4)$ is $Q(0,0)$.

18. Let r_1 and h_1 be the radius and altitude of the outer cone; r_2 and h_2 the radius and altitude of the inner cone.

$V_1=\frac{1}{3}\pi r_1^2 h_1$ is fixed. $r_1=\sqrt{\frac{3V_1}{\pi h_1}}$

By similar triangles $\frac{h_1-h_2}{h_1}=\frac{r_2}{r_1}$ (see figure).



$r_2=r_1\left(1-\frac{h_2}{h_1}\right)=\sqrt{\frac{3V_1}{\pi h_1}}\left(1-\frac{h_2}{h_1}\right)$

$$V_2 = \frac{1}{3}\pi r_2{}^2 h_2 = \frac{1}{3}\pi\left[\sqrt{\frac{3V_1}{\pi h_1}}\left(1-\frac{h_2}{h_1}\right)\right]^2 h_2$$

$$= \frac{\pi}{3}\cdot\frac{3V_1 h_2}{\pi h_1}\left(1-\frac{h_2}{h_1}\right)^2 = V_1\frac{h_2}{h_1}\left(1-\frac{h_2}{h_1}\right)^2$$

Let $k = \frac{h_2}{h_1}$, the ratio of the altitudes of the cones, then $V_2 = V_1 k(1-k)^2$.

$$\frac{dV_2}{dk} = V_1(1-k)^2 - 2kV_1(1-k) = V_1(1-k)(1-3k)$$

$0 < k < 1$ so $\frac{dV_2}{dk} = 0$ when $k = \frac{1}{3}$.

$$\frac{d^2V_2}{dk^2} = V_1(6k-4); \frac{d^2V_2}{dk^2} < 0 \text{ when } k = \frac{1}{3}$$

The altitude of the inner cone must be $\frac{1}{3}$ the altitude of the outer cone.

19. Let x be the distance from P to where the woman lands the boat. She must row a distance of $\sqrt{x^2+4}$ miles and walk $10 - x$ miles. This will take her $T(x) = \frac{\sqrt{x^2+4}}{3} + \frac{10-x}{4}$ hours;

$0 \le x \le 10$. $T'(x) = \frac{x}{3\sqrt{x^2+4}} - \frac{1}{4}; T'(x) = 0$ when $x = \frac{6}{\sqrt{7}}$.

$T(0) = \frac{19}{6}$ hr = 3 hr 10 min ≈ 3.17 hr ,

$T\left(\frac{6}{\sqrt{7}}\right) = \frac{15+\sqrt{7}}{6} \approx 2.94$ hr,

$T(10) = \frac{\sqrt{104}}{3} \approx 3.40$ hr

She should land the boat $\frac{6}{\sqrt{7}} \approx 2.27$ mi down the shore from P.

20. $T(x) = \frac{\sqrt{x^2+4}}{3} + \frac{10-x}{50}, 0 \le x \le 10.$

$T'(x) = \frac{x}{3\sqrt{x^2-4}} - \frac{1}{50}; T'(x) = 0$ when

$x = \frac{6}{\sqrt{2491}}$

$T(0) = \frac{13}{15} \approx 0.867$ hr; $T\left(\frac{6}{\sqrt{2491}}\right) \approx 0.865$ hr;

$T(10) \approx 3.399$ hr

She should land the boat $\frac{6}{\sqrt{2491}} \approx 0.12$ mi down the shore from P.

21. $T(x) = \frac{\sqrt{x^2+4}}{20} + \frac{10-x}{4}, 0 \le x \le 10.$

$T'(x) = \frac{x}{20\sqrt{x^2+4}} - \frac{1}{4}; T'(x) = 0$ has no solution.

$T(0) = \frac{2}{20} + \frac{10}{4} = \frac{13}{5}$ hr = 2 hr, 36 min

$T(10) = \frac{\sqrt{104}}{20} \approx 0.5$ hr

She should take the boat all the way to town.

22. Let x be the length of cable on land, $0 \le x \le L$. Let C be the cost.

$$C = a\sqrt{(L-x)^2 + w^2} + bx$$

$$\frac{dC}{dx} = -\frac{a(L-x)}{\sqrt{(L-x)^2+w^2}} + b$$

$-\frac{a(L-x)}{\sqrt{(L-x)^2+w^2}} + b = 0$ when

$$b^2[(L-x)^2 + w^2] = a^2(L-x)^2$$

$$(a^2 - b^2)(L-x)^2 = b^2w^2$$

$x = L - \frac{bw}{\sqrt{a^2-b^2}}$ ft on land;

$\frac{aw}{\sqrt{a^2-b^2}}$ ft under water

$\frac{d^2C}{dx^2} = \frac{aw^2}{[(L-x)^2+w^2]^{3/2}} > 0$ for all x, so this minimizes the cost.

23. Let the coordinates of the first ship at 7:00 a.m. be (0, 0). Thus, the coordinates of the second ship at 7:00 a.m. are (–60, 0). Let t be the time in hours since 7:00 a.m. The coordinates of the first and second ships at t are $(-20t, 0)$ and $\left(-60+15\sqrt{2}t, -15\sqrt{2}t\right)$ respectively. Let D be the square of the distances at t.

$$D = \left(-20t+60-15\sqrt{2}t\right)^2 + \left(0+15\sqrt{2}t\right)^2$$

$$= \left(1300+600\sqrt{2}\right)t^2 - \left(2400+1800\sqrt{2}\right)t + 3600$$

$$\frac{dD}{dt} = 2\left(1300+600\sqrt{2}\right)t - \left(2400+1800\sqrt{2}\right)$$

$2\left(1300+600\sqrt{2}\right)t - \left(2400+1800\sqrt{2}\right) = 0$ when

$t = \dfrac{12+9\sqrt{2}}{13+6\sqrt{2}} \approx 1.15$ hrs or 1 hr, 9 min

D is the minimum at $t = \dfrac{12+9\sqrt{2}}{13+6\sqrt{2}}$ since $\dfrac{d^2D}{dt^2} > 0$ for all t.

The ships are closest at 8:09 A.M.

24. Write y in terms of x: $y = \dfrac{b}{a}\sqrt{a^2-x^2}$ (positive square root since the point is in the first quadrant).

Compute the slope of the tangent line:

$$y' = -\frac{bx}{a\sqrt{a^2-x^2}}.$$

Find the y-intercept, y_0, of the tangent line through the point (x, y):

$$\frac{y_0-y}{0-x} = -\frac{bx}{a\sqrt{a^2-x^2}}$$

$$y_0 = \frac{bx^2}{a\sqrt{a^2-x^2}} + y = \frac{bx^2}{a\sqrt{a^2-x^2}} + \frac{b}{a}\sqrt{a^2-x^2}$$

$$= \frac{ab}{\sqrt{a^2-x^2}}$$

Find the x-intercept, x_0, of the tangent line through the point (x, y):

$$\frac{y-0}{x-x_0} = -\frac{bx}{a\sqrt{a^2-x^2}}$$

$$x_0 = \frac{ay\sqrt{a^2-x^2}}{bx} + x = \frac{a^2-x^2}{x} + x = \frac{a^2}{x}$$

Compute the Area A of the resulting triangle and maximize:

$$A = \frac{1}{2}x_0y_0 = \frac{a^3b}{2x\sqrt{a^2-x^2}} = \frac{a^3b}{2}\left(x\sqrt{a^2-x^2}\right)^{-1}$$

$$\frac{dA}{dx} = -\frac{a^3b}{2}\left(x\sqrt{a^2-x^2}\right)^{-2}\left(\sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}}\right)$$

$$= \frac{a^3b}{2x^2(a^2-x^2)^{3/2}}(2x^2-a^2)$$

$\dfrac{a^3b}{2x^2(a^2-x^2)^{3/2}}(2x^2-a^2) = 0$ when

$$x = \frac{a}{\sqrt{2}};\ y = \frac{b}{a}\sqrt{a^2-\left(\frac{a}{\sqrt{2}}\right)^2} = \frac{b}{\sqrt{2}}$$

$$y' = -\frac{b\left(\frac{a}{\sqrt{2}}\right)}{a\sqrt{a^2-\left(\frac{a}{\sqrt{2}}\right)^2}} = -\frac{b}{a}$$

Note that $\dfrac{dA}{dx} < 0$ on $\left(0, \dfrac{a}{\sqrt{2}}\right)$ and $\dfrac{dA}{dx} > 0$ on $\left(\dfrac{a}{\sqrt{2}}, a\right)$, so A is a minimum at $x = \dfrac{a}{\sqrt{2}}$. Then the equation of the tangent line is

$y = -\dfrac{b}{a}\left(x - \dfrac{a}{\sqrt{2}}\right) + \dfrac{b}{\sqrt{2}}$ or $bx + ay - ab\sqrt{2} = 0$.

25. Let x be the radius of the base of the cylinder and h the height.

$$V = \pi x^2h;\ r^2 = x^2 + \left(\frac{h}{2}\right)^2;\ x^2 = r^2 - \frac{h^2}{4}$$

$$V = \pi\left(r^2 - \frac{h^2}{4}\right)h = \pi hr^2 - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4};\ V' = 0 \text{ when } h = \pm\frac{2\sqrt{3}r}{3}$$

Since $\dfrac{d^2V}{dh^2} = -\dfrac{3\pi h}{2}$, the volume is maximized when $h = \dfrac{2\sqrt{3}r}{3}$.

$$V = \pi\left(\frac{2\sqrt{3}}{3}r\right)r^2 - \frac{\pi\left(\frac{2\sqrt{3}}{3}r\right)^3}{4}$$

$$= \frac{2\pi\sqrt{3}}{3}r^3 - \frac{2\pi\sqrt{3}}{9}r^3 = \frac{4\pi\sqrt{3}}{9}r^3$$

26. Let r be the radius of the circle, x the length of the rectangle, and y the width of the rectangle.

$$P = 2x + 2y;\ r^2 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2;\ r^2 = \frac{x^2}{4} + \frac{y^2}{4};$$

$$y = \sqrt{4r^2-x^2};\ P = 2x + 2\sqrt{4r^2-x^2}$$

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4r^2-x^2}};$$

$$2-\frac{2x}{\sqrt{4r^2-x^2}}=0;\ 2\sqrt{4r^2-x^2}=2x;$$

$16r^2-4x^2=4x^2;\ x=\pm\sqrt{2}r$

$$\frac{d^2P}{dx^2}=-\frac{8r^2}{(4r^2-x^2)^{3/2}}<0 \text{ when } x=\sqrt{2}r;$$

$y=\sqrt{4r^2-2r^2}=\sqrt{2}r$

The rectangle with maximum perimeter is a square with side length $\sqrt{2}r$.

27. Let x be the radius of the cylinder, r the radius of the sphere, and h the height of the cylinder.

$$A=2\pi xh;\ r^2=x^2+\frac{h^2}{4};\ x=\sqrt{r^2-\frac{h^2}{4}}$$

$$A=2\pi\sqrt{r^2-\frac{h^2}{4}}h=2\pi\sqrt{h^2r^2-\frac{h^4}{4}}$$

$$\frac{dA}{dh}=\frac{\pi\left(2r^2h-h^3\right)}{\sqrt{h^2r^2-\frac{h^4}{4}}};\ A'=0 \text{ when } h=0,\pm\sqrt{2}r$$

$\frac{dA}{dh}>0$ on $(0,\sqrt{2}r)$ and $\frac{dA}{dh}<0$ on $(\sqrt{2}r,2r)$,

so A is a maximum when $h=\sqrt{2}r$.

The dimensions are $h=\sqrt{2}r,\ x=\frac{r}{\sqrt{2}}$.

28. Let x be the distance from I_1.

$$Q=\frac{kI_1}{x^2}+\frac{kI_2}{(s-x)^2}$$

$$\frac{dQ}{dx}=\frac{-2kI_1}{x^3}+\frac{2kI_2}{(s-x)^3}$$

$$-\frac{2kI_1}{x^3}+\frac{2kI_2}{(s-x)^3}=0;\ \frac{x^3}{(s-x)^3}=\frac{I_1}{I_2};$$

$$x=\frac{s\sqrt[3]{I_1}}{\sqrt[3]{I_1}+\sqrt[3]{I_2}}$$

$$\frac{d^2Q}{dx^2}=\frac{6kI_1}{x^4}+\frac{6kI_2}{(s-x)^4}>0,$$ so this point minimizes the sum.

29. Let x be the length of a side of the square, so $\frac{100-4x}{3}$ is the side of the triangle, $0\le x\le 25$

$$A=x^2+\frac{1}{2}\left(\frac{100-4x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{100-4x}{3}\right)$$

$$=x^2+\frac{\sqrt{3}}{4}\left(\frac{10{,}000-800x+16x^2}{9}\right)$$

$$\frac{dA}{dx}=2x-\frac{200\sqrt{3}}{9}+\frac{8\sqrt{3}}{9}x$$

$$A'(x)=0 \text{ when } x=\frac{300\sqrt{3}}{11}-\frac{400}{11}\approx 10.874.$$

Critical points: $x = 0, 10.874, 25$

At $x = 0$, $A\approx 481$; at $x = 10.874$, $A\approx 272$; at $x = 25$, $A = 625$.

a. For minimum area, the cut should be approximately $4(10.874)\approx 43.50$ cm from one end and the shorter length should be bent to form the square.

b. For maximum area, the wire should not be cut; it should be bent to form a square.

30. Let x be the length of the sides of the base, y be the height of the box, and k be the cost per square inch of the material in the sides of the box.

$V=x^2y;$

The cost is $C=1.2kx^2+1.5kx^2+4kxy$

$$=2.7kx^2+4kx\left(\frac{V}{x^2}\right)=2.7kx^2+\frac{4kV}{x}$$

$$\frac{dC}{dx}=5.4kx-\frac{4kV}{x^2};\ \frac{dC}{dx}=0 \text{ when } x\approx 0.905\sqrt[3]{V}$$

$$y\approx\frac{V}{(0.905\sqrt[3]{V})^2}\approx 1.22\sqrt[3]{V}$$

31. Let r be the radius of the cylinder and h the height of the cylinder.

$$V=\pi r^2h+\frac{2}{3}\pi r^3;\ h=\frac{V-\frac{2}{3}\pi r^3}{\pi r^2}=\frac{V}{\pi r^2}-\frac{2}{3}r$$

Let k be the cost per square foot of the cylindrical wall. The cost is

$C=k(2\pi rh)+2k(2\pi r^2)$

$$=k\left(2\pi r\left(\frac{V}{\pi r^2}-\frac{2}{3}r\right)+4\pi r^2\right)=k\left(\frac{2V}{r}+\frac{8\pi r^2}{3}\right)$$

$$\frac{dC}{dr}=k\left(-\frac{2V}{r^2}+\frac{16\pi r}{3}\right);\ k\left(-\frac{2V}{r^2}+\frac{16\pi r}{3}\right)=0$$

when $r^3=\frac{3V}{8\pi},\ r=\frac{1}{2}\left(\frac{3V}{\pi}\right)^{1/3}$

$$h=\frac{4V}{\pi\left(\frac{3V}{\pi}\right)^{2/3}}-\frac{1}{3}\left(\frac{3V}{\pi}\right)^{1/3}=\left(\frac{3V}{\pi}\right)^{1/3}$$

For a given volume V, the height of the cylinder is $\left(\frac{3V}{\pi}\right)^{1/3}$ and the radius is $\frac{1}{2}\left(\frac{3V}{\pi}\right)^{1/3}$.

32. $\frac{dx}{dt} = 2\cos 2t - 2\sqrt{3}\sin 2t;$

$\frac{dx}{dt} = 0$ when $\tan 2t = \frac{1}{\sqrt{3}};$

$2t = \frac{\pi}{6} + \pi n$ for any integer n

$t = \frac{\pi}{12} + \frac{\pi}{2}n$

When $t = \frac{\pi}{12} + \frac{\pi}{2}n$,

$$|x| = \left|\sin\left(\frac{\pi}{6} + \pi n\right) + \sqrt{3}\cos\left(\frac{\pi}{6} + \pi n\right)\right|$$

$$= \left|\sin\frac{\pi}{6}\cos\pi n + \cos\frac{\pi}{6}\sin\pi n \right.$$

$$\left. + \sqrt{3}(\cos\frac{\pi}{6}\cos\pi n - \sin\frac{\pi}{6}\sin\pi n)\right|$$

$$= \left|(-1)^n\frac{1}{2} + (-1)^n\frac{3}{2}\right| = 2.$$

The farthest the weight gets from the origin is 2 units.

33. $A = \frac{r^2\theta}{2};\ \ \theta = \frac{2A}{r^2}$

The perimeter is

$Q = 2r + r\theta = 2r + \frac{2Ar}{r^2} = 2r + \frac{2A}{r}$

$\frac{dQ}{dr} = 2 - \frac{2A}{r^2}; Q' = 0$ when $r = \sqrt{A}$

$\theta = \frac{2A}{(\sqrt{A})^2} = 2$

$\frac{d^2Q}{dr^2} = \frac{4A}{r^3} > 0,$ so this minimizes the perimeter.

34. The distance from the fence to the base of the ladder is $\frac{h}{\tan\theta}$.

The length of the ladder is x.

$\cos\theta = \frac{\frac{h}{\tan\theta} + w}{x}; x\cos\theta = \frac{h}{\tan\theta} + w;$

$x = \frac{h}{\sin\theta} + \frac{w}{\cos\theta}$

$\frac{dx}{d\theta} = -\frac{h\cos\theta}{\sin^2\theta} + \frac{w\sin\theta}{\cos^2\theta}; \frac{w\sin^3\theta - h\cos^3\theta}{\sin^2\theta\cos^2\theta} = 0$

when $\tan^3\theta = \frac{h}{w}$

$\theta = \tan^{-1}\sqrt[3]{\frac{h}{w}}$

$\tan\theta = \frac{\sqrt[3]{h}}{\sqrt[3]{w}}; \sin\theta = \frac{\sqrt[3]{h}}{\sqrt{h^{2/3} + w^{2/3}}},$

$\cos\theta = \frac{\sqrt[3]{w}}{\sqrt{h^{2/3} + w^{2/3}}}$

$x = h\left(\frac{\sqrt{h^{2/3} + w^{2/3}}}{\sqrt[3]{h}}\right) + w\left(\frac{\sqrt{h^{2/3} + w^{2/3}}}{\sqrt[3]{w}}\right)$

$= (h^{2/3} + w^{2/3})^{3/2}$

35. x is limited by $0 \le x \le \sqrt{12}$.

$A = 2x(12 - x^2) = 24x - 2x^3; \frac{dA}{dx} = 24 - 6x^2;$

$24 - 6x^2 = 0; x = -2, 2$

Critical points: $0, 2, \sqrt{12}.$

When $x = 0$ or $\sqrt{12},\ A = 0.$

When $x = 2,\ y = 12 - (2)^2 = 8.$

The dimensions are $2x = 2(2) = 4$ by 8.

36. Let the x-axis lie on the diameter of the semicircle and the y-axis pass through the middle.

Then the equation $y = \sqrt{r^2 - x^2}$ describes the semicircle. Let (x, y) be the upper-right corner of the rectangle. x is limited by $0 \le x \le r$.

$A = 2xy = 2x\sqrt{r^2 - x^2}$

$\frac{dA}{dx} = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2}{\sqrt{r^2 - x^2}}(r^2 - 2x^2)$

$\frac{2}{\sqrt{r^2 - x^2}}(r^2 - 2x^2) = 0; x = \frac{r}{\sqrt{2}}$

Critical points: $0, \frac{r}{\sqrt{2}}, r$

When $x = 0$ or r, $A = 0$. When $x = \frac{r}{\sqrt{2}}, A = r^2.$

$y = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = \frac{r}{\sqrt{2}}$

The dimensions are $\frac{r}{\sqrt{2}}$ by $\frac{2r}{\sqrt{2}}$.

37. If the end of the cylinder has radius r and h is the height of the cylinder, the surface area is

$A = 2\pi r^2 + 2\pi rh$ so $h = \frac{A}{2\pi r} - r.$

The volume is

$V = \pi r^2 h = \pi r^2\left(\frac{A}{2\pi r} - r\right) = \frac{Ar}{2} - \pi r^3.$

$V'(r) = \frac{A}{2} - 3\pi r^2; V'(r) = 0$ when $r = \sqrt{\frac{A}{6\pi}},$

$V''(r) = -6\pi r,$ so the volume is maximum when

$r = \sqrt{\frac{A}{6\pi}}.$

$h = \frac{A}{2\pi r} - r = 2\sqrt{\frac{A}{6\pi}} = 2r$

38. The ellipse has equation

$y = \pm\sqrt{b^2 - \frac{b^2x^2}{a^2}} = \pm\frac{b}{a}\sqrt{a^2 - x^2}$

Let $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right-hand corner of the rectangle (use a and b positive). Then the dimensions of the rectangle are $2x$ by $\frac{2b}{a}\sqrt{a^2 - x^2}$ and the area is

$A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2}.$

$A'(x) = \frac{4b}{a}\sqrt{a^2 - x^2} - \frac{4bx^2}{a\sqrt{a^2 - x^2}} = \frac{4b(a^2 - 2x^2)}{a\sqrt{a^2 - x^2}};$

$A'(x) = 0$ when $x = \frac{a}{\sqrt{2}},$ so the corner is at

$\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right).$ The corners of the rectangle are at

$\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right),$

$\left(\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\right).$

The dimensions are $a\sqrt{2}$ and $b\sqrt{2}$.

39. If the rectangle has length l and width w, the diagonal is $d = \sqrt{l^2 + w^2}$, so $l = \sqrt{d^2 - w^2}$. The area is $A = lw = w\sqrt{d^2 - w^2}$.

$A'(w) = \sqrt{d^2 - w^2} - \frac{w^2}{\sqrt{d^2 - w^2}} = \frac{d^2 - 2w^2}{\sqrt{d^2 - w^2}};$

$A'(w) = 0$ when $w = \frac{d}{\sqrt{2}}$ and so

$l = \sqrt{d^2 - \frac{d^2}{2}} = \frac{d}{\sqrt{2}}.$ $A'(w) > 0$ on $\left(0, \frac{d}{\sqrt{2}}\right)$ and $A'(w) < 0$ on $\left(\frac{d}{\sqrt{2}}, d\right)$. Maximum area is for a square.

40. Note that $\cos t = \frac{h}{r},$ so $h = r\cos t,$

$\sin t = \frac{1}{r}\sqrt{r^2 - h^2}$, and $\sqrt{r^2 - h^2} = r\sin t$

Area of submerged region $= tr^2 - h\sqrt{r^2 - h^2}$

$= tr^2 - (r\cos t)(r\sin t) = r^2(t - \cos t\sin t)$

A = area of exposed wetted region

$= \pi r^2 - \pi h^2 - r^2(t - \cos t\sin t)$

$= r^2(\pi - \pi\cos^2 t - t + \cos t\sin t)$

$\frac{dA}{dt} = r^2(2\pi\cos t\sin t - 1 + \cos^2 t - \sin^2 t)$

$= r^2(2\pi\cos t\sin t - 2\sin^2 t)$

$= 2r^2\sin t(\pi\cos t - \sin t)$

Since $0 < t < \pi$, $\frac{dA}{dt} = 0$ only when

$\pi\cos t = \sin t$ or $\tan t = \pi$. In terms of r and h,

this is $\frac{\frac{1}{r}\sqrt{r^2 - h^2}}{\frac{h}{r}} = \pi$ or $h = \frac{r}{\sqrt{1 + \pi^2}}$.

41. The carrying capacity of the gutter is maximized when the area of the vertical end of the gutter is maximized. The height of the gutter is $3\sin\theta$. The area is

$A = 3(3\sin\theta) + 2\left(\frac{1}{2}\right)(3\cos\theta)(3\sin\theta)$

$= 9\sin\theta + 9\cos\theta\sin\theta$.

$\frac{dA}{d\theta} = 9\cos\theta + 9(-\sin\theta)\sin\theta + 9\cos\theta\cos\theta$

$= 9(\cos\theta - \sin^2\theta + \cos^2\theta)$

$= 9(2\cos^2\theta + \cos\theta - 1)$

$2\cos^2\theta + \cos\theta - 1 = 0;$ $\cos\theta = -1, \frac{1}{2}; \theta = \pi, \frac{\pi}{3}$

Since $0 \le \theta \le \frac{\pi}{2},$ the critical points are

$0, \frac{\pi}{3},$ and $\frac{\pi}{2}.$

When $\theta = 0$, $A = 0$.

When $\theta = \frac{\pi}{3}, A = \frac{27\sqrt{3}}{4} \approx 11.7.$

When $\theta = \frac{\pi}{2},$ $A = 9.$

The carrying capacity is maximized when $\theta = \frac{\pi}{3}.$

42. The circumference of the top of the tank is the circumference of the circular sheet minus the arc length of the sector, $20\pi - 10\theta$ meters. The radius of the top of the tank is $r = \frac{20\pi - 10\theta}{2\pi} = \frac{5}{\pi}(2\pi - \theta)$ meters. The slant height of the tank is 10 meters, so the height of the tank is

$$h = \sqrt{10^2 - \left(10 - \frac{5\theta}{\pi}\right)^2} = \frac{5}{\pi}\sqrt{4\pi\theta - \theta^2} \text{ meters.}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left[\frac{5}{\pi}(2\pi - \theta)\right]^2\left[\frac{5}{\pi}\sqrt{4\pi\theta - \theta^2}\right]$$

$$= \frac{125}{3\pi^2}(2\pi - \theta)^2\sqrt{4\pi\theta - \theta^2}$$

$$\frac{dV}{d\theta} = \frac{125}{3\pi^2}\left(2(2\pi - \theta)(-1)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2\left(\frac{1}{2}\right)(4\pi - 2\theta)}{\sqrt{4\pi\theta - \theta^2}}\right)$$

$$= \frac{125(2\pi - \theta)}{3\pi^2\sqrt{4\pi\theta - \theta^2}}(3\theta^2 - 12\pi\theta + 4\pi^2);$$

$$\frac{125(2\pi - \theta)}{3\pi^2\sqrt{4\pi\theta - \theta^2}}(3\theta^2 - 12\pi\theta + 4\pi^2) = 0$$

$$2\pi - \theta = 0 \text{ or } 3\theta^2 - 12\pi\theta + 4\pi^2 = 0$$

$$\theta = 2\pi, \theta = 2\pi - \frac{2\sqrt{6}}{3}\pi, \theta = 2\pi + \frac{2\sqrt{6}}{3}\pi$$

Since $0 < \theta < 2\pi$, the only critical point is $2\pi - \frac{2\sqrt{6}}{3}\pi$. A graph shows that this maximizes the volume.

43. Let V be the volume. $y = 4 - x$ and $z = 5 - 2x$. x is limited by $0 \le x \le 2.5$.

$$V = x(4 - x)(5 - 2x) = 20x - 13x^2 + 2x^3$$

$$\frac{dV}{dx} = 20 - 26x + 6x^2; 2(3x^2 - 13x + 10) = 0;$$

$$2(3x - 10)(x - 1) = 0;$$

$$x = 1, \frac{10}{3}$$

Critical points: 0, 1, 2.5

At $x = 0$ or 2.5, $V = 0$. At $x = 1$, $V = 9$.

Maximum volume when $x = 1$, $y = 4 - 1 = 3$, and $z = 5 - 2(1) = 3$.

44. Let x be the length of the edges of the cube. The surface area of the cube is $6x^2$ so $0 \le x \le \frac{1}{\sqrt{6}}$.

The surface area of the sphere is $4\pi r^2$, so

$$6x^2 + 4\pi r^2 = 1, r = \sqrt{\frac{1 - 6x^2}{4\pi}}$$

$$V = x^3 + \frac{4}{3}\pi r^3 = x^3 + \frac{1}{6\sqrt{\pi}}(1 - 6x^2)^{3/2}$$

$$\frac{dV}{dx} = 3x^2 - \frac{3}{\sqrt{\pi}}x\sqrt{1 - 6x^2} = 3x\left(x - \sqrt{\frac{1 - 6x^2}{\pi}}\right)$$

$$\frac{dV}{dx} = 0 \text{ when } x = 0, \frac{1}{\sqrt{6 + \pi}}$$

$$V(0) = \frac{1}{6\sqrt{\pi}} \approx 0.094 \text{ m}^3.$$

$$V\left(\frac{1}{\sqrt{6 + \pi}}\right) = (6 + \pi)^{-3/2} + \frac{1}{6\sqrt{\pi}}\left(1 - \frac{6}{6 + \pi}\right)^{3/2}$$

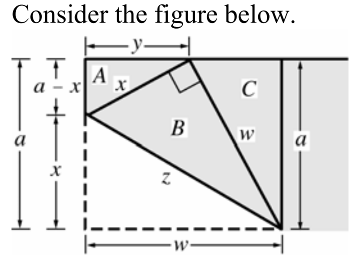
$$= \left(1 + \frac{\pi}{6}\right)(6 + \pi)^{-3/2} = \frac{1}{6\sqrt{6 + \pi}} \approx 0.055 \text{ m}^3$$

For maximum volume: no cube, a sphere of radius $\frac{1}{2\sqrt{\pi}} \approx 0.282$ meters.

For minimum volume: cube with sides of length $\frac{1}{\sqrt{6 + \pi}} \approx 0.331$ meters,

sphere of radius $\frac{1}{2\sqrt{6 + \pi}} \approx 0.165$ meters

45. Consider the figure below.



a. $y = \sqrt{x^2 - (a - x)^2} = \sqrt{2ax - a^2}$

Area of A $= A = \frac{1}{2}(a - x)y$

$$= \frac{1}{2}(a - x)\sqrt{2ax - a^2}$$

$$\frac{dA}{dx} = -\frac{1}{2}\sqrt{2ax - a^2} + \frac{\frac{1}{2}(a - x)\left(\frac{1}{2}\right)(2a)}{\sqrt{2ax - a^2}}$$

$$= \frac{a^2 - \frac{3}{2}ax}{\sqrt{2ax - a^2}}$$

$$\frac{a^2-\frac{3}{2}ax}{\sqrt{2ax-a^2}}=0 \text{ when } x=\frac{2a}{3}.$$

$\frac{dA}{dx}>0$ on $\left(\frac{a}{2},\frac{2a}{3}\right)$ and $\frac{dA}{dx}<0$ on $\left(\frac{2a}{3},a\right)$,

so $x=\frac{2a}{3}$ maximizes the area of triangle A.

b. Triangle A is similar to triangle C, so

$$w=\frac{ax}{y}=\frac{ax}{\sqrt{2ax-a^2}}$$

$$\text{Area of } B = B=\frac{1}{2}xw=\frac{ax^2}{2\sqrt{2ax-a^2}}$$

$$\frac{dB}{dx}=\frac{a}{2}\left(\frac{2x\sqrt{2ax-a^2}-x^2\frac{a}{\sqrt{2ax-a^2}}}{2ax-a^2}\right)$$

$$=\frac{a}{2}\left(\frac{2x(2ax-a^2)-ax^2}{(2ax-a^2)^{3/2}}\right)=\frac{a}{2}\left(\frac{3ax^2-2xa^2}{(2ax-a^2)^{3/2}}\right)$$

$$\frac{a^2}{2}\left(\frac{3x^2-2xa}{(2ax-a^2)^{3/2}}\right)=0 \text{ when } x=0,\frac{2a}{3}$$

Since $x=0$ is not possible, $x=\frac{2a}{3}$.

$\frac{dB}{dx}<0$ on $\left(\frac{a}{2},\frac{2a}{3}\right)$ and $\frac{dB}{dx}>0$ on $\left(\frac{2a}{3},a\right)$,

so $x=\frac{2a}{3}$ minimizes the area of triangle B.

c. $z=\sqrt{x^2+w^2}=\sqrt{x^2+\frac{a^2x^2}{2ax-a^2}}$

$$=\sqrt{\frac{2ax^3}{2ax-a^2}}$$

$$\frac{dz}{dx}=\frac{1}{2}\sqrt{\frac{2ax-a^2}{2ax^3}}\left(\frac{6ax^2(2ax-a^2)-2ax^3(2a)}{(2ax-a^2)^2}\right)$$

$$=\frac{4a^2x^3-3a^3x^2}{\sqrt{2ax^3(2ax-a^2)^3}}$$

$\frac{dz}{dx}=0$ when $x=0,\frac{3a}{4} \quad\to\quad x=\frac{3a}{4}$

$\frac{dz}{dx}<0$ on $\left(\frac{a}{2},\frac{3a}{4}\right)$ and $\frac{dz}{dx}>0$ on $\left(\frac{3a}{4},a\right)$,

so $x=\frac{3a}{4}$ minimizes length z.

46. Let $2x$ be the length of a bar and $2y$ be the width of a bar.

$$x=a\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=a\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}+\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right)=\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$$

$$y=a\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=a\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}-\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right)=\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)$$

Compute the area A of the cross and maximize.

$$A=2(2x)(2y)-(2y)^2=8\left[\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\right]\left[\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\right]-4\left[\frac{a}{\sqrt{2}}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\right]^2$$

$$=4a^2\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)-2a^2\left(1-2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)=4a^2\cos\theta-2a^2(1-\sin\theta)$$

$$\frac{dA}{d\theta}=-4a^2\sin\theta+2a^2\cos\theta;\quad -4a^2\sin\theta+2a^2\cos\theta=0 \text{ when } \tan\theta=\frac{1}{2};$$

$$\sin\theta=\frac{1}{\sqrt{5}},\cos\theta=\frac{2}{\sqrt{5}}$$

$\frac{d^2A}{d\theta^2}<0$ when $\tan\theta=\frac{1}{2}$, so this maximizes the area.

$$A=4a^2\left(\frac{2}{\sqrt{5}}\right)-2a^2\left(1-\frac{1}{\sqrt{5}}\right)=\frac{10a^2}{\sqrt{5}}-2a^2=2a^2(\sqrt{5}-1)$$

47. **a.** $L'(\theta)=15(9+25-30\cos\theta)^{-1/2}\sin\theta=15(34-30\cos\theta)^{-1/2}\sin\theta$

$L''(\theta)=-\frac{15}{2}(34-30\cos\theta)^{-3/2}(30\sin\theta)\sin\theta+15(34-30\cos\theta)^{-1/2}\cos\theta$

$=-225(34-30\cos\theta)^{-3/2}\sin^2\theta+15(34-30\cos\theta)^{-1/2}\cos\theta$

$=15(34-30\cos\theta)^{-3/2}[-15\sin^2\theta+(34-30\cos\theta)\cos\theta]$

$=15(34-30\cos\theta)^{-3/2}[-15\sin^2\theta+34\cos\theta-30\cos^2\theta]$

$=15(34-30\cos\theta)^{-3/2}[-15+34\cos\theta-15\cos^2\theta]$

$=-15(34-30\cos\theta)^{-3/2}[15\cos^2\theta-34\cos\theta+15]$

$L''=0$ when $\cos\theta=\frac{34\pm\sqrt{(34)^2-4(15)(15)}}{2(15)}=\frac{5}{3},\frac{3}{5}$

$\theta=\cos^{-1}\left(\frac{3}{5}\right)$

$L'\left(\cos^{-1}\left(\frac{3}{5}\right)\right)=15\left(9+25-30\left(\frac{3}{5}\right)\right)^{-1/2}\left(\frac{4}{5}\right)=3$

$L\left(\cos^{-1}\left(\frac{3}{5}\right)\right)=\left(9+25-30\left(\frac{3}{5}\right)\right)^{1/2}=4$

$\phi=90°$ since the resulting triangle is a 3-4-5 right triangle.

b. $L'(\theta)=65(25+169-130\cos\theta)^{-1/2}\sin\theta=65(194-130\cos\theta)^{-1/2}\sin\theta$

$L''(\theta)=-\frac{65}{2}(194-130\cos\theta)^{-3/2}(130\sin\theta)\sin\theta+65(194-130\cos\theta)^{-1/2}\cos\theta$

$=-4225(194-130\cos\theta)^{-3/2}\sin^2\theta+65(194-130\cos\theta)^{-1/2}\cos\theta$

$=65(194-130\cos\theta)^{-3/2}[-65\sin^2\theta+(194-130\cos\theta)\cos\theta]$

$=65(194-130\cos\theta)^{-3/2}[-65\sin^2\theta+194\cos\theta-130\cos^2\theta]$

$=65(194-130\cos\theta)^{-3/2}[-65\cos^2\theta+194\cos\theta-65]$

$=-65(194-130\cos\theta)^{-3/2}[65\cos^2\theta-194\cos\theta+65]$

$L''=0$ when $\cos\theta=\frac{194\pm\sqrt{(194)^2-4(65)(65)}}{2(65)}=\frac{13}{5},\frac{5}{13}$

$\theta=\cos^{-1}\left(\frac{5}{13}\right)$

$L'\left(\cos^{-1}\left(\frac{5}{13}\right)\right)=65\left(25+169-130\left(\frac{5}{13}\right)\right)^{1/2}\left(\frac{12}{13}\right)=5$

$L\left(\cos^{-1}\left(\frac{5}{13}\right)\right)=\left(25+169-130\left(\frac{5}{13}\right)\right)^{1/2}=12$

$\phi=90°$ since the resulting triangle is a 5-12-13 right triangle.

c. When the tips are separating most rapidly, $\phi=90°, L=\sqrt{m^2-h^2}, L'=h$

d. $L'(\theta) = hm(h^2 + m^2 - 2hm\cos\theta)^{-1/2}\sin\theta$

$L''(\theta) = -h^2m^2(h^2 + m^2 - 2hm\cos\theta)^{-3/2}\sin^2\theta + hm(h^2 + m^2 - 2hm\cos\theta)^{-1/2}\cos\theta$

$= hm(h^2 + m^2 - 2hm\cos\theta)^{-3/2}[-hm\sin^2\theta + (h^2 + m^2)\cos\theta - 2hm\cos^2\theta]$

$= hm(h^2 + m^2 - 2hm\cos\theta)^{-3/2}[-hm\cos^2\theta + (h^2 + m^2)\cos\theta - hm]$

$= -hm(h^2 + m^2 - 2hm\cos\theta)^{-3/2}[hm\cos^2\theta - (h^2 + m^2)\cos\theta + hm]$

$L'' = 0$ when $hm\cos^2\theta - (h^2 + m^2)\cos\theta + hm = 0$

$(h\cos\theta - m)(m\cos\theta - h) = 0$

$\cos\theta = \frac{m}{h}, \frac{h}{m}$

Since $h < m,\ \cos\theta = \frac{h}{m}$ so $\theta = \cos^{-1}\left(\frac{h}{m}\right)$.

$L'\left(\cos^{-1}\left(\frac{h}{m}\right)\right) = hm\left(h^2 + m^2 - 2hm\left(\frac{h}{m}\right)\right)^{-1/2}\frac{\sqrt{m^2 - h^2}}{m} = hm(m^2 - h^2)^{-1/2}\frac{\sqrt{m^2 - h^2}}{m} = h$

$L\left(\cos^{-1}\left(\frac{h}{m}\right)\right) = \left(h^2 + m^2 - 2hm\left(\frac{h}{m}\right)\right)^{1/2} = \sqrt{m^2 - h^2}$

Since $h^2 + L^2 = m^2, \phi = 90°$.

48. We are interested in finding the global extrema for the distance of the object from the observer. We will obtain this result by considering the squared distance instead. The squared distance can be expressed as

$D(x) = (x-2)^2 + \left(100 + x - \frac{1}{10}x^2\right)^2$

The first and second derivatives are given by

$D'(x) = \frac{1}{25}x^3 - \frac{3}{5}x^2 - 36x + 196$ and

$D''(x) = \frac{3}{25}\left(x^2 - 10x - 300\right)$

Using a computer package, we can solve the equation $D'(x) = 0$ to find the critical points. The critical points are $x \approx 5.1538, 36.148$. Using the second derivative we see that

$D''(5.1538) \approx -38.9972$ (max) and

$D''(36.148) \approx 77.4237$ (min)

Therefore, the position of the object closest to the observer is $\approx (36.148, 5.48)$ while the position of the object farthest from the person is $\approx (5.1538, 102.5)$.

(Remember to go back to the original equation for the path of the object once you find the critical points.)

49. Here we are interested in minimizing the distance between the earth and the asteroid. Using the coordinates P and Q for the two bodies, we can use the distance formula to obtain a suitable equation. However, for simplicity, we will minimize the squared distance to find the critical points. The squared distance between the objects is given by

$D(t) = (93\cos(2\pi t) - 60\cos[2\pi(1.51t - 1)])^2$

$\quad + (93\sin(2\pi t) - 120\sin[2\pi(1.51t - 1)])^2$

The first derivative is

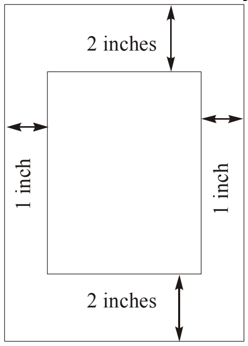
$D'(t) \approx -34359\left[\cos(2\pi t)\right]\left[\sin(9.48761t)\right]$

$\quad + \left[\cos(9.48761t)\right][(204932\sin(9.48761t)$

$\quad - 141643\sin(2\pi t))]$

Plotting the function and its derivative reveal a periodic relationship due to the orbiting of the objects. Careful examination of the graphs reveals that there is indeed a minimum squared distance (and hence a minimum distance) that occurs only once. The critical value for this occurrence is $t \approx 13.82790355$. This value gives a squared distance between the objects of ≈ 0.0022743 million miles. The actual distance is ≈ 0.047851 million miles $\approx 47{,}851$ miles.

50. Let x be the width and y the height of the flyer.



We wish to minimize the area of the flyer, $A = xy$.

As it stands, A is expressed in terms of two variables so we need to write one in terms of the other.

The printed area of the flyer has an area of 50 square inches. The equation for this area is $(x-2)(y-4)=50$

We can solve this equation for y to obtain

$$y = \frac{50}{x-2}+4$$

Substituting this expression for y in our equation for A, we get A in terms of a single variable, x.

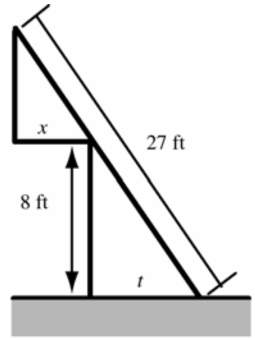
$$A = xy$$
$$= x\left(\frac{50}{x-2}+4\right) = \frac{50x}{x-2}+4x$$

The allowable values for x are $2 < x < \infty$; we want to minimize A on the open interval $(2,\infty)$.

$$\frac{dA}{dx} = \frac{(x-2)50-50x}{(x-2)^2}+4 = \frac{-100}{(x-2)^2}+4$$
$$= \frac{4x^2-16x-84}{(x-2)^2} = \frac{4(x-7)(x+3)}{(x-2)^2}$$

The only critical points are obtained by solving $\frac{dA}{dx}=0$; this yields $x=7$ and $x=-3$. We reject $x=-3$ because it is not in the feasible domain $(2,\infty)$. Since $\frac{dA}{dx}<0$ for x in $(2,7)$ and $\frac{dA}{dx}>0$ for x in $(7,\infty)$, we conclude that A attains its minimum value at $x=7$. This value of x makes $y=14$. So, the dimensions for the flyer that will use the least amount of paper are 7 inches by 14 inches.

51. Consider the following sketch.



By similar triangles, $\frac{x}{27-\sqrt{t^2+64}} = \frac{t}{\sqrt{t^2+64}}$.

$$x = \frac{27t}{\sqrt{t^2+64}}-t$$

$$\frac{dx}{dt} = \frac{27\sqrt{t^2+64}-\frac{27t^2}{\sqrt{t^2+64}}}{t^2+64}-1 = \frac{1728}{(t^2+64)^{3/2}}-1$$

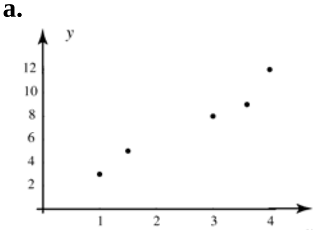
$\frac{1728}{(t^2+64)^{3/2}}-1=0$ when $t=4\sqrt{5}$

$$\frac{d^2x}{dt^2} = \frac{-5184t}{(t^2+64)^{5/2}};\left.\frac{d^2x}{dt^2}\right|_{t=4\sqrt{5}}<0$$

Therefore

$x = \frac{27\left(4\sqrt{5}\right)}{\sqrt{\left(4\sqrt{5}\right)^2+64}}-4\sqrt{5} = 5\sqrt{5} \approx 11.18$ ft is the maximum horizontal overhang.

52. a.



b. There are only a few data points, but they do seem fairly linear.

c. The data values can be entered into most scientific calculators to utilize the Least Squares Regression feature. Alternately one could use the formulas for the slope and intercept provided in the text. The resulting line should be $y = 0.56852+2.6074x$

d. Using the result from **c.**, the predicted number of surface imperfections on a sheet with area 2.0 square feet is
$y = 0.56852 + 2.6074(2.0) = 5.7833 \approx 6$
since we can't have partial imperfections

53. a. $\frac{dS}{db} = \frac{d}{db}\sum_{i=1}^{n}[y_i - (5 + bx_i)]^2$

$= \sum_{i=1}^{n}\frac{d}{db}[y_i - (5 + bx_i)]^2$

$= \sum_{i=1}^{n} 2(y_i - 5 - bx_i)(-x_i)$

$= 2\left[\sum_{i=1}^{n}\left(-x_i y_i + 5x_i + bx_i^2\right)\right]$

$= -2\sum_{i=1}^{n} x_i y_i + 10\sum_{i=1}^{n} x_i + 2b\sum_{i=1}^{n} x_i^2$

Setting $\frac{dS}{db} = 0$ gives

$0 = -2\sum_{i=1}^{n} x_i y_i + 10\sum_{i=1}^{n} x_i + 2b\sum_{i=1}^{n} x_i^2$

$0 = -\sum_{i=1}^{n} x_i y_i + 5\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$

$b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i - 5\sum_{i=1}^{n} x_i$

$$b = \frac{\sum_{i=1}^{n} x_i y_i - 5\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2}$$

You should check that this is indeed the value of b that minimizes the sum. Taking the second derivative yields

$\frac{d^2S}{db^2} = 2\sum_{i=1}^{n} x_i^2$

which is always positive (unless all the x values are zero). Therefore, the value for b above does minimize the sum as required.

b. Using the formula from **a.**, we get that

$b = \frac{(2037) - 5(52)}{590} \approx 3.0119$

c. The Least Squares Regression line is
$y = 5 + 3.0119x$
Using this line, the predicted total number of labor hours to produce a lot of 15 brass bookcases is
$y = 5 + 3.0119(15) \approx 50.179$ hours

54. $C(x) = 7000 + 100x$

55. $n = 100 + 10\frac{250 - p(n)}{5}$ so $p(n) = 300 - \frac{n}{2}$

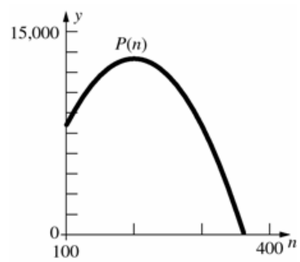
$R(n) = np(n) = 300n - \frac{n^2}{2}$

56. $P(n) = R(n) - C(n)$

$= 300n - \frac{n^2}{2} - (7000 + 100n)$

$= -7000 + 200n - \frac{n^2}{2}$

57.



Estimate $n \approx 200$
$P'(n) = 200 - n; 200 - n = 0$ when $n = 200$.
$P''(n) = -1$, so profit is maximum at $n = 200$.

58. $\frac{C(x)}{x} = \frac{100}{x} + 3.002 - 0.0001x$

When $x = 1600$, $\frac{C(x)}{x} = 2.9045$ or \$2.90 per unit.

$\frac{dC}{dx} = 3.002 - 0.0002x$

$C'(1600) = 2.682$ or \$2.68

59. $\frac{C(n)}{n} = \frac{1000}{n} + \frac{n}{1200}$

When $n = 800$, $\frac{C(n)}{n} \approx 1.9167$ or \$1.92 per unit.

$\frac{dC}{dn} = \frac{n}{600}$

$C'(800) \approx 1.333$ or \$1.33

60. a. $\frac{dC}{dx} = 33 - 18x + 3x^2$

$\frac{d^2C}{dx^2} = -18 + 6x; \frac{d^2C}{dx^2} = 0$ when $x = 3$

$\frac{d^2C}{dx^2} < 0$ on $(0, 3)$, $\frac{d^2C}{dx^2} > 0$ on $(3, \infty)$

Thus, the marginal cost is a minimum when $x = 3$ or 300 units.

b. $33 - 18(3) + 3(3)^2 = 6$

61. **a.** $R(x) = xp(x) = 20x + 4x^2 - \frac{x^3}{3}$

$\frac{dR}{dx} = 20 + 8x - x^2 = (10 - x)(x + 2)$

b. Increasing when $\frac{dR}{dx} > 0$

$20 + 8x - x^2 > 0$ on $[0, 10)$

Total revenue is increasing if $0 \le x \le 10$.

c. $\frac{d^2R}{dx^2} = 8 - 2x; \frac{d^2R}{dx^2} = 0$ when $x = 4$

$\frac{d^3R}{dx^3} = -2; \frac{dR}{dx}$ is maximum at $x = 4$.

62. $R(x) = x\left(182 - \frac{x}{36}\right)^{1/2}$

$\frac{dR}{dx} = x\frac{1}{2}\left(182 - \frac{x}{36}\right)^{-1/2}\left(-\frac{1}{36}\right) + \left(182 - \frac{x}{36}\right)^{1/2}$

$= \left(182 - \frac{x}{36}\right)^{-1/2}\left(182 - \frac{x}{24}\right)$

$\frac{dR}{dx} = 0$ when $x = 4368$

$x_1 = 4368; R(4368) \approx 34,021.83$

At $x_1, \frac{dR}{dx} = 0$.

63. $R(x) = \frac{800x}{x+3} - 3x$

$\frac{dR}{dx} = \frac{(x+3)(800) - 800x}{(x+3)^2} - 3 = \frac{2400}{(x+3)^2} - 3;$

$\frac{dR}{dx} = 0$ when $x = 20\sqrt{2} - 3 \approx 25$

$x_1 = 25; R(25) \approx 639.29$

At $x_1, \frac{dR}{dx} = 0$.

64. $p(x) = 12 - (0.20)\frac{(x - 400)}{10} = 20 - 0.02x$

$R(x) = 20x - 0.02x^2$

$\frac{dR}{dx} = 20 - 0.04x; \frac{dR}{dx} = 0$ when $x = 500$

Total revenue is maximized at $x_1 = 500$.

65. The revenue function would be $R(x) = x \cdot p(x) = 200x - 0.15x^2$. This, together with the cost function yields the following profit function:

$$P(x) = \begin{cases} -5000 + 194x - 0.148x^2 & \text{if } 0 \le x \le 500 \\ -9000 + 194x - 0.148x^2 & \text{if } 500 < x \le 750 \end{cases}$$

a. The only difference in the two pieces of the profit function is the constant. Since the derivative of a constant is 0, we can say that on the interval $0 < x < 750$,

$\frac{dP}{dx} = 194 - 0.296x$

There are no singular points in the given interval. To find stationary points, we solve

$\frac{dP}{dx} = 0$

$194 - 0.296x = 0$

$-0.296x = -194$

$x \approx 655$

Thus, the critical points are 0, 500, 655, and 750.

$P(0) = -5000$; $P(500) = 55,000$;

$P(655) = 54,574.30$; $P(750) = 53,250$

The profit is maximized if the company produces 500 chairs. The current machine can handle this work, so they should not buy the new machine.

b. Without the new machine, a production level of 500 chairs would yield a maximum profit of $55,000.

66. The revenue function would be $R(x) = x \cdot p(x) = 200x - 0.15x^2$. This, together with the cost function yields the following profit function:

$$P(x) = \begin{cases} -5000 + 194x - 0.148x^2 & \text{if } 0 \le x \le 500 \\ -8000 + 194x - 0.148x^2 & \text{if } 500 < x \le 750 \end{cases}$$

a. The only difference in the two pieces of the profit function is the constant. Since the derivative of a constant is 0, we can say that on the interval $0 < x < 750$,

$\frac{dP}{dx} = 194 - 0.296x$

There are no singular points in the given interval. To find stationary points, we solve

$\frac{dP}{dx} = 0$

$194 - 0.296x = 0$

$-0.296x = -194$

$x \approx 655$

Thus, the critical points are 0, 500, 655, and 750.
$P(0) = -5000$; $P(500) = 55{,}000$;
$P(655) = 55{,}574.30$; $P(750) = 54{,}250$
The profit is maximized if the company produces 655 chairs. The current machine cannot handle this work, so they should buy the new machine.

b. With the new machine, a production level of 655 chairs would yield a maximum profit of \$55,574.30.

67. $R(x) = 10x - 0.001x^2; 0 \le x \le 300$

$P(x) = (10x - 0.001x^2) - (200 + 4x - 0.01x^2)$

$= -200 + 6x + 0.009x^2$

$\frac{dP}{dx} = 6 + 0.018x; \frac{dP}{dx} = 0$ when $x \approx -333$

Critical numbers: $x = 0, 300$; $P(0) = -200$; $P(300) = 2410$; Maximum profit is \$2410 at $x = 300$.

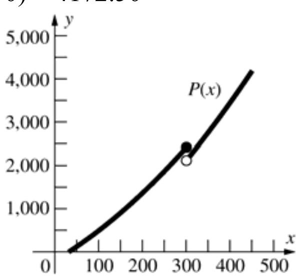
68. $C(x) = \begin{cases} 200 + 4x - 0.01x^2 & \text{if } 0 \le x \le 300 \\ 800 + 3x - 0.01x^2 & \text{if } 300 < x \le 450 \end{cases}$

$P(x) = \begin{cases} -200 + 6x + 0.009x^2 & \text{if } 0 \le x \le 300 \\ -800 + 7x + 0.009x^2 & \text{if } 300 < x \le 450 \end{cases}$

There are no stationary points on the interval [0, 300]. On [300, 450]:

$\frac{dP}{dx} = 7 + 0.018x; \frac{dP}{dx} = 0$ when $x \approx -389$

The critical numbers are 0, 300, 450.
$P(0) = -200$, $P(300) = 2410$, $P(450) = 4172.5$
Monthly profit is maximized at $x = 450$, $P(450) = 4172.50$



69. a. $ab \le \left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4}$

$= \frac{a^2}{4} + \frac{1}{2}ab + \frac{b^2}{4}$

This is true if

$0 \le \frac{a^2}{4} - \frac{1}{2}ab + \frac{b^2}{4} = \left(\frac{a}{2} - \frac{b}{2}\right)^2 = \left(\frac{a-b}{2}\right)^2$

Since a square can never be negative, this is always true.

b. $F(b) = \frac{a^2 + 2ab + b^2}{4b}$

As $b \to 0^+, a^2 + 2ab + b^2 \to a^2$ while $4b \to 0^+$, thus $\lim_{b \to 0^+} F(b) = \infty$ which is not close to a.

$\lim_{b \to \infty} \frac{a^2 + 2ab + b^2}{4b} = \lim_{b \to \infty} \frac{\frac{a^2}{b} + 2a + b}{4} = \infty$,

so when b is very large, $F(b)$ is not close to a.

$F'(b) = \frac{2(a+b)(4b) - 4(a+b)^2}{16b^2}$

$= \frac{4b^2 - 4a^2}{16b^2} = \frac{b^2 - a^2}{4b^2}$;

$F'(b) = 0$ when $b^2 = a^2$ or $b = a$ since a and b are both positive.

$F(a) = \frac{(a+a)^2}{4a} = \frac{4a^2}{4a} = a$

Thus $a \le \frac{(a+b)^2}{4b}$ for all $b > 0$ or

$ab \le \frac{(a+b)^2}{4}$ which leads to $\sqrt{ab} \le \frac{a+b}{2}$.

c. Let $F(b) = \frac{1}{b}\left(\frac{a+b+c}{3}\right)^3 = \frac{(a+b+c)^3}{27b}$

$F'(b) = \frac{3(a+b+c)^2(27b) - 27(a+b+c)^3}{27^2 b^2}$

$= \frac{(a+b+c)^2[3b - (a+b+c)]}{27b^2}$

$= \frac{(a+b+c)^2(2b - a - c)}{27b^2}$;

$F'(b) = 0$ when $b = \frac{a+c}{2}$.

$F\left(\frac{a+c}{2}\right) = \frac{2}{a+c} \cdot \left(\frac{a+c}{3} + \frac{a+c}{6}\right)^3$

$= \frac{2}{a+c}\left(\frac{3(a+c)}{6}\right)^3 = \frac{2}{a+c}\left(\frac{a+c}{2}\right)^3 = \left(\frac{a+c}{2}\right)^2$

Thus $\left(\frac{a+c}{2}\right)^2 \le \frac{1}{b}\left(\frac{a+b+c}{3}\right)^3$ for all $b > 0$.

From (b), $ac \le \left(\frac{a+c}{2}\right)^2$, thus

$ac \le \frac{1}{b}\left(\frac{a+b+c}{3}\right)^3$ or $abc \le \left(\frac{a+b+c}{3}\right)^3$

which gives the desired result

$(abc)^{1/3} \le \frac{a+b+c}{3}$.

70. Let $a = lw$, $b = lh$, and $c = hw$, then $S = 2(a + b + c)$ while $V^2 = abc$. By problem 69(c), $(abc)^{1/3} \le \frac{a+b+c}{3}$ so

$$(V^2)^{1/3} \le \frac{2(a+b+c)}{2\cdot 3} = \frac{S}{6}.$$

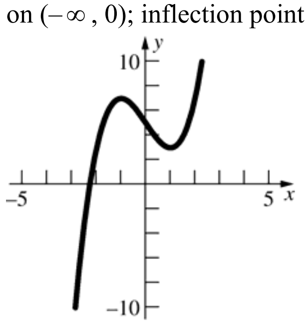
In problem 1c, the minimum occurs, hence equality holds, when $b = \frac{a+c}{2}$. In the result used from Problem 69(b), equality holds when $c = a$, thus $b = \frac{a+a}{2} = a$, so $a = b = c$. For the boxes, this means $l = w = h$, so the box is a cube.

3.5 Concepts Review

1. $f(x)$; $-f(x)$
2. decreasing; concave up
3. $x = -1$, $x = 2$, $x = 3$; $y = 1$
4. polynomial; rational.

Problem Set 3.5

1. Domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$
Neither an even nor an odd function.
y-intercept: 5; x-intercept: ≈ -2.3
$f'(x) = 3x^2 - 3;\ 3x^2 - 3 = 0$ when $x = -1, 1$
Critical points: $-1, 1$
$f'(x) > 0$ when $x < -1$ or $x > 1$
$f(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $[-1, 1]$.
Local minimum $f(1) = 3$;
local maximum $f(-1) = 7$
$f''(x) = 6x$; $f''(x) > 0$ when $x > 0$.
$f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; inflection point $(0, 5)$.



2. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function.
y-intercept: -10; x-intercept: 2
$f'(x) = 6x^2 - 3 = 3(2x^2 - 1);\ 2x^2 - 1 = 0$ when

$x = -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Critical points: $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

$f'(x) > 0$ when $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$

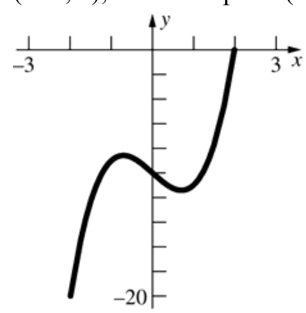
$f(x)$ is increasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and

decreasing on $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

Local minimum $f\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2} - 10 \approx -11.4$

Local maximum $f\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2} - 10 \approx -8.6$

$f''(x) = 12x$; $f''(x) > 0$ when $x > 0$. $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; inflection point $(0, -10)$.

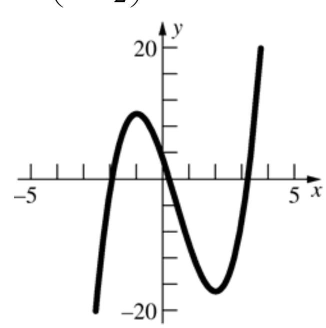


3. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function.
y-intercept: 3; x-intercepts: $\approx -2.0, 0.2, 3.2$
$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$;
$f'(x) = 0$ when $x = -1, 2$
Critical points: $-1, 2$
$f'(x) > 0$ when $x < -1$ or $x > 2$
$f(x)$ is increasing on $(-\infty, -1] \cup [2, \infty)$ and decreasing on $[-1, 2]$.
Local minimum $f(2) = -17$;
local maximum $f(-1) = 10$

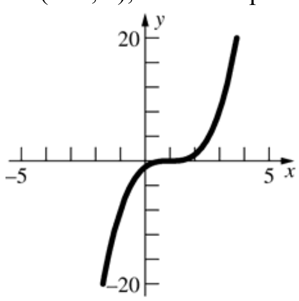
$f''(x) = 12x - 6 = 6(2x - 1)$; $f''(x) > 0$ when $x > \frac{1}{2}$.

$f(x)$ is concave up on $\left(\frac{1}{2}, \infty\right)$ and concave down

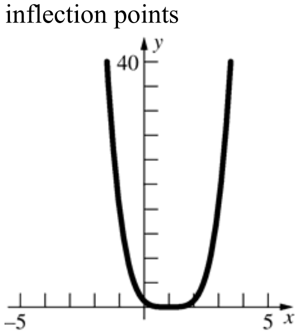
on $\left(-\infty, \frac{1}{2}\right)$; inflection point: $\left(\frac{1}{2}, -\frac{7}{2}\right)$



4. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function
y-intercept: –1; x-intercept: 1
$f'(x) = 3(x-1)^2$; $f'(x) = 0$ when $x = 1$
Critical point: 1
$f'(x) > 0$ for all $x \neq 1$
$f(x)$ is increasing on $(-\infty, \infty)$
No local minima or maxima
$f''(x) = 6(x-1)$; $f''(x) > 0$ when $x > 1$.
$f(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; inflection point $(1, 0)$



5. Domain: $(-\infty, \infty)$; range: $[0, \infty)$
Neither an even nor an odd function.
y-intercept: 1; x-intercept: 1
$G'(x) = 4(x-1)^3$; $G'(x) = 0$ when $x = 1$
Critical point: 1
$G'(x) > 0$ for $x > 1$
$G(x)$ is increasing on $[1, \infty)$ and decreasing on $(-\infty, 1]$.
Global minimum $f(1) = 0$; no local maxima
$G''(x) = 12(x-1)^2$; $G''(x) > 0$ for all $x \neq 1$
$G(x)$ is concave up on $(-\infty, 1) \cup (1, \infty)$; no inflection points



6. Domain: $(-\infty, \infty)$; range: $\left[-\frac{1}{4}, \infty\right)$

$H(-t) = (-t)^2[(-t)^2 - 1] = t^2(t^2 - 1) = H(t)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; t-intercepts: –1, 0, 1
$H'(t) = 4t^3 - 2t = 2t(2t^2 - 1)$; $H'(t) = 0$ when

$t = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

Critical points: $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

$H'(t) > 0$ for $-\frac{1}{\sqrt{2}} < t < 0$ or $\frac{1}{\sqrt{2}} < t$.

$H(t)$ is increasing on $\left[-\frac{1}{\sqrt{2}}, 0\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and

decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[0, \frac{1}{\sqrt{2}}\right]$

Global minima $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$, $f\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$;

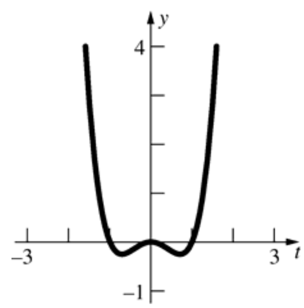
Local maximum $f(0) = 0$
$H''(t) = 12t^2 - 2 = 2(6t^2 - 1)$; $H'' > 0$ when

$t < -\frac{1}{\sqrt{6}}$ or $t > \frac{1}{\sqrt{6}}$

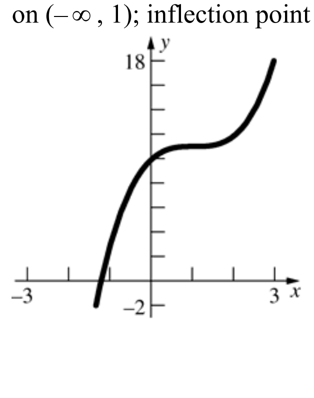
$H(t)$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{6}}\right) \cup \left(\frac{1}{\sqrt{6}}, \infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$; inflection

points $H\left(-\frac{1}{\sqrt{6}}, -\frac{5}{36}\right)$ and $H\left(\frac{1}{\sqrt{6}}, \frac{5}{36}\right)$



7. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
Neither an even nor an odd function.
y-intercept: 10; x-intercept: $1 - 11^{1/3} \approx -1.2$
$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$; $f'(x) = 0$ when $x = 1$.
Critical point: 1
$f'(x) > 0$ for all $x \neq 1$.
$f(x)$ is increasing on $(-\infty, \infty)$ and decreasing nowhere.
No local maxima or minima
$f''(x) = 6x - 6 = 6(x-1)$; $f''(x) > 0$ when $x > 1$.
$f(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; inflection point $(1, 11)$



8. Domain: $(-\infty, \infty)$; range: $\left[-\frac{16}{3}, \infty\right)$

$$F(-s) = \frac{4(-s)^4 - 8(-s)^2 - 12}{3} = \frac{4s^4 - 8s^2 - 12}{3}$$

$= F(s)$; even function; symmetric with respect to the y-axis

y-intercept: –4; s-intercepts: $-\sqrt{3}, \sqrt{3}$

$F'(s) = \frac{16}{3}s^3 - \frac{16}{3}s = \frac{16}{3}s(s^2 - 1)$; $F'(s) = 0$ when $s = -1, 0, 1$.

Critical points: –1, 0, 1

$F'(s) > 0$ when $-1 < x < 0$ or $x > 1$.

$F(s)$ is increasing on $[-1, 0] \cup [1, \infty)$ and decreasing on $(-\infty, -1] \cup [0, 1]$

Global minima $F(-1) = -\frac{16}{3}, F(1) = -\frac{16}{3}$; local maximum $F(0) = -4$

$F''(s) = 16s^2 - \frac{16}{3} = 16\left(s^2 - \frac{1}{3}\right)$; $F''(s) > 0$

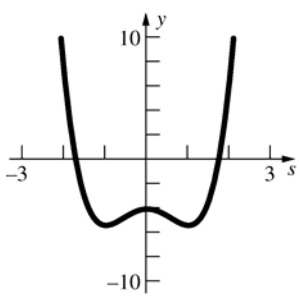
when $s < -\frac{1}{\sqrt{3}}$ or $s > \frac{1}{\sqrt{3}}$

$F(s)$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$

and concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$;

inflection points

$F\left(-\frac{1}{\sqrt{3}}, -\frac{128}{27}\right), F\left(\frac{1}{\sqrt{3}}, -\frac{128}{27}\right)$



9. Domain: $(-\infty, -1) \cup (-1, \infty)$;
range: $(-\infty, 1) \cup (1, \infty)$
Neither an even nor an odd function
y-intercept: 0; x-intercept: 0

$g'(x) = \frac{1}{(x+1)^2}$; $g'(x)$ is never 0.

No critical points

$g'(x) > 0$ for all $x \neq -1$.

$g(x)$ is increasing on $(-\infty, -1) \cup (-1, \infty)$.

No local minima or maxima

$g''(x) = -\frac{2}{(x+1)^3}$; $g''(x) > 0$ when $x < -1$.

$g(x)$ is concave up on $(-\infty, -1)$ and concave down on $(-1, \infty)$; no inflection points (–1 is not in the domain of g).

$$\lim_{x \to \infty} \frac{x}{x+1} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = 1;$$

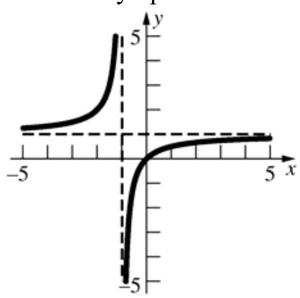
$$\lim_{x \to -\infty} \frac{x}{x+1} = \lim_{x \to -\infty} \frac{1}{1 + \frac{1}{x}} = 1;$$

horizontal asymptote: $y = 1$

As $x \to -1^-, x + 1 \to 0^-$ so $\lim_{x \to -1^-} \frac{x}{x+1} = \infty$;

as $x \to -1^+, x + 1 \to 0^+$ so $\lim_{x \to -1^+} \frac{x}{x+1} = -\infty$;

vertical asymptote: $x = -1$



10. Domain: $(-\infty, 0) \cup (0, \infty)$;

range: $(-\infty, -4\pi] \cup [0, \infty)$

Neither an even nor an odd function

No y-intercept; s-intercept: π

$g'(s) = \frac{s^2 - \pi^2}{s^2}$; $g'(s) = 0$ when $s = -\pi, \pi$

Critical points: $-\pi, \pi$

$g'(s) > 0$ when $s < -\pi$ or $s > \pi$

$g(s)$ is increasing on $(-\infty, -\pi] \cup [\pi, \infty)$ and decreasing on $[-\pi, 0) \cup (0, \pi]$.

Local minimum $g(\pi) = 0$;

local maximum $g(-\pi) = -4\pi$

$g''(s) = \frac{2\pi^2}{s^3}$; $g''(s) > 0$ when $s > 0$

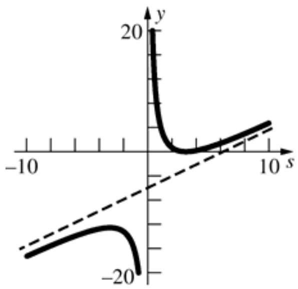
$g(s)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; no inflection points (0 is not in the domain of $g(s)$).

$g(s) = s - 2\pi + \frac{\pi^2}{s}$; $y = s - 2\pi$ is an oblique asymptote.

As $s \to 0^-, (s - \pi)^2 \to \pi^2$, so $\lim_{s \to 0^-} g(s) = -\infty$;

as $s \to 0^+, (s-\pi)^2 \to \pi^2$, so $\lim\limits_{s\to 0^+} g(s) = \infty$;

$s = 0$ is a vertical asymptote.



11. Domain: $(-\infty, \infty)$; range: $\left[-\frac{1}{4}, \frac{1}{4}\right]$

$f(-x) = \frac{-x}{(-x)^2+4} = -\frac{x}{x^2+4} = -f(x)$; odd function; symmetric with respect to the origin.

y-intercept: 0; x-intercept: 0

$f'(x) = \frac{4-x^2}{(x^2+4)^2}$; $f'(x) = 0$ when $x = -2, 2$

Critical points: –2, 2

$f'(x) > 0$ for $-2 < x < 2$

$f(x)$ is increasing on [–2, 2] and decreasing on $(-\infty, -2] \cup [2, \infty)$.

Global minimum $f(-2) = -\frac{1}{4}$; global maximum $f(2) = \frac{1}{4}$

$f''(x) = \frac{2x(x^2-12)}{(x^2+4)^3}$; $f''(x) > 0$ when $-2\sqrt{3} < x < 0$ or $x > 2\sqrt{3}$

$f(x)$ is concave up on $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$ and concave down on $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$;

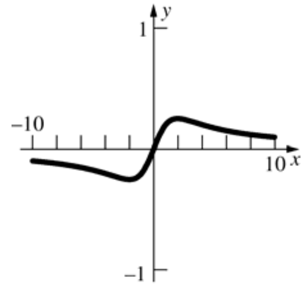
inflection points $\left(-2\sqrt{3}, -\frac{\sqrt{3}}{8}\right)$, $(0, 0)$, $\left(2\sqrt{3}, \frac{\sqrt{3}}{8}\right)$

$$\lim_{x\to\infty} \frac{x}{x^2+4} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{4}{x^2}} = 0;$$

$$\lim_{x\to-\infty} \frac{x}{x^2+4} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1+\frac{4}{x^2}} = 0;$$

$y = 0$ is a horizontal asymptote.

No vertical asymptotes



12. Domain: $(-\infty, \infty)$; range: [0, 1)

$\Lambda(-\theta) = \frac{(-\theta)^2}{(-\theta)^2+1} = \frac{\theta^2}{\theta^2+1} = \Lambda(\theta)$; even function; symmetric with respect to the y-axis.

y-intercept: 0; θ-intercept: 0

$\Lambda'(\theta) = \frac{2\theta}{(\theta^2+1)^2}$; $\Lambda'(\theta) = 0$ when $\theta = 0$

Critical point: 0

$\Lambda'(\theta) > 0$ when $\theta > 0$

$\Lambda(\theta)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

Global minimum $\Lambda(0) = 0$; no local maxima

$\Lambda''(\theta) = \frac{2(1-3\theta^2)}{(\theta^2+1)^3}$; $\Lambda''(\theta) > 0$ when $-\frac{1}{\sqrt{3}} < \theta < \frac{1}{\sqrt{3}}$

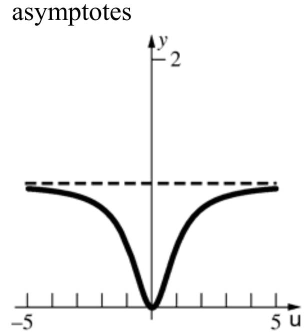
$\Lambda(\theta)$ is concave up on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and concave down on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$;

inflection points $\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$

$$\lim_{\theta\to\infty} \frac{\theta^2}{\theta^2+1} = \lim_{\theta\to\infty} \frac{1}{1+\frac{1}{\theta^2}} = 1;$$

$$\lim_{\theta\to-\infty} \frac{\theta^2}{\theta^2+1} = \lim_{\theta\to-\infty} \frac{1}{1+\frac{1}{\theta^2}} = 1;$$

$y = 1$ is a horizontal asymptote. No vertical asymptotes



13. Domain: $(-\infty, 1) \cup (1, \infty)$;
range $(-\infty, 1) \cup (1, \infty)$
Neither an even nor an odd function
y-intercept: 0; x-intercept: 0

$h(x) = -\frac{1}{(x-1)^2}$; $h'(x)$ is never 0.

No critical points
$h'(x) < 0$ for all $x \neq 1$.
$h(x)$ is increasing nowhere and decreasing on $(-\infty, 1) \cup (1, \infty)$.
No local maxima or minima

$h''(x) = \frac{2}{(x-1)^3}$; $h''(x) > 0$ when $x > 1$

$h(x)$ is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$; no inflection points (1 is not in the domain of $h(x)$)

$$\lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}} = 1;$$

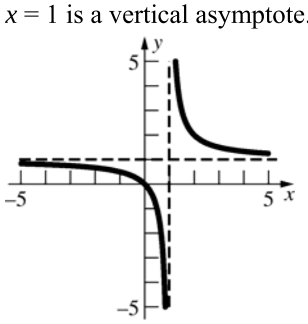
$$\lim_{x\to-\infty} \frac{x}{x-1} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x}} = 1;$$

$y = 1$ is a horizontal asymptote.

As $x \to 1^-$, $x-1 \to 0^-$ so $\lim_{x\to1^-} \frac{x}{x-1} = -\infty$;

as $x \to 1^+$, $x-1 \to 0^+$ so $\lim_{x\to1^+} \frac{x}{x-1} = \infty$;

$x = 1$ is a vertical asymptote.



14. Domain: $(-\infty, \infty)$

Range: $(0, 1]$

Even function since

$$P(-x) = \frac{1}{(-x)^2+1} = \frac{1}{x^2+1} = P(x)$$

so the function is symmetric with respect to the y-axis.
y-intercept: $y = 1$
x-intercept: none

$P'(x) = \frac{-2x}{(x^2+1)^2}$; $P'(x)$ is 0 when $x = 0$.

critical point: $x = 0$

$P'(x) > 0$ when $x < 0$ so $P(x)$ is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$. Global maximum $P(0) = 1$; no local minima.

$$P''(x) = \frac{6x^2-2}{(x^2+1)^3}$$

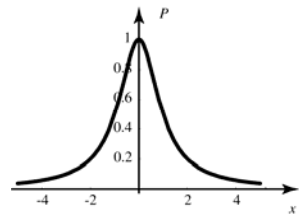
$P''(x) > 0$ on $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$ (concave up) and $P''(x) < 0$ on $(-1/\sqrt{3}, 1/\sqrt{3})$ (concave down).

Inflection points: $\left(\pm\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$

No vertical asymptotes.

$\lim_{x\to\infty} P(x) = 0$; $\lim_{x\to-\infty} P(x) = 0$

$y = 0$ is a horizontal asymptote.



15. Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$;
range: $(-\infty, \infty)$
Neither an even nor an odd function

y-intercept: $-\frac{3}{2}$; x-intercepts: 1, 3

$f'(x) = \frac{3x^2-10x+11}{(x+1)^2(x-2)^2}$; $f'(x)$ is never 0.

No critical points
$f'(x) > 0$ for all $x \neq -1, 2$
$f(x)$ is increasing on
$(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.
No local minima or maxima

$f''(x) = \frac{-6x^3+30x^2-66x+42}{(x+1)^3(x-2)^3}$; $f''(x) > 0$ when $x < -1$ or $1 < x < 2$

$f(x)$ is concave up on $(-\infty, -1) \cup (1, 2)$ and concave down on $(-1, 1) \cup (2, \infty)$;
inflection point $f(1) = 0$

$$\lim_{x\to\infty} \frac{(x-1)(x-3)}{(x+1)(x-2)} = \lim_{x\to\infty} \frac{x^2-4x+3}{x^2-x-2}$$

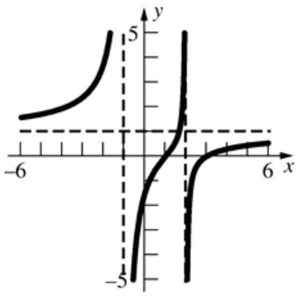
$$= \lim_{x\to\infty} \frac{1-\frac{4}{x}+\frac{3}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}} = 1;$$

$$\lim_{x\to-\infty} \frac{(x-1)(x-3)}{(x+1)(x-2)} = \lim_{x\to-\infty} \frac{1-\frac{4}{x}+\frac{3}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}} = 1;$$

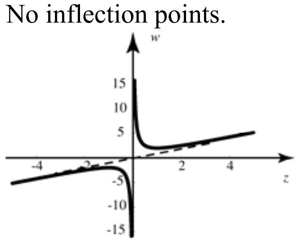
$y = 1$ is a horizontal asymptote.

As $x \to -1^-$, $x-1 \to -2$, $x-3 \to -4$, $x-2 \to -3$, and $x+1 \to 0^-$ so $\lim_{x\to-1^-} f(x) = \infty$;

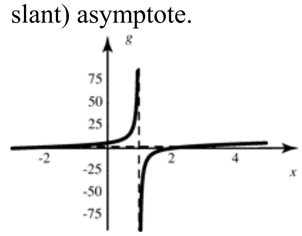
as $x \to -1^+, x-1 \to -2, x-3 \to -4,$
$x-2 \to -3,$ and $x+1 \to 0^+$, so
$\lim_{x \to -1^+} f(x) = -\infty$
As $x \to 2^-, x-1 \to 1, x-3 \to -1, x+1 \to 3,$ and
$x-2 \to 0^-$, so $\lim_{x \to 2^-} f(x) = \infty$; as
$x \to 2^+, x-1 \to 1, x-3 \to -1, x+1 \to 3,$ and
$x-2 \to 0^+$, so $\lim_{x \to 2^+} f(x) = -\infty$
$x = -1$ and $x = 2$ are vertical asymptotes.



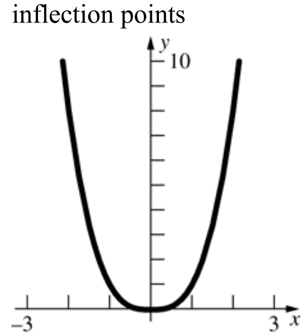
16. Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -2] \cup [2, \infty)$
Odd function since
$w(-z) = \frac{(-z)^2 + 1}{-z} = -\frac{z^2+1}{z} = -w(z)$; symmetric with respect to the origin.
y-intercept: none
x-intercept: none
$w'(z) = 1 - \frac{1}{z^2}$; $w'(z) = 0$ when $z = \pm 1$.
critical points: $z = \pm 1$. $w'(z) > 0$ on $(-\infty, -1) \cup (1, \infty)$ so the function is increasing on $(-\infty, -1] \cup [1, \infty)$. The function is decreasing on $[-1, 0) \cup (0, 1)$.
local minimum $w(1) = 2$ and local maximum $w(-1) = -2$. No global extrema.
$w''(z) = \frac{2}{z^3} > 0$ when $z > 0$. Concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.
No horizontal asymptote; $x = 0$ is a vertical asymptote; the line $y = z$ is an oblique (or slant) asymptote.
No inflection points.



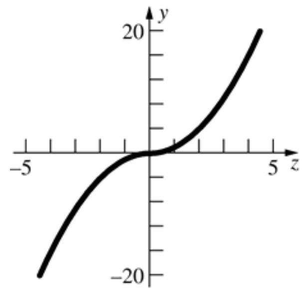
17. Domain: $(-\infty, 1) \cup (1, \infty)$
Range: $(-\infty, \infty)$
Neither even nor odd function.
y-intercept: $y = 6$; x-intercept: $x = -3, 2$
$g'(x) = \frac{x^2 - 2x + 5}{(x-1)^2}$; $g'(x)$ is never zero. No critical points.
$g'(x) > 0$ over the entire domain so the function is always increasing. No local extrema.
$f''(x) = \frac{-8}{(x-1)^3}$; $f''(x) > 0$ when $x < 1$ (concave up) and $f''(x) < 0$ when $x > 1$ (concave down); no inflection points.
No horizontal asymptote; $x = 1$ is a vertical asymptote; the line $y = x + 2$ is an oblique (or slant) asymptote.



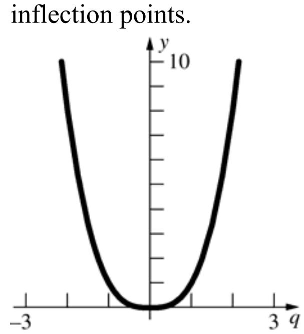
18. Domain: $(-\infty, \infty)$; range $[0, \infty)$
$f(-x) = |-x|^3 = |x|^3 = f(x)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; x-intercept: 0
$f'(x) = 3|x|^2 \left(\frac{x}{|x|}\right) = 3x|x|$; $f'(x) = 0$ when $x = 0$
Critical point: 0
$f'(x) > 0$ when $x > 0$
$f(x)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.
Global minimum $f(0) = 0$; no local maxima
$f''(x) = 3|x| + \frac{3x^2}{|x|} = 6|x|$ as $x^2 = |x|^2$;
$f''(x) > 0$ when $x \neq 0$
$f(x)$ is concave up on $(-\infty, 0) \cup (0, \infty)$; no inflection points



19. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$
$R(-z) = -z|-z| = -z|z| = -R(z)$; odd function; symmetric with respect to the origin.
y-intercept: 0; z-intercept: 0
$R'(z) = |z| + \frac{z^2}{|z|} = 2|z|$ since $z^2 = |z|^2$ for all z;
$R'(z) = 0$ when $z = 0$
Critical point: 0
$R'(z) > 0$ when $z \neq 0$
$R(z)$ is increasing on $(-\infty, \infty)$ and decreasing nowhere.
No local minima or maxima
$R''(z) = \frac{2z}{|z|}$; $R''(z) > 0$ when $z > 0$.
$R(z)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$; inflection point $(0, 0)$.



20. Domain: $(-\infty, \infty)$; range: $[0, \infty)$
$H(-q) = (-q)^2|-q| = q^2|q| = H(q)$; even function; symmetric with respect to the y-axis.
y-intercept: 0; q-intercept: 0
$H'(q) = 2q|q| + \frac{q^3}{|q|} = \frac{3q^3}{|q|} = 3q|q|$ as $|q|^2 = q^2$ for all q; $H'(q) = 0$ when $q = 0$
Critical point: 0
$H'(q) > 0$ when $q > 0$
$H(q)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.
Global minimum $H(0) = 0$; no local maxima
$H''(q) = 3|q| + \frac{3q^2}{|q|} = 6|q|$; $H''(q) > 0$ when $q \neq 0$.
$H(q)$ is concave up on $(-\infty, 0) \cup (0, \infty)$; no inflection points.



21. Domain: $(-\infty, \infty)$; range: $[0, \infty)$
Neither an even nor an odd function.
Note that for $x \le 0$, $|x| = -x$ so $|x| + x = 0$, while for $x > 0$, $|x| = x$ so $\frac{|x| + x}{2} = x$.

$$g(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 3x^2 + 2x & \text{if } x > 0 \end{cases}$$

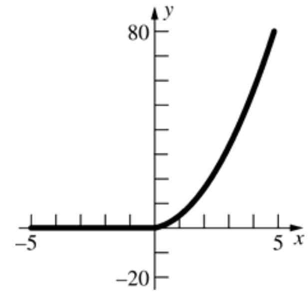
y-intercept: 0; x-intercepts: $(-\infty, 0]$

$$g'(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 6x + 2 & \text{if } x > 0 \end{cases}$$

No critical points for $x > 0$.
$g(x)$ is increasing on $[0, \infty)$ and decreasing nowhere.

$$g''(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 6 & \text{if } x > 0 \end{cases}$$

$g(x)$ is concave up on $(0, \infty)$; no inflection points



22. Domain: $(-\infty, \infty)$; range: $[0, \infty)$
Neither an even nor an odd function. Note that for $x < 0$, $|x| = -x$ so $\frac{|x| - x}{2} = -x$, while for $x \ge 0$, $|x| = x$ so $\frac{|x| - x}{2} = 0$.

$$h(x) = \begin{cases} -x^3 + x^2 - 6x & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

y-intercept: 0; x-intercepts: $[0, \infty)$

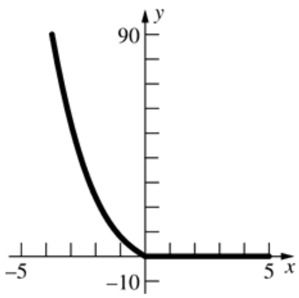
$$h'(x) = \begin{cases} -3x^2 + 2x - 6 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

No critical points for $x < 0$
$h(x)$ is increasing nowhere and decreasing on $(-\infty, 0]$.

$$h''(x) = \begin{cases} -6x + 2 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

$h(x)$ is concave up on $(-\infty, 0)$; no inflection

points



23. Domain: $(-\infty, \infty)$; range: [0, 1]

$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$; even function; symmetric with respect to the y-axis.

y-intercept: 0; x-intercepts: $k\pi$ where k is any integer.

$$f'(x) = \frac{\sin x}{|\sin x|}\cos x;\ f'(x) = 0 \text{ when } x = \frac{\pi}{2} + k\pi$$

and $f'(x)$ does not exist when $x = k\pi$, where k is any integer.

Critical points: $\frac{k\pi}{2}$ and $k\pi + \frac{\pi}{2}$, where k is any integer; $f'(x) > 0$ when $\sin x$ and $\cos x$ are either both positive or both negative.

$f(x)$ is increasing on $\left[k\pi, k\pi + \frac{\pi}{2}\right]$ and decreasing on $\left[k\pi + \frac{\pi}{2}, (k+1)\pi\right]$ where k is any integer.

Global minima $f(k\pi) = 0$; global maxima $f\left(k\pi + \frac{\pi}{2}\right) = 1$, where k is any integer.

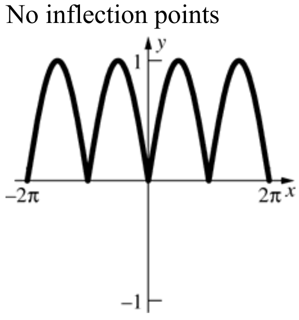
$$f''(x) = \frac{\cos^2 x}{|\sin x|} - \frac{\sin^2 x}{|\sin x|} + \sin x\cos x\left(-\frac{1}{|\sin x|^2}\right)\left(\frac{\sin x}{|\sin x|}\right)(\cos x)$$

$$= \frac{\cos^2 x}{|\sin x|} - \frac{\sin^2 x}{|\sin x|} - \frac{\cos^2 x}{|\sin x|} = -\frac{\sin^2 x}{|\sin x|} = -|\sin x|$$

$f''(x) < 0$ when $x \neq k\pi$, k any integer

$f(x)$ is never concave up and concave down on $(k\pi, (k+1)\pi)$ where k is any integer.

No inflection points



24. Domain: $[2k\pi, (2k+1)\pi]$ where k is any integer; range: [0, 1]

Neither an even nor an odd function

y-intercept: 0; x-intercepts: $k\pi$, where k is any integer.

$$f'(x) = \frac{\cos x}{2\sqrt{\sin x}};\ f'(x) = 0 \text{ when } x = 2k\pi + \frac{\pi}{2}$$

while $f'(x)$ does not exist when $x = k\pi$, k any integer.

Critical points: $k\pi, 2k\pi + \frac{\pi}{2}$ where k is any integer

$f'(x) > 0$ when $2k\pi < x < 2k\pi + \frac{\pi}{2}$

$f(x)$ is increasing on $\left[2k\pi, 2k\pi + \frac{\pi}{2}\right]$ and decreasing on $\left[2k\pi + \frac{\pi}{2}, (2k+1)\pi\right]$, k any integer.

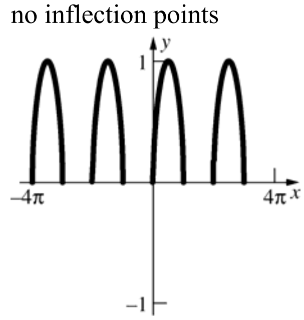
Global minima $f(k\pi) = 0$; global maxima $f\left(2k\pi + \frac{\pi}{2}\right) = 1$, k any integer

$$f''(x) = \frac{-\cos^2 x - 2\sin^2 x}{4\sin^{3/2} x} = \frac{-1 - \sin^2 x}{4\sin^{3/2} x}$$

$$= -\frac{1 + \sin^2 x}{4\sin^{3/2} x};$$

$f''(x) < 0$ for all x.

$f(x)$ is concave down on $(2k\pi, (2k+1)\pi)$; no inflection points



25. Domain: $(-\infty, \infty)$

Range: $[0,1]$

Even function since

$h(-t) = \cos^2(-t) = \cos^2 t = h(t)$

so the function is symmetric with respect to the y-axis.

y-intercept: $y = 1$; t-intercepts: $x = \frac{\pi}{2} + k\pi$

where k is any integer.

$h'(t) = -2\cos t \sin t$; $h'(t) = 0$ at $t = \frac{k\pi}{2}$.

Critical points: $t = \frac{k\pi}{2}$

$h'(t) > 0$ when $k\pi + \frac{\pi}{2} < t < (k+1)\pi$. The function is increasing on the intervals $[k\pi + (\pi/2), (k+1)\pi]$ and decreasing on the intervals $[k\pi, k\pi + (\pi/2)]$.

Global maxima $h(k\pi) = 1$

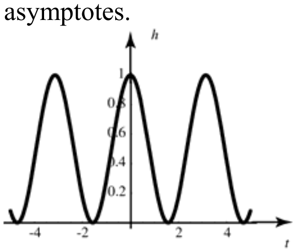
Global minima $h\left(\frac{\pi}{2} + k\pi\right) = 0$

$h''(t) = 2\sin^2 t - 2\cos^2 t = -2(\cos 2t)$

$h''(t) < 0$ on $\left(k\pi - \frac{\pi}{4}, k\pi + \frac{\pi}{4}\right)$ so h is concave down, and $h''(t) > 0$ on $\left(k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}\right)$ so h is concave up.

Inflection points: $\left(\frac{k\pi}{2} + \frac{\pi}{4}, \frac{1}{2}\right)$

No vertical asymptotes; no horizontal asymptotes.



26. Domain: all reals except $t = \frac{\pi}{2} + k\pi$

Range: $[0, \infty)$

y-intercepts: $y = 0$; t-intercepts: $t = k\pi$ where k is any integer.

Even function since

$g(-t) = \tan^2(-t) = (-\tan t)^2 = \tan^2 t$

so the function is symmetric with respect to the y-axis.

$g'(t) = 2\sec^2 t \tan t = \frac{2\sin t}{\cos^3 t}$; $g'(t) = 0$ when $t = k\pi$.

Critical points: $k\pi$

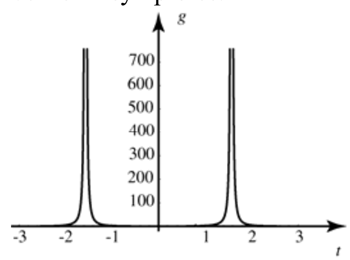
$g(t)$ is increasing on $\left[k\pi, k\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(k\pi - \frac{\pi}{2}, k\pi\right]$.

Global minima $g(k\pi) = 0$; no local maxima

$$g'(t) = 2\frac{\cos^4 t + \sin t(3)\cos^2 t \sin t}{\cos^6 t}$$
$$= 2\frac{\cos^2 t + 3\sin^2 t}{\cos^4 t}$$
$$= 2\frac{1 + 2\sin^2 t}{\cos^4 t} > 0$$

over the entire domain. Thus the function is concave up on $\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$; no inflection points.

No horizontal asymptotes; $t = \frac{\pi}{2} + k\pi$ are vertical asymptotes.



27. Domain: $\approx (-\infty, 0.44) \cup (0.44, \infty)$;

range: $(-\infty, \infty)$

Neither an even nor an odd function

y-intercept: 0; x-intercepts: 0, ≈ 0.24

$$f'(x) = \frac{74.6092x^3 - 58.2013x^2 + 7.82109x}{(7.126x - 3.141)^2};$$

$f'(x) = 0$ when $x = 0$, ≈ 0.17, ≈ 0.61

Critical points: 0, ≈ 0.17, ≈ 0.61

$f'(x) > 0$ when $0 < x < 0.17$ or $0.61 < x$

$f(x)$ is increasing on $\approx [0, 0.17] \cup [0.61, \infty)$

and decreasing on

$(-\infty, 0] \cup [0.17, 0.44) \cup (0.44, 0.61]$

Local minima $f(0) = 0$, $f(0.61) \approx 0.60$; local maximum $f(0.17) \approx 0.01$

$$f''(x) = \frac{531.665x^3 - 703.043x^2 + 309.887x - 24.566}{(7.126x - 3.141)^3};$$

$f''(x) > 0$ when $x < 0.10$ or $x > 0.44$

$f(x)$ is concave up on $(-\infty, 0.10) \cup (0.44, \infty)$

and concave down on $(0.10, 0.44)$;

inflection point $\approx (0.10, 0.003)$

$$\lim_{x\to\infty} \frac{5.235x^3 - 1.245x^2}{7.126x - 3.141} = \lim_{x\to\infty} \frac{5.235x^2 - 1.245x}{7.126 - \frac{3.141}{x}} = \infty$$

so $f(x)$ does not have a horizontal asymptote.

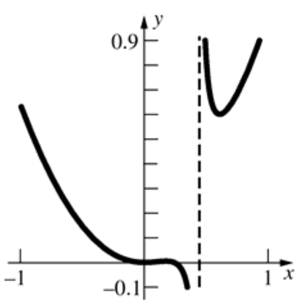
As $x \to 0.44^-, 5.235x^3 - 1.245x^2 \to 0.20$ while

$7.126x - 3.141 \to 0^-$, so $\lim\limits_{x\to 0.44^-} f(x) = -\infty$;

as $x \to 0.44^+, 5.235x^3 - 1.245x^2 \to 0.20$ while

$7.126x - 3.141 \to 0^+$, so $\lim\limits_{x\to 0.44^+} f(x) = \infty$;

$x \approx 0.44$ is a vertical asymptote of $f(x)$.



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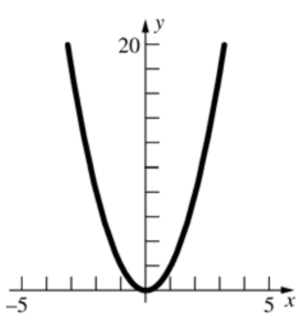

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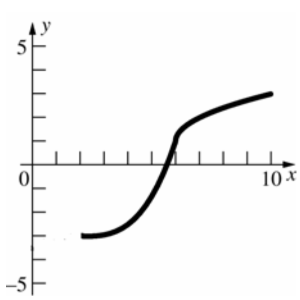

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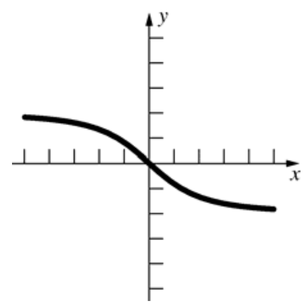

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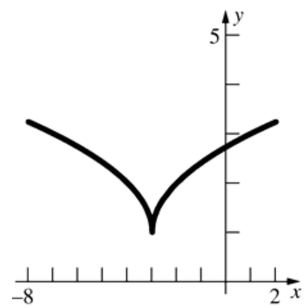

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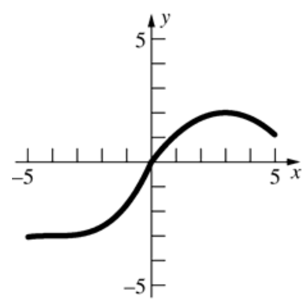

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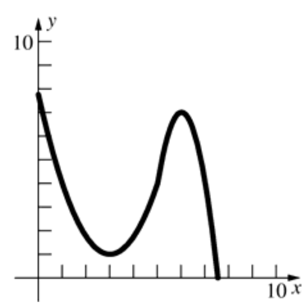

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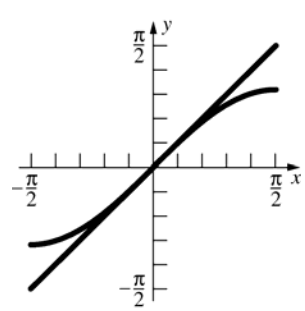

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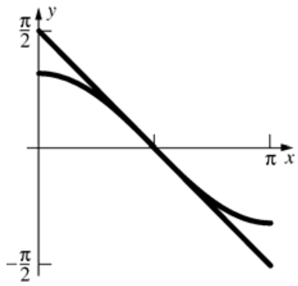
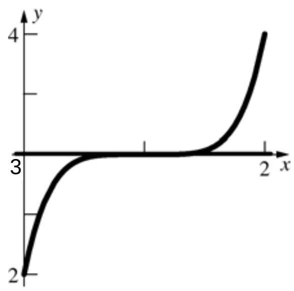


36. $y' = 5(x-1)^4$; $y'' = 20(x-1)^3$; $y''(x) > 0$ when $x > 1$; inflection point (1, 3)
At $x = 1$, $y' = 0$, so the linear approximation is a horizontal line.



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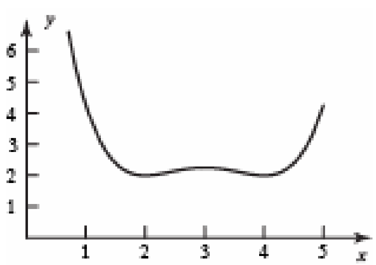

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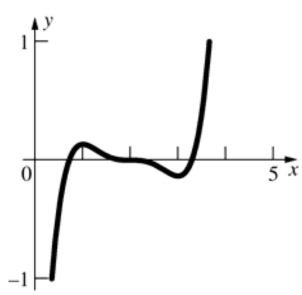

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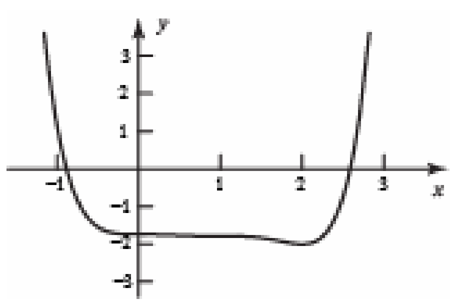
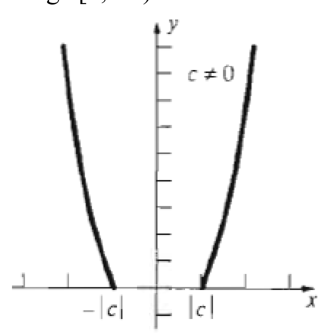


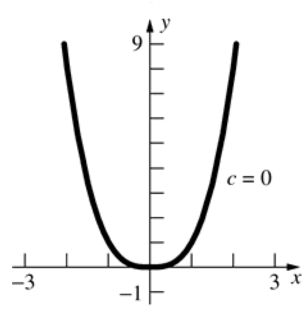
40. Let $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$ and $f''(x) = 2a$. An inflection point occurs where $f''(x)$ changes from positive to negative, but $2a$ is either always positive or always negative, so $f(x)$ does not have any inflection points.
($f''(x) = 0$ only when $a = 0$, but then $f(x)$ is not a quadratic curve.)

41. Let $f(x) = ax^3 + bx^2 + cx + d$, then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. As long as $a \neq 0$, $f''(x)$ will be positive on one side of $x = \dfrac{b}{3a}$ and negative on the other side.
$x = \dfrac{b}{3a}$ is the only inflection point.

42. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + c$, then
$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$ and
$f''(x) = 12ax^2 + 6bx + 2c = 2(6ax^2 + 3bx + c)$
Inflection points can only occur when $f''(x)$ changes sign from positive to negative and $f''(x) = 0$. $f''(x)$ has at most 2 zeros, thus $f(x)$ has at most 2 inflection points.

43. Since the c term is squared, the only difference occurs when $c = 0$. When $c = 0$,
$y = x^2\sqrt{x^2} = |x|^3$ which has domain $(-\infty, \infty)$ and range $[0, \infty)$. When $c \neq 0$, $y = x^2\sqrt{x^2 - c^2}$ has domain $(-\infty, -|c|] \cup [|c|, \infty)$ and range $[0, \infty)$.





The only extremum points are $\pm|c|$. For $c = 0$, there is one minimum, for $c \neq 0$ there are two. No maxima, independent of *c*. No inflection points, independent of *c*.

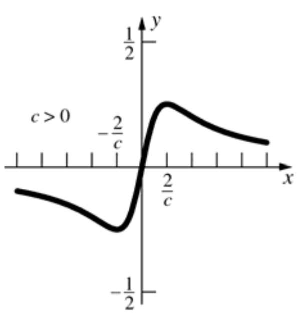
44. $f(x) = \dfrac{cx}{4+(cx)^2} = \dfrac{cx}{4+c^2x^2}$

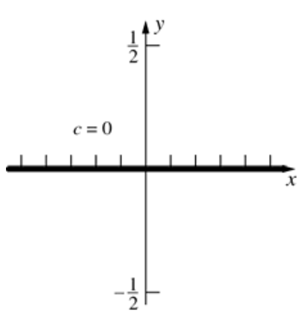
$f'(x) = \dfrac{c(4-c^2x^2)}{(4+c^2x^2)^2}$; $f'(x) = 0$ when $x = \pm\dfrac{2}{c}$ unless $c = 0$, in which case $f(x) = 0$ and $f'(x) = 0$.

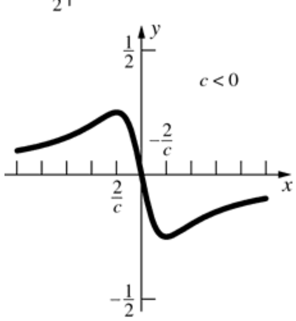
If $c > 0$, $f(x)$ is increasing on $\left[-\dfrac{2}{c}, \dfrac{2}{c}\right]$ and decreasing on $\left(-\infty, -\dfrac{2}{c}\right] \cup \left[\dfrac{2}{c}, \infty\right)$, thus, $f(x)$ has a global minimum at $f\left(-\dfrac{2}{c}\right) = -\dfrac{1}{4}$ and a global maximum of $f\left(\dfrac{2}{c}\right) = \dfrac{1}{4}$.

If $c < 0$, $f(x)$ is increasing on $\left(-\infty, \dfrac{2}{c}\right] \cup \left[-\dfrac{2}{c}, \infty\right)$ and decreasing on $\left[\dfrac{2}{c}, -\dfrac{2}{c}\right]$. Thus, $f(x)$ has a global minimum at $f\left(-\dfrac{2}{c}\right) = -\dfrac{1}{4}$ and a global maximum at $f\left(\dfrac{2}{c}\right) = \dfrac{1}{4}$.

$f''(x) = \dfrac{2c^3x(c^2x^2-12)}{(4+c^2x^2)^3}$, so $f(x)$ has inflection points at $x = 0$, $\pm\dfrac{2\sqrt{3}}{c}$, $c \neq 0$







45. $f(x) = \dfrac{1}{(cx^2-4)^2+cx^2}$, then

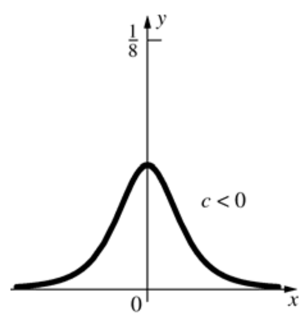
$f'(x) = \dfrac{2cx(7-2cx^2)}{[(cx^2-4)^2+cx^2]^2}$;

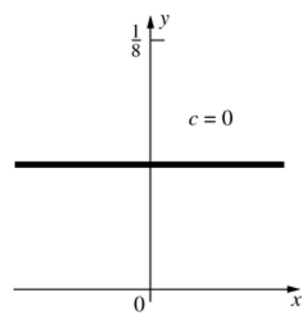
If $c > 0$, $f'(x) = 0$ when $x = 0$, $\pm\sqrt{\dfrac{7}{2c}}$.

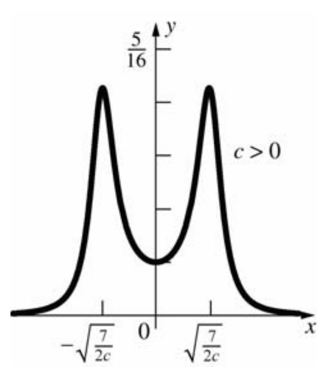
If $c < 0$, $f'(x) = 0$ when $x = 0$.

Note that $f(x) = \dfrac{1}{16}$ (a horizontal line) if $c = 0$.

If $c > 0$, $f'(x) > 0$ when $x < -\sqrt{\dfrac{7}{2c}}$ and $0 < x < \sqrt{\dfrac{7}{2c}}$, so $f(x)$ is increasing on $\left(-\infty, -\sqrt{\dfrac{7}{2c}}\right] \cup \left[0, \sqrt{\dfrac{7}{2c}}\right]$ and decreasing on $\left[-\sqrt{\dfrac{7}{2c}}, 0\right] \cup \left[\sqrt{\dfrac{7}{2c}}, \infty\right)$. Thus, $f(x)$ has local maxima $f\left(-\sqrt{\dfrac{7}{2c}}\right) = \dfrac{4}{15}$, $f\left(\sqrt{\dfrac{7}{2c}}\right) = \dfrac{4}{15}$ and local minimum $f(0) = \dfrac{1}{16}$. If $c < 0$, $f'(x) > 0$ when $x < 0$, so $f(x)$ is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$. Thus, $f(x)$ has a local maximum $f(0) = \dfrac{1}{16}$. Note that $f(x) > 0$ and has horizontal asymptote $y = 0$.

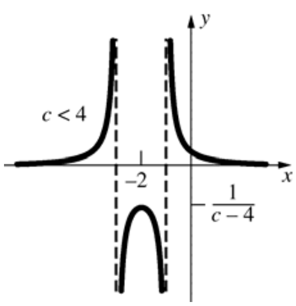


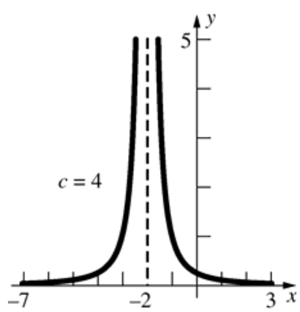


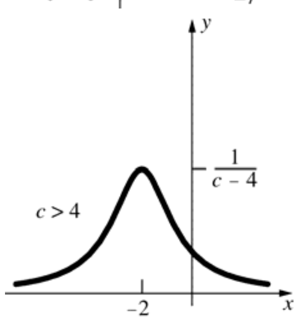


46. $f(x) = \dfrac{1}{x^2 + 4x + c}$. By the quadratic formula, $x^2 + 4x + c = 0$ when $x = -2 \pm \sqrt{4-c}$. Thus $f(x)$ has vertical asymptote(s) at $x = -2 \pm \sqrt{4-c}$ when $c \le 4$. $f'(x) = \dfrac{-2x-4}{(x^2+4x+c)^2}$; $f'(x) = 0$ when $x = -2$, unless $c = 4$ since then $x = -2$ is a vertical asymptote.

For $c \ne 4$, $f'(x) > 0$ when $x < -2$, so $f(x)$ is increasing on $(-\infty, -2]$ and decreasing on $[-2, \infty)$ (with the asymptotes excluded). Thus $f(x)$ has a local maximum at $f(-2) = \dfrac{1}{c-4}$. For $c = 4$, $f'(x) = -\dfrac{2}{(x+2)^3}$ so $f(x)$ is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.







47. $f(x) = c + \sin cx$.

Since c is constant for all x and $\sin cx$ is continuous everywhere, the function $f(x)$ is continuous everywhere.

$f'(x) = c \cdot \cos cx$

$f'(x) = 0$ when $cx = \left(k + \frac{1}{2}\right)\pi$ or $x = \left(k + \frac{1}{2}\right)\frac{\pi}{c}$

where k is an integer.

$f''(x) = -c^2 \cdot \sin cx$

$f''\left(\left(k + \frac{1}{2}\right)\frac{\pi}{c}\right) = -c^2 \cdot \sin\left(c \cdot \left(k + \frac{1}{2}\right)\frac{\pi}{c}\right) = -c^2 \cdot (-1)^k$

In general, the graph of f will resemble the graph of $y = \sin x$. The period will decrease as $|c|$ increases and the graph will shift up or down depending on whether c is positive or negative.

If $c = 0$, then $f(x) = 0$.

If $c < 0$:

$f(x)$ is decreasing on $\left[\dfrac{(4k+1)\pi}{2c}, \dfrac{(4k-1)\pi}{2c}\right]$

$f(x)$ is increasing on $\left[\dfrac{(4k-1)\pi}{2c}, \dfrac{(4k-3)\pi}{2c}\right]$

$f(x)$ has local minima at $x = \dfrac{(4k-1)}{2c}\pi$ and local maxima at $x = \dfrac{(4k-3)\pi}{2c}$ where k is an integer.

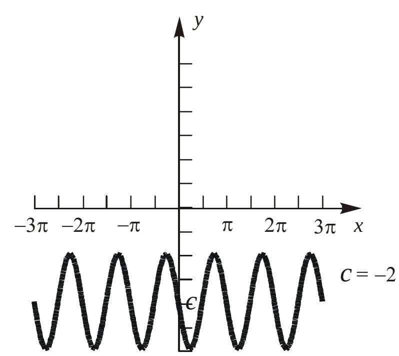
If $c = 0$, $f(x) = 0$ and there are no extrema.

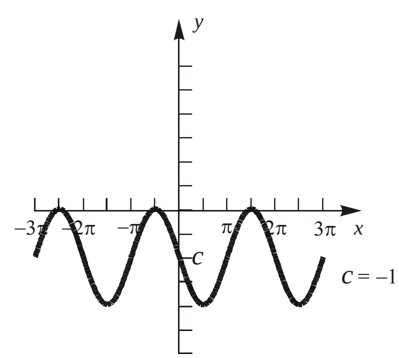
If $c > 0$:

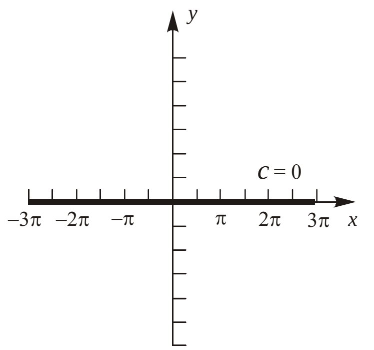
$f(x)$ is decreasing on $\left[\dfrac{(4k-3)\pi}{2c}, \dfrac{(4k-1)\pi}{2c}\right]$

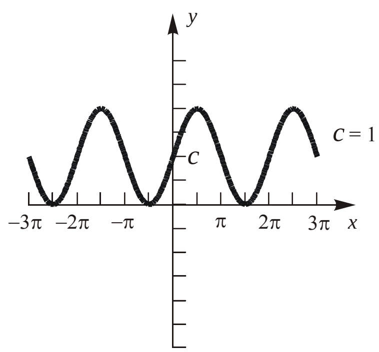
$f(x)$ is increasing on $\left[\dfrac{(4k-1)\pi}{2c}, \dfrac{(4k+1)\pi}{2c}\right]$

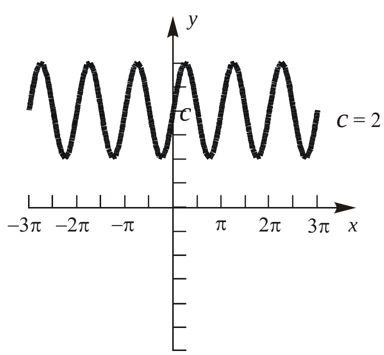
$f(x)$ has local minima at $x = \dfrac{(4k-1)}{2c}\pi$ and local maxima at $x = \dfrac{(4k-3)\pi}{2c}$ where k is an integer.





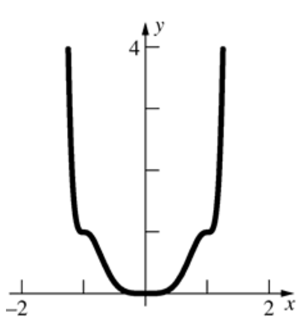






48. Since we have $f'''(c) > 0$, we know that $f'(x)$ is concave up in a neighborhood around $x = c$. Since $f'(c) = 0$, we then know that the graph of $f'(x)$ must be positive in that neighborhood. This means that the graph of *f* must be increasing to the left of *c* and increasing to the right of *c*. Therefore, there is a point of inflection at *c*.

49.



Justification:

$f(1) = g(1) = 1$

$f(-x) = g((-x)^4) = g(x^4) = f(x)$

f is an even function; symmetric with respect to the *y*-axis.

$f'(x) = g'(x^4)4x^3$

$f'(x) > 0$ for x on $(0,1) \cup (1,\infty)$

$f'(x) < 0$ for x on $(-\infty,-1) \cup (-1,0)$

$f'(x) = 0$ for $x = -1, 0, 1$ since f' is continuous.

$f''(x) = g''(x^4)16x^6 + g'(x)12x^2$

$f''(x) = 0$ for $x = -1, 0, 1$

$f''(x) > 0$ for x on $(0, x_0) \cup (1,\infty)$

$f''(x) < 0$ for x on $(x_0, 1)$

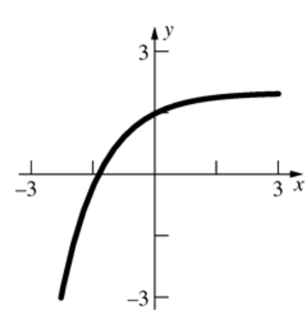
Where x_0 is a root of $f''(x) = 0$ (assume that there is only one root on (0, 1)).

50. Suppose $H'''(1) < 0,$ then $H''(x)$ is decreasing in a neighborhood around $x = 1$. Thus, $H''(x) > 0$ to the left of 1 and $H''(x) < 0$ to the right of 1, so $H(x)$ is concave up to the left of 1 and concave down to the right of 1. Suppose $H'''(1) > 0,$ then $H''(x)$ is increasing in a neighborhood around $x = 1$. Thus, $H''(x) < 0$ to the left of 1 and $H''(x) > 0$ to the right of 1, so $H(x)$ is concave up to the right of 1 and concave down to the left of 1. In either case, $H(x)$ has a point of inflection at $x = 1$ and not a local max or min.

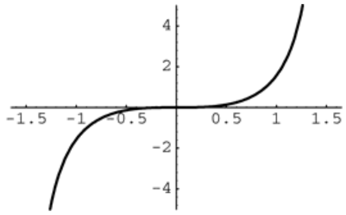
51. **a.** Not possible; $F'(x) > 0$ means that $F(x)$ is increasing. $F''(x) > 0$ means that the rate at which $F(x)$ is increasing never slows down. Thus the values of *F* must eventually become positive.

b. Not possible; If $F(x)$ is concave down for all *x*, then $F(x)$ cannot always be positive.

c.

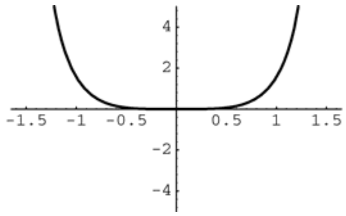


52. a.



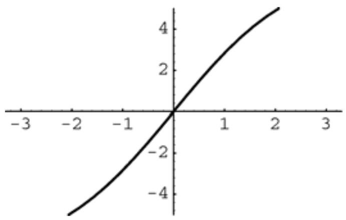
No global extrema; inflection point at (0, 0)

b.



No global maximum; global minimum at (0, 0); no inflection points

c.



Global minimum

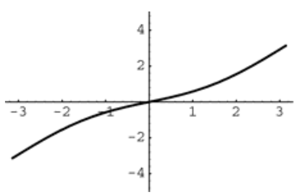
$f(-\pi) = -2\pi + \sin(-\pi) = -2\pi \approx -6.3;$

global maximum

$f(\pi) = 2\pi + \sin \pi = 2\pi \approx 6.3$;

inflection point at (0, 0)

d.



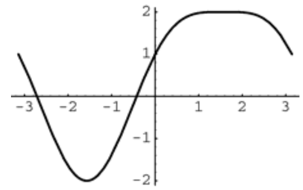
Global minimum

$f(-\pi) = -\pi - \dfrac{\sin(-\pi)}{2} = -\pi \approx 3.1;$ global

maximum $f(\pi) = \pi + \dfrac{\sin \pi}{2} = \pi \approx 3.1;$

inflection point at (0, 0).

53. a.



$f'(x) = 2\cos x - 2\cos x \sin x$

$= 2\cos x(1 - \sin x);$

$f'(x) = 0$ when $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

$f''(x) = -2\sin x - 2\cos^2 x + 2\sin^2 x$

$= 4\sin^2 x - 2\sin x - 2;$ $f''(x) = 0$ when

$\sin x = -\dfrac{1}{2}$ or $\sin x = 1$ which occur when

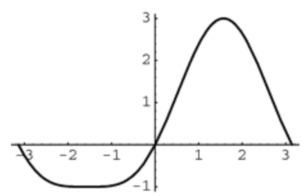
$x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6}, \dfrac{\pi}{2}$

Global minimum $f\left(-\dfrac{\pi}{2}\right) = -2;$ global

maximum $f\left(\dfrac{\pi}{2}\right) = 2;$ inflection points

$f\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{4}, f\left(-\dfrac{5\pi}{6}\right) = -\dfrac{1}{4}$

b.



$f'(x) = 2\cos x + 2\sin x \cos x$

$= 2\cos x(1 + \sin x);$ $f'(x) = 0$ when

$x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

$f''(x) = -2\sin x + 2\cos^2 x - 2\sin^2 x$

$= -4\sin^2 x - 2\sin x + 2;$ $f''(x) = 0$ when

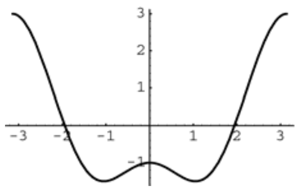
$\sin x = -1$ or $\sin x = \dfrac{1}{2}$ which occur when

$x = -\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Global minimum $f\left(-\dfrac{\pi}{2}\right) = -1;$ global

maximum $f\left(\dfrac{\pi}{2}\right) = 3;$ inflection points

$f\left(\dfrac{\pi}{6}\right) = \dfrac{5}{4}, f\left(\dfrac{5\pi}{6}\right) = \dfrac{5}{4}.$

c. 

$f'(x) = -2\sin 2x + 2\sin x$
$= -4\sin x\cos x + 2\sin x = 2\sin x(1 - 2\cos x);$

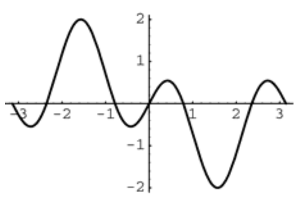
$f'(x) = 0$ when $x = -\pi, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \pi$

$f''(x) = -4\cos 2x + 2\cos x;\quad f''(x) = 0$ when
$x \approx -2.206, -0.568, 0.568, 2.206$

Global minimum $f\left(-\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) = -1.5;$

Global maximum $f(-\pi) = f(\pi) = 3;$

Inflection points: $\approx(-2.206, 0.890)$, $(-0.568, -1.265)$, $(0.568, -1.265)$, $(2.206, 0.890)$

d. 

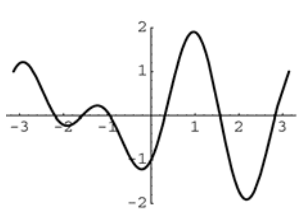
$f'(x) = 3\cos 3x - \cos x;\ f'(x) = 0$ when
$3\cos 3x = \cos x$ which occurs when
$x = -\frac{\pi}{2}, \frac{\pi}{2}$ and when
$x \approx -2.7, -0.4, 0.4, 2.7$

$f''(x) = -9\sin 3x + \sin x$ which occurs when
$x = -\pi$, 0, π and when
$x \approx -2.126, -1.016, 1.016, 2.126$

Global minimum $f\left(\frac{\pi}{2}\right) = -2;$

global maximum $f\left(-\frac{\pi}{2}\right) = 2;$

Inflection points: $\approx(-2.126, 0.755)$, $(-1.016, 0.755)$, $(0,0)$, $(1.016, -0.755)$, $(2.126, -0.755)$

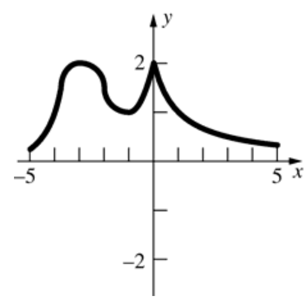
e. 

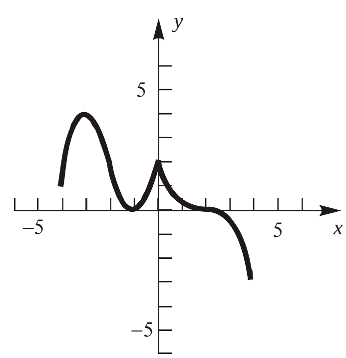
$f'(x) = 2\cos 2x + 3\sin 3x$

Using the graphs, $f(x)$ has a global minimum at $f(2.17) \approx -1.9$ and a global maximum at $f(0.97) \approx 1.9$

$f''(x) = -4\sin 2x + 9\cos 3x;\ f''(x) = 0$ when
$x = -\frac{\pi}{2}, \frac{\pi}{2}$ and when
$x \approx -2.469, -0.673, 0.413, 2.729.$

Inflection points: $\left(-\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, 0\right)$,
$\approx(-2.469, 0.542), (-0.673, -0.542)$, $(0.413, 0.408)$, $(2.729, -0.408)$

54. 

55. 

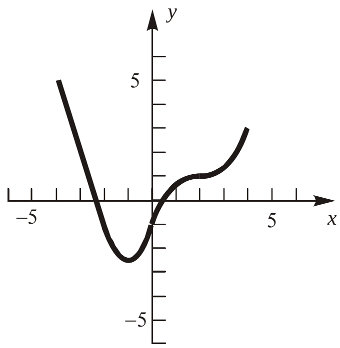
a. f is increasing on the intervals $(-\infty, -3]$ and $[-1, 0]$.
f is decreasing on the intervals $[-3, -1]$ and $[0, \infty)$.

b. f is concave down on the intervals $(-\infty, -2)$ and $(2, \infty)$.
f is concave up on the intervals $(-2, 0)$ and $(0, 2)$.

c. f attains a local maximum at $x = -3$ and $x = 0$.
f attains a local minimum at $x = -1$.

d. f has a point of inflection at $x = -2$ and $x = 2$.

56.



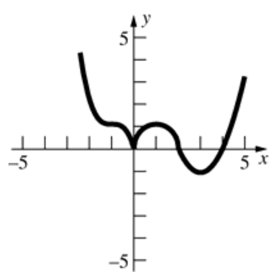
a. f is increasing on the interval $[-1, \infty)$.
f is decreasing on the interval $(-\infty, -1]$

b. f is concave up on the intervals $(-2, 0)$ and $(2, \infty)$.
f is concave down on the interval $(0, 2)$.

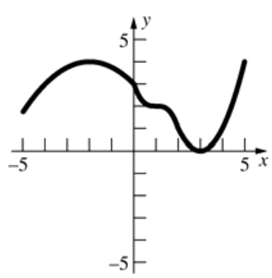
c. f does not have any local maxima.
f attains a local minimum at $x = -1$.

d. f has inflection points at $x = 0$ and $x = 2$.

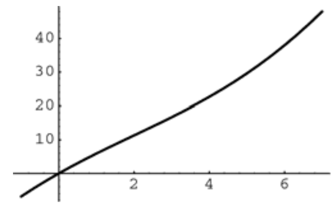
57.



58.



59. a.



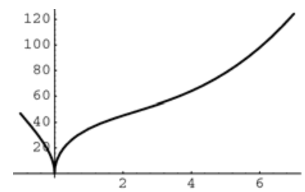
$$f'(x) = \frac{2x^2 - 9x + 40}{\sqrt{x^2 - 6x + 40}};\ f'(x) \text{ is never } 0,$$

and always positive, so $f(x)$ is increasing for all x. Thus, on [–1, 7], the global minimum is $f(-1) \approx -6.9$ and the global maximum if $f(7) \approx 48.0$.

$$f''(x) = \frac{2x^3 - 18x^2 + 147x - 240}{(x^2 - 6x + 40)^{3/2}};\ f''(x) = 0$$

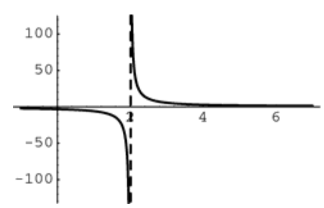
when $x \approx 2.02$; inflection point $f(2.02) \approx 11.4$

b.



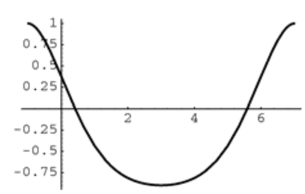
Global minimum $f(0) = 0$; global maximum $f(7) \approx 124.4$; inflection point at $x \approx 2.34$, $f(2.34) \approx 48.09$

c.

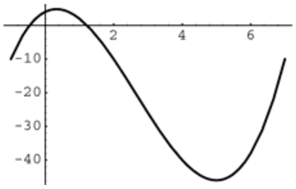


No global minimum or maximum; no inflection points

d.



Global minimum $f(3) \approx -0.9$;
global maximum $f(-1) \approx 1.0$ or $f(7) \approx 1.0$;
Inflection points at $x \approx 0.05$ and $x \approx 5.9$, $f(0.05) \approx 0.3$, $f(5.9) \approx 0.3$.

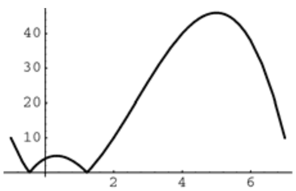
60. a. 

$f'(x) = 3x^2 - 16x + 5$; $f'(x) = 0$ when $x = \frac{1}{3}, 5.$

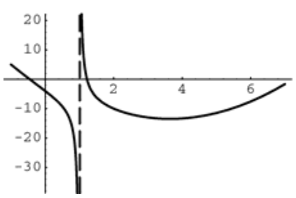
Global minimum $f(5) = -46$;
global maximum $f\left(\frac{1}{3}\right) \approx 4.8$

$f''(x) = 6x - 16$; $f''(x) = 0$ when $x = \frac{8}{3}$;

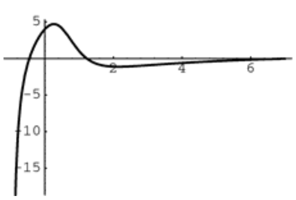
inflection point: $\left(\frac{8}{3}, -20.6\right)$

b. 

Global minimum when $x \approx -0.5$ and $x \approx 1.2$, $f(-0.5) \approx 0$, $f(1.2) \approx 0$;
global maximum $f(5) = 46$
Inflection point: $(-0.5, 0)$, $(1.2, 0)$, $\left(\frac{8}{3}, 20.6\right)$

c. 

No global minimum or maximum;
inflection point at
$x \approx -0.26$, $f(-0.26) \approx -1.7$

d. 

No global minimum, global maximum when $x \approx 0.26$, $f(0.26) \approx 4.7$
Inflection points when $x \approx 0.75$ and $x \approx 3.15$, $f(0.75) \approx 2.6$, $f(3.15) \approx -0.88$

3.6 Concepts Review

1. continuous; (a, b); $f(b) - f(a) = f'(c)(b - a)$

2. $f'(0)$ does not exist.

3. $F(x) = G(x) + C$

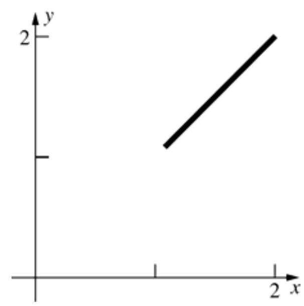
4. $x^4 + C$

Problem Set 3.6

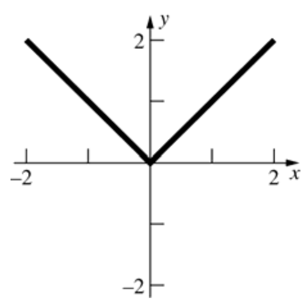
1. $f'(x) = \frac{x}{|x|}$

$\frac{f(2) - f(1)}{2 - 1} = \frac{2 - 1}{1} = 1$

$\frac{c}{|c|} = 1$ for all $c > 0$, hence for all c in (1, 2)



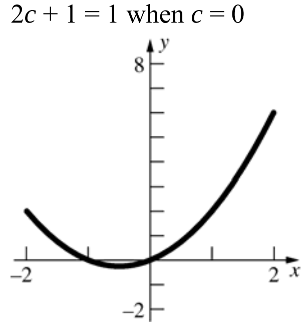
2. The Mean Value Theorem does not apply because $g'(0)$ does not exist.



3. $f'(x) = 2x + 1$

$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 - 2}{4} = 1$

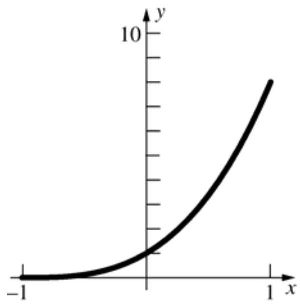
$2c + 1 = 1$ when $c = 0$



4. $g'(x) = 3(x+1)^2$

$$\frac{g(1)-g(-1)}{1-(-1)} = \frac{8-0}{2} = 4$$

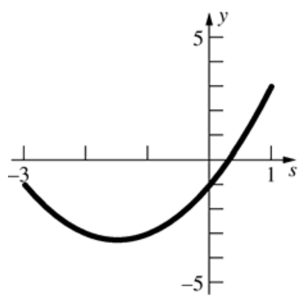
$3(c+1)^2 = 4$ when $c = -1 + \frac{2}{\sqrt{3}} \approx 0.15$



5. $H'(s) = 2s + 3$

$$\frac{H(1)-H(-3)}{1-(-3)} = \frac{3-(-1)}{1-(-3)} = 1$$

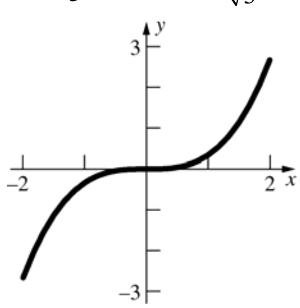
$2c + 3 = 1$ when $c = -1$



6. $F'(x) = x^2$

$$\frac{F(2)-F(-2)}{2-(-2)} = \frac{\frac{8}{3}-\left(-\frac{8}{3}\right)}{4} = \frac{4}{3}$$

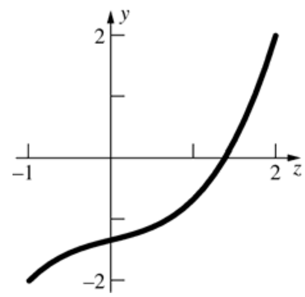
$c^2 = \frac{4}{3}$ when $c = \pm\frac{2}{\sqrt{3}} \approx \pm 1.15$



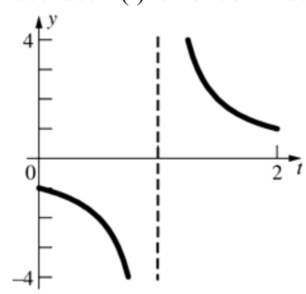
7. $f'(z) = \frac{1}{3}(3z^2+1) = z^2 + \frac{1}{3}$

$$\frac{f(2)-f(-1)}{2-(-1)} = \frac{2-(-2)}{3} = \frac{4}{3}$$

$c^2 + \frac{1}{3} = \frac{4}{3}$ when $c = -1, 1$, but -1 is not in $(-1, 2)$ so $c = 1$ is the only solution.



8. The Mean Value Theorem does not apply because $F(t)$ is not continuous at $t = 1$.

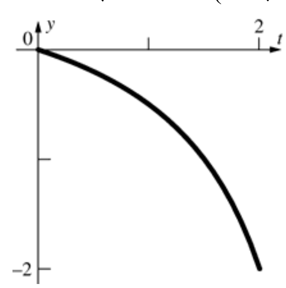


9. $h'(x) = -\frac{3}{(x-3)^2}$

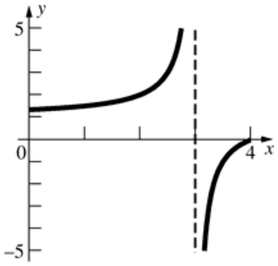
$$\frac{h(2)-h(0)}{2-0} = \frac{-2-0}{2} = -1$$

$-\frac{3}{(c-3)^2} = -1$ when $c = 3 \pm \sqrt{3}$,

$c = 3 - \sqrt{3} \approx 1.27$ $(3+\sqrt{3}$ is not in $(0, 2)$.)



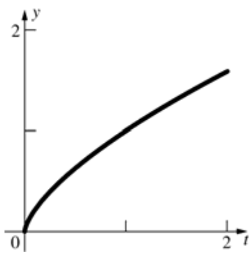
10. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 3$.



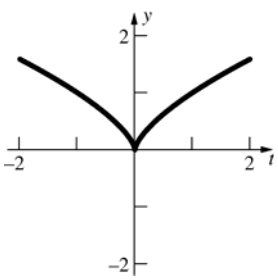
11. $h'(t) = \dfrac{2}{3t^{1/3}}$

$$\frac{h(2)-h(0)}{2-0} = \frac{2^{2/3}-0}{2} = 2^{-1/3}$$

$$\frac{2}{3c^{1/3}} = 2^{-1/3} \text{ when } c = \frac{16}{27} \approx 0.59$$



12. The Mean Value Theorem does not apply because $h'(0)$ does not exist.

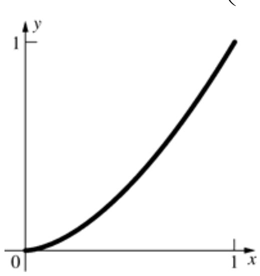


13. $g'(x) = \dfrac{5}{3}x^{2/3}$

$$\frac{g(1)-g(0)}{1-0} = \frac{1-0}{1} = 1$$

$$\frac{5}{3}c^{2/3} = 1 \text{ when } c = \pm\left(\frac{3}{5}\right)^{3/2},$$

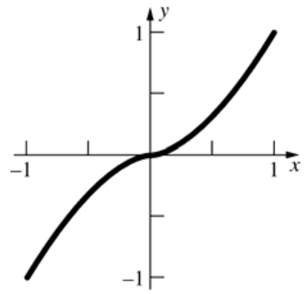
$$c = \left(\frac{3}{5}\right)^{3/2} \approx 0.46, \left(-\left(\frac{3}{5}\right)^{3/2} \text{ is not in } (0, 1).\right)$$



14. $g'(x) = \dfrac{5}{3}x^{2/3}$

$$\frac{g(1)-g(-1)}{1-(-1)} = \frac{1-(-1)}{2} = 1$$

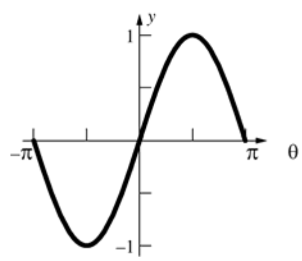
$$\frac{5}{3}c^{2/3} = 1 \text{ when } c = \pm\left(\frac{3}{5}\right)^{3/2} \approx \pm 0.46$$



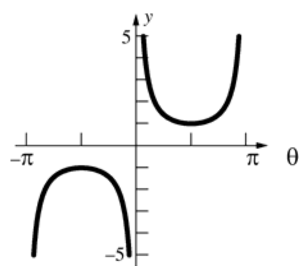
15. $S'(\theta) = \cos\theta$

$$\frac{S(\pi)-S(-\pi)}{\pi-(-\pi)} = \frac{0-0}{2\pi} = 0$$

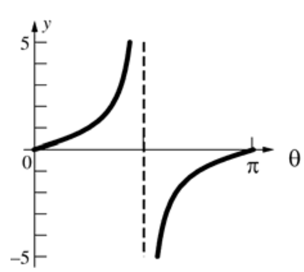
$\cos c = 0$ when $c = \pm\dfrac{\pi}{2}$.



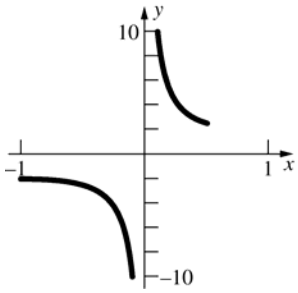
16. The Mean Value Theorem does not apply because $C(\theta)$ is not continuous at $\theta = -\pi, 0, \pi$.



17. The Mean Value Theorem does not apply because $T(\theta)$ is not continuous at $\theta = \dfrac{\pi}{2}$.



18. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 0$.

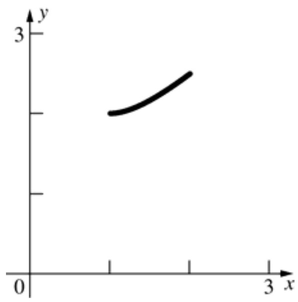


19. $f'(x) = 1 - \frac{1}{x^2}$

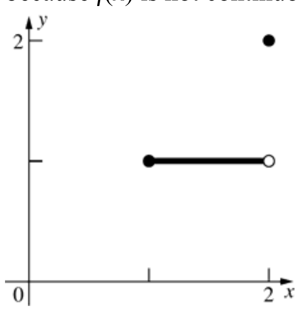
$\frac{f(2)-f(1)}{2-1} = \frac{\frac{5}{2}-2}{1} = \frac{1}{2}$

$1 - \frac{1}{c^2} = \frac{1}{2}$ when $c = \pm\sqrt{2}$, $c = \sqrt{2} \approx 1.41$

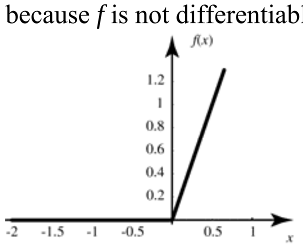
($c = -\sqrt{2}$ is not in (1, 2).)



20. The Mean Value Theorem does not apply because $f(x)$ is not continuous at $x = 2$.



21. The Mean Value Theorem does not apply because f is not differentiable at $x = 0$.



22. By the Mean Value Theorem

$\frac{f(b)-f(a)}{b-a} = f'(c)$ for some c in (a, b).

Since $f(b) = f(a)$, $\frac{0}{b-a} = f'(c)$; $f'(c) = 0$.

23. $\frac{f(8)-f(0)}{8-0} = -\frac{1}{4}$

There are three values for c such that

$f'(c) = -\frac{1}{4}$.

They are approximately 1.5, 3.75, and 7.

24. $f'(x) = 2\alpha x + \beta$

$\frac{f(b)-f(a)}{b-a} = \frac{1}{b-a}[\alpha(b^2 - a^2) + \beta(b-a)]$

$= \alpha(a+b) + \beta$

$2\alpha c + \beta = \alpha(a+b) + \beta$ when $c = \frac{a+b}{2}$ which is the midpoint of $[a, b]$.

25. By the Monotonicity Theorem, f is increasing on the intervals (a, x_0) and (x_0, b).

To show that $f(x_0) > f(x)$ for x in (a, x_0), consider f on the interval $(a, x_0]$.

f satisfies the conditions of the Mean Value Theorem on the interval $[x, x_0]$ for x in (a, x_0).

So for some c in (x, x_0),

$f(x_0) - f(x) = f'(c)(x_0 - x)$.

Because

$f'(c) > 0$ and $x_0 - x > 0$, $f(x_0) - f(x) > 0$,

so $f(x_0) > f(x)$.

Similar reasoning shows that

$f(x) > f(x_0)$ for x in (x_0, b).

Therefore, f is increasing on (a, b).

26. **a.** $f'(x) = 3x^2 > 0$ except at $x = 0$ in $(-\infty, \infty)$.

$f(x) = x^3$ is increasing on $(-\infty, \infty)$ by Problem 25.

b. $f'(x) = 5x^4 > 0$ except at $x = 0$ in $(-\infty, \infty)$.

$f(x) = x^5$ is increasing on $(-\infty, \infty)$ by Problem 25.

c. $f'(x) = \begin{cases} 3x^2 & x \le 0 \\ 1 & x > 0 \end{cases} > 0$ except at $x = 0$ in $(-\infty, \infty)$.

$f(x) = \begin{cases} x^3 & x \le 0 \\ x & x > 0 \end{cases}$ is increasing on $(-\infty, \infty)$ by Problem 25.

27. $s(t)$ is defined in any interval not containing $t = 0$.
$s'(c) = -\frac{1}{c^2} < 0$ for all $c \neq 0$. For any a, b with $a < b$ and both either positive or negative, the Mean Value Theorem says
$s(b) - s(a) = s'(c)(b - a)$ for some c in (a, b).
Since $a < b$, $b - a > 0$ while $s'(c) < 0$, hence $s(b) - s(a) < 0$, or $s(b) < s(a)$.
Thus, $s(t)$ is decreasing on any interval not containing $t = 0$.

28. $s'(c) = -\frac{2}{c^3} < 0$ for all $c > 0$. If $0 < a < b$, the Mean Value Theorem says
$s(b) - s(a) = s'(c)(b - a)$ for some c in (a, b).
Since $a < b$, $b - a > 0$ while $s'(c) < 0$, hence $s(b) - s(a) < 0$, or $s(b) < s(a)$. Thus, $s(t)$ is decreasing on any interval to the right of the origin.

29. $F'(x) = 0$ and $G(x) = 0; G'(x) = 0$.
By Theorem B,
$F(x) = G(x) + C$, so $F(x) = 0 + C = C$.

30. $F(x) = \cos^2 x + \sin^2 x; F(0) = 1^2 + 0^2 = 1$
$F'(x) = 2\cos x(-\sin x) + 2\sin x(\cos x) = 0$
By Problem 29, $F(x) = C$ for all x.
Since $F(0) = 1$, $C = 1$, so $\sin^2 x + \cos^2 x = 1$ for all x.

31. Let $G(x) = Dx; F'(x) = D$ and $G'(x) = D$.
By Theorem B, $F(x) = G(x) + C$; $F(x) = Dx + C$.

32. $F'(x) = 5; F(0) = 4$
$F(x) = 5x + C$ by Problem 31.
$F(0) = 4$ so $C = 4$.
$F(x) = 5x + 4$

33. Since $f(a)$ and $f(b)$ have opposite signs, 0 is between $f(a)$ and $f(b)$. $f(x)$ is continuous on $[a, b]$, since it has a derivative. Thus, by the Intermediate Value Theorem, there is at least one point c,
$a < c < b$ with $f(c) = 0$.
Suppose there are two points, c and $c', c < c'$ in (a, b) with $f(c) = f(c') = 0$. Then by Rolle's Theorem, there is at least one number d in (c, c') with $f'(d) = 0$. This contradicts the given information that $f'(x) \neq 0$ for all x in $[a, b]$, thus there cannot be more than one x in $[a, b]$ where $f(x) = 0$.

34. $f'(x) = 6x^2 - 18x = 6x(x - 3)$; $f'(x) = 0$ when $x = 0$ or $x = 3$.
$f(-1) = -10$, $f(0) = 1$ so, by Problem 33, $f(x) = 0$ has exactly one solution on $(-1, 0)$.
$f(0) = 1$, $f(1) = -6$ so, by Problem 33, $f(x) = 0$ has exactly one solution on $(0, 1)$.
$f(4) = -15$, $f(5) = 26$ so, by Problem 33, $f(x) = 0$ has exactly one solution on $(4, 5)$.

35. Suppose there is more than one zero between successive distinct zeros of f'. That is, there are a and b such that $f(a) = f(b) = 0$ with a and b between successive distinct zeros of f'. Then by Rolle's Theorem, there is a c between a and b such that $f'(c) = 0$. This contradicts the supposition that a and b lie between successive distinct zeros.

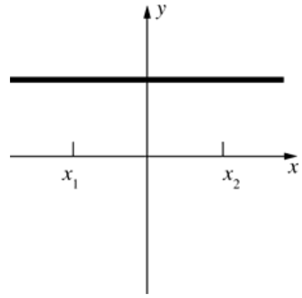
36. Let x_1, x_2, and x_3 be the three values such that $g(x_1) = g(x_2) = g(x_3) = 0$ and $a \le x_1 < x_2 < x_3 \le b$. By applying Rolle's Theorem (see Problem 22) there is at least one number x_4 in (x_1, x_2) and one number x_5 in (x_2, x_3) such that $g'(x_4) = g'(x_5) = 0$. Then by applying Rolle's Theorem to $g'(x)$, there is at least one number x_6 in (x_4, x_5) such that $g''(x_6) = 0$.

37. $f(x)$ is a polynomial function so it is continuous on $[0, 4]$ and $f''(x)$ exists for all x on $(0, 4)$.
$f(1) = f(2) = f(3) = 0$, so by Problem 36, there are at least two values of x in $[0, 4]$ where $f'(x) = 0$ and at least one value of x in $[0, 4]$ where $f''(x) = 0$.

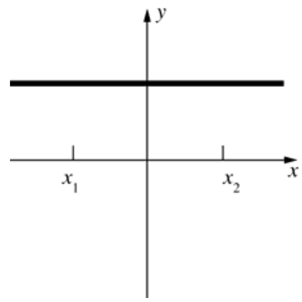
38. By applying the Mean Value Theorem and taking the absolute value of both sides,
$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} = |f'(c)|$, for some c in (x_1, x_2).
Since $|f'(x)| \le M$ for all x in (a, b),
$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \le M; |f(x_2) - f(x_1)| \le M|x_2 - x_1|$.

39. $f'(x) = 2\cos 2x; |f'(x)| \le 2$
$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} = |f'(x)|; \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \le 2$
$|f(x_2) - f(x_1)| \le 2|x_2 - x_1|$;
$|\sin 2x_2 - \sin 2x_1| \le 2|x_2 - x_1|$

40. **a.**



b.



41. Suppose $f'(x) \geq 0$. Let a and b lie in the interior of I such that $b > a$. By the Mean Value Theorem, there is a point c between a and b such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}; \dfrac{f(b) - f(a)}{b - a} \geq 0$.

Since $a < b$, $f(b) \geq f(a)$, so f is nondecreasing.

Suppose $f'(x) \leq 0$. Let a and b lie in the interior of I such that $b > a$. By the Mean Value Theorem, there is a point c between a and b such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}; \ \dfrac{f(b) - f(a)}{b - a} \leq 0$. Since $a < b$, $f(a) \geq f(b)$, so f is nonincreasing.

42. $[f^2(x)]' = 2f(x)f'(x)$

Because $f(x) \geq 0$ and $f'(x) \geq 0$ on $I, [f^2(x)]' \geq 0$ on I.

As a consequence of the Mean Value Theorem, $f^2(x_2) - f^2(x_1) \geq 0$ for all $x_2 > x_1$ on I.

Therefore f^2 is nondecreasing.

43. Let $f(x) = h(x) - g(x)$.

$f'(x) = h'(x) - g'(x)$; $f'(x) \geq 0$ for all x in (a, b) since $g'(x) \leq h'(x)$ for all x in (a, b), so f is nondecreasing on (a, b) by Problem 41. Thus

$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$;

$h(x_1) - g(x_1) \leq h(x_2) - g(x_2)$;

$g(x_2) - g(x_1) \leq h(x_2) - h(x_1)$ for all x_1 and x_2 in (a, b).

44. Let $f(x) = \sqrt{x}$ so $f'(x) = \dfrac{1}{2\sqrt{x}}$. Apply the Mean Value Theorem to f on the interval $[x, x + 2]$ for $x > 0$.

Thus $\sqrt{x+2} - \sqrt{x} = \dfrac{1}{2\sqrt{c}}(2) = \dfrac{1}{\sqrt{c}}$ for some c in $(x, x + 2)$. Observe $\dfrac{1}{\sqrt{x+2}} < \dfrac{1}{\sqrt{c}} < \dfrac{1}{\sqrt{x}}$.

Thus as $x \to \infty, \dfrac{1}{\sqrt{c}} \to 0$.

Therefore $\lim\limits_{x\to\infty}\left(\sqrt{x+2} - \sqrt{x}\right) = \lim\limits_{x\to\infty}\dfrac{1}{\sqrt{c}} = 0$.

45. Let $f(x) = \sin x$. $f'(x) = \cos x$, so $|f'(x)| = |\cos x| \leq 1$ for all x.

By the Mean Value Theorem,

$\dfrac{f(x) - f(y)}{x - y} = f'(c)$ for some c in (x, y).

Thus, $\dfrac{|f(x) - f(y)|}{|x - y|} = |f'(c)| \leq 1$;

$|\sin x - \sin y| \leq |x - y|$.

46. Let d be the difference in distance between horse A and horse B as a function of time t.

Then d' is the difference in speeds.

Let t_0 and t_1 and be the start and finish times of the race.

$d(t_0) = d(t_1) = 0$

By the Mean Value Theorem,

$\dfrac{d(t_1) - d(t_0)}{t_1 - t_0} = d'(c)$ for some c in (t_0, t_1).

Therefore $d'(c) = 0$ for some c in (t_0, t_1).

47. Let s be the difference in speeds between horse A and horse B as function of time t.

Then s' is the difference in accelerations.

Let t_2 be the time in Problem 46 at which the horses had the same speeds and let t_1 be the finish time of the race.

$s(t_2) = s(t_1) = 0$

By the Mean Value Theorem,

$\dfrac{s(t_1) - s(t_2)}{t_1 - t_2} = s'(c)$ for some c in (t_2, t_1).

Therefore $s'(c) = 0$ for some c in (t_2, t_1).

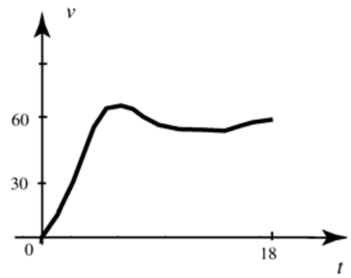
48. Suppose $x > c$. Then by the Mean Value Theorem,
$f(x) - f(c) = f'(a)(x-c)$ for some a in (c, x).
Since f is concave up, $f'' > 0$ and by the Monotonicity Theorem f' is increasing.
Therefore $f'(a) > f'(c)$ and
$$f(x) - f(c) = f'(a)(x-c) > f'(c)(x-c)$$
$$f(x) > f(c) + f'(c)(x-c), x > c$$
Suppose $x < c$. Then by the Mean Value Theorem,
$f(c) - f(x) = f'(a)(c-x)$ for some a in (x, c).
Since f is concave up, $f'' > 0$, and by the Monotonicity Theorem f' is increasing.
Therefore, $f'(c) > f'(a)$ and
$$f(c) - f(x) = f'(a)(c-x) < f'(c)(c-x).$$
$$-f(x) < -f(c) + f'(c)(c-x)$$
$$f(x) > f(c) - f'(c)(c-x)$$
$$f(x) > f(c) + f'(c)(x-c), x < c$$
Therefore $f(x) > f(c) + f'(c)(x-c), x \neq c$.

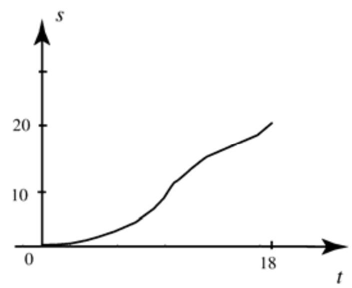
49. Fix an arbitrary x.
$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0, \text{ since}$$
$$\left| \frac{f(y) - f(x)}{y - x} \right| \le M |y - x|.$$
So, $f' \equiv 0 \to f = \text{constant}$.

50. $f(x) = x^{1/3}$ on $[0, a]$ or $[-a, 0]$ where a is any positive number. $f'(0)$ does not exist, but $f(x)$ has a vertical tangent line at $x = 0$.

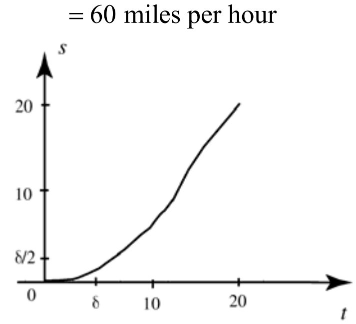
51. Let $f(t)$ be the distance traveled at time t.
$$\frac{f(2) - f(0)}{2 - 0} = \frac{112 - 0}{2} = 56$$
By the Mean Value Theorem, there is a time c such that $f'(c) = 56$.
At some time during the trip, Johnny must have gone 56 miles per hour.

52. s is differentiable with $s(0) = 0$ and $s(18) = 20$ so we can apply the Mean Value Theorem. There exists a c in the interval $(0, 18)$ such that
$$v(c) = s'(c) = \frac{(20-0)}{(18-0)} \approx 1.11 \text{ miles per minute}$$
$$\approx 66.67 \text{ miles per hour}$$





53. Since the car is stationary at $t = 0$, and since v is continuous, there exists a δ such that $v(t) < \frac{1}{2}$ for all t in the interval $[0, \delta]$. $v(t)$ is therefore less than $\frac{1}{2}$ and $s(\delta) < \delta \cdot \frac{1}{2} = \frac{\delta}{2}$. By the Mean Value Theorem, there exists a c in the interval $(\delta, 20)$ such that
$$v(c) = s'(c) = \frac{\left(20 - \frac{\delta}{2}\right)}{(20 - \delta)}$$
$$> \frac{20 - \delta}{20 - \delta}$$
$$= 1 \text{ mile per minute}$$
$$= 60 \text{ miles per hour}$$



54. Given the position function $s(t) = at^2 + bt + c$, the car's instantaneous velocity is given by the function $s'(t) = 2at + b$.

The midpoint of the interval $[A, B]$ is $\frac{A+B}{2}$.

Thus, the car's instantaneous velocity at the midpoint of the interval is given by

$$s'\left(\frac{A+B}{2}\right) = 2a\left(\frac{A+B}{2}\right) + b$$
$$= a(A+B) + b$$

The car's average velocity will be its change in position divided by the length of the interval. That is,

$$\frac{s(B) - s(A)}{B - A} = \frac{(aB^2 + bB + c) - (aA^2 + bA + c)}{B - A}$$
$$= \frac{aB^2 - aA^2 + bB - bA}{B - A}$$
$$= \frac{a(B^2 - A^2) + b(B - A)}{B - A}$$
$$= \frac{a(B - A)(B + A) + b(B - A)}{B - A}$$
$$= a(B + A) + b$$
$$= a(A + B) + b$$

This is the same result as the instantaneous velocity at the midpoint.

3.7 Concepts Review

1. slowness of convergence

2. root; Intermediate Value

3. algorithms

4. fixed point

Problem Set 3.7

1. Let $f(x) = x^3 + 2x - 6$.
$f(1) = -3$, $f(2) = 6$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	0.375
2	0.25	1.25	–1.546875
3	0.125	1.375	–0.650391
4	0.0625	1.4375	–0.154541
5	0.03125	1.46875	0.105927
6	0.015625	1.45312	–0.0253716
7	0.0078125	1.46094	0.04001
8	0.00390625	1.45703	0.00725670
9	0.00195312	1.45508	–0.00907617

$r \approx 1.46$

2. Let $f(x) = x^4 + 5x^3 + 1$.
$f(-1) = -3$, $f(0) = 1$

n	h_n	m_n	$f(m_n)$
1	0.5	–0.5	0.4375
2	0.25	–0.75	–0.792969
3	0.125	–0.625	–0.0681152
4	0.0625	–0.5625	0.21022
5	0.03125	–0.59375	0.0776834
6	0.015625	–0.609375	0.00647169
7	0.0078125	–0.617187	–0.0303962
8	0.00390625	–0.613281	–0.011854
9	0.00195312	–0.611328	–0.00266589

$r \approx -0.61$

3. Let $f(x) = 2\cos x - \sin x$.
$f(1) \approx 0.23913$; $f(2) \approx -1.74159$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	–0.856021
2	0.25	1.25	–0.318340
3	0.125	1.125	–0.039915
4	0.0625	1.0625	0.998044
5	0.03125	1.09375	0.029960
6	0.01563	1.109375	–0.004978

$r \approx 1.11$

4. Let $f(x) = x - 2 + 2\cos x$

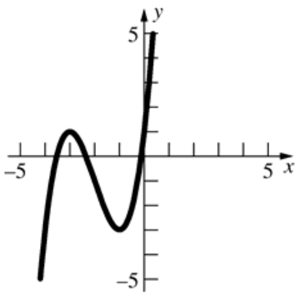
$f(1) = 1 - 2 + 2\cos(1) \approx 0.080605$

$f(2) = 2 - 2 + 2\cos(2) \approx -0.832294$

n	h_n	m_n	$f(m_n)$
1	0.5	1.5	–0.358526
2	0.25	1.25	–0.119355
3	0.125	1.125	–0.012647
4	0.0625	1.0625	0.035879
5	0.03125	1.09375	0.012065
6	0.01563	1.109375	–0.000183

$r \approx 1.11$

5. Let $f(x) = x^3 + 6x^2 + 9x + 1 = 0$.

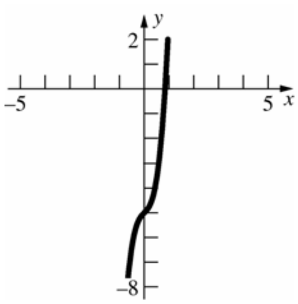


$f'(x) = 3x^2 + 12x + 9$

n	x_n
1	0
2	–0.1111111
3	–0.1205484
4	–0.1206148
5	–0.1206148

$r \approx -0.12061$

6. Let $f(x) = 7x^3 + x - 5$

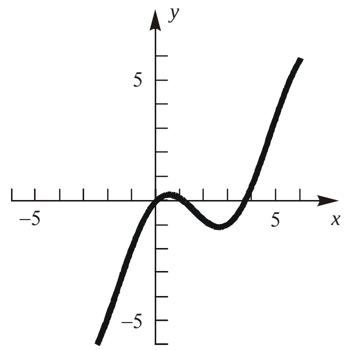


$f'(x) = 21x^2 + 1$

n	x_n
1	1
2	0.8636364
3	0.8412670
4	0.8406998
5	0.8406994
6	0.8406994

$r \approx 0.84070$

7. Let $f(x) = x - 2 + 2\cos x$.

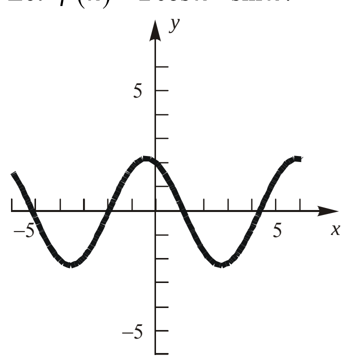


$f'(x) = 1 - 2\sin x$

n	x_n
1	4
2	3.724415
3	3.698429
4	3.698154
5	3.698154

$r \approx 3.69815$

8. Let $f(x) = 2\cos x - \sin x$.

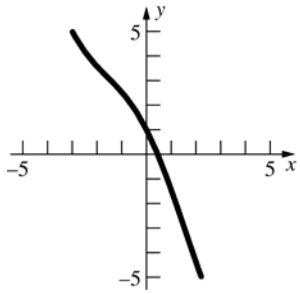


$f'(x) = -2\sin x - \cos x$

n	x_n
1	0.5
2	1.1946833
3	1.1069244
4	1.1071487
5	1.1071487

$r \approx 1.10715$

9. Let $f(x) = \cos x - 2x$.

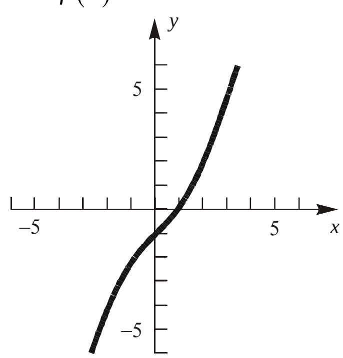


$f'(x) = -\sin x - 2$

n	x_n
1	0.5
2	0.4506267
3	0.4501836
4	0.4501836

$r \approx 0.45018$

10. Let $f(x) = 2x - \sin x - 1$.

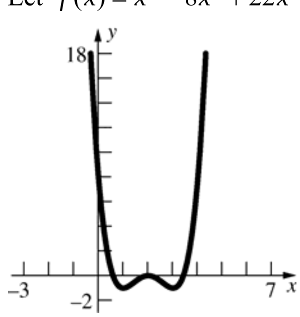


$f'(x) = 2 - \cos x$

n	x_n
1	1
2	0.891396
3	0.887866
4	0.887862
5	0.887862

$r \approx 0.88786$

11. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 8$.



$f'(x) = 4x^3 - 24x^2 + 44x - 24$

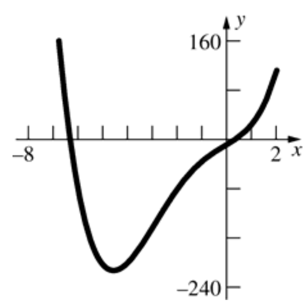
Note that $f(2) = 0$.

n	x_n
1	0.5
2	0.575
3	0.585586
4	0.585786

n	x_n
1	3.5
2	3.425
3	3.414414
4	3.414214
5	3.414214

$r = 2$, $r \approx 0.58579$, $r \approx 3.41421$

12. Let $f(x) = x^4 + 6x^3 + 2x^2 + 24x - 8$.



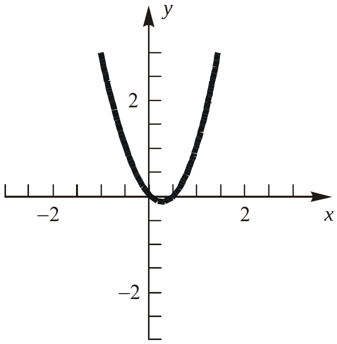
$f'(x) = 4x^3 + 18x^2 + 4x + 24$

n	x_n
1	–6.5
2	–6.3299632
3	–6.3167022
4	–6.3166248
5	–6.3166248

n	x_n
1	0.5
2	0.3286290
3	0.3166694
4	0.3166248
5	0.3166248

$r \approx -6.31662$, $r \approx 0.31662$

13. Let $f(x) = 2x^2 - \sin x$.

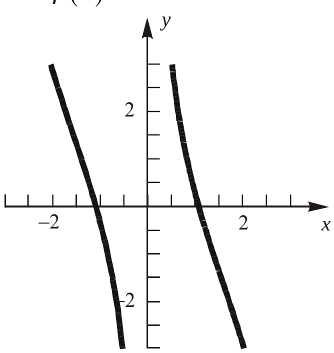


$f'(x) = 4x - \cos x$

n	x_n
1	0.5
2	0.481670
3	0.480947
4	0.480946

$r \approx 0.48095$

14. Let $f(x) = 2\cot x - x$.



$f'(x) = -2\csc^2 x - 1$

n	x_n
1	1
2	1.074305
3	1.076871
4	1.076874

$r \approx 1.07687$

15. Let $f(x) = x^3 - 6$.

$f'(x) = 3x^2$

n	x_n
1	1.5
2	1.888889
3	1.819813
4	1.817125
5	1.817121
6	1.817121

$\sqrt[3]{6} \approx 1.81712$

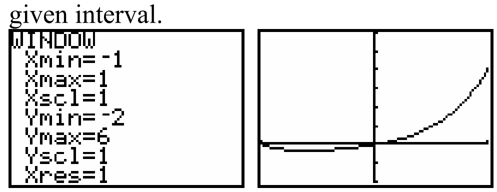
16. Let $f(x) = x^4 - 47$.

$f'(x) = 4x^3$

n	x_n
1	2.5
2	2.627
3	2.618373
4	2.618330
5	2.618330

$\sqrt[4]{47} \approx 2.61833$

17. $f(x) = x^4 + x^3 + x^2 + x$ is continuous on the given interval.



From the graph of *f*, we see that the maximum value of the function on the interval occurs at the right endpoint. The minimum occurs at a stationary point within the interval. To find where the minimum occurs, we solve $f'(x) = 0$ on the interval $[-1,1]$.

$f'(x) = 4x^3 + 3x^2 + 2x + 1 = g(x)$

Using Newton's Method to solve $g(x) = 0$, we get:

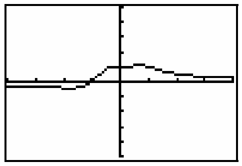
n	x_n
1	0
2	−0.5
3	−0.625
4	−0.60638
5	−0.60583
6	−0.60583

Minimum: $f(-0.60583) \approx -0.32645$

Maximum: $f(1) = 4$

18. $f(x)=\dfrac{x^3+1}{x^4+1}$ is continuous on the given interval.

WINDOW
Xmin=-4
Xmax=4
Xscl=1
Ymin=-5
Ymax=5
Yscl=1
Xres=1



From the graph of *f*, we see that the maximum and minimum will both occur at stationary points within the interval. The minimum appears to occur at about $x=-1.5$ while the maximum appears to occur at about $x=0.8$. To find the stationary points, we solve $f'(x)=0$.

$$f'(x)=\frac{-x^2\left(x^4+4x-3\right)}{\left(x^4+1\right)^2}=g(x)$$

Using Newton's method to solve $g(x)=0$ on the interval, we use the starting values of -1.5 and 0.8.

n	x_n
1	−1.5
2	−1.680734
3	−1.766642
4	−1.783766
5	−1.784357
6	−1.784358
7	−1.784358

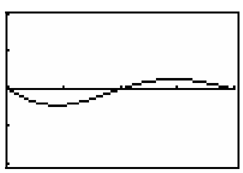
n	x_n
1	0.8
2	0.694908
3	0.692512
4	0.692505
5	0.692505

Maximum: $f(0.692505)\approx 1.08302$

Minimum: $f(-1.78436)\approx -0.42032$

19. $f(x)=\dfrac{\sin x}{x}$ is continuous on the given interval.

WINDOW
Xmin=3.1415926…
Xmax=9.4247779…
Xscl=1.5707963…
Ymin=-1
Ymax=1
Yscl=.5
Xres=1



From the graph of *f*, we see that the minimum value and maximum value on the interval will occur at stationary points within the interval. To find these points, we need to solve $f'(x)=0$ on the interval.

$$f'(x)=\frac{x\cos x-\sin x}{x^2}=g(x)$$

Using Newton's method to solve $g(x)=0$ on the interval, we use the starting values of $\dfrac{3\pi}{2}$ and $\dfrac{5\pi}{2}$.

n	x_n
1	4.712389
2	4.479179
3	4.793365
4	4.493409
5	4.493409

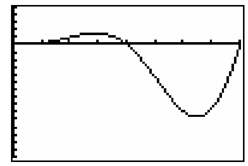
n	x_n
1	7.853982
2	7.722391
3	7.725251
4	7.725252
5	7.725252

Minimum: $f(4.493409)\approx -0.21723$

Maximum: $f(7.725252)\approx 0.128375$

20. $f(x)=x^2\sin\dfrac{x}{2}$ is continuous on the given interval.

WINDOW
Xmin=0
Xmax=12.566370…
Xscl=1.5707963…
Ymin=-150
Ymax=50
Yscl=10
Xres=1



From the graph of *f*, we see that the minimum value and maximum value on the interval will occur at stationary points within the interval. To find these points, we need to solve $f'(x)=0$ on the interval.

$$f'(x)=\frac{x^2\cos\frac{x}{2}+4x\sin\frac{x}{2}}{2}=g(x)$$

Using Newton's method to solve $g(x)=0$ on the interval, we use the starting values of $\dfrac{3\pi}{2}$ and $\dfrac{13\pi}{4}$.

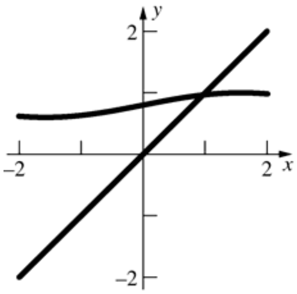
n	x_n
1	4.712389
2	4.583037
3	4.577868
4	4.577859
5	4.577859

n	x_n
1	10.210176
2	10.174197
3	10.173970
4	10.173970

Minimum: $f(10.173970)\approx -96.331841$

Maximum: $f(4.577859)\approx 15.78121$

21. Graph $y = x$ and $y = 0.8 + 0.2\sin x$.



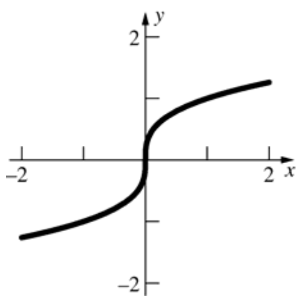
$x_{n+1} = 0.8 + 0.2\sin x_n$

Let $x_1 = 1$.

n	x_n
1	1
2	0.96829
3	0.96478
4	0.96439
5	0.96434
6	0.96433
7	0.96433

$x \approx 0.9643$

22.



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$$

$= x_n - 3x_n = -2x_n$

Thus, every iteration of Newton's Method gets further from zero. Note that $x_{n+1} = (-2)^{n+1}x_0$. Newton's Method is based on approximating f by its tangent line near the root. This function has a vertical tangent at the root.

23. a. For Tom's car, $P = 2000$, $R = 100$, and $k = 24$, thus

$$2000 = \frac{100}{i}\left[1 - \frac{1}{(1+i)^{24}}\right] \text{ or}$$

$20i = 1 - \dfrac{1}{(1+i)^{24}}$, which is equivalent to

$20i(1+i)^{24} - (1+i)^{24} + 1 = 0$.

b. Let

$f(i) = 20i(1+i)^{24} - (1+i)^{24} + 1$

$= (1+i)^{24}(20i - 1) + 1$.

Then

$f'(i) = 20(1+i)^{24} + 480i(1+i)^{23} - 24(1+i)^{23}$

$= (1+i)^{23}(500i - 4)$, so

$$i_{n+1} = i_n - \frac{f(i_n)}{f'(i_n)} = i_n - \frac{(1+i_n)^{24}(20i_n - 1) + 1}{(1+i_n)^{23}(500i_n - 4)}$$

$$= i_n - \left[\frac{20i_n^2 + 19i_n - 1 + (1+i_n)^{-23}}{500i_n - 4}\right].$$

c.

n	i_n
1	0.012
2	0.0165297
3	0.0152651
4	0.0151323
5	0.0151308
6	0.0151308

$i = 0.0151308$

$r = 18.157\%$

24. From Newton's algorithm, $x_{n+1} - x_n = -\dfrac{f(x_n)}{f'(x_n)}$.

$$\lim_{x_n \to \bar{x}}(x_{n+1} - x_n) = \lim_{x_n \to \bar{x}} x_{n+1} - \lim_{x_n \to \bar{x}} x_n$$

$= \bar{x} - \bar{x} = 0$

$\lim_{x_n \to \bar{x}} \dfrac{f(x_n)}{f'(x_n)}$ exists if f and f' are continuous at $\bar{x}$ and $f'(\bar{x}) \neq 0$.

Thus, $\lim_{x_n \to \bar{x}} \dfrac{f(x_n)}{f'(x_n)} = \dfrac{f(\bar{x})}{f'(\bar{x})} = 0$, so $f(\bar{x}) = 0$.

$\bar{x}$ is a solution of $f(x) = 0$.

25. $x_{n+1} = \dfrac{x_n + 1.5\cos x_n}{2}$

n	x_n
1	1
2	0.905227
3	0.915744
4	0.914773

n	x_n
5	0.914864
6	0.914856
7	0.914857

$x \approx 0.91486$

26. $x_{n+1} = 2 - \sin x$

n	x_n	n	x_n	n	x_n
1	2	5	1.10746	9	1.10603
2	1.09070	6	1.10543	10	1.10607
3	1.11305	7	1.10634	11	1.10606
4	1.10295	8	1.10612	12	1.10606

$x \approx 1.10606$

27. $x_{n+1} = \sqrt{2.7 + x_n}$

n	x_n
1	1
2	1.923538
3	2.150241
4	2.202326
5	2.214120
6	2.216781
7	2.217382
8	2.217517
9	2.217548
10	2.217554
11	2.217556
12	2.217556

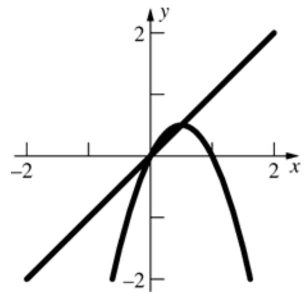
$x \approx 2.21756$

28. $x_{n+1} = \sqrt{3.2 + x_n}$

n	x_n
1	47
2	7.085196
3	3.207054
4	2.531216
5	2.393996
6	2.365163
7	2.359060
8	2.357766
9	2.357491
10	2.357433
11	2.357421
12	2.357418
13	2.357418

$x \approx 2.35742$

29. a.



$x \approx 0.5$

b. $x_{n+1} = 2(x_n - x_n^2)$

n	x_n
1	0.7
2	0.42
3	0.4872
4	0.4996723
5	0.4999998
6	0.5
7	0.5

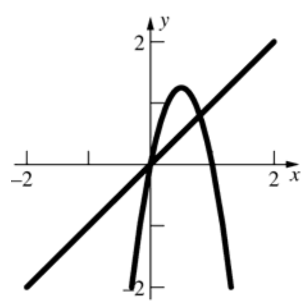
c. $x = 2(x - x^2)$

$2x^2 - x = 0$

$x(2x - 1) = 0$

$x = 0, \; x = \dfrac{1}{2}$

30. a.



$x \approx 0.8$

b. $x_{n+1} = 5(x_n - x_n^2)$

n	x_n
1	0.7
2	1.05
3	–0.2625
4	–1.657031
5	–22.01392
6	–2533.133

c. $x = 5(x - x^2)$

$5x^2 - 4x = 0$

$x(5x - 4) = 0$

$x = 0,\ x = \frac{4}{5}$

31. a. $x_1 = 0$

$x_2 = \sqrt{1} = 1$

$x_3 = \sqrt{1+\sqrt{1}} = \sqrt{2} \approx 1.4142136$

$x_4 = \sqrt{1+\sqrt{1+\sqrt{1}}} \approx 1.553774$

$x_5 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} \approx 1.5980532$

b. $x = \sqrt{1+x}$

$x^2 = 1 + x$

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{1+4\cdot 1\cdot 1}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Taking the minus sign gives a negative solution for x, violating the requirement that $x \ge 0$. Hence, $x = \frac{1+\sqrt{5}}{2} \approx 1.618034$.

c. Let $x = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$. Then x satisfies the equation $x = \sqrt{1+x}$. From part (b) we know that x must equal $\left(1+\sqrt{5}\right)/2 \approx 1.618034$.

32. a. $x_1 = 0$

$x_2 = \sqrt{5} \approx 2.236068$

$x_3 = \sqrt{5+\sqrt{5}} \approx 2.689994$

$x_4 = \sqrt{5+\sqrt{5+\sqrt{5}}} \approx 2.7730839$

$x_5 = \sqrt{5+\sqrt{5+\sqrt{5+\sqrt{5}}}} \approx 2.7880251$

b. $x = \sqrt{5+x}$, and x must satisfy $x \ge 0$

$x^2 = 5 + x$

$x^2 - x - 5 = 0$

$x = \frac{1 \pm \sqrt{1+4\cdot 1\cdot 5}}{2} = \frac{1 \pm \sqrt{21}}{2}$

Taking the minus sign gives a negative solution for x, violating the requirement that $x \ge 0$. Hence,

$x = \left(1+\sqrt{21}\right)/2 \approx 2.7912878$

c. Let $x = \sqrt{5+\sqrt{5+\sqrt{5+\ldots}}}$. Then x satisfies the equation $x = \sqrt{5+x}$.
From part (b) we know that x must equal $\left(1+\sqrt{21}\right)/2 \approx 2.7912878$

33. a. $x_1 = 1$

$x_2 = 1 + \frac{1}{1} = 2$

$x_3 = 1 + \frac{1}{1+\frac{1}{1}} = \frac{3}{2} = 1.5$

$x_4 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}} = \frac{5}{3} \approx 1.6666667$

$x_5 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} = \frac{8}{5} = 1.6$

b. $x = 1 + \frac{1}{x}$

$x^2 = x + 1$

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{1+4\cdot 1\cdot 1}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Taking the minus sign gives a negative solution for x, violating the requirement that $x \ge 0$. Hence, $x = \frac{1+\sqrt{5}}{2} \approx 1.618034$.

c. Let

$x = 1 + \frac{1}{1+\frac{1}{1+\cdots}}$.

Then x satisfies the equation $x = 1 + \frac{1}{x}$.

From part (b) we know that x must equal $\left(1+\sqrt{5}\right)/2 \approx 1.618034$.

34. a. Suppose r is a root. Then $r = r - \frac{f(r)}{f'(r)}$.

$\frac{f(r)}{f'(r)} = 0$, so $f(r) = 0$.

Suppose $f(r) = 0$. Then $r - \frac{f(r)}{f'(r)} = r - 0 = r$,

so r is a root of $x = x - \frac{f(x)}{f'(x)}$.

b. If we want to solve $f(x) = 0$ and $f'(x) \neq 0$ in $[a, b]$, then $\frac{f(x)}{f'(x)} = 0$ or

$$x = x - \frac{f(x)}{f'(x)} = g(x).$$

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)}{[f'(x)]^2} f''(x)$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

and $g'(r) = \frac{f(r)f''(r)}{[f'(r)]^2} = 0.$

35. a. The algorithm computes the root of $\frac{1}{x} - a = 0$ for x_1 close to $\frac{1}{a}$.

b. Let $f(x) = \frac{1}{x} - a.$

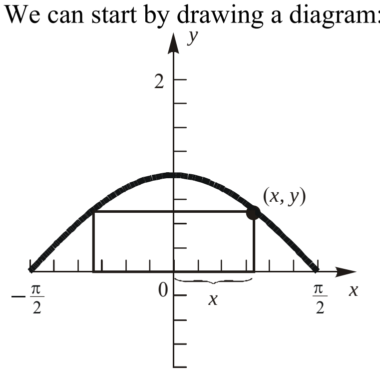
$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(x)}{f'(x)} = -x + ax^2$$

The recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2x_n - ax_n^2.$$

36. We can start by drawing a diagram:



From symmetry, maximizing the area of the entire rectangle is equivalent to maximizing the area of the rectangle in quadrant I. The area of the rectangle in quadrant I is given by

$$A = xy$$
$$= x\cos x$$

To find the maximum area, we first need the stationary points on the interval $\left(0, \frac{\pi}{2}\right)$.

$$A'(x) = \cos x - x\sin x$$

Therefore, we need to solve

$$A'(x) = 0$$
$$\cos x - x\sin x = 0$$

On the interval $\left(0, \frac{\pi}{2}\right)$, there is only one stationary point (check graphically). We will use Newton's Method to find the stationary point, starting with $x = \frac{\pi}{4} \approx 0.785398$.

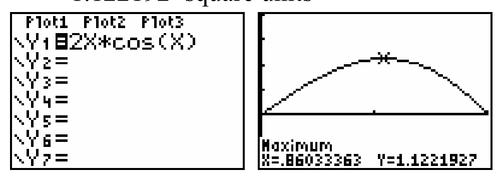
n	x_n
1	$\frac{\pi}{4} \approx 0.785398$
2	0.862443
3	0.860335
4	0.860334
5	0.860334

$x \approx 0.860334$ will maximize the area of the rectangle in quadrant I, and subsequently the larger rectangle as well.

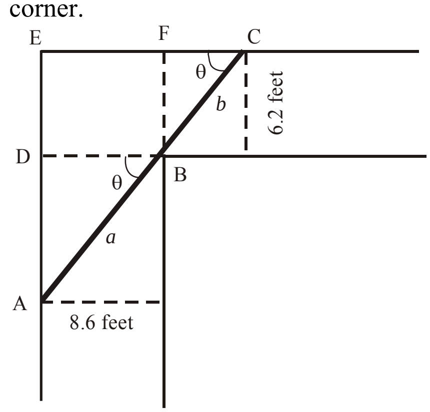
$y = \cos x = \cos(0.860334) \approx 0.652184$

The maximum area of the larger rectangle is

$$A_L = (2x)y \approx 2(0.860334)(0.652184)$$
$$\approx 1.122192 \text{ square units}$$



37. The rod that barely fits around the corner will touch the outside walls as well as the inside corner.



As suggested in the diagram, let a and b represent the lengths of the segments AB and BC, and let θ denote the angles $\angle DBA$ and $\angle FCB$. Consider the two similar triangles ΔADB and ΔBFC; these have hypotenuses a and b respectively. A little trigonometry applied to these angles gives

$$a = \frac{8.6}{\cos\theta} = 8.6\sec\theta \quad \text{and} \quad b = \frac{6.2}{\sin\theta} = 6.2\csc\theta$$

Note that the angle θ determines the position of the rod. The total length of the rod is then

$$L = a + b = 8.6\sec\theta + 6.2\csc\theta$$

The domain for θ is the open interval $\left(0, \frac{\pi}{2}\right)$.

The derivative of L is

$$L'(\theta)=\frac{8.6\sin^3\theta-6.2\cos^3\theta}{\sin^2\theta\cdot\cos^2\theta}$$

Thus, $L'(\theta)=0$ provided

$$8.6\sin^3\theta-6.2\cos^3\theta=0$$

$$8.6\sin^3\theta=6.2\cos^3\theta$$

$$\frac{\sin^3\theta}{\cos^3\theta}=\frac{6.2}{8.6}$$

$$\tan^3\theta=\frac{6.2}{8.6}$$

$$\tan\theta=\sqrt[3]{\frac{6.2}{8.6}}$$

On the interval $\left(0,\frac{\pi}{2}\right)$, there will only be one solution to this equation. We will use Newton's method to solve $\tan\theta-\sqrt[3]{\frac{6.2}{8.6}}=0$ starting with

$\theta_1=\frac{\pi}{4}$.

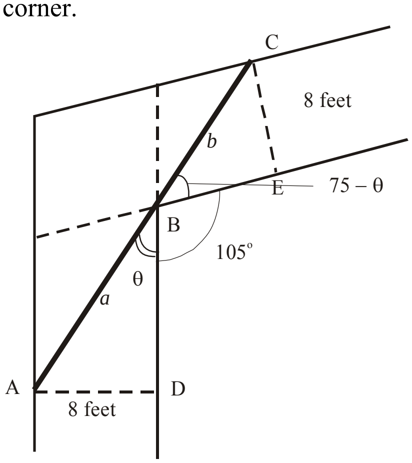
n	θ_n
1	$\frac{\pi}{4}\approx 0.78540$
2	0.73373
3	0.73098
4	0.73097
5	0.73097

Note that $\theta\approx 0.73097$ minimizes the length of the rod that does *not* fit around the corner, which in turn maximizes the length of the rod that will fit around the corner (verify by using the Second Derivative Test).

$$L(0.73097)=8.6\sec(0.73097)+6.2\csc(0.73097)$$
$$\approx 20.84$$

Thus, the length of the longest rod that will fit around the corner is about 20.84 feet.

38. The rod that barely fits around the corner will touch the outside walls as well as the inside corner.



As suggested in the diagram, let a and b represent the lengths of the segments AB and BC, and let θ denote the angle $\angle ABD$. Consider the two right triangles ΔADB and ΔCEB; these have hypotenuses a and b respectively. A little trigonometry applied to these angles gives

$$a=\frac{8}{\sin\theta}=8\csc\theta \quad\text{and}$$

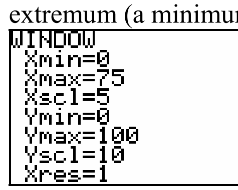
$$b=\frac{8}{\sin(75-\theta)}=8\csc(75-\theta)$$

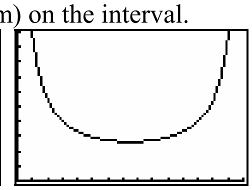
Note that the angle θ determines the position of the rod. The total length of the rod is then

$$L=a+b=8\csc\theta+8\csc(75-\theta)$$

The domain for θ is the open interval $(0,75)$.

A graph of of L indicates there is only one extremum (a minimum) on the interval.





The derivative of L is

$$L'(\theta)=\frac{8\left(\sin^2\theta\cdot\cos(\theta-75)-\cos\theta\cdot\sin^2(\theta-75)\right)}{\sin^2\theta\cdot\sin^2(\theta-75)}$$

We will use Newton's method to solve $L'(\theta)=0$ starting with $\theta_1=40$.

n	θ_n
1	40
2	37.54338
3	37.50000
4	37.5

Note that $\theta=37.5°$ minimizes the length of the rod that does *not* fit around the corner, which in turn maximizes the length of the rod that will fit around the corner (verify by using the Second Derivative Test).

$$L(37.5)=8\csc(37.5)+8\csc(75-37.5)$$
$$=16\csc(37.5)$$
$$\approx 26.28$$

Thus, the length of the longest rod that will fit around the corner is about 26.28 feet.

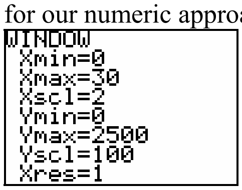
39. We can solve the equation $-\frac{2x^2}{25}+x+42=0$ to find the value for x when the object hits the ground. We want the value to be positive, so we use the quadratic formula, keeping only the positive solution.

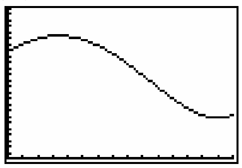
$$x=\frac{-1-\sqrt{1^2-4(-0.08)(42)}}{2(-0.08)}=30$$

We are interested in the global extrema for the distance of the object from the observer. We obtain the same extrema by considering the squared distance

$$D(x)=(x-3)^2+(42+x-.08x^2)^2$$

A graph of D will help us identify a starting point for our numeric approach.





From the graph, it appears that D (and thus the distance from the observer) is maximized at about $x=7$ feet and minimized just before the object hits the ground at about $x=28$ feet. The first derivative is given by

$$D'(x)=\frac{16}{625}x^3-\frac{12}{25}x^2-\frac{236}{25}x+78.$$

a. We will use Newton's method to find the stationary point that yields the minimum distance, starting with $x_1=28$.

n	x_n
1	28
2	28.0280
3	28.0279
4	28.0279

$x\approx 28.0279;\quad y\approx 7.1828$

The object is closest to the observer when it is at the point $(28.0279, 7.1828)$.

b. We will use Newton's method to find the stationary point that yields the maximum distance, starting with $x_1=7$.

n	x_n
1	7
2	6.7726
3	6.7728
4	6.7728

$x\approx 6.7728;\quad y\approx 45.1031$

The object is closest to the observer when it is at the point $(6.7728, 45.1031)$.

3.8 Concepts Review

1. $rx^{r-1};\ \frac{x^{r+1}}{r+1}+C,\ r\neq -1$

2. $r[f(x)]^{r-1}f'(x);\ [f(x)]^r f'(x)$

3. $u=x^4+3x^2+1,\ du=(4x^3+6x)dx$

$$\int (x^4+3x^2+1)^8(4x^3+6x)dx=\int u^8du$$
$$=\frac{u^9}{9}+C=\frac{(x^4+3x^2+1)^9}{9}+C$$

4. $c_1\int f(x)dx+c_2\int g(x)dx$

Problem Set 3.8

1. $\int 5dx=5x+C$

2. $\int (x-4)dx=\int xdx-4\int 1dx$
$$=\frac{x^2}{2}-4x+C$$

3. $\int (x^2+\pi)dx=\int x^2dx+\pi\int 1dx=\frac{x^3}{3}+\pi x+C$

4. $\int\left(3x^2+\sqrt{3}\right)dx=3\int x^2dx+\sqrt{3}\int 1dx$
$$=3\frac{x^3}{3}+\sqrt{3}\,x+C=x^3+\sqrt{3}\,x+C$$

5. $\int x^{5/4}dx=\frac{x^{9/4}}{\frac{9}{4}}+C=\frac{4}{9}x^{9/4}+C$

6. $\int 3x^{2/3}dx=3\int x^{2/3}dx=3\left(\frac{x^{5/3}}{\frac{5}{3}}+C_1\right)$
$$=\frac{9}{5}x^{5/3}+C$$

7. $\int\frac{1}{\sqrt[3]{x^2}}dx=\int x^{-2/3}dx=3x^{1/3}+C=3\sqrt[3]{x}+C$

8. $\int 7x^{-3/4}dx=7\int x^{-3/4}dx=7(4x^{1/4}+C_1)$
$$=28x^{1/4}+C$$

9. $\int (x^2-x)dx=\int x^2dx-\int x\,dx=\frac{x^3}{3}-\frac{x^2}{2}+C$

10. $\int (3x^2 - \pi x)dx = 3\int x^2 dx - \pi \int x\,dx$
$$= 3\left(\frac{x^3}{3} + C_1\right) - \pi\left(\frac{x^2}{2} + C_2\right)$$
$$= x^3 - \frac{\pi x^2}{2} + C$$

11. $\int (4x^5 - x^3)dx = 4\int x^5 dx - \int x^3 dx$
$$= 4\left(\frac{x^6}{6} + C_1\right) - \left(\frac{x^4}{4} + C_2\right)$$
$$= \frac{2x^6}{3} - \frac{x^4}{4} + C$$

12. $\int (x^{100} + x^{99})dx = \int x^{100} dx + \int x^{99} dx$
$$= \frac{x^{101}}{101} + \frac{x^{100}}{100} + C$$

13. $\int (27x^7 + 3x^5 - 45x^3 + \sqrt{2}\,x)dx$
$$= 27\int x^7 dx + 3\int x^5 dx - 45\int x^3\,dx + \sqrt{2}\int x\,dx$$
$$= \frac{27x^8}{8} + \frac{x^6}{2} - \frac{45x^4}{4} + \frac{\sqrt{2}\,x^2}{2} + C$$

14. $\int \left[x^2\left(x^3 + 5x^2 - 3x + \sqrt{3}\right)\right]dx$
$$= \int \left(x^5 + 5x^4 - 3x^3 + \sqrt{3}\,x^2\right)dx$$
$$= \int x^5\,dx + 5\int x^4\,dx - 3\int x^3\,dx + \sqrt{3}\int x^2\,dx$$
$$= \frac{x^6}{6} + x^5 - \frac{3x^4}{4} + \frac{\sqrt{3}\,x^3}{3} + C$$

15. $\int \left(\frac{3}{x^2} - \frac{2}{x^3}\right)dx = \int (3x^{-2} - 2x^{-3})\,dx$
$$= 3\int x^{-2}\,dx - 2\int x^{-3}\,dx$$
$$= \frac{3x^{-1}}{-1} - \frac{2x^{-2}}{-2} + C$$
$$= -\frac{3}{x} + \frac{1}{x^2} + C$$

16. $\int \left(\frac{\sqrt{2x}}{x} + \frac{3}{x^5}\right)dx = \int \left(\sqrt{2}\,x^{-1/2} + 3x^{-5}\right)dx$
$$= \frac{\sqrt{2}\,x^{1/2}}{\frac{1}{2}} + \frac{3x^{-4}}{-4} + C$$
$$= 2\sqrt{2x} - \frac{3}{4x^4} + C$$

17. $\int \frac{4x^6 + 3x^4}{x^3}\,dx = \int (4x^3 + 3x)\,dx$
$$= 4\int x^3\,dx + 3\int x\,dx$$
$$= x^4 + \frac{3x^2}{2} + C$$

18. $\int \frac{x^6 - x}{x^3}\,dx = \int (x^3 - x^{-2})\,dx$
$$= \int x^3\,dx - \int x^{-2}\,dx = \frac{x^4}{4} - \frac{x^{-1}}{-1} + C$$
$$= \frac{x^4}{4} + \frac{1}{x} + C$$

19. $\int (x^2 + x)\,dx = \int x^2\,dx + \int x\,dx = \frac{x^3}{3} + \frac{x^2}{2} + C$

20. $\int \left(x^3 + \sqrt{x}\right)dx = \int x^3\,dx + \int x^{1/2}\,dx$
$$= \frac{x^4}{4} + \frac{x^{3/2}}{\frac{3}{2}} + C = \frac{x^4}{4} + \frac{2\sqrt{x^3}}{3} + C$$

21. Let $u = x + 1$; then $du = dx$.
$$\int (x+1)^2\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{(x+1)^3}{3} + C$$

22. $\int \left(z + \sqrt{2}\,z\right)^2 dz = \int \left[\left(1+\sqrt{2}\right)z\right]^2 dz$
$$= \left(1+\sqrt{2}\right)^2 \int z^2\,dz = \frac{\left(1+\sqrt{2}\right)^2 z^3}{3} + C$$

23. $\int \frac{(z^2+1)^2}{\sqrt{z}}\,dz = \int \frac{z^4 + 2z^2 + 1}{\sqrt{z}}\,dz$
$$= \int z^{7/2}\,dz + 2\int z^{3/2} + \int z^{-1/2}\,dz$$
$$= \frac{2}{9}z^{9/2} + \frac{4}{5}z^{5/2} + 2z^{1/2} + C$$

24. $\int \frac{s(s+1)^2}{\sqrt{s}}\,ds = \int \frac{s^3 + 2s^2 + s}{\sqrt{s}}\,ds$
$$= \int s^{5/2} ds + 2\int s^{3/2} ds + \int s^{1/2} ds$$
$$= \frac{2s^{7/2}}{7} + \frac{4s^{5/2}}{5} + \frac{2s^{3/2}}{3} + C$$

25. $\int (\sin\theta - \cos\theta)d\theta = \int \sin\theta\,d\theta - \int \cos\theta\,d\theta$
$$= -\cos\theta - \sin\theta + C$$

26. $\int (t^2 - 2\cos t)dt = \int t^2\, dt - 2\int \cos t\, dt$

$= \frac{t^3}{3} - 2\sin t + C$

27. Let $g(x) = \sqrt{2}\, x + 1$; then $g'(x) = \sqrt{2}$.

$\int \left(\sqrt{2}\, x + 1\right)^3 \sqrt{2}\, dx = \int [g(x)]^3\, g'(x) dx$

$= \frac{[g(x)]^4}{4} + C = \frac{\left(\sqrt{2}\, x + 1\right)^4}{4} + C$

28. Let $g(x) = \pi x^3 + 1$; then $g'(x) = 3\pi x^2$.

$\int (\pi x^3 + 1)^4\, 3\pi x^2\, dx = \int [g(x)]^4\, g'(x)\, dx$

$= \frac{[g(x)]^5}{5} + C = \frac{(\pi x^3 + 1)^5}{5} + C$

29. Let $u = 5x^3 + 3x - 8$; then $du = (15x^2 + 3)\, dx$.

$\int (5x^2 + 1)(5x^3 + 3x - 8)^6\, dx$

$= \int \frac{1}{3}(15x^2 + 3)(5x^3 + 3x - 8)^6\, dx$

$= \frac{1}{3}\int u^6\, du = \frac{1}{3}\left(\frac{u^7}{7} + C_1\right)$

$= \frac{(5x^3 + 3x - 8)^7}{21} + C$

30. Let $u = 5x^3 + 3x - 2$; then $du = (15x^2 + 3)dx$.

$\int (5x^2 + 1)\sqrt{5x^3 + 3x - 2}\, dx$

$= \int \frac{1}{3}(15x^2 + 3)\sqrt{5x^3 + 3x - 2}\, dx$

$= \frac{1}{3}\int u^{1/2}\, du = \frac{1}{3}\left(\frac{2}{3}u^{3/2} + C_1\right)$

$= \frac{2}{9}(5x^3 + 3x - 2)^{3/2} + C$

$= \frac{2}{9}\sqrt{(5x^3 + 3x - 2)^3} + C$

31. Let $u = 2t^2 - 11$; then $du = 4t\, dt$.

$\int 3t\sqrt[3]{2t^2 - 11}\, dt = \int \frac{3}{4}(4t)(2t^2 - 11)^{1/3}\, dt$

$= \frac{3}{4}\int u^{1/3} du = \frac{3}{4}\left(\frac{3}{4}u^{4/3} + C_1\right)$

$= \frac{9}{16}(2t^2 - 11)^{4/3} + C$

$= \frac{9}{16}\sqrt[3]{(2t^2 - 11)^4} + C$

32. Let $u = 2y^2 + 5$; then $du = 4y\, dy$

$\int \frac{3y}{\sqrt{2y^2 + 5y}}\, dy = \int \frac{3}{4}(4y)(2y^2 + 5)^{-1/2}\, dy$

$= \frac{3}{4}\int u^{-1/2} du = \frac{3}{4}(2u^{1/2} + C_1)$

$= \frac{3}{2}\sqrt{2y^2 + 5} + C$

33. Let $u = x^3 + 4$; then $du = 3x^2 dx$.

$\int x^2\sqrt{x^3 + 4}\, dx = \int \frac{1}{3} 3x^2\sqrt{x^3 + 4}\, dx$

$= \frac{1}{3}\int \sqrt{u}\, du = \frac{1}{3}\int u^{1/2}\, du$

$= \frac{1}{3}\left(\frac{2}{3}u^{3/2} + C_1\right)$

$= \frac{2}{9}\left(x^3 + 4\right)^{3/2} + C$

34. Let $u = x^4 + 2x^2$; then

$du = \left(4x^3 + 4x\right)dx = 4\left(x^3 + x\right)dx$.

$\int \left(x^3 + x\right)\sqrt{x^4 + 2x^2}\, dx$

$= \int \frac{1}{4}\cdot 4\left(x^3 + x\right)\sqrt{x^4 + 2x^2}\, dx$

$= \frac{1}{4}\int \sqrt{u}\, du = \frac{1}{4}\int u^{1/2}\, du$

$= \frac{1}{4}\left(\frac{2}{3}u^{3/2} + C_1\right)$

$= \frac{1}{6}\left(x^4 + 2x^2\right)^{3/2} + C$

35. Let $u = 1 + \cos x$; then $du = -\sin x\, dx$.

$\int \sin x\left(1 + \cos x\right)^4\, dx = -\int -\sin x\left(1 + \cos x\right)^4\, dx$

$= -\int u^4\, du = -\left(\frac{1}{5}u^5 + C_1\right)$

$= -\frac{1}{5}\left(1 + \cos x\right)^5 + C$

36. Let $u = 1 + \sin^2 x$; then $du = 2\sin x\cos x\, dx$.

$\int \sin x\cos x\sqrt{1 + \sin^2 x}\, dx$

$= \int \frac{1}{2}\cdot 2\sin x\cos x\sqrt{1 + \sin^2 x}\, dx$

$= \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\int u^{1/2}\, du$

$= \frac{1}{2}\left(\frac{2}{3}u^{3/2} + C_1\right)$

$= \frac{1}{3}\left(1 + \sin^2 x\right)^{3/2} + C$

37. $f'(x)=\int(3x+1)dx=\frac{3}{2}x^2+x+C_1$

$f(x)=\int\left(\frac{3}{2}x^2+x+C_1\right)dx$

$=\frac{1}{2}x^3+\frac{1}{2}x^2+C_1x+C_2$

38. $f'(x)=\int(-2x+3)\,dx=-x^2+3x+C_1$

$f(x)=\int(-x^2+3x+C_1)dx$

$=-\frac{1}{3}x^3+\frac{3}{2}x^2+C_1x+C_2$

39. $f'(x)=\int x^{1/2}dx=\frac{2}{3}x^{3/2}+C_1$

$f(x)=\int\left(\frac{2}{3}x^{3/2}+C_1\right)dx$

$=\frac{4}{15}x^{5/2}+C_1x+C_2$

40. $f'(x)=\int x^{4/3}\,dx=\frac{3}{7}x^{7/3}+C_1$

$f(x)=\int\left(\frac{3}{7}x^{7/3}+C_1\right)dx=\frac{9}{70}x^{10/3}+C_1x+C_2$

41. $f''(x)=x+x^{-3}$

$f'(x)=\int(x+x^{-3})dx=\frac{x^2}{2}-\frac{x^{-2}}{2}+C_1$

$f(x)=\int\left(\frac{1}{2}x^2-\frac{1}{2}x^{-2}+C_1\right)dx$

$=\frac{1}{6}x^3+\frac{1}{2}x^{-1}+C_1\,x+C_2$

$=\frac{1}{6}x^3+\frac{1}{2x}+C_1\,x+C_2$

42. $f'(x)=2\int(x+1)^{1/3}\,dx=\frac{3}{2}(x+1)^{4/3}+C_1$

$f(x)=\int\left[\frac{3}{2}(x+1)^{4/3}+C_1\right]dx$

$=\frac{9}{14}(x+1)^{7/3}+C_1x+C_2$

43. The Product Rule for derivatives says

$\frac{d}{dx}[f(x)g(x)+C]=f(x)g'(x)+f'(x)g(x)$.

Thus,

$\int[f(x)g'(x)+f'(x)g(x)]dx=f(x)g(x)+C$.

44. The Quotient Rule for derivatives says

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}+C\right]=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$.

Thus, $\int\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}dx=\frac{f(x)}{g(x)}+C$.

45. Let $f(x)=x^2,\,g(x)=\sqrt{x-1}$.

$f'(x)=2x,\,g'(x)=\frac{1}{2\sqrt{x-1}}$

$\int\left[\frac{x^2}{2\sqrt{x-1}}+2x\sqrt{x-1}\right]dx$

$=\int[f(x)g'(x)+f'(x)g(x)]dx=f(x)g(x)+C$

$=x^2\sqrt{x-1}+C$

46. Let $f(x)=x^3,\,g(x)=(2x+5)^{-1/2}$.

$f'(x)=3x^2,\,g'(x)=-(2x+5)^{-3/2}$

$=-\frac{1}{(2x+5)^{3/2}}$

$\int\left[\frac{-x^3}{(2x+5)^{3/2}}+\frac{3x^2}{\sqrt{2x+5}}\right]dx$

$=\int[f(x)g'(x)+g(x)f'(x)]dx$

$=f(x)g(x)+C=x^3(2x+5)^{-1/2}+C$

$=\frac{x^3}{\sqrt{2x+5}}+C$

47. $\int f''(x)dx=\int\frac{d}{dx}f'(x)dx=f'(x)+C$

$f'(x)=\sqrt{x^3+1}+\frac{3x^3}{2\sqrt{x^3+1}}=\frac{5x^3+2}{2\sqrt{x^3+1}}$ so

$\int f''(x)dx=\frac{5x^3+2}{2\sqrt{x^3+1}}+C$.

48. $\frac{d}{dx}\left(\frac{f(x)}{\sqrt{g(x)}}+C\right)$

$=\frac{\sqrt{g(x)}f'(x)-f(x)\frac{1}{2}[g(x)]^{-1/2}g'(x)}{g(x)}$

$=\frac{2g(x)f'(x)-f(x)g'(x)}{2[g(x)]^{3/2}}$

Thus,

$\int\frac{2g(x)f'(x)-f(x)g'(x)}{2[g(x)]^{3/2}}=\frac{f(x)}{\sqrt{g(x)}}+C$

49. The Product Rule for derivatives says that

$$\frac{d}{dx}[f^m(x)g^n(x)+C]$$
$$= f^m(x)[g^n(x)]' + [f^m(x)]'g^n(x)$$
$$= f^m(x)[ng^{n-1}(x)g'(x)] + [mf^{m-1}(x)f'(x)]g^n(x)$$
$$= f^{m-1}(x)g^{n-1}(x)[nf(x)g'(x) + mg(x)f'(x)].$$

Thus,

$$\int f^{m-1}(x)g^{n-1}(x)[nf(x)g'(x)+mg(x)f'(x)]dx$$
$$= f^m(x)g^n(x)+C.$$

50. Let $u = \sin[(x^2+1)^4]$;

then $du = \cos\left[(x^2+1)^4\right]4(x^2+1)^3(2x)dx$.

$$du = 8x\cos\left[(x^2+1)^4\right](x^2+1)^3\,dx$$
$$\int \sin^3\left[(x^2+1)^4\right]\cos\left[(x^2+1)^4\right](x^2+1)^3 x\,dx$$
$$= \int u^3 \cdot \frac{1}{8}du = \frac{1}{8}\int u^3\,du = \frac{1}{8}\left(\frac{u^4}{4}+C_1\right)$$
$$= \frac{\sin^4\left[(x^2+1)^4\right]}{32}+C$$

51. If $x \geq 0$, then $|x| = x$ and $\int |x|\,dx = \frac{1}{2}x^2 + C$.

If $x < 0$, then $|x| = -x$ and $\int |x|\,dx = -\frac{1}{2}x^2 + C$.

$$\int |x|\,dx = \begin{cases} \frac{1}{2}x^2 + C & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{if } x < 0 \end{cases}$$

52. Using $\sin^2\frac{u}{2} = \frac{1-\cos u}{2}$,

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

53. Different software may produce different, but equivalent answers. These answers were produced by Mathematica.

a. $\int 6\sin(3(x-2))\,dx = -2\cos(3(x-2)) + C$

b. $\int \sin^3\left(\frac{x}{6}\right)dx = \frac{1}{2}\cos\left(\frac{x}{2}\right) - \frac{9}{2}\cos\left(\frac{x}{6}\right) + C$

c. $\int (x^2\cos 2x + x\sin 2x)dx = \frac{x^2\sin 2x}{2} + C$

54. a. $F_1(x) = \int (x\sin x)dx = \sin x - x\cos x + C_1$

$$F_2(x) = \int (\sin x - x\cos x + C_1)dx$$
$$= -2\cos x - x\sin x + C_1x + C_2$$
$$F_3(x) = \int (-2\cos x - x\sin x + C_1x + C_2)dx$$
$$= x\cos x - 3\sin x + \frac{1}{2}C_1x^2 + C_2x + C_3$$
$$F_4(x) = \int (x\cos x - 3\sin x + \frac{1}{2}C_1x^2 + C_2x + C_3)dx$$
$$= x\sin x + 4\cos x + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

b. $F_{16}(x) = x\sin x + 16\cos x + \sum_{n=1}^{16}\frac{C_n x^{16-n}}{(16-n)!}$

3.9 Concepts Review

1. differential equation

2. function

3. separate variables

4. $-32t + v_0; -16t^2 + v_0t + s_0$

Problem Set 3.9

1. $\frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} + \frac{x}{y} = \frac{-x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = 0$$

2. $\frac{dy}{dx} = C$

$$-x\frac{dy}{dx} + y = -Cx + Cx = 0$$

3. $\frac{dy}{dx} = C_1\cos x - C_2\sin x;$

$$\frac{d^2y}{dx^2} = -C_1\sin x - C_2\cos x$$
$$\frac{d^2y}{dx^2} + y$$
$$= (-C_1\sin x - C_2\cos x) + (C_1\sin x + C_2\cos x) = 0$$

4. For $y = \sin(x + C)$, $\frac{dy}{dx} = \cos(x+C)$

$$\left(\frac{dy}{dx}\right)^2 + y^2 = \cos^2(x+C) + \sin^2(x+C) = 1$$

For $y = \pm 1$, $\frac{dy}{dx} = 0$.

$$\left(\frac{dy}{dx}\right)^2 + y^2 = 0^2 + (\pm 1)^2 = 1$$

5. $\frac{dy}{dx} = x^2 + 1$

$dy = (x^2 + 1)\,dx$

$\int dy = \int (x^2 + 1)\,dx$

$y + C_1 = \frac{x^3}{3} + x + C_2$

$y = \frac{x^3}{3} + x + C$

At $x = 1$, $y = 1$:

$1 = \frac{1}{3} + 1 + C; C = -\frac{1}{3}$

$y = \frac{x^3}{3} + x - \frac{1}{3}$

6. $\frac{dy}{dx} = x^{-3} + 2$

$dy = (x^{-3} + 2)\,dx$

$\int dy = \int (x^{-3} + 2)\,dx$

$y + C_1 = -\frac{x^{-2}}{2} + 2x + C_2$

$y = -\frac{1}{2x^2} + 2x + C$

At $x = 1$, $y = 3$:

$3 = -\frac{1}{2} + 2 + C; C = \frac{3}{2}$

$y = -\frac{1}{2x^2} + 2x + \frac{3}{2}$

7. $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} + C_1 = \frac{x^2}{2} + C_2$

$y^2 = x^2 + C$

$y = \pm\sqrt{x^2 + C}$

At $x = 1$, $y = 1$:

$1 = \pm\sqrt{1 + C}; C = 0$ and the square root is positive.

$y = \sqrt{x^2}$ or $y = x$

8. $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$

$\int \sqrt{y}\,dy = \int \sqrt{x}\,dx$

$\frac{2}{3}y^{3/2} + C_1 = \frac{2}{3}x^{3/2} + C_2$

$y^{3/2} = x^{3/2} + C$

$y = (x^{3/2} + C)^{2/3}$

At $x = 1$, $y = 4$:

$4 = (1 + C)^{2/3}; C = 7$

$y = (x^{3/2} + 7)^{2/3}$

9. $\frac{dz}{dt} = t^2 z^2$

$\int z^{-2}dz = \int t^2\,dt$

$-z^{-1} + C_1 = \frac{t^3}{3} + C_2$

$\frac{1}{z} = -\frac{t^3}{3} + C_3 = \frac{C - t^3}{3}$

$z = \frac{3}{C - t^3}$

At $t = 1$, $z = \frac{1}{3}$:

$\frac{1}{3} = \frac{3}{C - 1}; C - 1 = 9; C = 10$

$z = \frac{3}{10 - t^3}$

10. $\frac{dy}{dt} = y^4$

$\int y^{-4} dy = \int dt$

$-\frac{1}{3y^3} + C_1 = t + C_2$

$y = -\frac{1}{\sqrt[3]{3t+C}}$

At $t = 0$, $y = 1$:
$C = -1$

$y = -\frac{1}{\sqrt[3]{3t-1}}$

11. $\frac{ds}{dt} = 16t^2 + 4t - 1$

$\int ds = \int (16t^2 + 4t - 1)\, dt$

$s + C_1 = \frac{16}{3}t^3 + 2t^2 - t + C_2$

$s = \frac{16}{3}t^3 + 2t^2 - t + C$

At $t = 0$, $s = 100$: $C = 100$

$s = \frac{16}{3}t^3 + 2t^2 - t + 100$

12. $\frac{du}{dt} = u^3(t^3 - t)$

$\int u^{-3}\, du = \int (t^3 - t)\, dt$

$-\frac{1}{2u^2} + C_1 = \frac{t^4}{4} - \frac{t^2}{2} + C_2$

$u^{-2} = t^2 - \frac{t^4}{2} + C$

$u = \left(t^2 - \frac{t^4}{2} + C\right)^{-1/2}$

At $t = 0$, $u = 4$:

$4 = C^{-1/2}; C = \frac{1}{16}$

$u = \left(t^2 - \frac{t^4}{2} + \frac{1}{16}\right)^{-1/2}$

13. $\frac{dy}{dx} = (2x+1)^4$

$y = \int (2x+1)^4\, dx = \frac{1}{2}\int (2x+1)^4\, 2\, dx$

$= \frac{1}{2}\frac{(2x+1)^5}{5} + C = \frac{(2x+1)^5}{10} + C$

At $x = 0$, $y = 6$:

$6 = \frac{1}{10} + C; C = \frac{59}{10}$

$y = \frac{(2x+1)^5}{10} + \frac{59}{10} = \frac{(2x+1)^5 + 59}{10}$

14. $\frac{dy}{dx} = -y^2 x(x^2+2)^4$

$-\int y^{-2} dy = \frac{1}{2}\int 2x(x^2+2)^4\, dx$

$\frac{1}{y} + C_1 = \frac{1}{2}\frac{(x^2+2)^5}{5} + C_2$

$\frac{1}{y} = \frac{(x^2+2)^5 + C}{10}$

$y = \frac{10}{(x^2+2)^5 + C}$

At $x = 0$, $y = 1$:

$1 = \frac{10}{32 + C}; C = 10 - 32 = -22$

$y = \frac{10}{(x^2+2)^5 - 22}$

15. $\frac{dy}{dx} = 3x$

$y = \int 3x\, dx = \frac{3}{2}x^2 + C$

At (1, 2):

$2 = \frac{3}{2} + C$

$C = \frac{1}{2}$

$y = \frac{3}{2}x^2 + \frac{1}{2} = \frac{3x^2 + 1}{2}$

16. $\frac{dy}{dx}=3y^2$

$\int y^{-2}\,dy=3\int dx$

$-\frac{1}{y}+C_1=3x+C_2$

$\frac{1}{y}=-3x+C$

$y=\frac{1}{C-3x}$

At (1, 2):

$2=\frac{1}{C-3}$

$C=\frac{7}{2}$

$y=\frac{1}{\frac{7}{2}-3x}=\frac{2}{7-6x}$

17. $v=\int t\,dt=\frac{t^2}{2}+v_0$

$v=\frac{t^2}{2}+3$

$s=\int\left(\frac{t^2}{2}+3\right)dt=\frac{t^3}{6}+3t+s_0$

$s=\frac{t^3}{6}+3t+0=\frac{t^3}{6}+3t$

At $t = 2$:

$v = 5$ cm/s

$s=\frac{22}{3}$ cm

18. $v=\int(1+t)^{-4}\,dt=-\frac{1}{3(1+t)^3}+C$

$v_0=0:0=-\frac{1}{3(1+0)^3}+C;C=\frac{1}{3}$

$v=-\frac{1}{3(1+t)^3}+\frac{1}{3}$

$s=\int\left(-\frac{1}{3(1+t)^3}+\frac{1}{3}\right)dt=\frac{1}{6(1+t)^2}+\frac{1}{3}t+C$

$s_0=10:10=\frac{1}{6(1+0)^2}+\frac{1}{3}(0)+C;C=\frac{59}{6}$

$s=\frac{1}{6(1+t)^2}+\frac{1}{3}t+\frac{59}{6}$

At $t = 2$:

$v=-\frac{1}{81}+\frac{1}{3}=\frac{26}{81}$ cm/s

$s=\frac{1}{54}+\frac{2}{3}+\frac{59}{6}=\frac{284}{27}$ cm

19. $v=\int(2t+1)^{1/3}\,dt=\frac{1}{2}\int(2t+1)^{1/3}\,2dt$

$=\frac{3}{8}(2t+1)^{4/3}+C_1$

$v_0=0:0=\frac{3}{8}+C_1;C_1=-\frac{3}{8}$

$v=\frac{3}{8}(2t+1)^{4/3}-\frac{3}{8}$

$s=\frac{3}{8}\int(2t+1)^{4/3}\,dt-\frac{3}{8}\int 1dt$

$=\frac{3}{16}\int(2t+1)^{4/3}\,2dt-\frac{3}{8}\int 1dt$

$=\frac{9}{112}(2t+1)^{7/3}-\frac{3}{8}t+C_2$

$s_0=10:10=\frac{9}{112}+C_2;C_2=\frac{1111}{112}$

$s=\frac{9}{112}(2t+1)^{7/3}-\frac{3}{8}t+\frac{1111}{112}$

At $t = 2$: $v=\frac{3}{8}(5)^{4/3}-\frac{3}{8}\approx 2.83$

$s=\frac{9}{112}(5)^{7/3}-\frac{6}{8}+\frac{1111}{112}\approx 12.6$

20. $v=\int(3t+1)^{-3}\,dt=\frac{1}{3}\int(3t+1)^{-3}\,3dt$

$=-\frac{1}{6}(3t+1)^{-2}+C_1$

$v_0=4:4=-\frac{1}{6}+C_1;C_1=\frac{25}{6}$

$v=-\frac{1}{6}(3t+1)^{-2}+\frac{25}{6}$

$s=-\frac{1}{6}\int(3t+1)^{-2}\,dt+\int\frac{25}{6}\,dt$

$=-\frac{1}{18}\int(3t+1)^{-2}\,3dt+\frac{25}{6}\int dt$

$=\frac{1}{18}(3t+1)^{-1}+\frac{25}{6}t+C_2$

$s_0=0:0=\frac{1}{18}+C_2;C_2=-\frac{1}{18}$

$s=\frac{1}{18}(3t+1)^{-1}+\frac{25}{6}t-\frac{1}{18}$

At $t = 2$: $v=-\frac{1}{6}(7)^{-2}+\frac{25}{6}\approx 4.16$

$s=\frac{1}{18}(7)^{-1}+\frac{25}{3}-\frac{1}{18}\approx 8.29$

21. $v = -32t + 96,$

$s = -16t^2 + 96t + s_0 = -16t^2 + 96t$

$v = 0$ at $t = 3$

At $t = 3,\ s = -16(3^2) + 96(3) = 144$ ft

22. $a = \dfrac{dv}{dt} = k$

$v = \int k\,dt = kt + v_0 = \dfrac{ds}{dt};$

$s = \int (kt + v_0)dt = \dfrac{k}{2}t^2 + v_0 t + s_0 = \dfrac{k}{2}t^2 + v_0 t$

$v = 0$ when $t = -\dfrac{v_0}{k}$. Then

$s = \dfrac{k}{2}\left(-\dfrac{v_0}{k}\right)^2 + \left(-\dfrac{v_0^2}{k}\right) = -\dfrac{v_0^2}{2k}.$

23. $\dfrac{dv}{dt} = -5.28$

$\int dv = -\int 5.28dt$

$v = \dfrac{ds}{dt} = -5.28t + v_0 = -5.28t + 56$

$\int ds = \int (-5.28t + 56)dt$

$s = -2.64t^2 + 56t + s_0 = -2.64t^2 + 56t + 1000$

When $t = 4.5,\ v = 32.24$ ft/s and $s = 1198.54$ ft

24. $v = 0$ when $t = \dfrac{-56}{-5.28} \approx 10.6061$. Then

$s \approx -2.64(10.6061)^2 + 56(10.6061) + 1000$

≈ 1296.97 ft

25. $\dfrac{dV}{dt} = -kS$

Since $V = \dfrac{4}{3}\pi r^3$ and $S = 4\pi r^2$,

$4\pi r^2 \dfrac{dr}{dt} = -k4\pi r^2$ so $\dfrac{dr}{dt} = -k$.

$\int dr = -\int k\,dt$

$r = -kt + C$

$2 = -k(0) + C$ and $0.5 = -k(10) + C$, so

$C = 2$ and $k = \dfrac{3}{20}$. Then, $r = -\dfrac{3}{20}t + 2$.

26. Solving $v = -136 = -32t$ yields $t = \dfrac{17}{4}$.

Then $s = 0 = -16\left(\dfrac{17}{4}\right)^2 + (0)\left(\dfrac{17}{4}\right) + s_0$, so

$s_0 = 289$ ft.

27. $v_{\text{esc}} = \sqrt{2gR}$

For the Moon, $v_{\text{esc}} \approx \sqrt{2(0.165)(32)(1080 \cdot 5280)}$

≈ 7760 ft/s ≈ 1.470 mi/s.

For Venus, $v_{\text{esc}} \approx \sqrt{2(0.85)(32)(3800 \cdot 5280)}$

$\approx 33{,}038$ ft/s ≈ 6.257 mi/s.

For Jupiter, $v_{\text{esc}} \approx 194{,}369$ ft/s ≈ 36.812 mi/s.

For the Sun, $v_{\text{esc}} \approx 2{,}021{,}752$ ft/s

≈ 382.908 mi/s.

28. $v_0 =$ 60 mi/h = 88 ft/s

v = 0 = –11t + 88; t = 8 sec

$s(t) = -\dfrac{11}{2}t^2 + 88t$

$s(8) = -\dfrac{11}{2}(8)^2 + 88(8) = 352$ feet

The shortest distance in which the car can be braked to a halt is 352 feet.

29. $a = \dfrac{dv}{dt} = \dfrac{\Delta v}{\Delta t} = \dfrac{60-45}{10} = 1.5$ mi/h/s = 2.2 ft/s^2

30. $75 = \dfrac{8}{2}(3.75)^2 + v_0(3.75) + 0;\ v_0 = 5$ ft/s

31. For the first 10 s, $a = \dfrac{dv}{dt} = 6t, v = 3t^2$, and $s = t^3$. So $v(10) = 300$ and $s(10) = 1000$. After 10 s, $a = \dfrac{dv}{dt} = -10$, $v = -10(t - 10) + 300$, and $s = -5(t-10)^2 + 300(t-10) + 1000$. $v = 0$ at $t = 40$, at which time $s = 5500$ m.

32. **a.** After accelerating for 8 seconds, the velocity is $8 \cdot 3 = 24$ m/s.

b. Since acceleration and deceleration are constant, the average velocity during those times is

$\dfrac{24}{2} = 12$ m/s. Solve $0 = -4t + 24$ to get the time spent decelerating. $t = \dfrac{24}{4} = 6$ s;

$d = (12)(8) + (24)(100) + (12)(6) = 2568$ m.

3.10 Chapter Review

Concepts Test

1. True: Max-Min Existence Theorem

2. True: Since c is an interior point and f is differentiable ($f'(c)$ exists), by the Critical Point Theorem, c is a stationary point ($f'(c)$ = 0).

3. True: For example, let $f(x) = \sin x$.

4. False: $f(x) = x^{1/3}$ is continuous and increasing for all x, but $f'(x)$ does not exist at $x = 0$.

5. True: $f'(x) = 18x^5 + 16x^3 + 4x$;
$f''(x) = 90x^4 + 48x^2 + 4$, which is greater than zero for all x.

6. False: For example, $f(x) = x^3$ is increasing on [–1, 1] but $f'(0) = 0$.

7. True: When $f'(x) > 0$, $f(x)$ is increasing.

8. False: If $f''(c) = 0$, c is a candidate, but not necessarily an inflection point. For example, if $f(x) = x^4$, $P''(0) = 0$ but $x = 0$ is not an inflection point.

9. True: $f(x) = ax^2 + bx + c$;
$f'(x) = 2ax + b$; $f''(x) = 2a$

10. True: If $f(x)$ is increasing for all x in $[a, b]$, the maximum occurs at b.

11. False: $\tan^2 x$ has a minimum value of *0*. This occurs whenever $x = k\pi$ where k is an integer.

12. True: $\lim_{x\to\infty}(2x^3 + x) = \infty$ while
$\lim_{x\to-\infty}(2x^3 + x) = -\infty$

13. True: $\lim_{x\to\frac{\pi}{2}^-}(2x^3 + x + \tan x) = \infty$ while
$\lim_{x\to-\frac{\pi}{2}^+}(2x^3 + x + \tan x) = -\infty.$

14. False: At $x = 3$ there is a removable discontinuity.

15. True: $\lim_{x\to\infty}\frac{x^2+1}{1-x^2} = \lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{\frac{1}{x^2}-1}$
$= \frac{1}{-1} = -1$ and
$\lim_{x\to-\infty}\frac{x^2+1}{1-x^2} = \lim_{x\to-\infty}\frac{1+\frac{1}{x^2}}{\frac{1}{x^2}-1}$
$= \frac{1}{-1} = -1.$

16. True: $\frac{3x^2 + 2x + \sin x}{x} - (3x+2) = \frac{\sin x}{x}$;
$\lim_{x\to\infty}\frac{\sin x}{x} = 0$ and $\lim_{x\to-\infty}\frac{\sin x}{x} = 0$.

17. True: The function is differentiable on (0, 2).

18. False: $f'(x) = \frac{x}{|x|}$ so $f'(0)$ does not exist.

19. False: There are two points: $x = -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$.

20. True: Let $g(x) = D$ where D is any number. Then $g'(x) = 0$ and so, by Theorem B of Section 3.6, $f(x) = g(x) + C = D + C$, which is a constant, for all x in (a, b).

21. False: For example if $f(x) = x^4$, $f'(0) = f''(0) = 0$ but f has a minimum at $x = 0$.

22. True: $\frac{dy}{dx} = \cos x; \frac{d^2y}{dx^2} = -\sin x$; $-\sin x = 0$ has infinitely many solutions.

23. False: The rectangle will have *minimum* perimeter if it is a square.
$A = xy = K;\ y = \frac{K}{x}$
$P = 2x + \frac{2K}{x}; \frac{dP}{dx} = 2 - \frac{2K}{x^2}; \frac{d^2P}{dx^2} = \frac{4K}{x^3}$
$\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} > 0$
when $x = \sqrt{K}, y = \sqrt{K}$.

24. True: By the Mean Value Theorem, the derivative must be zero between each pair of distinct x-intercepts.

25. True: If $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ for $x_1 < x_2$, $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $f + g$ is increasing.

26. False: Let $f(x) = g(x) = 2x$, $f'(x) > 0$ and $g'(x) > 0$ for all x, but $f(x)g(x) = 4x^2$ is decreasing on $(-\infty, 0)$.

27. True: Since $f''(x) > 0$, $f'(x)$ is increasing for $x \geq 0$. Therefore, $f'(x) > 0$ for x in $[0, \infty)$, so $f(x)$ is increasing.

28. False: If $f(3) = 4$, the Mean Value Theorem requires that at some point c in [0, 3], $f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{4-1}{3-0} = 1$ which does not contradict that $f'(x) \leq 2$ for all x in $[0, 3]$.

29. True: If the function is nondecreasing, $f'(x)$ must be greater than or equal to zero, and if $f'(x) \geq 0$, f is nondecreasing. This can be seen using the Mean Value Theorem.

30. True: However, if the constant is 0, the functions are the same.

31. False: For example, let $f(x) = e^x$. $\lim_{x \to -\infty} e^x = 0$, so $y = 0$ is a horizontal asymptote.

32. True: If $f(c)$ is a global maximum then $f(c)$ is the maximum value of f on $(a, b) \leftrightarrow S$ where (a, b) is any interval containing c and S is the domain of f. Hence, $f(c)$ is a local maximum value.

33. True: $f'(x) = 3ax^2 + 2bx + c$; $f'(x) = 0$ when $x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$ by the Quadratic Formula. $f''(x) = 6ax + 2b$ so

$$f''\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) = \pm 2\sqrt{b^2 - 3ac}.$$

Thus, if $b^2 - 3ac > 0$, one critical point is a local maximum and the other is a local minimum.
(If $b^2 - 3ac = 0$ the only critical point is an inflection point while if $b^2 - 3ac < 0$ there are no critical points.)
On an open interval, no local maxima can come from endpoints, so there can be at most one local maximum in an open interval.

34. True: $f'(x) = a \neq 0$ so $f(x)$ has no local minima or maxima. On an open interval, no local minima or maxima can come from endpoints, so $f(x)$ has no local minima.

35. True: Intermediate Value Theorem

36. False: The Bisection Method can be very slow to converge.

37. False: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n$.

38. False: Newton's method can fail to exist for several reasons (e.g. if f'(x) is 0 at or near r). It may be possible to achieve convergence by selecting a different starting value.

39. True: From the Fixed-point Theorem, if g is continuous on $[a, b]$ and $a \leq g(x) \leq b$ whenever $a \leq x \leq b$, then there is at least one fixed point on $[a, b]$. The given conditions satisfy these criteria.

40. True: The Bisection Method always converges as long as the function is continuous and the values of the function at the endpoints are of opposite sign.

41. True: Theorem 3.8.C

42. True: Obtained by integrating both sides of the Product Rule

43. True: $(-\sin x)^2 = \sin^2 x = 1 - \cos^2 x$

44. True: If $F(x) = \int f(x)\,dx$, $f(x)$ is a derivative of $F(x)$.

45. False: $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 7x - 5$ are a counter-example.

46. False: The two sides will in general differ by a constant term.

47. True: At any given height, speed on the downward trip is the negative of speed on the upward.

Sample Test Problems

1. $f'(x) = 2x - 2$; $2x - 2 = 0$ when $x = 1$.
Critical points: 0, 1, 4
$f(0) = 0$, $f(1) = -1$, $f(4) = 8$
Global minimum $f(1) = -1$;
global maximum $f(4) = 8$

2. $f'(t) = -\frac{1}{t^2}; -\frac{1}{t^2}$ is never 0.
Critical points: 1, 4
$f(1) = 1$, $f(4) = \frac{1}{4}$
Global minimum $f(4) = \frac{1}{4}$;
global maximum $f(1) = 1$.

3. $f'(z) = -\frac{2}{z^3}; -\frac{2}{z^3}$ is never 0.
Critical points: $-2, -\frac{1}{2}$
$f(-2) = \frac{1}{4}, f\left(-\frac{1}{2}\right) = 4$
Global minimum $f(-2) = \frac{1}{4}$;
global maximum $f\left(-\frac{1}{2}\right) = 4$.

4. $f'(x) = -\frac{2}{x^3}; -\frac{2}{x^3}$ is never 0.
Critical point: –2
$f(-2) = \frac{1}{4}$
$f'(x) > 0$ for $x < 0$, so f is increasing.
Global minimum $f(-2) = \frac{1}{4}$; no global maximum.

5. $f'(x) = \frac{x}{|x|}$; $f'(x)$ does not exist at $x = 0$.
Critical points: $-\frac{1}{2}, 0, 1$
$f\left(-\frac{1}{2}\right) = \frac{1}{2}, f(0) = 0, f(1) = 1$
Global minimum $f(0) = 0$;
global maximum $f(1) = 1$

6. $f'(s) = 1 + \frac{s}{|s|}$; $f'(s)$ does not exist when $s = 0$.
For $s < 0$, $|s| = -s$ so $f(s) = s - s = 0$ and $f'(s) = 1 - 1 = 0$.
Critical points: 1 and all s in [–1, 0]
$f(1) = 2$, $f(s) = 0$ for s in [–1, 0]
Global minimum $f(s) = 0, -1 \le s \le 0$;
global maximum $f(1) = 2$.

7. $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$; $f'(x) = 0$ when $x = 0, 1$
Critical points: –2, 0, 1, 3
$f(-2) = 80$, $f(0) = 0$, $f(1) = -1$, $f(3) = 135$
Global minimum $f(1) = -1$;
global maximum $f(3) = 135$

8. $f'(u) = \frac{u(7u-12)}{3(u-2)^{2/3}}$; $f'(u) = 0$ when $u = 0, \frac{12}{7}$
$f'(2)$ does not exist.
Critical points: $-1, 0, \frac{12}{7}, 2, 3$
$f(-1) = \sqrt[3]{-3} \approx -1.44$, $f(0) = 0$,
$f\left(\frac{12}{7}\right) = \frac{144}{49}\sqrt[3]{-\frac{2}{7}} \approx -1.94$, $f(2) = 0$, $f(3) = 9$
Global minimum $f\left(\frac{12}{7}\right) \approx -1.94$;
global maximum $f(3) = 9$

9. $f'(x) = 10x^4 - 20x^3 = 10x^3(x-2)$;
$f'(x) = 0$ when $x = 0, 2$
Critical points: –1, 0, 2, 3
$f(-1) = 0$, $f(0) = 7$, $f(2) = -9$, $f(3) = 88$
Global minimum $f(2) = -9$;
global maximum $f(3) = 88$

10. $f'(x) = 3(x-1)^2(x+2)^2 + 2(x-1)^3(x+2)$
$= (x-1)^2(x+2)(5x+4)$; $f'(x) = 0$ when
$x = -2, -\frac{4}{5}, 1$

Critical points: $-2, -\frac{4}{5}, 1, 2$

$f(-2) = 0$, $f\left(-\frac{4}{5}\right) = -\frac{26{,}244}{3125} \approx -8.40$,
$f(1) = 0$, $f(2) = 16$

Global minimum $f\left(-\frac{4}{5}\right) \approx -8.40$;
global maximum $f(2) = 16$

11. $f'(\theta) = \cos\theta$; $f'(\theta) = 0$ when $\theta = \frac{\pi}{2}$ in
$\left[\frac{\pi}{4}, \frac{4\pi}{3}\right]$

Critical points: $\frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{3}$

$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.71$, $f\left(\frac{\pi}{2}\right) = 1$,
$f\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \approx -0.87$

Global minimum $f\left(\frac{4\pi}{3}\right) \approx -0.87$;
global maximum $f\left(\frac{\pi}{2}\right) = 1$

12. $f'(\theta) = 2\sin\theta\cos\theta - \cos\theta = \cos\theta(2\sin\theta - 1)$;
$f'(\theta) = 0$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ in $[0, \pi]$

Critical points: $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

$f(0) = 0$, $f\left(\frac{\pi}{6}\right) = -\frac{1}{4}$, $f\left(\frac{\pi}{2}\right) = 0$,
$f\left(\frac{5\pi}{6}\right) = -\frac{1}{4}$, $f(\pi) = 0$

Global minimum $f\left(\frac{\pi}{6}\right) = -\frac{1}{4}$ or $f\left(\frac{5\pi}{6}\right) = -\frac{1}{4}$;
global maximum $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 0$, or
$f(\pi) = 0$

13. $f'(x) = 3 - 2x$; $f'(x) > 0$ when $x < \frac{3}{2}$.
$f''(x) = -2$; $f''(x)$ is always negative.
$f(x)$ is increasing on $\left(-\infty, \frac{3}{2}\right]$ and concave down
on $(-\infty, \infty)$.

14. $f'(x) = 9x^8$; $f'(x) > 0$ for all $x \neq 0$.
$f''(x) = 72x^7$; $f''(x) < 0$ when $x < 0$.
$f(x)$ is increasing on $(-\infty, \infty)$ and concave down on $(-\infty, 0)$.

15. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$; $f'(x) > 0$ when
$x < -1$ or $x > 1$.
$f''(x) = 6x$; $f''(x) < 0$ when $x < 0$.
$f(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and concave down on $(-\infty, 0)$.

16. $f'(x) = -6x^2 - 6x + 12 = -6(x+2)(x-1)$;
$f'(x) > 0$ when $-2 < x < 1$.
$f''(x) = -12x - 6 = -6(2x+1)$; $f''(x) < 0$ when
$x > -\frac{1}{2}$.
$f(x)$ is increasing on $[-2, 1]$ and concave down on $\left(-\frac{1}{2}, \infty\right)$.

17. $f'(x) = 4x^3 - 20x^4 = 4x^3(1-5x)$; $f'(x) > 0$
when $0 < x < \frac{1}{5}$.
$f''(x) = 12x^2 - 80x^3 = 4x^2(3-20x)$; $f''(x) < 0$
when $x > \frac{3}{20}$.
$f(x)$ is increasing on $\left[0, \frac{1}{5}\right]$ and concave down on $\left(\frac{3}{20}, \infty\right)$.

18. $f'(x) = 3x^2 - 6x^4 = 3x^2(1-2x^2)$; $f'(x) > 0$
when $-\frac{1}{\sqrt{2}} < x < 0$ and $0 < x < \frac{1}{\sqrt{2}}$.
$f''(x) = 6x - 24x^3 = 6x(1-4x^2)$; $f''(x) < 0$ when
$-\frac{1}{2} < x < 0$ or $x > \frac{1}{2}$.
$f(x)$ is increasing on $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ and concave down on $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$.

19. $f'(x) = 3x^2 - 4x^3 = x^2(3-4x)$; $f'(x) > 0$ when $x < \frac{3}{4}$.

$f''(x) = 6x - 12x^2 = 6x(1-2x)$; $f''(x) < 0$ when $x < 0$ or $x > \frac{1}{2}$.

$f(x)$ is increasing on $\left(-\infty, \frac{3}{4}\right]$ and concave down on $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$.

20. $g'(t) = 3t^2 - \frac{1}{t^2}$; $g'(t) > 0$ when $3t^2 > \frac{1}{t^2}$ or $t^4 > \frac{1}{3}$, so $t < -\frac{1}{3^{1/4}}$ or $t > \frac{1}{3^{1/4}}$.

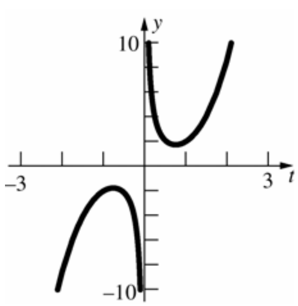
$g'(t)$ is increasing on $\left(-\infty, -\frac{1}{3^{1/4}}\right] \cup \left[\frac{1}{3^{1/4}}, \infty\right)$ and decreasing on $\left[-\frac{1}{3^{1/4}}, 0\right) \cup \left(0, \frac{1}{3^{1/4}}\right]$.

Local minimum $g\left(\frac{1}{3^{1/4}}\right) = \frac{1}{3^{3/4}} + 3^{1/4} \approx 1.75$;

local maximum

$g\left(-\frac{1}{3^{1/4}}\right) = -\frac{1}{3^{3/4}} - 3^{1/4} \approx -1.75$

$g''(t) = 6t + \frac{2}{t^3}$; $g''(t) > 0$ when $t > 0$. $g(t)$ has no inflection point since $g(0)$ does not exist.



21. $f'(x) = 2x(x-4) + x^2 = 3x^2 - 8x = x(3x-8)$;

$f'(x) > 0$ when $x < 0$ or $x > \frac{8}{3}$

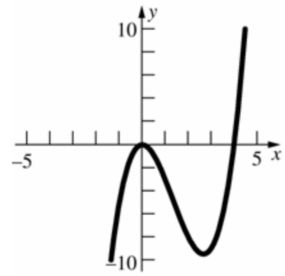
$f(x)$ is increasing on $(-\infty, 0] \cup \left[\frac{8}{3}, \infty\right)$ and decreasing on $\left[0, \frac{8}{3}\right]$

Local minimum $f\left(\frac{8}{3}\right) = -\frac{256}{27} \approx -9.48$;

local maximum $f(0) = 0$

$f''(x) = 6x - 8$; $f''(x) > 0$ when $x > \frac{4}{3}$.

$f(x)$ is concave up on $\left(\frac{4}{3}, \infty\right)$ and concave down on $\left(-\infty, \frac{4}{3}\right)$; inflection point $\left(\frac{4}{3}, -\frac{128}{27}\right)$



22. $f'(x) = -\frac{8x}{(x^2+1)^2}$; $f'(x) = 0$ when $x = 0$.

$f''(x) = \frac{8(3x^2-1)}{(x^2+1)^3}$; $f''(0) = -8$, so $f(0) = 6$ is a local maximum. $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$ so $f(0) = 6$ is a global maximum value. $f(x)$ has no minimum value.

23. $f'(x) = 4x^3 - 2$; $f'(x) = 0$ when $x = \frac{1}{\sqrt[3]{2}}$.

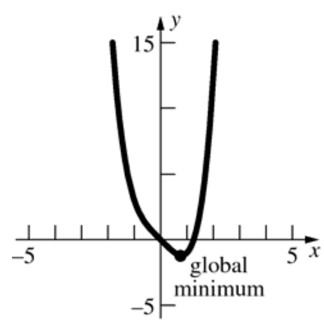
$f''(x) = 12x^2$; $f''(x) = 0$ when $x = 0$.

$f''\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{12}{2^{2/3}} > 0$, so

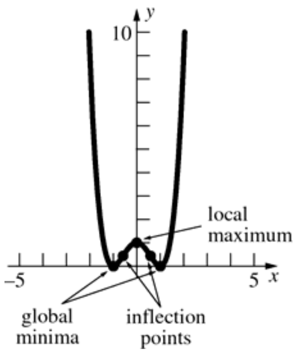
$f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{1}{2^{4/3}} - \frac{2}{2^{1/3}} = -\frac{3}{2^{4/3}}$ is a global minimum.

$f''(x) > 0$ for all $x \neq 0$; no inflection points

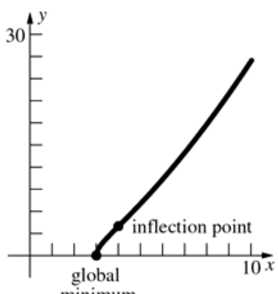
No horizontal or vertical asymptotes



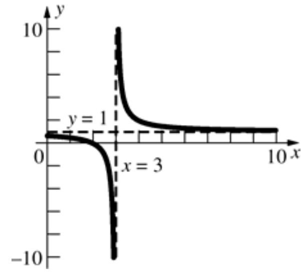
24. $f'(x) = 2(x^2 - 1)(2x) = 4x(x^2 - 1) = 4x^3 - 4x;$
$f'(x) = 0$ when $x = -1, 0, 1.$
$f''(x) = 12x^2 - 4 = 4(3x^2 - 1);\ f''(x) = 0$ when
$x = \pm\frac{1}{\sqrt{3}}.$
$f''(-1) = 8,\ f''(0) = -4,\ f''(1) = 8$
Global minima $f(-1) = 0$, $f(1) = 0$;
local maximum $f(0) = 1$
Inflection points $\left(\pm\frac{1}{\sqrt{3}}, \frac{4}{9}\right)$
No horizontal or vertical asymptotes



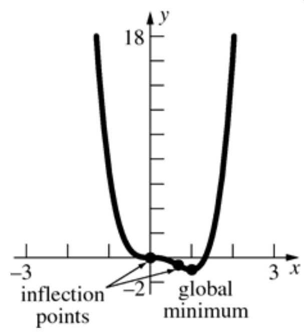
25. $f'(x) = \frac{3x - 6}{2\sqrt{x - 3}};\ f'(x) = 0$ when $x = 2$, but $x = 2$ is not in the domain of $f(x)$. $f'(x)$ does not exist when $x = 3$.
$f''(x) = \frac{3(x - 4)}{4(x - 3)^{3/2}};\ f''(x) = 0$ when $x = 4.$
Global minimum $f(3) = 0$; no local maxima
Inflection point (4, 4)
No horizontal or vertical asymptotes.



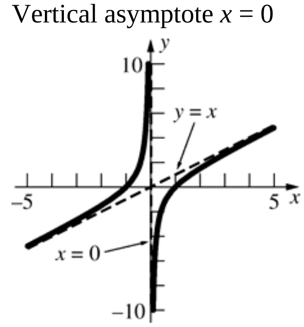
26. $f'(x) = -\frac{1}{(x - 3)^2};\ f'(x) < 0$ for all $x \neq 3.$
$f''(x) = \frac{2}{(x - 3)^3};\ f''(x) > 0$ when $x > 3.$
No local minima or maxima
No inflection points
$$\lim_{x\to\infty} \frac{x - 2}{x - 3} = \lim_{x\to\infty} \frac{1 - \frac{2}{x}}{1 - \frac{3}{x}} = 1$$
Horizontal asymptote $y = 1$
Vertical asymptote $x = 3$



27. $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1);\ f'(x) = 0$ when $x = 0, 1.$
$f''(x) = 36x^2 - 24x = 12x(3x - 2);\ f''(x) = 0$ when $x = 0, \frac{2}{3}.$
$f''(1) = 12$, so $f(1) = -1$ is a minimum.
Global minimum $f(1) = -1$; no local maxima
Inflection points (0, 0), $\left(\frac{2}{3}, -\frac{16}{27}\right)$
No horizontal or vertical asymptotes.



28. $f'(x) = 1 + \frac{1}{x^2};\ f'(x) > 0$ for all $x \neq 0.$
$f''(x) = -\frac{2}{x^3};\ f''(x) > 0$ when $x < 0$ and $f''(x) < 0$ when $x > 0.$
No local minima or maxima
No inflection points
$f(x) = x - \frac{1}{x}$, so
$\lim_{x\to\infty} [f(x) - x] = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0$ and $y = x$ is an oblique asymptote.
Vertical asymptote $x = 0$



29. $f'(x) = 3 + \frac{1}{x^2}$; $f'(x) > 0$ for all $x \neq 0$.

$f''(x) = -\frac{2}{x^3}$; $f''(x) > 0$ when $x < 0$ and

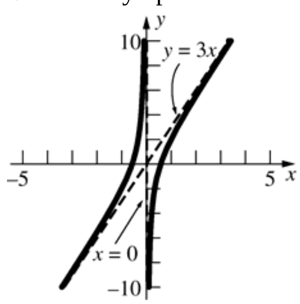
$f''(x) < 0$ when $x > 0$

No local minima or maxima
No inflection points

$f(x) = 3x - \frac{1}{x}$, so

$\lim_{x\to\infty} [f(x) - 3x] = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0$ and $y = 3x$ is an oblique asymptote.

Vertical asymptote $x = 0$



30. $f'(x) = -\frac{4}{(x+1)^3}$; $f'(x) > 0$ when $x < -1$ and

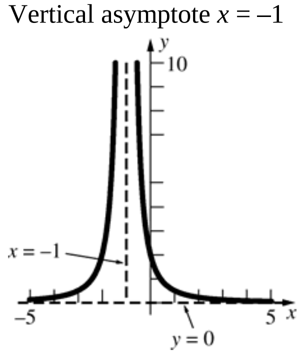
$f'(x) < 0$ when $x > -1$.

$f''(x) = \frac{12}{(x+1)^4}$; $f''(x) > 0$ for all $x \neq -1$.

No local minima or maxima
No inflection points

$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

Vertical asymptote $x = -1$



31. $f'(x) = -\sin x - \cos x$; $f'(x) = 0$ when

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$.

$f''(x) = -\cos x + \sin x$; $f''(x) = 0$ when

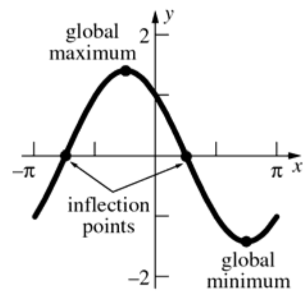
$x = -\frac{3\pi}{4}, \frac{\pi}{4}$.

$f''\left(-\frac{\pi}{4}\right) = -\sqrt{2}$, $f''\left(\frac{3\pi}{4}\right) = \sqrt{2}$

Global minimum $f\left(\frac{3\pi}{4}\right) = -\sqrt{2}$;

global maximum $f\left(-\frac{\pi}{4}\right) = \sqrt{2}$

Inflection points $\left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right)$



32. $f'(x) = \cos x - \sec^2 x$; $f'(x) = 0$ when $x = 0$

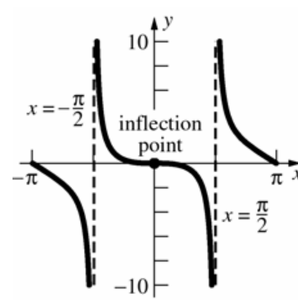
$f''(x) = -\sin x - 2\sec^2 x \tan x$

$= -\sin x(1 + 2\sec^3 x)$

$f''(x) = 0$ when $x = 0$

No local minima or maxima
Inflection point $f(0) = 0$

Vertical asymptotes $x = -\frac{\pi}{2}, \frac{\pi}{2}$

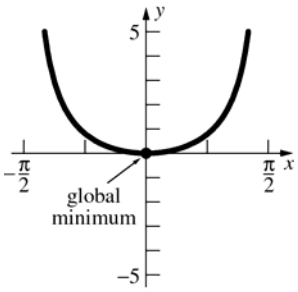


33. $f'(x) = x\sec^2 x + \tan x$; $f'(x) = 0$ when $x = 0$

$f''(x) = 2\sec^2 x(1 + x\tan x)$; $f''(x)$ is never 0 on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$f''(0) = 2$

Global minimum $f(0) = 0$

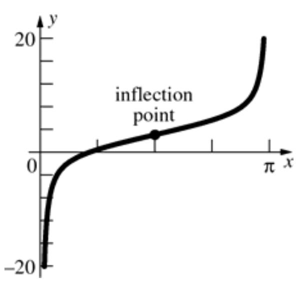


34. $f'(x) = 2 + \csc^2 x$; $f'(x) > 0$ on $(0, \pi)$

$f''(x) = -2\cot x\csc^2 x$; $f''(x) = 0$ when

$x = \frac{\pi}{2}$; $f''(x) > 0$ on $\left(\frac{\pi}{2}, \pi\right)$

Inflection point $\left(\frac{\pi}{2}, \pi\right)$



35. $f'(x) = \cos x - 2\cos x\sin x = \cos x(1 - 2\sin x)$;

$f'(x) = 0$ when $x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

$f''(x) = -\sin x + 2\sin^2 x - 2\cos^2 x$; $f''(x) = 0$

when $x \approx -2.51, -0.63, 1.00, 2.14$

$f''\left(-\frac{\pi}{2}\right) = 3$, $f''\left(\frac{\pi}{6}\right) = -\frac{3}{2}$, $f''\left(\frac{\pi}{2}\right) = 1$,

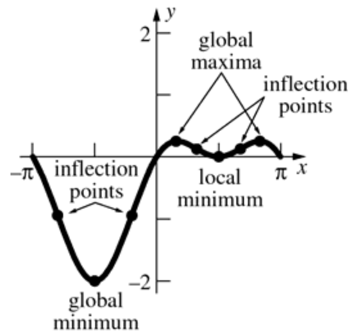
$f''\left(\frac{5\pi}{6}\right) = -\frac{3}{2}$

Global minimum $f\left(-\frac{\pi}{2}\right) = -2$,

local minimum $f\left(\frac{\pi}{2}\right) = 0$;

global maxima $f\left(\frac{\pi}{6}\right) = \frac{1}{4}$, $f\left(\frac{5\pi}{6}\right) = \frac{1}{4}$

Inflection points $(-2.51, -0.94)$, $(-0.63, -0.94)$, $(1.00, 0.13)$, $(2.14, 0.13)$



36. $f'(x) = -2\sin x - 2\cos x$; $f'(x) = 0$ when

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$.

$f''(x) = -2\cos x + 2\sin x$; $f''(x) = 0$ when

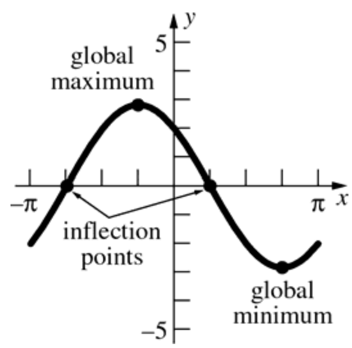
$x = -\frac{3\pi}{4}, \frac{\pi}{4}$.

$f''\left(-\frac{\pi}{4}\right) = -2\sqrt{2}$, $f''\left(\frac{3\pi}{4}\right) = 2\sqrt{2}$

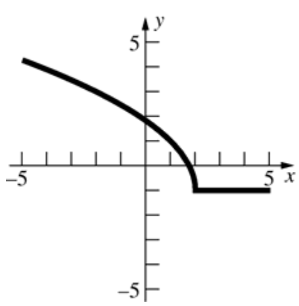
Global minimum $f\left(\frac{3\pi}{4}\right) = -2\sqrt{2}$;

global maximum $f\left(-\frac{\pi}{4}\right) = 2\sqrt{2}$

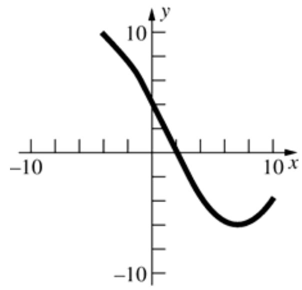
Inflection points $\left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right)$



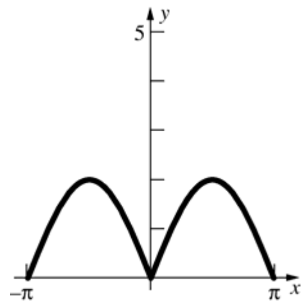
37.



38.



39.



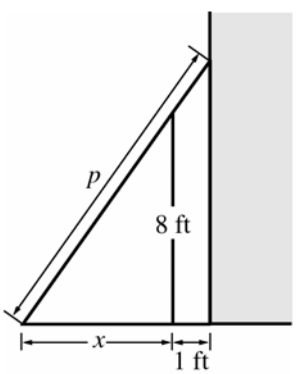
40. Let x be the length of a turned up side and let l be the (fixed) length of the sheet of metal.

$V = x(16-2x)l = 16xl - 2x^2 l$

$\frac{dV}{dx} = 16l - 4xl; V' = 0$ when $x = 4$

$\frac{d^2V}{dx^2} = -4l;$ 4 inches should be turned up for each side.

41. Let p be the length of the plank and let x be the distance from the fence to where the plank touches the ground.
See the figure below.



By properties of similar triangles,

$\frac{p}{x+1} = \frac{\sqrt{x^2+64}}{x}$

$p = \left(1+\frac{1}{x}\right)\sqrt{x^2+64}$

Minimize p:

$\frac{dp}{dx} = -\frac{1}{x^2}\sqrt{x^2+64} + \left(1+\frac{1}{x}\right)\frac{x}{\sqrt{x^2+64}}$

$= \frac{1}{x^2\sqrt{x^2+64}}\left(-(x^2+64)+\left(1+\frac{1}{x}\right)x^3\right)$

$= \frac{x^3-64}{x^2\sqrt{x^2+64}}$

$\frac{x^3-64}{x^2\sqrt{x^2+64}} = 0; x = 4$

$\frac{dp}{dx} < 0$ if $x < 4$, $\frac{dp}{dx} > 0$ if $x > 4$

When $x = 4$, $p = \left(1+\frac{1}{4}\right)\sqrt{16+64} \approx 11.18$ ft.

42. Let x be the width and y the height of a page. $A = xy$. Because of the margins,

$(y-4)(x-3) = 27$ or $y = \frac{27}{x-3}+4$

$A = \frac{27x}{x-3}+4x;$

$\frac{dA}{dx} = \frac{(x-3)(27)-27x}{(x-3)^2}+4 = -\frac{81}{(x-3)^2}+4$

$\frac{dA}{dx} = 0$ when $x = -\frac{3}{2}, \frac{15}{2}$

$\frac{d^2A}{dx^2} = \frac{162}{(x-3)^3};\ \frac{d^2A}{dx^2} > 0$ when $x = \frac{15}{2}$

$x = \frac{15}{2};\ y = 10$

43. $\frac{1}{2}\pi r^2 h = 128\pi$

$h = \frac{256}{r^2}$

Let S be the surface area of the trough.

$S = \pi r^2 + \pi r h = \pi r^2 + \frac{256\pi}{r}$

$\frac{dS}{dr} = 2\pi r - \frac{256\pi}{r^2}$

$2\pi r - \frac{256\pi}{r^2} = 0; r^3 = 128, r = 4\sqrt[3]{2}$

Since $\frac{d^2S}{dr^2} > 0$ when $r = 4\sqrt[3]{2}$, $r = 4\sqrt[3]{2}$ minimizes S.

$h = \frac{256}{\left(4\sqrt[3]{2}\right)^2} = 8\sqrt[3]{2}$

44. $f'(x) = \begin{cases} \frac{x}{2} + \frac{3}{2} & \text{if } -2 < x < 0 \\ -\frac{x+2}{3} & \text{if } 0 < x < 2 \end{cases}$

$\frac{x}{2} + \frac{3}{2} = 0;\ x = -3$, which is not in the domain.

$-\frac{x+2}{3} = 0;\ x = -2,$ which is not in the domain.

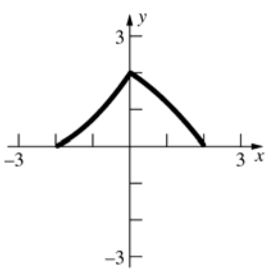
Critical points: $x = -2, 0, 2$

$f(-2) = 0,\ f(0) = 2,\ f(2) = 0$

Minima $f(-2) = 0,\ f(2) = 0$, maximum $f(0) = 2$.

$f''(x) = \begin{cases} \frac{1}{2} & \text{if } -2 < x < 0 \\ -\frac{1}{3} & \text{if } 0 < x < 2 \end{cases}$

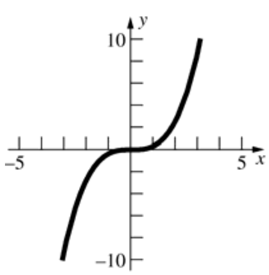
Concave up on $(-2, 0)$, concave down on $(0, 2)$



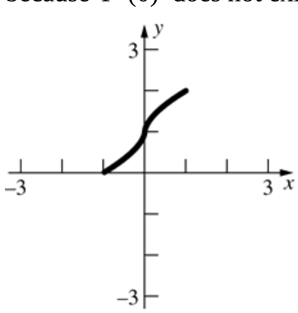
45. **a.** $f'(x) = x^2$

$\frac{f(3) - f(-3)}{3 - (-3)} = \frac{9+9}{6} = 3$

$c^2 = 3;\ c = -\sqrt{3}, \sqrt{3}$



b. The Mean Value Theorem does not apply because $F'(0)$ does not exist.

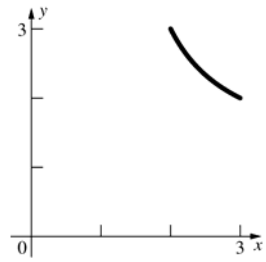


c. $g'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$

$\frac{g(3)-g(2)}{3-2} = \frac{2-3}{1} = -1$

$\frac{-2}{(c-1)^2} = -1;\ c = 1 \pm \sqrt{2}$

Only $c = 1 + \sqrt{2}$ is in the interval (2, 3).



46. $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x - 3$

$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24;\ 12(x^2 - 3x + 2) = 0$ when $x = 1, 2$

Inflection points: $x = 1$, $y = 5$ and $x = 2$, $y = 11$

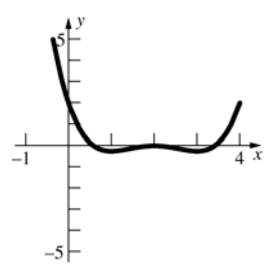
Slope at $x = 1$: $\left.\frac{dy}{dx}\right|_{x=1} = 7$

Tangent line: $y - 5 = 7(x - 1);\ y = 7x - 2$

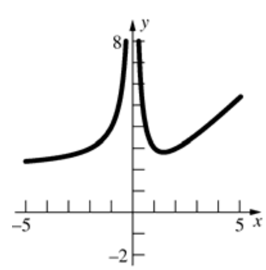
Slope at $x = 2$: $\left.\frac{dy}{dx}\right|_{x=2} = 5$

Tangent line: $y - 11 = 5(x - 2);\ y = 5x + 1$

47.



48.



49. Let $f(x) = 3x - \cos 2x$; $a_1 = 0$, $b_1 = 1$.

$f(0) = -1$; $f(1) \approx 3.4161468$

n	h_n	m_n	$f(m_n)$
1	0.5	0.5	0.9596977
2	0.25	0.25	–0.1275826
3	0.125	0.375	0.3933111
4	0.0625	0.3125	0.1265369
5	0.03125	0.28125	–0.0021745
6	0.015625	0.296875	0.0617765
7	0.0078125	0.2890625	0.0296988
8	0.0039063	0.2851563	0.0137364
9	0.0019532	0.2832031	0.0057745
10	0.0009766	0.2822266	0.0017984
11	0.0004883	0.2817383	–0.0001884
12	0.0002442	0.2819824	0.0008049
13	0.0001221	0.2818604	0.0003082
14	0.0000611	0.2817994	0.0000600
15	0.0000306	0.2817689	–0.0000641
16	0.0000153	0.2817842	–0.0000018
17	0.0000077	0.2817918	0.0000293
18	0.0000039	0.2817880	0.0000138
19	0.0000020	0.2817861	0.0000061
20	0.0000010	0.2817852	0.0000022
21	0.0000005	0.2817847	0.0000004
22	0.0000003	0.2817845	–0.0000006
23	0.0000002	0.2817846	–0.0000000

$x \approx 0.281785$

50. $f(x) = 3x - \cos 2x$, $f'(x) = 3 + 2\sin 2x$

Let $x_1 = 0.5$.

n	x_n
1	0.5
2	0.2950652
3	0.2818563
4	0.2817846
5	0.2817846

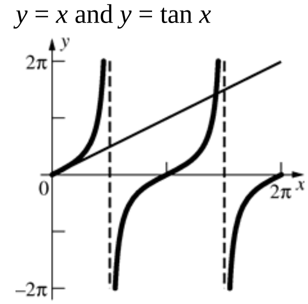
$x \approx 0.281785$

51. $x_{n+1} = \dfrac{\cos 2x_n}{3}$

n	x_n
1	0.5
2	0.18010
3	0.311942
4	0.270539
5	0.285718
6	0.280375
7	0.282285
8	0.281606
9	0.281848
10	0.281762
11	0.281793
12	0.281782
13	0.281786
14	0.281784
15	0.281785
16	0.281785

$x \approx 0.2818$

52. $y = x$ and $y = \tan x$



Let $x_1 = \dfrac{11\pi}{8}$.

$f(x) = x - \tan x$, $f'(x) = 1 - \sec^2 x$.

n	x_n
1	$\dfrac{11\pi}{8}$
2	4.64661795
3	4.60091050
4	4.54662258
5	4.50658016
6	4.49422443
7	4.49341259
8	4.49340946

$x \approx 4.4934$

53. $\int \left(x^3 - 3x^2 + 3\sqrt{x}\right)dx$

$= \int \left(x^3 - 3x^2 + 3x^{1/2}\right)dx$

$= \frac{1}{4}x^4 - x^3 + 3 \cdot \frac{2}{3}x^{3/2} + C$

$= \frac{1}{4}x^4 - x^3 + 2x^{3/2} + C$

54. $\int \frac{2x^4 - 3x^2 + 1}{x^2}dx$

$= \int \left(2x^2 - 3 + x^{-2}\right)dx$

$= \frac{2}{3}x^3 - 3x - x^{-1} + C$

$= \frac{2x^3}{3} - 3x - \frac{1}{x} + C$ or $\frac{2x^4 - 9x^2 - 3}{3x} + C$

55. $\int \frac{y^3 - 9y\sin y + 26y^{-1}}{y}dy$

$= \int \left(y^2 - 9\sin y + 26\right)dy$

$= \frac{1}{3}y^3 + 9\cos y + 26y + C$

56. Let $u = y^2 - 4$; then $du = 2y\,dy$ or $\frac{1}{2}du = y\,dy$.

$\int y\sqrt{y^2 - 4}\,dy = \int \sqrt{u} \cdot \frac{1}{2}du$

$= \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2} \cdot \frac{2}{3}u^{3/2} + C$

$= \frac{1}{3}\left(y^2 - 4\right)^{3/2} + C$

57. Let $u = 2z^2 - 3$; then $du = 4z\,dz$ or $\frac{1}{4}du = z\,dz$.

$\int z\left(2z^2 - 3\right)^{1/3}dz = \int u^{1/3} \cdot \frac{1}{4}du$

$= \frac{1}{4}\int u^{1/3}\,du$

$= \frac{1}{4} \cdot \frac{3}{4}u^{4/3} + C$

$= \frac{3}{16}\left(2z^2 - 3\right)^{4/3} + C$

58. Let $u = \cos x$; then $du = -\sin x\,dx$ or $-du = \sin x\,dx$.

$\int \cos^4 x \sin x\,dx = \int (\cos x)^4 \sin x\,dx$

$= \int u^4 \cdot -du$

$= -\int u^4 du$

$= -\frac{1}{5}u^5 + C$

$= -\frac{1}{5}\cos^5 x + C$

59. $u = \tan(3x^2 + 6x), du = (6x + 6)\sec^2(3x^2 + 6x)$

$\int (x+1)\tan^2\left(3x^2 + 6x\right)\sec^2\left(3x^2 + 6x\right)dx$

$= \frac{1}{6}\int u^2 du = \frac{1}{18}u^3 + C$

$= \frac{1}{18}\tan^3\left(3x^2 + 6x\right) + C$

60. $u = t^4 + 9, du = 4t^3 dt$

$\int \frac{t^3}{\sqrt{t^4 + 9}}dt = \int \frac{\frac{1}{4}du}{\sqrt{u}}$

$= \frac{1}{4}\int u^{-1/2}\,du$

$= \frac{1}{4} \cdot 2u^{1/2} + C$

$= \frac{1}{2}\sqrt{t^4 + 9} + C$

61. Let $u = t^5 + 5$; then $du = 5t^4 dt$ or $\frac{1}{5}du = t^4 dt$.

$\int t^4\left(t^5 + 5\right)^{2/3}dt = \int \frac{1}{5}u^{2/3}du$

$= \frac{1}{5}\int u^{2/3}\,du$

$= \frac{1}{5} \cdot \frac{3}{5}u^{5/3} + C$

$= \frac{3}{25}\left(t^5 + 5\right)^{5/3} + C$

62. Let $u = x^2 + 4$; then $du = 2x\,dx$ or $\frac{1}{2}du = xdx$.

$$\int \frac{x}{\sqrt{x^2+4}}dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}$$
$$= \frac{1}{2}\int u^{-1/2}du$$
$$= \frac{1}{2}\cdot 2u^{1/2} + C$$
$$= \sqrt{x^2+4} + C$$

63. Let $u = x^3 + 9$; then $du = 3x^2dx$ or $\frac{1}{3}du = x^2dx$.

$$\int \frac{x^2}{\sqrt{x^3+9}}dx = \frac{1}{3}\int \frac{du}{\sqrt{u}}$$
$$= \frac{1}{3}\int u^{-1/2}du$$
$$= \frac{1}{3}\cdot 2u^{1/2} + C$$
$$= \frac{2}{3}\sqrt{x^3+9} + C$$

64. Let $u = y + 1$; then $du = dy$.

$$\int \frac{1}{(y+1)^2}dy = \int \frac{1}{u^2}du$$
$$= \int u^{-2}du$$
$$= -u^{-1} + C$$
$$= -\frac{1}{y+1} + C$$

65. Let $u = 2y - 1$; then $du = 2dy$.

$$\int \frac{2}{(2y-1)^3}dy = \int \frac{du}{u^3}$$
$$= \int u^{-3}du$$
$$= -\frac{1}{2}u^{-2} + C$$
$$= -\frac{1}{2(2y-1)^2} + C$$

66. Let $u = y^3 - 3y$; then
$du = \left(3y^2 - 3\right)dy = 3\left(y^2 - 1\right)dy$.

$$\int \frac{y^2-1}{\left(y^3-3y\right)^2}dy = \frac{1}{3}\int \frac{du}{u^2}$$
$$= \frac{1}{3}\int u^{-2}du$$
$$= \frac{1}{3}\cdot -u^{-1} + C$$
$$= -\frac{1}{3}\cdot\frac{1}{y^3-3y} + C$$
$$= -\frac{1}{3y^3-9y} + C$$

67. $u = 2y^3 + 3y^2 + 6y,\ du = (6y^2 + 6y + 6)\,dy$

$$\frac{1}{6}\int u^{-1/5}du = \frac{5}{24}(2y^3 + 3y^2 + 6y)^{4/5} + C$$

68. $\int dy = \int \sin x\,dx$

$y = -\cos x + C$

$y = -\cos x + 3$

69. $\int dy = \int \frac{1}{\sqrt{x+1}}dx$

$y = 2\sqrt{x+1} + C$

$y = 2\sqrt{x+1} + 14$

70. $\int \sin y\,dy = \int dx$

$-\cos y = x + C$

$x = -1 - \cos y$

71. $\int dy = \int \sqrt{2t-1}\,dt$

$y = \frac{1}{3}(2t-1)^{3/2} + C$

$y = \frac{1}{3}(2t-1)^{3/2} - 1$

72. $\int y^{-4}dy = \int t^2dt$

$-\frac{1}{3y^3} = \frac{t^3}{3} + C$

$-\frac{1}{3y^3} = \frac{t^3}{3} - \frac{2}{3}$

$y = \sqrt[3]{\frac{1}{2-t^3}}$

73. $\int 2y\,dy = \int (6x - x^3)dx$

$y^2 = 3x^2 - \frac{1}{4}x^4 + C$

$y^2 = 3x^2 - \frac{1}{4}x^4 + 9$

$y = \sqrt{3x^2 - \frac{1}{4}x^4 + 9}$

74. $\int \cos y\,dy = \int x\,dx$

$\sin y = \frac{x^2}{2} + C$

$y = \sin^{-1}\left(\frac{x^2}{2}\right)$

75. $s(t) = -16t^2 + 48t + 448;\ s = 0$ at $t = 7$;

$v(t) = s'(t) = -32t + 48$

when $t = 7$, $v = -32(7) + 48 = -176$ ft/s

Review and Preview Problems

1. $A_{\text{region}} = \frac{1}{2}bh = \frac{1}{2}aa\sin 60^{\text{o}} = \frac{\sqrt{3}}{4}a^2$

2. $A_{\text{region}} = 6\left(\frac{1}{2}\text{base} \times \text{height}\right) = 6\left(\frac{1}{2}a\right)\left(\frac{\sqrt{3}}{2}a\right)$

$= \frac{3\sqrt{3}}{2}a^2$

3. $A_{\text{region}} = 10\left(\frac{1}{2}\text{base} \times \text{height}\right) = 5\frac{a^2}{4}\cot 36^\circ$

$= \frac{5}{4}a^2 \cot 36^\circ$

4. $A_{\text{region}} = A_{\text{rect}} + A_{\text{tri}} = 17(8.5) + \frac{1}{2}17\left(\frac{8.5}{\tan 45^{\text{o}}}\right)$

$= 216.75$

5. $A_{\text{region}} = A_{\text{rect}} + A_{\text{semic.}} = 3.6 \cdot 5.8 + \frac{1}{2}\pi(1.8)^2$

≈ 25.97

6. $A_{\text{region}} = A_{\#5} + 2A_{\text{tri}} = 25.97 + 2\left(\frac{1}{2} \cdot 1.2\right)5.8$

$= 32.93$

7. $A_{\text{region}} = 0.5(1 + 1.5 + 2 + 2.5) = 3.5$

8. $A_{\text{region}} = 0.5(1.5 + 2 + 2.5 + 3) = 4.5$

9. $A_{\text{region}} = A_{\text{rect}} + A_{\text{tri}} = 1x + \frac{1}{2}x \cdot x = \frac{1}{2}x^2 + x$

10. $A_{\text{region}} = \frac{1}{2}bh = \frac{1}{2}x \cdot xt = \frac{1}{2}x^2 t$

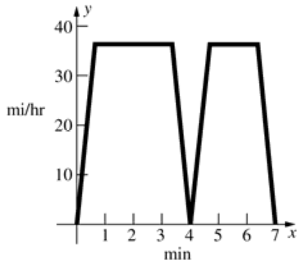
11. $y = 5 - x;\ A_{\text{region}} = A_{\text{rect}} + A_{\text{tri}}$

$= 2(2) + \frac{1}{2}(2)(2) = 6$

12. $A_{\text{region}} = A_{\text{rect}} + A_{\text{tri}}$

$= 1(1) + \frac{1}{2}(1)(7) = 4.5$

33. a.



b. Since the trip that involves 1 min more travel time at speed v_m is 0.6 mi longer,

$v_m = 0.6$ mi/min
$= 36$ mi/h.

c. From part b, $v_m = 0.6$ mi/min. Note that the average speed during acceleration and deceleration is $\frac{v_m}{2} = 0.3$ mi/min. Let t be the time spent between stop C and stop D at the constant speed v_m, so

$0.6t + 0.3(4 - t) = 2$ miles. Therefore,

$t = 2\frac{2}{3}$ min and the time spent accelerating

is $\frac{4 - 2\frac{2}{3}}{2} = \frac{2}{3}$ min.

$$a = \frac{0.6 - 0}{\frac{2}{3}} = 0.9 \text{ mi/min}^2.$$

34. For the balloon, $\frac{dh}{dt} = 4$, so $h(t) = 4t + C_1$. Set $t = 0$ at the time when Victoria threw the ball, and height 0 at the ground, then $h(t) = 4t + 64$. The height of the ball is given by $s(t) = -16t^2 + v_0 t$, since $s_0 = 0$. The maximum height of the ball is when $t = \frac{v_0}{32}$, since then $s'(t) = 0$. At this time

$h(t) = s(t)$ or $4\left(\frac{v_0}{32}\right) + 64 = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right)$.

Solve this for v_0 to get $v_0 \approx 68.125$ feet per second.

35. a. $\frac{dV}{dt} = C_1\sqrt{h}$ where h is the depth of the water. Here, $V = \pi r^2 h = 100h$, so $h = \frac{V}{100}$.

Hence $\frac{dV}{dt} = C_1\frac{\sqrt{V}}{10}$, $V(0) = 1600$, $V(40) = 0$.

b. $\int 10V^{-1/2}dV = \int C_1 dt;\ 20\sqrt{V} = C_1 t + C_2;$

$V(0) = 1600$: $C_2 = 20 \cdot 40 = 800$;

$V(40) = 0$: $C_1 = -\frac{800}{40} = -20$

$$V(t) = \frac{1}{400}(-20t + 800)^2 = (40 - t)^2$$

c. $V(10) = (40 - 10)^2 = 900 \text{ cm}^3$

36. a. $\frac{dP}{dt} = C_1\sqrt[3]{P}$, $P(0) = 1000$, $P(10) = 1700$ where t is the number of years since 1980.

b. $\int P^{-1/3}dP = \int C_1 dt;\ \frac{3}{2}P^{2/3} = C_1 t + C_2$

$P(0) = 1000$: $C_2 = \frac{3}{2} \cdot 1000^{2/3} = 150$

$$P(10) = 1700:\ C_1 = \frac{\frac{3}{2} \cdot 1700^{2/3} - 150}{10} \approx 6.3660$$

$P = (4.2440t + 100)^{3/2}$

c. $4000 = (4.2440t + 100)^{3/2}$

$$t = \frac{4000^{2/3} - 100}{4.2440} \approx 35.812$$

$t \approx 36$ years, so the population will reach 4000 by 2016.

37. Initially, $v = -32t$ and $s = -16t^2 + 16$. $s = 0$ when $t = 1$. Later, the ball falls 9 ft in a time given by $0 = -16t^2 + 9$, or $\frac{3}{4}$ s, and on impact has a velocity of $-32\left(\frac{3}{4}\right) = -24$ ft/s. By symmetry, 24 ft/s must be the velocity right after the first bounce. So

a. $v(t) = \begin{cases} -32t & \text{for } 0 \le t < 1 \\ -32(t-1) + 24 & \text{for } 1 < t \le 2.5 \end{cases}$

b. $9 = -16t^2 + 16 \Rightarrow t \approx 0.66$ sec; s also equals 9 at the apex of the first rebound at $t = 1.75$ sec.

CHAPTER 4

The Definite Integral

4.1 Concepts Review

1. $2\cdot\frac{5(6)}{2}=30;\ 2(5)=10$

2. 3(9) – 2(7) = 13; 9 + 4(10) = 49

3. inscribed; circumscribed

4. $0+1+2+3=6$

Problem Set 4.1

1. $\sum_{k=1}^{6}(k-1)=\sum_{k=1}^{6}k-\sum_{k=1}^{6}1$
$=\frac{6(7)}{2}-6(1)$
$=15$

2. $\sum_{i=1}^{6}i^2=\frac{6(7)(13)}{6}=91$

3. $\sum_{k=1}^{7}\frac{1}{k+1}=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}$
$+\frac{1}{4+1}+\frac{1}{5+1}+\frac{1}{6+1}+\frac{1}{7+1}$
$=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$
$=\frac{1443}{840}$
$=\frac{481}{280}$

4. $\sum_{l=3}^{8}(l+1)^2=4^2+5^2+6^2+7^2+8^2+9^2=271$

5. $\sum_{m=1}^{8}(-1)^m 2^{m-2}$
$=(-1)^1 2^{-1}+(-1)^2 2^0+(-1)^3 2^1$
$+(-1)^4 2^2+(-1)^5 2^3+(-1)^6 2^4$
$+(-1)^7 2^5+(-1)^8 2^6$
$=-\frac{1}{2}+1-2+4-8+16-32+64$
$=\frac{85}{2}$

6. $\sum_{k=3}^{7}\frac{(-1)^k 2^k}{(k+1)}$
$=\frac{(-1)^3 2^3}{4}+\frac{(-1)^4 2^4}{5}$
$+\frac{(-1)^5 2^5}{6}+\frac{(-1)^6 2^6}{7}+\frac{(-1)^7 2^7}{8}$
$=-\frac{1154}{105}$

7. $\sum_{n=1}^{6}n\cos(n\pi)=\sum_{n=1}^{6}(-1)^n\cdot n$
= –1 + 2 – 3 + 4 – 5 + 6
= 3

8. $\sum_{k=-1}^{6}k\sin\left(\frac{k\pi}{2}\right)$
$=-\sin\left(-\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)+2\sin(\pi)$
$+3\sin\left(\frac{3\pi}{2}\right)+4\sin(2\pi)\ +5\sin\left(\frac{5\pi}{2}\right)+6\sin(3\pi)$
= 1 + 1 + 0 – 3 + 0 + 5 + 0
= 4

9. $1+2+3+\cdots+41=\sum_{i=1}^{41}i$

10. $2+4+6+8+\cdots+50=\sum_{i=1}^{25}2i$

11. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}=\sum_{i=1}^{100}\frac{1}{i}$

12. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{100}=\sum_{i=1}^{100}\frac{(-1)^{i+1}}{i}$

13. $a_1+a_3+a_5+a_7+\cdots+a_{99}=\sum_{i=1}^{50}a_{2i-1}$

14. $f(w_1)\Delta x+f(w_2)\Delta x+\cdots+f(w_n)\Delta x$
$=\sum_{i=1}^{n}f(w_i)\Delta x$

15. $\sum_{i=1}^{10}(a_i+b_i)$
$=\sum_{i=1}^{10}a_i+\sum_{i=1}^{10}b_i$
$=40+50$
$=90$

16. $\sum_{n=1}^{10}(3a_n+2b_n)$
$=3\sum_{n=1}^{10}a_n+2\sum_{n=1}^{10}b_n$
$= 3(40) + 2(50)$
$= 220$

17. $\sum_{p=0}^{9}(a_{p+1}-b_{p+1})$
$=\sum_{p=1}^{10}a_p-\sum_{p=1}^{10}b_p$
$=40-50$
$=-10$

18. $\sum_{q=1}^{10}(a_q-b_q-q)$
$=\sum_{q=1}^{10}a_q-\sum_{q=1}^{10}b_q-\sum_{q=1}^{10}q$
$=40-50-\frac{10(11)}{2}$
$=-65$

19. $\sum_{i=1}^{100}(3i-2)$
$=3\sum_{i=1}^{100}i-\sum_{i=1}^{100}2$
$=3(5050)-2(100)$
$= 14{,}950$

20. $\sum_{i=1}^{10}\left[(i-1)(4i+3)\right]$
$=\sum_{i=1}^{10}(4i^2-i-3)$
$=4\sum_{i=1}^{10}i^2-\sum_{i=1}^{10}i-\sum_{i=1}^{10}3$
$= 4(385) - 55 - 3(10)$
$= 1455$

21. $\sum_{k=1}^{10}(k^3-k^2)=\sum_{k=1}^{10}k^3-\sum_{k=1}^{10}k^2$
$=3025-385$
$= 2640$

22. $\sum_{k=1}^{10}5k^2(k+4)=\sum_{k=1}^{10}(5k^3+20k^2)$
$=5\sum_{k=1}^{10}k^3+20\sum_{k=1}^{10}k^2$
$= 5(3025) + 20(385)$
$= 22{,}825$

23. $\sum_{i=1}^{n}(2i^2-3i+1)=2\sum_{i=1}^{n}i^2-3\sum_{i=1}^{n}i+\sum_{i=1}^{n}1$
$=\frac{2n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{2}+n$
$=\frac{2n^3+3n^2+n}{3}-\frac{3n^2+3n}{2}+n$
$=\frac{4n^3-3n^2-n}{6}$

24. $\sum_{i=1}^{n}(2i-3)^2=\sum_{i=1}^{n}(4i^2-12i+9)$
$=4\sum_{i=1}^{n}i^2-12\sum_{i=1}^{n}i+\sum_{i=1}^{n}9$
$=\frac{4n(n+1)(2n+1)}{6}-\frac{12n(n+1)}{2}+9n$
$=\frac{4n^3-12n^2+11n}{3}$

25.
$$\begin{aligned} S &= 1+2+3+\cdots+(n-2)+(n-1)+n \\ +\;S &= n+(n-1)+(n-2)+\cdots+3+2+1 \\ \hline 2S &= (n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1) \end{aligned}$$
$2S = n(n + 1)$
$S=\frac{n(n+1)}{2}$

26.
$$\begin{aligned}S - rS &= a + ar + ar^2 + \cdots + ar^n \\ &- \underline{\quad (ar + ar^2 + \cdots + ar^n + ar^{n+1})} \\ &= a - ar^{n+1} \\ &= S(1-r);\ S = \frac{a - ar^{n+1}}{1-r}\end{aligned}$$

27. a. $\sum_{k=0}^{10}\left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^{11}}{\frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{10}$, so

$\sum_{k=1}^{10}\left(\frac{1}{2}\right)^k = 1 - \left(\frac{1}{2}\right)^{10} = \frac{1023}{1024}$.

b. $\sum_{k=0}^{10} 2^k = \frac{1-2^{11}}{-1} = 2^{11} - 1$, so

$\sum_{k=1}^{10} 2^k = 2^{11} - 2 = 2046$.

28.
$$\begin{aligned}&S = a + (a+d) + (a+2d) + \cdots\cdots + [a + (n-2)d] + [a + (n-1)d] + (a + nd) \\ &\underline{+S = (a+nd) + [a + (n-1)d] + [a + (n-2)d] + \cdots + (a+2d) + (a+d) + a} \\ &2S = (2a+nd) + (2a+nd) + (2a+nd) + \cdots + (2a+nd) + (2a+nd) + (2a+nd) \\ &2S = (n+1)(2a+nd) \\ &S = \frac{(n+1)(2a+nd)}{2}\end{aligned}$$

29.
$$\begin{aligned}(i+1)^3 - i^3 &= 3i^2 + 3i + 1 \\ \sum_{i=1}^{n}\left[(i+1)^3 - i^3\right] &= \sum_{i=1}^{n}\left(3i^2 + 3i + 1\right) \\ (n+1)^3 - 1^3 &= 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ n^3 + 3n^2 + 3n &= 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n \\ 2n^3 + 6n^2 + 6n &= 6\sum_{i=1}^{n} i^2 + 3n^2 + 3n + 2n \\ \frac{2n^3 + 3n^2 + n}{6} &= \sum_{i=1}^{n} i^2 \\ \frac{n(n+1)(2n+1)}{6} &= \sum_{i=1}^{n} i^2\end{aligned}$$

30. $$(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$$

$$\sum_{i=1}^{n}\left[(i+1)^4 - i^4\right] = \sum_{i=1}^{n}\left(4i^3 + 6i^2 + 4i + 1\right)$$

$$(n+1)^4 - 1^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum_{i=1}^{n} i^3 + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n$$

Solving for $\sum_{i=1}^{n} i^3$ gives

$$4\sum_{i=1}^{n} i^3 = n^4 + 4n^3 + 6n^2 + 4n - \left(2n^3 + 3n^2 + n\right) - \left(2n^2 + 2n\right) - n$$

$$4\sum_{i=1}^{n} i^3 = n^4 + 2n^3 + n^2$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

31. $$(i+1)^5 - i^5 = 5i^4 + 10i^3 + 10i^2 + 5i + 1$$

$$\sum_{i=1}^{n}\left[(i+1)^5 - i^5\right] = 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$(n+1)^5 - 1^5 = 5\sum_{i=1}^{n} i^4 + 10\frac{n^2(n+1)^2}{4} + 10\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2} + n$$

$$n^5 + 5n^4 + 10n^3 + 10n^2 + 5n = 5\sum_{i=1}^{n} i^4 + \tfrac{5}{2}n^2(n+1)^2 + \tfrac{10}{6}n(n+1)(2n+1) + \frac{5}{2}n(n+1) + n$$

Solving for $\sum_{i=1}^{n} i^4$ yields

$$\sum_{i=1}^{n} i^4 = \tfrac{1}{5}\left[n^5 + \tfrac{5}{2}n^4 + \tfrac{5}{3}n^3 - \tfrac{1}{6}n\right] = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

32. Suppose we have a $(n+1)\times n$ grid. Shade in $n + 1 - k$ boxes in the kth column. There are n columns, and the shaded area is $1 + 2 + \cdots + n$. The shaded area is also half the area of the grid or $\frac{n(n+1)}{2}$. Thus, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Suppose we have a square grid with sides of length $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. From the diagram the area is $1^3 + 2^3 + \cdots + n^3$ or $\left[\frac{n(n+1)}{2}\right]^2$. Thus, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

33. $$\bar{x} = \frac{1}{7}(2+5+7+8+9+10+14) = \frac{55}{7} \approx 7.86$$

$$s^2 = \frac{1}{7}\left[\left(2 - \frac{55}{7}\right)^2 + \left(5 - \frac{55}{7}\right)^2 + \left(7 - \frac{55}{7}\right)^2 + \left(8 - \frac{55}{7}\right)^2 + \left(9 - \frac{55}{7}\right)^2 + \left(10 - \frac{55}{7}\right)^2 + \left(14 - \frac{55}{7}\right)^2\right] = \frac{608}{49} \approx 12.4$$

34. **a.** $\overline{x}=1,\ s^2=0$

b. $\overline{x}=1001,\ s^2=0$

c. $\overline{x}=2$

$$s^2=\tfrac{1}{3}\left[(1-2)^2+(2-2)^2+(3-2)^2\right]=\tfrac{1}{3}\left[(-1)^2+0^2+1^2\right]=\tfrac{1}{3}(2)=\tfrac{2}{3}$$

d. $\overline{x}=1,000,002$

$$s^2=\tfrac{1}{3}\left[(-1)^2+0^2+1^2\right]=\tfrac{2}{3}$$

35. **a.** $\sum_{i=1}^{n}(x_i-\overline{x})=\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\overline{x}=n\overline{x}-n\overline{x}=0$

b. $$s^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2=\frac{1}{n}\sum_{i=1}^{n}(x_i^2-2\overline{x}\,x_i+\overline{x}^2)$$

$$=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{2\overline{x}}{n}\sum_{i=1}^{n}x_i+\frac{1}{n}\sum_{i=1}^{n}\overline{x}^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{2\overline{x}}{n}(n\overline{x})+\frac{1}{n}(n\overline{x}^2)$$

$$=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-2\overline{x}^2+\overline{x}^2=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-\overline{x}^2$$

36. The variance of n identical numbers is 0. Let c be the constant. Then

$$s^2=\tfrac{1}{n}\left[(c-c)^2+(c-c)^2+\cdots+(c-c)^2\right]=0$$

37. Let $S(c)=\sum_{i=1}^{n}(x_i-c)^2$. Then

$$S'(c)=\frac{d}{dc}\sum_{i=1}^{n}(x_i-c)^2$$

$$=\sum_{i=1}^{n}\frac{d}{dc}(x_i-c)^2$$

$$=\sum_{i=1}^{n}2(x_i-c)(-1)$$

$$=-2\sum_{i=1}^{n}x_i+2nc$$

$S''(c)=2n$

Set $S'(c)=0$ and solve for c:

$$-2\sum_{i=1}^{n}x_i+2nc=0$$

$$c=\tfrac{1}{n}\sum_{i=1}^{n}x_i=\overline{x}$$

Since $S''(\overline{x})=2n>0$ we know that $\overline{x}$ minimizes $S(c)$.

38. **a.** The number of gifts given on the nth day is $\sum_{m=1}^{i} m = \frac{i(i+1)}{2}$.

The total number of gifts is $\sum_{i=1}^{12} \frac{i(i+1)}{2} = 364$.

b. For n days, the total number of gifts is $\sum_{i=1}^{n} \frac{i(i+1)}{2}$.

$$\sum_{i=1}^{n} \frac{i(i+1)}{2} = \sum_{i=1}^{n} \frac{i^2}{2} + \sum_{i=1}^{n} \frac{i}{2} = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{2}\left[\frac{n(n+1)}{2}\right]$$
$$= \frac{1}{4}n(n+1)\left(\frac{2n+1}{3}+1\right) = \frac{1}{12}n(n+1)(2n+4) = \frac{1}{6}n(n+1)(n+2)$$

39. The bottom layer contains 10 · 16 = 160 oranges, the next layer contains 9 · 15 = 135 oranges, the third layer contains 8 · 14 = 112 oranges, and so on, up to the top layer, which contains 1 · 7 = 7 oranges. The stack contains
1 · 7 + 2 · 8+ ··· + 9· 15 + 10 · 16
$= \sum_{i=1}^{10} i(6+i) = 715$ oranges.

40. If the bottom layer is 50 oranges by 60 oranges, the stack contains $\sum_{i=1}^{50} i(10+i) = 55,675$.

41. For a general stack whose base is m rows of n oranges with $m \le n$, the stack contains

$$\sum_{i=1}^{m} i(n-m+i) = (n-m)\sum_{i=1}^{m} i + \sum_{i=1}^{m} i^2$$
$$= (n-m)\frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6}$$
$$= \frac{m(m+1)(3n-m+1)}{6}$$

42. $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$
$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$

43. $A = \frac{1}{2}\left[1 + \frac{3}{2} + 2 + \frac{5}{2}\right] = \frac{7}{2}$

44. $A = \frac{1}{4}\left[1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2 + \frac{9}{4} + \frac{5}{2} + \frac{11}{4}\right] = \frac{15}{4}$

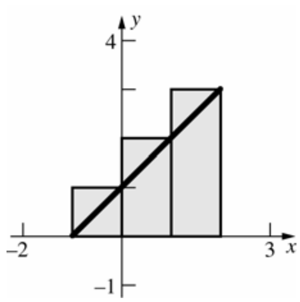
45. $A = \frac{1}{2}\left[\frac{3}{2} + 2 + \frac{5}{2} + 3\right] = \frac{9}{2}$

46. $A = \frac{1}{4}\left[\frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2 + \frac{9}{4} + \frac{5}{2} + \frac{11}{4} + 3\right] = \frac{17}{4}$

47. $A=\frac{1}{2}\left[\left(\frac{1}{2}\cdot 0^2+1\right)+\left(\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2+1\right)+\left(\frac{1}{2}\cdot 1^2+1\right)+\left(\frac{1}{2}\cdot\left(\frac{3}{2}\right)^2+1\right)\right]=\frac{1}{2}\left(1+\frac{9}{8}+\frac{3}{2}+\frac{17}{8}\right)=\frac{23}{8}$

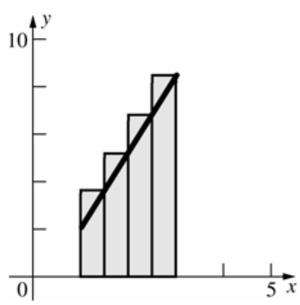
48. $A=\frac{1}{2}\left[\left(\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2+1\right)+\left(\frac{1}{2}\cdot 1^2+1\right)+\left(\frac{1}{2}\cdot\left(\frac{3}{2}\right)^2+1\right)+\left(\frac{1}{2}\cdot 2^2+1\right)\right]=\frac{1}{2}\left(\frac{9}{8}+\frac{3}{2}+\frac{17}{8}+3\right)=\frac{31}{8}$

49.



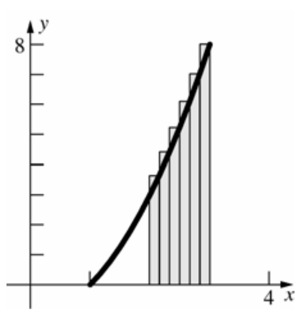
$A = 1(1 + 2 + 3) = 6$

50.



$A=\frac{1}{2}\left[\left(3\cdot\frac{3}{2}-1\right)+(3\cdot 2-1)+\left(3\cdot\frac{5}{2}-1\right)+(3\cdot 3-1)\right]=\frac{1}{2}\left(\frac{7}{2}+5+\frac{13}{2}+8\right)=\frac{23}{2}$

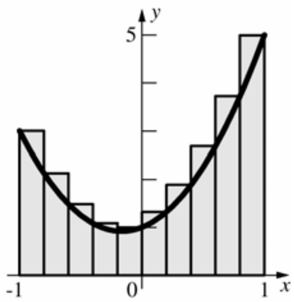
51.



$A=\frac{1}{6}\left[\left(\left(\frac{13}{6}\right)^2-1\right)+\left(\left(\frac{7}{3}\right)^2-1\right)+\left(\left(\frac{5}{2}\right)^2-1\right)+\left(\left(\frac{8}{3}\right)^2-1\right)+\left(\left(\frac{17}{6}\right)^2-1\right)+(3^2-1)\right]$

$=\frac{1}{6}\left(\frac{133}{36}+\frac{40}{9}+\frac{21}{4}+\frac{55}{9}+\frac{253}{36}+8\right)=\frac{1243}{216}$

52.



$$A=\frac{1}{5}\left[(3(-1)^2+(-1)+1)+\left(3\left(-\frac{4}{5}\right)^2+\left(-\frac{4}{5}\right)+1\right)+\left(3\left(-\frac{3}{5}\right)^2+\left(-\frac{3}{5}\right)+1\right)+\left(3\left(-\frac{2}{5}\right)^2+\left(-\frac{2}{5}\right)+1\right)+(3(0)^2+0+1)\right.$$

$$\left.+\left(3\left(\frac{1}{5}\right)^2+\frac{1}{5}+1\right)+\left(3\left(\frac{2}{5}\right)^2+\frac{2}{5}+1\right)+\left(3\left(\frac{3}{5}\right)^2+\frac{3}{5}+1\right)\left(3\left(\frac{4}{5}\right)^2+\frac{4}{5}+1\right)+(3(1)^2+1+1)\right]$$

$$=\frac{1}{5}[3+2.12+1.48+1.08+1+1.32+1.88+2.68+3.72+5]=4.656$$

53. $\Delta x=\dfrac{1}{n},\ x_i=\dfrac{i}{n}$

$$f(x_i)\Delta x=\left(\frac{i}{n}+2\right)\left(\frac{1}{n}\right)=\frac{i}{n^2}+\frac{2}{n}$$

$$A(S_n)=\left[\left(\frac{1}{n^2}+\frac{2}{n}\right)+\left(\frac{2}{n^2}+\frac{2}{n}\right)+\cdots+\left(\frac{n}{n^2}+\frac{2}{n}\right)\right]=\frac{1}{n^2}(1+2+3+\cdots+n)+2\ =\frac{n(n+1)}{2n^2}+2\ =\frac{1}{2n}+\frac{5}{2}$$

$$\lim_{n\to\infty}A(S_n)=\lim_{n\to\infty}\left(\frac{1}{2n}+\frac{5}{2}\right)=\frac{5}{2}$$

54. $\Delta x=\dfrac{1}{n},\ x_i=\dfrac{i}{n}$

$$f(x_i)\Delta x=\left[\frac{1}{2}\cdot\left(\frac{i}{n}\right)^2+1\right]\left(\frac{1}{n}\right)=\frac{i^2}{2n^3}+\frac{1}{n}$$

$$A(S_n)=\left[\left(\frac{1^2}{2n^3}+\frac{1}{n}\right)+\left(\frac{2^2}{2n^3}+\frac{1}{n}\right)+\cdots+\left(\frac{n^2}{2n^3}+\frac{1}{n}\right)\right]=\frac{1}{2n^3}(1^2+2^2+3^2+\cdots+n^2)+1$$

$$=\frac{1}{2n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+1\ =\frac{1}{12}\left[\frac{2n^3+3n^2+n}{n^3}\right]+1\ =\frac{1}{12}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]+1$$

$$\lim_{n\to\infty}A(S_n)=\lim_{n\to\infty}\left[\frac{1}{12}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+1\right]=\frac{7}{6}$$

55. $\Delta x = \frac{2}{n}, x_i = -1 + \frac{2i}{n}$

$$f(x_i)\Delta x = \left[2\left(-1+\frac{2i}{n}\right)+2\right]\left(\frac{2}{n}\right) = \frac{8i}{n^2}$$

$$A(S_n) = \left[\left(\frac{8}{n^2}\right)+\left(\frac{16}{n^2}\right)+\cdots+\left(\frac{8n}{n^2}\right)\right]$$

$$= \frac{8}{n^2}(1+2+3+\cdots+n) \quad = \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 4\left[\frac{n^2+n}{n^2}\right] \quad = 4+\frac{4}{n}$$

$$\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty}\left(4+\frac{4}{n}\right) = 4$$

56. First, consider $a = 0$ and $b = 2$.

$\Delta x = \frac{2}{n}, x_i = \frac{2i}{n}$

$$f(x_i)\Delta x = \left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \frac{8i^2}{n^3}$$

$$A(S_n) = \left[\left(\frac{8}{n^3}\right)+\left(\frac{8(2^2)}{n^3}\right)+\cdots+\left(\frac{8n^2}{n^3}\right)\right]$$

$$= \frac{8}{n^3}(1^2+2^2+\cdots+n^2) = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{4}{3}\left[\frac{2n^3+3n^2+n}{n^3}\right] = \frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}$$

$$\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\right) = \frac{8}{3}.$$

By symmetry, $A = 2\left(\frac{8}{3}\right) = \frac{16}{3}$.

57. $\Delta x = \frac{1}{n}, x_i = \frac{i}{n}$

$$f(x_i)\Delta x = \left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \frac{i^3}{n^4}$$

$$A(S_n) = \left[\frac{1}{n^4}(1^3)+\frac{1}{n^4}(2^3)+\cdots+\frac{1}{n^4}(n^3)\right]$$

$$= \frac{1}{n^4}(1^3+2^3+\cdots+n^3) \quad = \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

$$= \frac{1}{n^4}\left[\frac{n^4+2n^3+n^2}{4}\right] \quad = \frac{1}{4}\left[1+\frac{2}{n}+\frac{1}{n^2}\right]$$

$$\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty}\frac{1}{4}\left[1+\frac{2}{n}+\frac{1}{n^2}\right] = \frac{1}{4}$$

58. $\Delta x = \frac{1}{n}, x_i = \frac{i}{n}$

$$f(x_i)\Delta x = \left[\left(\frac{i}{n}\right)^3+\frac{i}{n}\right]\left(\frac{1}{n}\right) = \frac{i^3}{n^4}+\frac{i}{n^2}$$

$$A(S_n) = \frac{1}{n^4}(1^3+2^3+\cdots+n^3)+\frac{1}{n^2}(1+2+\cdots+n)$$

$$= \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2+\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{n^2+2n+1}{4n^2}+\frac{n^2+n}{2n^2} = \frac{3n^2+4n+1}{4n^2} = \frac{3}{4}+\frac{1}{n}+\frac{1}{4n^2}$$

$$\lim_{n\to\infty} A(S_n) = \frac{3}{4}$$

59. $f(t_i)\Delta t = \left[\frac{i}{n}+2\right]\frac{1}{n} = \frac{i}{n^2}+\frac{2}{n}$

$$A(S_n) = \sum_{i=1}^{n}\left(\frac{i}{n^2}+\frac{2}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} i+\sum_{i=1}^{n}\frac{2}{n}$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]+2$$

$$= \left[\frac{n^2+n}{2n^2}\right]+2$$

$$= \left(\frac{1}{2}+\frac{1}{2n}\right)+2$$

$$\lim_{n\to\infty} A(S_n) = \frac{1}{2}+2 = \frac{5}{2}$$

The object traveled $2\frac{1}{2}$ ft.

60. $f(t_i)\Delta t = \left[\frac{1}{2}\left(\frac{i}{n}\right)^2+1\right]\frac{1}{n} = \frac{i^2}{2n^3}+\frac{1}{n}$

$$A(S_n) = \sum_{i=1}^{n}\left(\frac{1i^2}{2n^3}+\frac{1}{n}\right) = \frac{1}{2n^3}\sum_{i=1}^{n} i^2+\sum_{i=1}^{n}\frac{1}{n}$$

$$= \frac{1}{2n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+1 \quad = \frac{1}{12}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]+1$$

$$\lim_{n\to\infty} A(S_n) = \frac{1}{12}(2)+1 = \frac{7}{6} \approx 1.17$$

The object traveled about 1.17 feet.

61. **a.** $f(x_i)\Delta x = \left(\frac{ib}{n}\right)^2\left(\frac{b}{n}\right) = \frac{b^3 i^2}{n^3}$

$A_0^b = \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 = \frac{b^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{b^3}{6}\left[2 + \frac{3}{n} + \frac{1}{n^2}\right]$

$\lim_{n\to\infty} A_0^b = \frac{2b^3}{6} = \frac{b^3}{3}$

b. Since $a \geq 0$, $A_0^b = A_0^a + A_a^b$, or

$A_a^b = A_0^b - A_0^a = \frac{b^3}{3} - \frac{a^3}{3}$.

62. $A_3^5 = \frac{5^3}{3} - \frac{3^3}{3} = \frac{98}{3} \approx 32.7$

The object traveled about 32.7 m.

63. **a.** $A_0^5 = \frac{5^3}{3} = \frac{125}{3}$

b. $A_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$

c. $A_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{117}{3} = 39$

64. **a.** $\Delta x = \frac{b}{n}$, $x_i = \frac{bi}{n}$

$f(x_i)\Delta x = \left(\frac{bi}{n}\right)^m\left(\frac{b}{n}\right) = \frac{b^{m+1} i^m}{n^{m+1}}$

$A(S_n) = \frac{b^{m+1}}{n^{m+1}}\sum_{i=1}^{n} i^m$

$= \frac{b^{m+1}}{n^{m+1}}\left[\frac{n^{m+1}}{m+1} + C_n\right]$

$= \frac{b^{m+1}}{m+1} + \frac{b^{m+1}C_n}{n^{m+1}}$

$A_0^b(x^m) = \lim_{n\to\infty} A(S_n) = \frac{b^{m+1}}{m+1}$

$\lim_{n\to\infty} \frac{C_n}{n^{m+1}} = 0$ since C_n is a polynomial in n of degree m.

b. Notice that $A_a^b(x^m) = A_0^b(x^m) - A_0^a(x^m)$.

Thus, using part a, $A_a^b(x^m) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}$.

65. **a.** $A_0^2(x^3) = \frac{2^{3+1}}{3+1} = 4$

b. $A_1^2(x^3) = \frac{2^{3+1}}{3+1} - \frac{1^{3+1}}{3+1} = 4 - \frac{1}{4} = \frac{15}{4}$

c. $A_1^2(x^5) = \frac{2^{5+1}}{5+1} - \frac{1^{5+1}}{5+1} = \frac{32}{3} - \frac{1}{6} = \frac{63}{6}$

$= \frac{21}{2} = 10.5$

d. $A_0^2(x^9) = \frac{2^{9+1}}{9+1} = \frac{1024}{10} = 102.4$

66. Inscribed:

Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The angle at the center is $\frac{2\pi}{n}$. The length of the base is $2r\sin\frac{\pi}{n}$. The height is $r\cos\frac{\pi}{n}$. Thus the area of the triangle is $r^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{1}{2}r^2\sin\frac{2\pi}{n}$.

$A_n = n\left(\frac{1}{2}r^2\sin\frac{2\pi}{n}\right) = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

Circumscribed:

Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The angle at the center is $\frac{2\pi}{n}$. The length of the base is $2r\tan\frac{\pi}{n}$. The height is r. Thus the area of the triangle is $r^2\tan\frac{\pi}{n}$.

$B_n = n\left(r^2\tan\frac{\pi}{n}\right) = nr^2\tan\frac{\pi}{n}$

$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{1}{2}nr^2\sin\frac{2\pi}{n} = \lim_{n\to\infty}\pi r^2\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$

$= \pi r^2$

$\lim_{n\to\infty} B_n = \lim_{n\to\infty} nr^2\tan\frac{\pi}{n} = \lim_{n\to\infty}\frac{\pi r^2}{\cos\frac{\pi}{n}}\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)$

$= \pi r^2$

4.2 Concepts Review

1. Riemann sum

2. definite integral; $\int_a^b f(x)dx$

3. $A_{\text{up}} - A_{\text{down}}$

4. $8 - \frac{1}{2} = \frac{15}{2}$

Problem Set 4.2

1. $R_P = f(2)(2.5-1) + f(3)(3.5-2.5) + f(4.5)(5-3.5)$ = 4(1.5) + 3(1) + (–2.25)(1.5) = 5.625

2. $R_P = f(0.5)(0.7 – 0) + f(1.5)(1.7 – 0.7) + f(2)(2.7 – 1.7) + f(3.5)(4 – 2.7)$
= 1.25(0.7) + (–0.75)(1) + (–1)(1) + 1.25(1.3) = 0.75

3. $R_P = \sum_{i=1}^{5} f(\overline{x}_i)\Delta x_i = f(3)(3.75-3) + f(4)(4.25-3.75) + f(4.75)(5.5-4.25) + f(6)(6-5.5) + f(6.5)(7-6)$
= 2(0.75) + 3(0.5) + 3.75(1.25) + 5(0.5) + 5.5(1) = 15.6875

4. $R_P = \sum_{i=1}^{4} f(\overline{x}_i)\Delta x_i = f(-2)(-1.3+3) + f(-0.5)(0+1.3) + f(0)(0.9-0) + f(2)(2-0.9)$
= 4(1.7) + 3.25(1.3) + 3(0.9) + 2(1.1) = 15.925

5. $R_P = \sum_{i=1}^{8} f(\overline{x}_i)\Delta x_i = \left[f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)\right](0.5)$
= [–0.21875 – 0.46875 – 0.46875 – 0.21875 + 0.28125 + 1.03125 + 2.03125 + 3.28125](0.5) = 2.625

6. $R_P = \sum_{i=1}^{6} f(\overline{x}_i)\Delta x_i = \left[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)\right](0.5)$
= [1.5 + 5 + 14.5 + 33 + 63.5 + 109](0.5) = 113.25

7. $\int_1^3 x^3\, dx$

8. $\int_0^2 (x+1)^3\, dx$

9. $\int_{-1}^1 \frac{x^2}{1+x}\, dx$

10. $\int_0^\pi (\sin x)^2\, dx$

11. $\Delta x = \frac{2}{n}, \overline{x}_i = \frac{2i}{n}$

$f(\overline{x}_i) = \overline{x}_i + 1 = \frac{2i}{n} + 1$

$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\frac{2}{n}$

$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{4}{n^2}\sum_{i=1}^{n} i = \frac{2}{n}(n) + \frac{4}{n^2}\left[\frac{n(n+1)}{2}\right]$

$= 2 + 2\left(1 + \frac{1}{n}\right)$

$\int_0^2 (x+1)dx = \lim_{n\to\infty}\left[2 + 2\left(1 + \frac{1}{n}\right)\right] = 4$

12. $\Delta x = \frac{2}{n}, \overline{x}_i = \frac{2i}{n}$

$$f(\overline{x}_i) = \left(\frac{2i}{n}\right)^2 + 1 = \frac{4i^2}{n^2} + 1$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i^2\left(\frac{4}{n^2}\right)\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{2}{n}(n) + \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 2 + \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty}\left[2 + \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = \frac{14}{3}$$

13. $\Delta x = \frac{3}{n}, \overline{x}_i = -2 + \frac{3i}{n}$

$$f(\overline{x}_i) = 2\left(-2 + \frac{3i}{n}\right) + \pi = \pi - 4 + \frac{6i}{n}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[\pi - 4 + \frac{6i}{n}\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n}(\pi - 4) + \frac{18}{n^2}\sum_{i=1}^{n} i = 3(\pi - 4) + \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 3\pi - 12 + 9\left(1 + \frac{1}{n}\right)$$

$$\int_{-2}^{1} (2x+\pi)\,dx = \lim_{n\to\infty}\left[3\pi - 12 + 9\left(1 + \frac{1}{n}\right)\right]$$

$$= 3\pi - 3$$

14. $\Delta x = \frac{3}{n}, \overline{x}_i = -2 + \frac{3i}{n}$

$$f(\overline{x}_i) = 3\left(-2 + \frac{3i}{n}\right)^2 + 2 = 14 - \frac{36i}{n} + \frac{27i^2}{n^2}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[14 - \left(\frac{36}{n}\right)i + \left(\frac{27}{n^2}\right)i^2\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n} 14 - \frac{108}{n^2}\sum_{i=1}^{n} i + \frac{81}{n^3}\sum_{i=1}^{n} i^2$$

$$= 42 - \frac{108}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{81}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 42 - 54\left(1 + \frac{1}{n}\right) + \frac{27}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_{-2}^{1} (3x^2+2)\,dx$$

$$= \lim_{n\to\infty}\left[42 - 54\left(1 + \frac{1}{n}\right) + \frac{27}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = 15$$

15. $\Delta x = \frac{5}{n}, \overline{x}_i = \frac{5i}{n}$

$$f(\overline{x}_i) = 1 + \frac{5i}{n}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i\left(\frac{5}{n}\right)\right]\frac{5}{n}$$

$$= \frac{5}{n}\sum_{i=1}^{n} 1 + \frac{25}{n^2}\sum_{i=1}^{n} i = 5 + \frac{25}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 5 + \frac{25}{2}\left(1 + \frac{1}{n}\right)$$

$$\int_0^5 (x+1)\,dx = \lim_{n\to\infty}\left[5 + \frac{25}{2}\left(1 + \frac{1}{n}\right)\right] = \frac{35}{2}$$

16. $\Delta x = \frac{20}{n}, \overline{x}_i = -10 + \frac{20i}{n}$

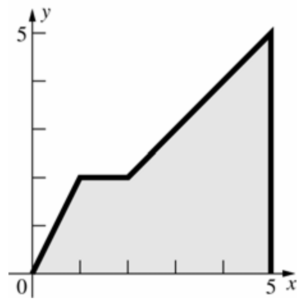
$$f(\overline{x}_i) = \left(-10 + \frac{20i}{n}\right)^2 + \left(-10 + \frac{20i}{n}\right) = 90 - \frac{380i}{n} + \frac{400i^2}{n^2}$$

$$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[90 - i\left(\frac{380}{n}\right) + i^2\left(\frac{400}{n^2}\right)\right]\frac{20}{n} = \frac{20}{n}\sum_{i=1}^{n} 90 - \frac{7600}{n^2}\sum_{i=1}^{n} i + \frac{8000}{n^3}\sum_{i=1}^{n} i^2$$

$$= 1800 - \frac{7600}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{8000}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = 1800 - 3800\left(1 + \frac{1}{n}\right) + \frac{4000}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_{-10}^{10} (x^2+x)\,dx = \lim_{n\to\infty}\left[1800 - 3800\left(1 + \frac{1}{n}\right) + \frac{4000}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = \frac{2000}{3}$$

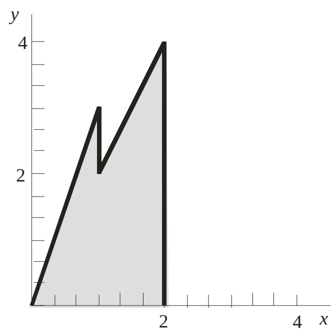
17.



$\int_0^5 f(x)\,dx$

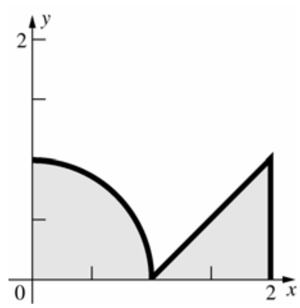
$= \frac{1}{2}(1)(2) + 1(2) + 3(2) + \frac{1}{2}(3)(3) = \frac{27}{2}$

18.



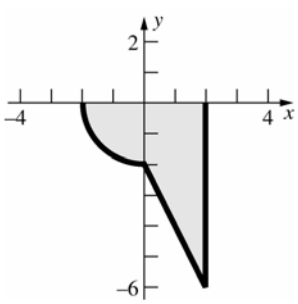
$\int_0^2 f(x)\,dx = \frac{1}{2}(1)(3) + (1)(2) + \frac{1}{2}(1)(2) = \frac{9}{2}$

19.



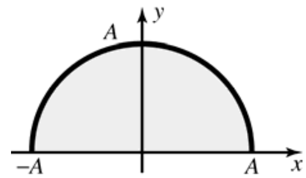
$\int_0^2 f(x)\,dx = \frac{1}{4}(\pi \cdot 1^2) + \frac{1}{2}(1)(1) = \frac{1}{2} + \frac{\pi}{4}$

20.

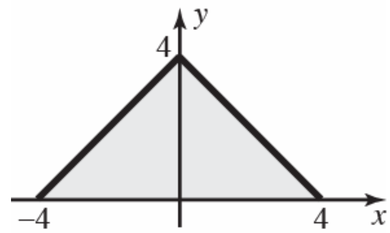


$\int_{-2}^2 f(x)\,dx = -\frac{1}{4}(\pi \cdot 2^2) - (2)(2) - \frac{1}{2}(2)(4)$
$= -\pi - 8$

21. The area under the curve is equal to the area of a semi-circle: $\int_{-A}^{A} \sqrt{A^2 - x^2}\,dx = \frac{1}{2}\pi A^2$.



22. The area under the curve is equal to the area of a triangle:



$\int_{-4}^{4} f(x)\,dx = 2\left(\frac{1}{2}\right)4 \cdot 4 = 16$

23. $s(4) = \int_0^4 v(t)\,dt = \frac{1}{2}4\left(\frac{4}{60}\right) = \frac{2}{15}$

24. $s(4) = \int_0^4 v(t)\,dt = 4 + \frac{1}{2}4(9-1) = 20$

25. $s(4) = \int_0^4 v(t)\,dt = \frac{1}{2}2(1) + 2(1) = 3$

26. $s(4) = \int_0^4 v(t)\,dt = \frac{1}{4}\pi(2)^2 + 0 = \pi$

27.

t	$s(t)$
20	40
40	80
60	120
80	160
100	200
120	240

28.

t	$s(t)$
20	10
40	40
60	90
80	160
100	250
120	360

29.

t	$s(t)$
20	20
40	80
60	160
80	240
100	320
120	400

30.

t	$s(t)$
20	20
40	60
60	80
80	60
100	0
120	-100

31. a. $\int_{-3}^{3} [\![x]\!]\,dx = (-3-2-1+0+1+2)(1) = -3$

b. $\int_{-3}^{3} [\![x]\!]^2\,dx = [(-3)^2 + (-2)^2$
$+(-1)^2 + 0 + 1 + 4](1) = 19$

c. $\int_{-3}^{3} \left(x - [\![x]\!]\right) dx = 6\left[\frac{1}{2}(1)(1)\right] = 3$

d. $\int_{-3}^{3} \left(x - [\![x]\!]\right)^2 dx = 6\int_0^1 x^2 dx = 6 \cdot \frac{1^3}{3} = 2$

e. $\int_{-3}^{3} |x|\,dx = \frac{1}{2}(3)(3) + \frac{1}{2}(3)(3) = 9$

f. $\int_{-3}^{3} x|x|\,dx = \frac{(-3)^3}{3} + \frac{(3)^3}{3} = 0$

g. $\int_{-1}^{2} |x|[\![x]\!]\,dx = -\int_{-1}^{0}|x|\,dx + 0\int_0^1 |x|\,dx + \int_1^2 |x|\,dx$

$= -\frac{1}{2}(1)(1) + 1(1) + \frac{1}{2}(1)(1) = 1$

h. $\int_{-1}^{2} x^2 [\![x]\!]\,dx = -\int_{-1}^{0} x^2\,dx + 0\int_0^1 x^2\,dx$
$+\int_1^2 x^2\,dx$

$= -\frac{1^3}{3} + \left(\frac{2^3}{3} - \frac{1^3}{3}\right) = 2$

32. a. $\int_{-1}^{1} f(x)\,dx = 0$ because this is an odd function.

b. $\int_{-1}^{1} g(x)\,dx = 3 + 3 = 6$

c. $\int_{-1}^{1} |f(x)|\,dx = 3 + 3 = 6$

d. $\int_{-1}^{1} [-g(x)]\,dx = -3 + (-3) = -6$

e. $\int_{-1}^{1} xg(x)\,dx = 0$ because $xg(x)$ is an odd function.

f. $\int_{-1}^{1} f^3(x)g(x)\,dx = 0$ because $f^3(x)g(x)$ is an odd function.

33. $R_P = \frac{1}{2}\sum_{i=1}^{n}(x_i + x_{i-1})(x_i - x_{i-1})$

$= \frac{1}{2}\sum_{i=1}^{n}\left(x_i^2 - x_{i-1}^2\right)$

$= \frac{1}{2}\Big[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + (x_3^2 - x_2^2)$
$+ \cdots + (x_n^2 - x_{n-1}^2)\Big]$

$= \frac{1}{2}(x_n^2 - x_0^2)$

$= \frac{1}{2}(b^2 - a^2)$

$\lim_{n\to\infty} \frac{1}{2}(b^2 - a^2) = \frac{1}{2}(b^2 - a^2)$

34. Note that $\overline{x}_i = \left[\frac{1}{3}\left(x_{i-1}^2 + x_{i-1}x_i + x_i^2\right)\right]^{1/2}$

$\geq \left[\frac{1}{3}(x_{i-1}^2 + x_{i-1}^2 + x_{i-1}^2\right]^{1/2} = x_{i-1}$ and

$\overline{x}_i = \left[\frac{1}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2\right]^{1/2}$

$\leq \left[\frac{1}{3}(x_i^2 + x_i^2 + x_i^2)\right]^{1/2} = x_i$.

$$R_p = \sum_{i=1}^{n} \overline{x}_i^2 \Delta x_i$$

$$= \sum_{i=1}^{n} \frac{1}{3}(x_i^2 + x_{i-1}\, x_i + x_{i-1}^2)(x_i - x_{i-1})$$

$$= \frac{1}{3}\sum_{i=1}^{n}(x_i^3 - x_{i-1}^3)$$

$$= \frac{1}{3}\Big[(x_1^3 - x_0^3) + (x_2^3 - x_1^3) + (x_3^3 - x_2^3) + \cdots + (x_n^3 - x_{n-1}^3)\Big]$$

$$= \frac{1}{3}(x_n^3 - x_0^3) = \frac{1}{3}(b^3 - a^3)$$

35. Left: $\int_0^2 (x^3 + 1)\, dx = 5.24$

Right: $\int_0^2 (x^3 + 1)\, dx = 6.84$

Midpoint: $\int_0^2 (x^3 + 1)\, dx = 5.98$

36. Left: $\int_0^1 \tan x\, dx \approx 0.5398$

Right: $\int_0^1 \tan x\, dx \approx 0.6955$

Midpoint: $\int_0^1 \tan x\, dx \approx 0.6146$

37. Left: $\int_0^1 \cos x\, dx \approx 0.8638$

Right: $\int_0^1 \cos x\, dx \approx 0.8178$

Midpoint: $\int_0^1 \cos x\, dx \approx 0.8418$

38. Left: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.1682$

Right: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.0349$

Midpoint: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.0971$

39. Partition [0, 1] into n regular intervals, so $\|P\| = \frac{1}{n}$.

If $\overline{x}_i = \frac{i}{n} + \frac{1}{2n}$, $f(\overline{x}_i) = 1$.

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} = 1$$

If $\overline{x}_i = \frac{i}{n} + \frac{1}{\pi n}$, $f(\overline{x}_i) = 0$.

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} 0 = 0$$

Thus f is not integrable on [0, 1].

4.3 Concepts Review

1. 4(4 – 2) = 8; 16(4 – 2) = 32

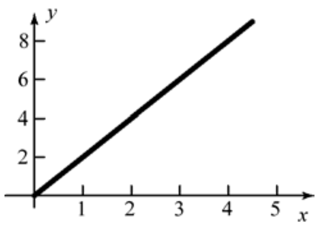
2. $\sin^3 x$

3. $\int_1^4 f(x)\,dx$; $\int_2^5 \sqrt{x}\,dx$

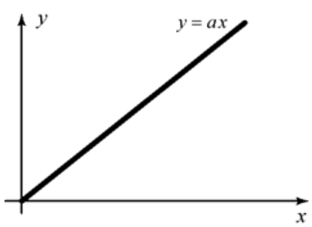
4. 5

Problem Set 4.3

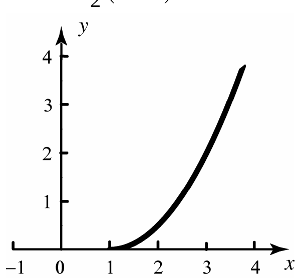
1. $A(x) = 2x$



2. $A(x) = ax$

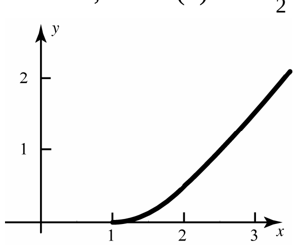


3. $A(x) = \frac{1}{2}(x-1)^2, \quad x \ge 1$

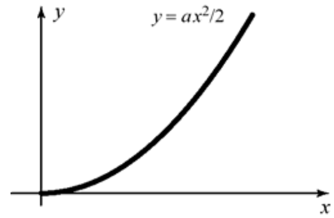


4. If $1 \le x \le 2$, then $A(x) = \frac{1}{2}(x-1)^2$.

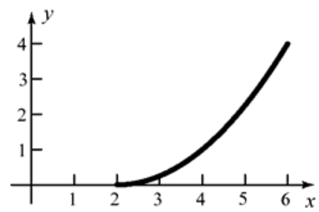
If $2 \le x$, then $A(x) = x - \frac{3}{2}$



5. $A(x) = \dfrac{1}{2}x(ax) = \dfrac{ax^2}{2}$

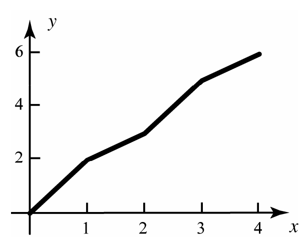


6. $A(x) = \dfrac{1}{2}(x-2)(-1+x/2) = \dfrac{1}{4}(x-2)^2, \quad x \ge 2$



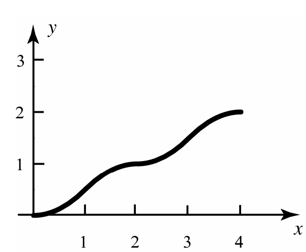
7.

$$A(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 2 + (x-1) & 1 < x \le 2 \\ 3 + 2(x-2) & 2 < x \le 3 \\ 5 + (x-3) & 3 < x \le 4 \\ \text{etc.} \end{cases}$$



8.

$$A(x) = \begin{cases} \frac{1}{2}x^2 & 0 \le x \le 1 \\ \frac{1}{2} + \frac{1}{2}(3-x)(x-1) & 1 < x \le 2 \\ 1 + \frac{1}{2}(x-2)^2 & 2 < x \le 3 \\ \frac{3}{2} + \frac{1}{2}(5-x)(x-3) & 3 < x \le 4 \\ 2 + \frac{1}{2}(x-4)^2 & 4 < x \le 5 \\ \text{etc.} \end{cases}$$



9. $\int_1^2 2f(x)\,dx = 2\int_1^2 f(x)\,dx = 2(3) = 6$

10. $\int_0^2 2f(x)\,dx = 2\int_0^2 f(x)\,dx$
$= 2\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)dx\right] = 2(2+3) = 10$

11. $\int_0^2 [2f(x)+g(x)]\,dx = 2\int_0^2 f(x)\,dx + \int_0^2 g(x)\,dx$
$= 2\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx\right] + \int_0^2 g(x)\,dx$
= 2(2 + 3) + 4 = 14

12. $\int_0^1 [2f(s)+g(s)]\,ds = 2\int_0^1 f(s)\,ds + \int_0^1 g(s)\,ds$
= 2(2) + (–1) = 3

13. $\int_2^1 [2f(s)+5g(s)]\,ds = -2\int_1^2 f(s)\,ds - 5\int_1^2 g(s)\,ds$
$= -2(3) - 5\left[\int_0^2 g(s)\,ds - \int_0^1 g(s)\,ds\right]$
= –6 – 5[4 + 1] = –31

14. $\int_1^1 [3f(x)+2g(x)]\,dx = 0$

15. $\int_0^2 [3f(t)+2g(t)]\,dt$
$= 3\left[\int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt\right] + 2\int_0^2 g(t)\,dt$
= 3(2 + 3) + 2(4) = 23

16. $\int_0^2 \left[\sqrt{3}\,f(t) + \sqrt{2}\,g(t) + \pi\right]dt$
$= \sqrt{3}\left[\int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt\right] + \sqrt{2}\int_0^2 g(t)\,dt$
$+\pi\int_0^2 dt$
$= \sqrt{3}(2+3) + \sqrt{2}(4) + 2\pi = 5\sqrt{3} + 4\sqrt{2} + 2\pi$

17. $G'(x) = D_x\left[\int_1^x 2t\,dt\right] = 2x$

18. $G'(x) = D_x\left[\int_x^1 2t\,dt\right] = D_x\left[-\int_1^x 2t\,dt\right] = -2x$

19. $G'(x) = D_x\left[\int_0^x \left(2t^2 + \sqrt{t}\right)dt\right] = 2x^2 + \sqrt{x}$

20. $G'(x) = D_x\left[\int_1^x \cos^3(2t)\tan(t)\,dt\right]$
$= \cos^3(2x)\tan(x)$

21. $G'(x) = D_x\left[\int_x^{\pi/4} (s-2)\cot(2s)ds\right]$
$= D_x\left[-\int_{\pi/4}^x (s-2)\cot(2s)ds\right]$
$= -(x-2)\cot(2x)$

22. $G'(x) = D_x\left[\int_1^x xt\,dt\right] = D_x\left[x\int_1^x t\,dt\right]$
$= D_x\left[x\left[\frac{t^2}{2}\right]_1^x\right] = D_x\left[x\left(\frac{x^2-1}{2}\right)\right]$
$= D_x\left(\frac{x^3}{2} - \frac{x}{2}\right) = \frac{3}{2}x^2 - \frac{1}{2}$

23. $G'(x) = D_x\left[\int_1^{x^2} \sin t\,dt\right] = 2x\sin(x^2)$

24. $G'(x) = D_x\left[\int_1^{x^2+x} \sqrt{2z+\sin z}\,dz\right]$
$= (2x+1)\sqrt{2(x^2+x)+\sin(x^2+x)}$

25.
$G(x) = \int_{-x^2}^x \frac{t^2}{1+t^2}\,dt$
$= \int_{-x^2}^0 \frac{t^2}{1+t^2}\,dt + \int_0^x \frac{t^2}{1+t^2}\,dt$
$= -\int_0^{-x^2} \frac{t^2}{1+t^2}\,dt + \int_0^x \frac{t^2}{1+t^2}\,dt$
$G'(x) = -\frac{\left(-x^2\right)^2}{1+\left(-x^2\right)^2}(-2x) + \frac{x^2}{1+x^2}$
$= \frac{2x^5}{1+x^4} + \frac{x^2}{1+x^2}$

26. $G(x) = D_x\left[\int_{\cos x}^{\sin x} t^5 dt\right]$
$= D_x\left[\int_0^{\sin x} t^5 dt + \int_{\cos x}^0 t^5 dt\right]$
$= D_x\left[\int_0^{\sin x} t^5 dt - \int_0^{\cos x} t^5 dt\right]$
$= \sin^5 x\cos x + \cos^5 x\sin x$

27. $f'(x) = \frac{x}{\sqrt{1+x^2}}$; $f''(x) = \frac{1}{\left(x^2+1\right)^{3/2}}$
So, $f(x)$ is increasing on $[0,\infty)$ and concave up on $(0,\infty)$.

28. $f'(x)=\dfrac{1+x}{1+x^2}$

$f''(x)=\dfrac{(1+x^2)-(1+x)2x}{(x^2+1)^2}=-\dfrac{x^2+2x-1}{(x^2+1)^2}$

So, $f(x)$ is increasing on $[0,\infty)$ and concave up on $\left(0,-1+\sqrt{2}\right)$.

29. $f'(x)=\cos x;\quad f''(x)=-\sin x$

So, $f(x)$ is increasing on $\left[0,\frac{\pi}{2}\right],\left[\frac{3\pi}{2},\frac{5\pi}{2}\right],\ldots$ and concave up on $(\pi,2\pi),(3\pi,4\pi),\ldots$.

30. $f'(x)=x+\sin x;\ f''(x)=1+\cos x$

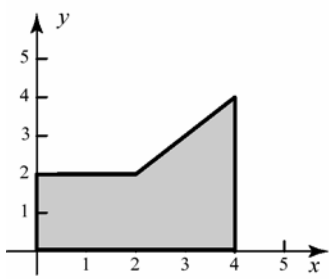
So, $f(x)$ is increasing on $(0,\infty)$ and concave up on $(0,\infty)$.

31. $f'(x)=\dfrac{1}{x};\quad f''(x)=-\dfrac{1}{x^2}$

So, $f(x)$ is increasing on $(0,\infty)$ and never concave up.

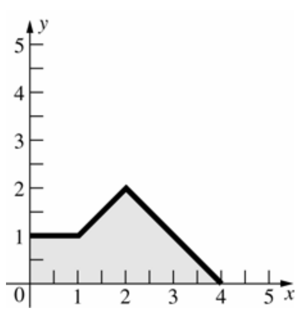
32. $f(x)$ is increasing on $x\ge 0$ and concave up on $(0,1),(2,3),\ldots$

33.



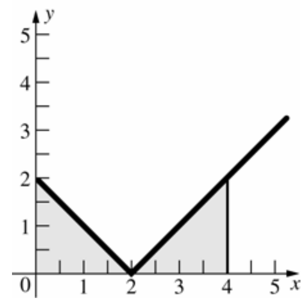
$$\int_0^4 f(x)\,dx=\int_0^2 2\,dx+\int_2^4 x\,dx=4+6=10$$

34.



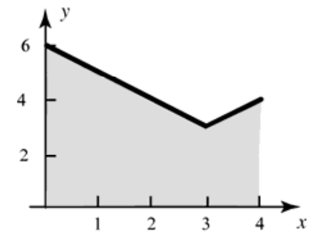
$$\int_0^4 f(x)\,dx=\int_0^1 dx+\int_1^2 x\,dx+\int_2^4 (4-x)\,dx$$
$$=1+1.5+2.0=4.5$$

35.



$$\int_0^4 f(x)\,dx=\int_0^2 (2-x)dx+\int_2^4 (x-2)\,dx$$
$$=2+2=4$$

36.



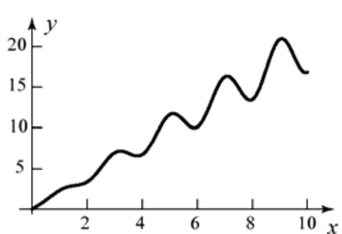
$$\int_0^4\left(3+|x-3|\right)dx$$
$$=\int_0^3\left(3+|x-3|\right)dx+\int_3^4\left(3+|x-3|\right)dx$$
$$=\int_0^3(6-x)\,dx+\int_3^4 x\,dx=\frac{27}{2}+\frac{7}{2}=17$$

37. a. Local minima at 0, ≈ 3.8, ≈ 5.8, ≈ 7.9, ≈ 9.9;
local maxima at ≈ 3.1, ≈ 5, ≈ 7.1, ≈ 9, 10

b. Absolute minimum at 0, absolute maximum at ≈ 9

c. $\approx$(0.7, 1.5), (2.5, 3.5), (4.5, 5.5), (6.5, 7.5), (8.5, 9.5)

d.

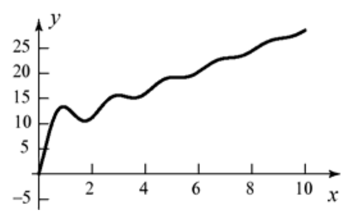


38. a. Local minima at 0, ≈ 1.8, ≈ 3.8, ≈ 5.8;
local maxima at ≈ 1, ≈ 2.9, ≈ 5.2, ≈ 10

b. Absolute minimum at 0, absolute maximum at 10

c. (0.5, 1.5), (2.2, 3.2), (4.2,5.2), (6.2,7.2), (8.2, 9.2)

d.



39. **a.** $F(0)=\int_0^0\left(t^4+1\right)dt=0$

b. $y=F(x)$

$\frac{dy}{dx}=F'(x)=x^4+1$

$dy=\left(x^4+1\right)dx$

$y=\frac{1}{5}x^5+x+C$

c. Now apply the initial condition $y(0)=0$:

$0=\frac{1}{5}0^5+0+C$

$C=0$

Thus $y=F(x)=\frac{1}{5}x^5+x$

d. $\int_0^1\left(x^4+1\right)dx=F(1)=\frac{1}{5}1^5+1=\frac{6}{5}.$

40. **a.** $G(x)=\int_0^x \sin t\,dt$

$G(0)=\int_0^0 \sin t\,dt=0$

$G(2\pi)=\int_0^{2\pi} \sin t\,dt=0$

b. Let $y=G(x)$. Then

$\frac{dy}{dx}=G'(x)=\sin x.$

$dy=\sin x\,dx$

$y=-\cos x+C$

c. Apply the initial condition $0=y(0)=-\cos 0+C$. Thus, $C=1$, and hence $y=G(x)=1-\cos x$.

d. $\int_0^{\pi}\sin x\,dx=G(\pi)=1-\cos\pi=2$

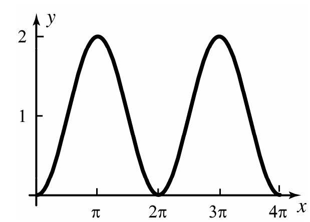
e. G attains the maximum of 2 when $x=\pi,3\pi$.

G attains the minimum of 0 when $x=0,2\pi,4\pi$

Inflection points of G occur at

$x=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$

f.



41. For $t\ge 1$, $\sqrt{t}\le t$. Since $1+x^4\ge 1$ for all x, $1\le\sqrt{1+x^4}\le 1+x^4$.

$\int_0^1 dx\le\int_0^1\sqrt{1+x^4}\,dx\le\int_0^1(1+x^4)\,dx$

By problem 39d, $1\le\int_0^1\sqrt{1+x^4}\,dx\le\frac{6}{5}$

42. On the interval [0,1], $2\le\sqrt{4+x^4}\le 4+x^4$. Thus

$\int_0^1 2\,dx\le\int_0^1\sqrt{4+x^2}\,dx\le\int_0^1\left(4+x^2\right)dx$

$2\le\int_0^1\sqrt{4+x^2}\,dx\le\frac{21}{5}$

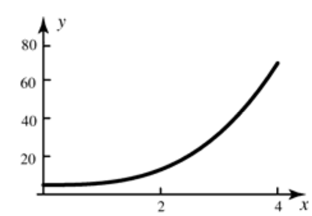
Here, we have used the result from problem 39:

$$\int_0^1\left(4+x^4\right)dx=\int_0^1\left(3+1+x^4\right)dx$$
$$=\int_0^1 3\,dx+\int_0^1\left(1+x^4\right)dx$$
$$=3+\tfrac{6}{5}=\tfrac{21}{5}$$

43. $5\le f(x)\le 69$ so

$4\cdot 5\le\int_0^4\left(5+x^3\right)dx\le 4\cdot 69$

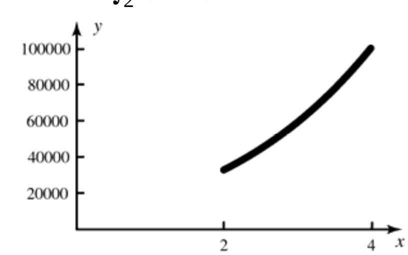
$20\le\int_0^4\left(5+x^3\right)dx\le 276$



44. On [2,4], $8^5\le(x+6)^5\le 10^5$. Thus,

$2\cdot 8^5\le\int_2^4(x+6)^5\,dx\le 2\cdot 10^5$

$65{,}536\le\int_2^4(x+6)^5\,dx\le 200{,}000$

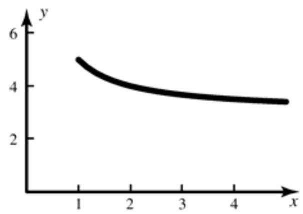


45. On [1,5],

$$3+\frac{2}{5}\le 3+\frac{2}{x}\le 3+\frac{2}{1}$$

$$4\left(\frac{17}{5}\right)\le\int_1^5\left(3+\frac{2}{x}\right)dx\le 4\cdot 5$$

$$\frac{68}{5}\le\int_1^5\left(3+\frac{2}{x}\right)dx\le 20$$



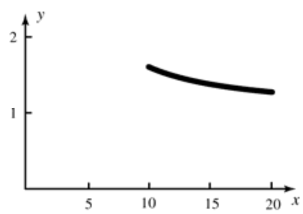
46. On [10, 20],

$$\left(1+\frac{1}{20}\right)^5\le\left(1+\frac{1}{x}\right)^5\le\left(1+\frac{1}{10}\right)^5$$

$$10\left(\frac{21}{20}\right)^5\le\int_{10}^{20}\left(1+\frac{1}{x}\right)^5 dx\le 10\left(\frac{11}{10}\right)^5$$

$$\frac{4,084,101}{320,000}\le\int_{10}^{20}\left(1+\frac{1}{x}\right)^5 dx\le\frac{161,051}{10,000}$$

$$12.7628\le\int_{10}^{20}\left(1+\frac{1}{x}\right)^5 dx\le 16.1051$$

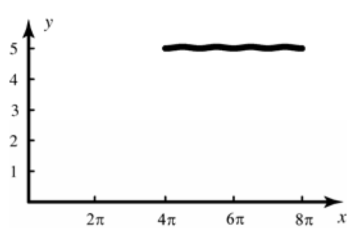


47. On $[4\pi, 8\pi]$

$$5\le 5+\tfrac{1}{20}\sin^2 x\le 5+\tfrac{1}{20}$$

$$(4\pi)(5)\le\int_{4\pi}^{8\pi}\left(5+\tfrac{1}{20}\sin^2 x\right)dx\le(4\pi)\left(5+\tfrac{1}{20}\right)$$

$$20\pi\le\int_{4\pi}^{8\pi}\left(5+\tfrac{1}{20}\sin^2 x\right)dx\le\frac{101}{5}\pi$$



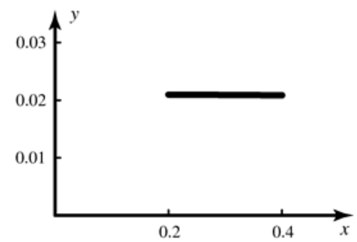
48. On [0.2,0.4],

$$0.002+0.0001\cos^2 0.4\le 0.002+0.0001\cos^2 x$$
$$\le 0.002+0.0001\cos^2 0.2$$

$$0.2\left(0.002+0.0001\cos^2 0.4\right)$$
$$\le\int_{0.2}^{0.4}\left(0.002+0.0001\cos^2 x\right)dx$$
$$\le 0.2\left(0.002+0.0001\cos^2 0.2\right)$$

Thus,

$$0.000417\le\int_{0.2}^{0.4}\left(0.002+0.0001\cos^2 x\right)dx$$
$$\le 0.000419$$



49. Let $F(x)=\int_0^x\frac{1+t}{2+t}dt$. Then

$$\lim_{x\to 0}\frac{1}{x}\int_0^x\frac{1+t}{2+t}dt=\lim_{x\to 0}\frac{F(x)-F(0)}{x-0}$$
$$=F'(0)=\frac{1+0}{2+0}=\frac{1}{2}$$

50. $$\lim_{x\to 1}\frac{1}{x-1}\int_1^x\frac{1+t}{2+t}dt$$
$$=\lim_{x\to 1}\frac{1}{x-1}\left[\int_0^x\frac{1+t}{2+t}dt-\int_0^1\frac{1+t}{2+t}dt\right]$$
$$=\lim_{x\to 1}\frac{F(x)-F(1)}{x-1}$$
$$=F'(1)=\frac{1+1}{2+1}=\frac{2}{3}$$

51. $$\int_1^x f(t)\,dt=2x-2$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\int_1^x f(t)\,dt=\frac{d}{dx}(2x-2)$$
$$f(x)=2$$

If such a function exists, it must satisfy $f(x)=2$, but both sides of the first equality may differ by a constant yet still have equal derivatives. When $x=1$ the left side is $\int_1^1 f(t)\,dt=0$ and the right side is $2\cdot 1-2=0$. Thus the function $f(x)=2$ satisfies

$$\int_1^x f(t)\,dt=2x-2.$$

52. $\int_0^x f(t)\,dt = x^2$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}x^2$$
$$f(x) = 2x$$

53. $\int_0^{x^2} f(t)\,dt = \frac{1}{3}x^3$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\int_0^{x^2} f(t)\,dt = \frac{d}{dx}\left(\frac{1}{3}x^3\right)$$
$$f\left(x^2\right)(2x) = x^2$$
$$f\left(x^2\right) = \frac{x}{2}$$
$$f(x) = \frac{\sqrt{x}}{2}$$

54. No such function exists. When $x = 0$ the left side is 0, whereas the right side is 1

55. True; by Theorem B (Comparison Property)

56. False. $a = -1$, $b = 2$, $f(x) = x$ is a counterexample.

57. False. $a = -1$, $b = 1$, $f(x) = x$ is a counterexample.

58. False; A counterexample is $f(x) = 0$ for all x, except $f(1) = 1$. Thus, $\int_0^2 f(x)\,dx = 0$, but f is not identically zero.

59. True. $\int_a^b f(x)dx - \int_a^b g(x)dx$
$= \int_a^b [f(x) - g(x)]dx$

60. False. $a = 0$, $b = 1$, $f(x) = 0$, $g(x) = -1$ is a counterexample.

61. $$v(t) = \begin{cases} 2 + (t-2), & t \le 2 \\ 2 - (t-2), & t > 2 \end{cases}$$
$$= \begin{cases} t, & t \le 2 \\ 4 - t, & t > 2 \end{cases}$$
$$s(t) = \int_0^t v(u)\,du$$
$$= \begin{cases} \int_0^t u\,du, & 0 \le t \le 2 \\ \int_0^2 u\,du + \int_2^t (4-u)\,du, & t > 2 \end{cases}$$
$$= \begin{cases} \frac{t^2}{2}, & 0 \le t \le 2 \\ 2 + \left[4t - \frac{t^2}{2}\right], & t > 2 \end{cases}$$
$$= \begin{cases} \frac{t^2}{2}, & 0 \le t \le 2 \\ -4 + 4t - \frac{t^2}{2} & t > 2 \end{cases}$$
$$\frac{t^2}{2} - 4t + 4 = 0;\ t = 4 + 2\sqrt{2} \approx 6.83$$

62. **a.** $$s(t) = \begin{cases} \int_0^t 5\,du, & 0 \le t \le 100 \\ \int_0^{100} 5\,du + \int_{100}^t \left(6 - \frac{u}{100}\right)du & 100 < t \le 700 \\ \int_0^{100} 5\,du + \int_{100}^{700}\left(6 - \frac{u}{100}\right)du + \int_{700}^t (-1)\,du, & t > 700 \end{cases}$$
$$= \begin{cases} 5t, & 0 \le t \le 100 \\ 500 + \left[6u - \frac{u^2}{200}\right]_{100}^t & 100 < t \le 700 \\ 500 + \left[6u - \frac{u^2}{200}\right]_{100}^{700} - (t - 700) & t > 700 \end{cases}$$
$$= \begin{cases} 5t, & 0 \le t \le 100 \\ -50 + 6t - \frac{t^2}{200}, & 100 < t \le 700 \\ 2400 - t, & t > 700 \end{cases}$$

b. $v(t) > 0$ for $0 \le t < 600$ and $v(t) < 0$ for $t > 600$. So, $t = 600$ is the point at which the object is farthest to the right of the origin. At $t = 600$, $s(t) = 1750$.

c. $s(t) = 0 = 2400 - t;\ t = 2400$

63. $-|f(x)| \le f(x) \le |f(x)|$, so

$$\int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \Rightarrow$$

$$\int_a^b |f(x)|\,dx \ge -\int_a^b f(x)\,dx$$

and combining this with

$$\int_a^b |f(x)|\,dx \ge \int_a^b f(x)\,dx,$$

we can conclude that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

64. If $x > a$, $\int_a^x |f'(x)|\,dx \le M(x-a)$ by the Boundedness Property. If $x < a$,

$\int_x^a |f(x)|\,dx = -\int_a^x |f'(x)|\,dx \ge -M(x-a)$ by the Boundedness Property. Thus

$\int_a^x |f'(x)|\,dx \le M|x-a|$.

From Problem 63, $\int_a^x |f'(x)|\,dx \ge \left|\int_a^x f'(x)\,dx\right|$.

$$\left|\int_a^x f'(x)\,dx\right| = |f(x) - f(a)| \ge |f(x)| - |f(a)|$$

Therefore, $|f(x)| - |f(a)| \le M|x-a|$ or $|f(x)| \le |f(a)| + M|x-a|$.

4.4 Concepts Review

1. antiderivative; $F(b) - F(a)$

2. $F(b) - F(a)$

3. $F(d) - F(c)$

4. $\int_1^2 \frac{1}{3}u^4\,du$

Problem Set 4.4

1. $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4 - 0 = 4$

2. $\int_{-1}^2 x^4\,dx = \left[\frac{x^5}{5}\right]_{-1}^2 = \frac{32}{5} + \frac{1}{5} = \frac{33}{5}$

3. $\int_{-1}^2 (3x^2 - 2x + 3)\,dx = \left[x^3 - x^2 + 3x\right]_{-1}^2$
$= (8 - 4 + 6) - (-1 - 1 - 3) = 15$

4. $\int_1^2 (4x^3 + 7)\,dx = \left[x^4 + 7x\right]_1^2$
$= (16 + 14) - (1 + 7) = 22$

5. $\int_1^4 \frac{1}{w^2}\,dw = \left[-\frac{1}{w}\right]_1^4 = \left(-\frac{1}{4}\right) - (-1) = \frac{3}{4}$

6. $\int_1^3 \frac{2}{t^3}\,dt = \left[-\frac{1}{t^2}\right]_1^3 = \left(-\frac{1}{9}\right) - (-1) = \frac{8}{9}$

7. $\int_0^4 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^4 = \left(\frac{2}{3}\cdot 8\right) - 0 = \frac{16}{3}$

8. $\int_1^8 \sqrt[3]{w}\,dw = \left[\frac{3}{4}w^{4/3}\right]_1^8 = \left(\frac{3}{4}\cdot 16\right) - \left(\frac{3}{4}\cdot 1\right) = \frac{45}{4}$

9. $\int_{-4}^{-2}\left(y^2 + \frac{1}{y^3}\right)dy = \left[\frac{y^3}{3} - \frac{1}{2y^2}\right]_{-4}^{-2}$
$= \left(-\frac{8}{3} - \frac{1}{8}\right) - \left(-\frac{64}{3} - \frac{1}{32}\right) = \frac{1783}{96}$

10. $\int_1^4 \frac{s^4 - 8}{s^2}\,ds = \int_1^4 (s^2 - 8s^{-2})\,ds = \left[\frac{s^3}{3} + \frac{8}{s}\right]_1^4$
$= \left(\frac{64}{3} + 2\right) - \left(\frac{1}{3} + 8\right) = 15$

11. $\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$

12. $\int_{\pi/6}^{\pi/2} 2\sin t\,dt = [-2\cos t]_{\pi/6}^{\pi/2} = 0 + \sqrt{3} = \sqrt{3}$

13. $\int_0^1 (2x^4 - 3x^2 + 5)\,dx = \left[\frac{2}{5}x^5 - x^3 + 5x\right]_0^1$
$= \left(\frac{2}{5} - 1 + 5\right) - 0 = \frac{22}{5}$

14. $\int_0^1 (x^{4/3} - 2x^{1/3})\,dx = \left[\frac{3}{7}x^{7/3} - \frac{3}{2}x^{4/3}\right]_0^1$
$= \left(\frac{3}{7} - \frac{3}{2}\right) - 0 = -\frac{15}{14}$

15. $u = 3x + 2,\ du = 3\,dx$

$$\int \sqrt{u} \cdot \frac{1}{3}du = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(3x+2)^{3/2} + C$$

16. $u = 2x - 4,\ du = 2\,dx$

$$\int u^{1/3} \cdot \frac{1}{2}du = \frac{3}{8}u^{4/3} + C = \frac{3}{8}(2x-4)^{4/3} + C$$

17. $u = 3x + 2,\ du = 3\,dx$

$$\int \cos(u) \cdot \frac{1}{3}du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3x+2) + C$$

18. $u = 2x - 4,\ du = 2\,dx$

$$\int \sin u \cdot \frac{1}{2}du = -\frac{1}{2}\cos u + C$$

$$= -\frac{1}{2}\cos(2x-4) + C$$

19. $u = 6x - 7,\ du = 6dx$

$$\int \sin u \cdot \frac{1}{6}du = -\frac{1}{6}\cos u + C$$

$$= -\frac{1}{6}\cos(6x-7) + C$$

20. $u = \pi v - \sqrt{7},\ du = \pi\,dv$

$$\int \cos u \cdot \frac{1}{\pi}du = \frac{1}{\pi}\sin u + C = \frac{1}{\pi}\sin(\pi v - \sqrt{7}) + C$$

21. $u = x^2 + 4,\ du = 2x\,dx$

$$\int \sqrt{u} \cdot \frac{1}{2}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+4)^{3/2} + C$$

22. $u = x^3 + 5,\ du = 3x^2\,dx$

$$\int u^9 \cdot \frac{1}{3}du = \frac{1}{30}u^{10} + C = \frac{1}{30}(x^3+5)^{10} + C$$

23. $u = x^2 + 3,\ du = 2x\,dx$

$$\int u^{-12/7} \cdot \frac{1}{2}du = -\frac{7}{10}u^{-5/7} + C$$

$$= -\frac{7}{10}(x^2+3)^{-5/7} + C$$

24. $u = \sqrt{3}v^2 + \pi,\ du = 2\sqrt{3}v\,dv$

$$\int u^{7/8} \cdot \frac{1}{2\sqrt{3}}du = \frac{4}{15\sqrt{3}}u^{15/8} + C$$

$$= \frac{4}{15\sqrt{3}}\left(\sqrt{3}v^2 + \pi\right)^{15/8} + C$$

25. $u = x^2 + 4,\ du = 2x\,dx$

$$\int \sin(u) \cdot \frac{1}{2}du = -\frac{1}{2}\cos u + C$$

$$= -\frac{1}{2}\cos(x^2+4) + C$$

26. $u = x^3 + 5,\ du = 3x^2\,dx$

$$\int \cos u \cdot \frac{1}{3}du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(x^3+5) + C$$

27. $u = \sqrt{x^2+4},\ du = \dfrac{x}{\sqrt{x^2+4}}dx$

$$\int \sin u\,du = -\cos u + C = -\cos\sqrt{x^2+4} + C$$

28. $u = \sqrt[3]{z^2+3},\ du = \dfrac{2z}{3\left(\sqrt[3]{z^2+3}\right)^2}dz$

$$\int \cos u \cdot \frac{3}{2}du = \frac{3}{2}\sin u + C = \frac{3}{2}\sin\sqrt[3]{z^2+3} + C$$

29. $u = (x^3+5)^9$,

$du = 9(x^3+5)^8(3x^2)dx = 27x^2(x^3+5)^8\,dx$

$$\int \cos u \cdot \frac{1}{27}du = \frac{1}{27}\sin u + C$$

$$= \frac{1}{27}\sin\left[(x^3+5)^9\right] + C$$

30. $u = (7x^7+\pi)^9,\ du = 441x^6(7x^7+\pi)^8\,dx$

$$\int \sin u \cdot \frac{1}{441}du = -\frac{1}{441}\cos u + C$$

$$= -\frac{1}{441}\cos(7x^7+\pi)^9 + C$$

31. $u = \sin(x^2+4),\ du = 2x\cos(x^2+4)\,dx$

$$\int \sqrt{u} \cdot \frac{1}{2}du = \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}\left[\sin(x^2+4)\right]^{3/2} + C$$

32. $u = \cos(3x^7+9)$

$du = -21x^6\sin(3x^7+9)\,dx$

$$\int \sqrt[3]{u} \cdot \left(-\frac{1}{21}\right)du = -\frac{1}{28}u^{4/3} + C$$

$$= -\frac{1}{28}\left[\cos(3x^7+9)\right]^{4/3} + C$$

33. $u = \cos(x^3+5),\ du = -3x^2\sin(x^3+5)\,dx$

$$\int u^9 \cdot \left(-\frac{1}{3}\right) du = -\frac{1}{30}u^{10} + C$$

$$= -\frac{1}{30}\cos^{10}(x^3+5) + C$$

34. $u = \tan(x^{-3}+1)$, $du = -3x^{-4}\sec^2(x^{-3}+1)\,dx$

$$\int \sqrt[5]{u} \cdot \left(-\frac{1}{3}\right) du = -\frac{5}{18}u^{6/5} + C$$

$$= -\frac{5}{18}\left[\tan(x^{-3}+1)\right]^{6/5} + C$$

35. $u = x^2+1,\ du = 2x\,dx$

$$\int_0^1 (x^2+1)^{10}(2x)dx = \int_1^2 u^{10}du = \left[\frac{u^{11}}{11}\right]_1^2$$

$$= \left[\frac{1}{11}(2)^{11}\right] - \left[\frac{1}{11}(1)^{11}\right] = \frac{2047}{11}$$

36. $u = x^3+1,\ du = 3x^2\,dx$

$$\int_{-1}^0 \sqrt{x^3+1}\,(3x^2)\,dx = \int_0^1 \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^1$$

$$= \left(\frac{2}{3}\cdot 1^{3/2}\right) - \left(\frac{2}{3}\cdot 0\right) = \frac{2}{3}$$

37. $u = t + 2,\ du = dt$

$$\int_{-1}^3 \frac{1}{(t+2)^2}\,dt = \int_1^5 u^{-2}du = \left[-\frac{1}{u}\right]_1^5$$

$$= \left[-\frac{1}{5}\right] - [-1] = \frac{4}{5}$$

38. $u = y - 1,\ du = dy$

$$\int_2^{10} \sqrt{y-1}\,dy = \int_1^9 \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^9$$

$$= \left[\frac{2}{3}(27)\right] - \left[\frac{2}{3}(1)\right] = \frac{52}{3}$$

39. $u = 3x + 1,\ du = 3\,dx$

$$\int_5^8 \sqrt{3x+1}\,dx = \frac{1}{3}\int_5^8 \sqrt{3x+1}\cdot 3dx = \frac{1}{3}\int_{16}^{25}\sqrt{u}\,du$$

$$= \left[\frac{2}{9}u^{3/2}\right]_{16}^{25} = \left[\frac{2}{9}(125)\right] - \left[\frac{2}{9}(64)\right] = \frac{122}{9}$$

40. $u = 2x + 2,\ du = 2\,dx$

$$\int_1^7 \frac{1}{\sqrt{2x+2}}\,dx = \frac{1}{2}\int_1^7 \frac{2}{\sqrt{2x+2}}\,dx$$

$$= \frac{1}{2}\int_4^{16} u^{-1/2}du = \left[\sqrt{u}\right]_4^{16} = 4 - 2 = 2$$

41. $u = 7 + 2t^2,\ du = 4t\,dt$

$$\int_{-3}^3 \sqrt{7+2t^2}\,(8t)\,dt = 2\int_{-3}^3 \sqrt{7+2t^2}\cdot(4t)\,dt$$

$$= 2\int_{25}^{25}\sqrt{u}\,du = \left[\frac{4}{3}u^{3/2}\right]_{25}^{25}$$

$$= \left[\frac{4}{3}(125)\right] - \left[\frac{4}{3}(125)\right] = 0$$

42. $u = x^3 + 3x,\ du = (3x^2+3)\,dx$

$$\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}}\,dx = \frac{1}{3}\int_1^3 \frac{3x^2+3}{\sqrt{x^3+3x}}\,dx$$

$$= \frac{1}{3}\int_4^{16} u^{-1/2}du = \left[\frac{2}{3}u^{1/2}\right]_4^{36}$$

$$= \left(\frac{2}{3}\cdot 6\right) - \left(\frac{2}{3}\cdot 2\right) = \frac{8}{3}$$

43. $u = \cos x,\ du = -\sin x\,dx$

$$\int_0^{\pi/2} \cos^2 x\sin x\,dx = -\int_0^{\pi/2}\cos^2 x(-\sin x)\,dx$$

$$= -\int_1^0 u^2du = \left[-\frac{u^3}{3}\right]_1^0$$

$$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$$

44. $u = \sin 3x,\ du = 3\cos 3x\,dx$

$$\int_0^{\pi/2} \sin^2 3x\cos 3x\,dx$$

$$= \frac{1}{3}\int_0^{\pi/2}\sin^2 3x(3\cos 3x)\,dx = \frac{1}{3}\int_0^{-1}u^2du$$

$$= \left[\frac{u^3}{9}\right]_0^{-1} = \left(-\frac{1}{9}\right) - 0 = -\frac{1}{9}$$

45. $u = x^2 + 2x,\ du = (2x+2)\,dx = 2(x+1)\,dx$

$$\int_0^1 (x+1)(x^2+2x)^2\,dx$$

$$= \int_0^1 \frac{1}{2}(x^2+2x)^2\,2(x+1)\,dx$$

$$= \frac{1}{2}\int_0^3 u^2\,du = \left[\frac{u^3}{6}\right]_0^3 = \frac{9}{2}$$

46. $u = \sqrt{x} - 1,\ du = \frac{1}{2\sqrt{x}}\,dx$

$$\int_1^4 \frac{(\sqrt{x}-1)^3}{\sqrt{x}}\,dx = 2\int_1^4 \frac{(\sqrt{x}-1)^3}{2\sqrt{x}}\,dx$$

$$= 2\int_0^1 u^3\,du = 2\left[\frac{u^4}{4}\right]_0^1 = \frac{1}{2}$$

47. $u = \sin\theta, du = \cos\theta\, d\theta$

$$\int_0^{1/2} u^3 du = \left[\frac{u^4}{4}\right]_0^{1/2} = \frac{1}{64} - 0 = \frac{1}{64}$$

48. $u = \cos\theta, du = -\sin\theta\, d\theta$

$$-\int_1^{\sqrt{3}/2} u^{-3} du = \frac{1}{2}\left[u^{-2}\right]_1^{\sqrt{3}/2} = \frac{1}{2}\left(\frac{4}{3} - 1\right) = \frac{1}{6}$$

49. $u = 3x - 3, du = 3dx$

$$\frac{1}{3}\int_{-3}^{0} \cos u\, du = \frac{1}{3}[\sin u]_{-3}^{0} = \frac{1}{3}(0 - \sin(-3))$$

$$= \frac{\sin 3}{3}$$

50. $u = 2\pi x, du = 2\pi dx$

$$\frac{1}{2\pi}\int_0^{\pi} \sin u\, du = -\frac{1}{2\pi}[\cos u]_0^{\pi} = -\frac{1}{2\pi}(-1-1)$$

$$= \frac{1}{\pi}$$

51. $u = \pi x^2, du = 2\pi x\, dx$

$$\frac{1}{2\pi}\int_0^{\pi} \sin u\, du = -\frac{1}{2\pi}[\cos u]_0^{\pi} = -\frac{1}{2\pi}(-1-1)$$

$$= \frac{1}{\pi}$$

52. $u = 2x^5, du = 10x^4 dx$

$$\frac{1}{10}\int_0^{2\pi^5} \cos u\, du = \frac{1}{10}[\sin u]_0^{2\pi^5}$$

$$= \frac{1}{10}(\sin(2\pi^5) - 0) = \frac{1}{10}\sin(2\pi^5)$$

53. $u = 2x, du = 2dx$

$$\frac{1}{2}\int_0^{\pi/2} \cos u\, du + \frac{1}{2}\int_0^{\pi/2} \sin u\, du$$

$$= \frac{1}{2}[\sin u]_0^{\pi/2} - \frac{1}{2}[\cos u]_0^{\pi/2}$$

$$= \frac{1}{2}(1-0) - \frac{1}{2}(0-1) = 1$$

54. $u = 3x, du = 3dx; v = 5x, dv = 5dx$

$$\frac{1}{3}\int_{-3\pi/2}^{3\pi/2} \cos u\, du + \frac{1}{5}\int_{-5\pi/2}^{5\pi/2} \sin v\, dv$$

$$= \frac{1}{3}[\sin u]_{-3\pi/2}^{3\pi/2} - \frac{1}{5}[\cos v]_{-5\pi/2}^{5\pi/2}$$

$$= \frac{1}{3}[(-1) - 1] - \frac{1}{5}[0 - 0] = -\frac{2}{3}$$

55. $u = \cos x, du = -\sin x\, dx$

$$-\int_1^0 \sin u\, du = [\cos u]_1^0 = 1 - \cos 1$$

56. $u = \pi\sin\theta, du = \pi\cos\theta\, d\theta$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos u\, du = \frac{1}{\pi}[\sin u]_{-\pi}^{\pi} = 0$$

57. $u = \cos(x^2),\ du = -2x\sin(x^2)dx$

$$-\frac{1}{2}\int_1^{\cos 1} u^3 du = -\frac{1}{2}\left[\frac{u^4}{4}\right]_1^{\cos 1} = -\frac{\cos^4 1}{8} + \frac{1}{8}$$

$$= \frac{1 - \cos^4 1}{8}$$

58. $u = \sin(x^3), du = 3x^2\cos(x^3)dx$

$$\frac{1}{3}\int_{-\sin(\pi^3/8)}^{\sin(\pi^3/8)} u^2\, du = \frac{1}{9}\left[u^3\right]_{-\sin(\pi^3/8)}^{\sin(\pi^3/8)}$$

$$= \frac{2\sin^3\left(\frac{\pi^3}{8}\right)}{9}$$

59. **a.** Between 0 and 3, $f(x) > 0$. Thus, $\int_0^3 f(x)\, dx > 0$.

b. Since f is an antiderivative of f',

$$\int_0^3 f'(x)\, dx = f(3) - f(0)$$
$$= 0 - 2 = -2 < 0$$

c. $$\int_0^3 f''(x)\, dx = f'(3) - f'(0)$$
$$= -1 - 0 = -1 < 0$$

d. Since f is concave down at 0, $f''(0) < 0$.

$$\int_0^3 f'''(x)\, dx = f''(3) - f''(0)$$
$$= 0 - (\text{negative number}) > 0$$

60. **a.** On $[0, 4]$, $f(x) > 0$. Thus, $\int_0^4 f(x)\, dx > 0$.

b. Since f is an antiderivative of f',

$$\int_0^4 f'(x)\, dx = f(4) - f(0)$$
$$= 1 - 2 = -1 < 0$$

c. $$\int_0^4 f''(x)\, dx = f'(4) - f'(0)$$
$$= \frac{1}{4} - (-2) = \frac{9}{4} > 0$$

d. $$\int_0^4 f'''(x)\, dx = f''(4) - f''(0)$$
$$= (\text{negative}) - (\text{positive}) < 0$$

61. $V(t)=\int V'(t)=\int (20-t)\,dt=20t-\frac{1}{2}t^2+C$

$V(0)=C=0$ since no water has leaked out at time $t=0$. Thus, $V(t)=20t-\frac{1}{2}t^2$, so

$V(20)-V(10)=200-150=50$ gallons.

Time to drain: $20t-\frac{1}{2}t^2=200;\ t=20$ hours.

62. $V(1)-V(0)=\int_0^1 V'(t)\,dt=\left[t-\frac{t^2}{220}\right]_0^1=\frac{219}{220}$

$V(10)-V(9)=\int_9^{10}\left(1-\frac{t}{110}\right)dt=\frac{201}{220}$

$55=V(T)-V(0)=\int_0^T\left(1-\frac{t}{110}\right)dt=T-\frac{T^2}{220}$

$T\approx 110$ hrs

63. Use a midpoint Riemann sum with $n=12$ partitions.

$$V=\sum_{i=1}^{12} f(x_i)\Delta x_i$$

$$\approx 1(5.4+6.3+6.4+6.5+6.9+7.5+8.4 +8.4+8.0+7.5+7.0+6.5)$$

$$=84.8$$

64. Use a midpoint Riemann sum with $n=10$ partitions.

$$V=\sum_{i=1}^{10} f(x_i)\Delta x_i$$

$$\approx 1\begin{pmatrix}6200+6300+6500+6500+6600\\ +6700+6800+7000+7200+7200\end{pmatrix}$$

$$=67{,}000$$

65. Use a midpoint Riemann sum with $n=12$ partitions.

$$E=\sum_{i=0}^{12} P(t_i)\Delta t_i$$

$$\approx 2(3.0+3.0+3.8+5.8+7.8+6.9 +6.5+6.3+7.2+8.2+8.7+5.4)$$

$$=145.2$$

66. $\delta(x)=m'(x)=1+\frac{x}{4}$

mass $=\int_0^2\delta(x)\,dx=m(2)=\frac{5}{2}$

67. **a.** $\int_a^b x^n dx=B_n;\ \int_{a^n}^{b^n}\sqrt[n]{y}\,dy=A_n$

Using Figure 3 of the text,

$(a)(a^n)+A_n+B_n=(b)(b^n)$ or

$B_n+A_n=b^{n+1}-a^{n+1}$. Thus

$\int_a^b x^n dx+\int_{a^n}^{b^n}\sqrt[n]{y}\,dy=b^{n+1}-a^{n+1}$

b. $\int_a^b x^n\,dx+\int_{a^n}^{b^n}\sqrt[n]{y}\,dy$

$$=\left[\frac{x^{n+1}}{n+1}\right]_a^b+\left[\frac{n}{n+1}y^{(n+1)/n}\right]_{a^n}^{b^n}$$

$$=\left(\frac{b^{n+1}}{n+1}-\frac{a^{n+1}}{n+1}\right)+\left(\frac{n}{n+1}b^{n+1}-\frac{n}{n+1}a^{n+1}\right)$$

$$=\frac{(n+1)b^{n+1}-(n+1)a^{n+1}}{n+1}=b^{n+1}-a^{n+1}$$

c. $B_n=\int_a^b x^n\,dx=\frac{1}{n+1}\left[x^{n+1}\right]_a^b$

$$=\frac{1}{n+1}(b^{n+1}-a^{n+1})$$

$$A_n=\int_{a^n}^{b^n}\sqrt[n]{y}\,dy=\left[\frac{n}{n+1}y^{(n+1)/n}\right]_{a^n}^{b^n}$$

$$=\frac{n}{n+1}\left(b^{n+1}-a^{n+1}\right)$$

$$nB_n=\frac{n}{n+1}(b^{n+1}-a^{n+1})=A_n$$

68. Let $y=G(x)=\int_a^x f(t)\,dt$. Then

$\frac{dy}{dx}=G'(x)=f(x)$

$dy=f(x)\,dx$

Let F be any antiderivative of f. Then $G(x)=F(x)+C$. When $x=a$, we must have $G(a)=0$. Thus, $C=-F(a)$ and $G(x)=F(x)-F(a)$. Now choose $x=b$ to obtain

$\int_a^b f(t)\,dt=G(b)=F(b)-F(a)$

69. $\int_0^3 x^2dx=\left[\frac{x^3}{3}\right]_0^3=9-0=9$

70. $\int_0^2 x^3\,dx=\left[\frac{x^4}{4}\right]_0^2=4-0=4$

71. $\int_0^{\pi} \sin x\, dx = [-\cos x]_0^{\pi} = 1+1 = 2$

72. $\int_0^2 (1+x+x^2)\,dx = \left[x+\frac{1}{2}x^2+\frac{1}{3}x^3\right]_0^2$

$= \left(2+2+\frac{8}{3}\right) - 0 = \frac{20}{3}$

73. The right-endpoint Riemann sum is

$\sum_{i=1}^{n}\left(0+\frac{1-0}{n}i\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n} i^2$, which for

$n = 10$ equals $\frac{77}{200} = 0.385$.

$\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3} = 0.\overline{333}$

74 $\int_{-2}^{4}\left(2[\![x]\!]-3|x|\right)dx = 2\int_{-2}^{4}[\![x]\!]\,dx - 3\int_{-2}^{4}|x|\,dx$

$= 2[(-2-1+0+1+2+3)(1)]$

$-3\left[\frac{1}{2}(2)(2)+\frac{1}{2}(4)(4)\right]$

$= -24$

75. $\frac{d}{dx}\left(\frac{1}{2}x|x|\right) = \frac{1}{2}x\left(\frac{|x|}{x}\right)+\frac{|x|}{2} = |x|$

$\int_a^b |x|\,dx = \left[\frac{1}{2}x|x|\right]_a^b = \frac{1}{2}\left(b|b|-a|a|\right)$

76. For $b > 0$, if b is an integer,

$\int_0^b [\![x]\!]\,dx = 0+1+2+\cdots+(b-1)$

$= \sum_{i=1}^{b-1} i = \frac{(b-1)b}{2}.$

If b is not an integer, let $n = [\![b]\!]$. Then

$\int_0^b [\![x]\!]\,dx = 0+1+2+\cdots+(n-1)+n(b-n)$

$= \frac{(n-1)n}{2}+n(b-n)$

$= \frac{([\![b]\!]-1)[\![b]\!]}{2}+[\![b]\!](b-[\![b]\!])$.

77. **a.** Let c be in (a,b). Then $G'(c) = f(c)$ by the First Fundamental Theorem of Calculus. Since G is differentiable at c, G is continuous there. Now suppose $c = a$.

Then $\lim_{x\to c} G(x) = \lim_{x\to a}\int_a^x f(t)\,dt$. Since f is continuous on $[a,b]$, there exist (by the Min-Max Existence Theorem) m and M such that $f(m) \le f(x) \le f(M)$ for all x in $[a,b]$.

Then

$\int_a^x f(m)\,dt \le \int_a^x f(t)\,dt \le \int_a^x f(M)\,dt$

$(x-a)f(m) \le G(x) \le (x-a)f(M)$

By the Squeeze Theorem

$\lim_{x\to a^+}(x-a)f(m) \le \lim_{x\to a^+} G(x)$

$\le \lim_{x\to a^+}(x-a)f(M)$

Thus,

$\lim_{x\to a^+} G(x) = 0 = \int_a^a f(t)\,dt = G(a)$

Therefore G is right-continuous at $x = a$.

Now, suppose $c = b$. Then

$\lim_{x\to b^-} G(x) = \lim_{x\to b^-}\int_x^b f(t)\,dt$

As before,

$(b-x)f(m) \le G(x) \le (b-x)f(M)$ so we can apply the Squeeze Theorem again to obtain

$\lim_{x\to b^-}(b-x)f(m) \le \lim_{x\to b^-} G(x)$

$\le \lim_{x\to b^-}(b-x)f(M)$

Thus

$\lim_{x\to b^-} G(x) = 0 = \int_b^b f(t)\,dt = G(b)$

Therefore, G is left-continuous at $x = b$.

b. Let F be any antiderivative of f. Note that G is also an antiderivative of f. Thus, $F(x) = G(x)+C$. We know from part (a) that $G(x)$ is continuous on $[a,b]$. Thus $F(x)$, being equal to $G(x)$ plus a constant, is also continuous on $[a,b]$.

78. Let $f(x)=\begin{cases}1, & x>0\\ 0, & x\le 0\end{cases}$ and $F(x)=\int_{-1}^{x} f(t)\,dt$.

If $x<0$, then $F(x)=0$. If $x\ge 0$, then

$$F(x)=\int_{-1}^{x} f(t)\,dt$$
$$=\int_{-1}^{0} 0\,dt+\int_{0}^{x} 1\,dt$$
$$=0+x=x$$

Thus,

$$F(x)=\begin{cases}x, & x\ge 0\\ 0, & x<0\end{cases}$$

which is continuous everywhere even though $f(x)$ is not continuous everywhere.

4.5 Concepts Review

1. $\frac{1}{b-a}\int_a^b f(x)\,dx$

2. $f(c)$

3. 0; $2\int_0^2 f(x)\,dx$

4. $f(x+p)=f(x)$; period

Problem Set 4.5

1. $\frac{1}{3-1}\int_1^3 4x^3 dx=\frac{1}{2}\left[x^4\right]_1^3=40$

2. $\frac{1}{4-1}\int_1^4 5x^2 dx=\frac{1}{3}\left[\frac{5}{3}x^3\right]_1^4=35$

3. $\frac{1}{3-0}\int_0^3 \frac{x}{\sqrt{x^2+16}}dx=\frac{1}{3}\left[\sqrt{x^2+16}\right]_0^3=\frac{1}{3}$

4. $\frac{1}{2-0}\int_0^2 \frac{x^2}{\sqrt{x^3+16}}dx=\frac{1}{2}\left[\frac{2}{3}\sqrt{x^3+16}\right]_0^2$

$=\frac{1}{3}\left(\sqrt{24}-4\right)=\frac{2}{3}\left(\sqrt{6}-2\right)$

5. $\frac{1}{1+2}\int_{-2}^{1}\left(2+|x|\right)dx$

$=\frac{1}{3}\left[\int_{-2}^{0}(2-x)\,dx+\int_0^1(2+x)\,dx\right]$

$=\frac{1}{3}\left\{\left[2x-\tfrac{1}{2}x^2\right]_{-2}^{0}+\left[2x+\tfrac{1}{2}x^2\right]_0^1\right\}$

$=\frac{1}{3}\left(-2(-2)+\tfrac{1}{2}(-2)^2+2+\tfrac{1}{2}\right)=\frac{17}{6}$

6. $\frac{1}{2+3}\int_{-3}^{2}\left(x+|x|\right)dx$

$=\frac{1}{5}\left(\int_{-3}^{0}(-x+x)\,dx+\int_0^2 2x\,dx\right)$

$=\frac{1}{5}\left[x^2\right]_0^2=\frac{4}{5}$

7. $\frac{1}{\pi}\int_0^{\pi}\cos x\,dx=\frac{1}{\pi}\left[\sin x\right]_0^{\pi}$

$=\frac{1}{\pi}\left[\sin\pi-\sin 0\right]=0$

8. $\frac{1}{\pi-0}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}\left(-\cos x\right)_0^{\pi}$

$=-\frac{1}{\pi}(-1-1)=\frac{2}{\pi}$

9. $\frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} x\cos x^2\,dx=\frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\sin x^2\right)_0^{\sqrt{\pi}}$

$=\frac{1}{\sqrt{\pi}}(0-0)=0$

10. $\frac{1}{\pi/2-0}\int_0^{\pi/2}\sin^2 x\cos x\,dx$

$=\frac{2}{\pi}\left[\frac{1}{3}\sin^3 x\right]_0^{\pi/2}=\frac{2}{3\pi}$

11. $\frac{1}{2-1}\int_1^2 y\left(1+y^2\right)^3 dy=\left[\frac{1}{8}\left(1+y^2\right)^4\right]_1^2$

$=\frac{625}{8}-2=\frac{609}{8}=76.125$

12. $\frac{1}{\pi/4-1}\int_0^{\pi/4}\tan x\sec^2 x=\frac{1}{\pi/4-1}\left[\frac{1}{2}\tan^2 x\right]_0^{\pi/4}$

$=\frac{2}{\pi-4}(1-0)=\frac{2}{\pi-4}$

13. $\frac{1}{\pi/4}\int_{\pi/4}^{\pi/2}\frac{\sin\sqrt{z}}{\sqrt{z}}dz=\frac{4}{\pi}\left[-2\cos\sqrt{z}\right]_{\pi/4}^{\pi/2}$

$=\frac{8}{\pi}\left(\cos\sqrt{\pi/4}-\cos\sqrt{\pi/2}\right)\approx 0.815$

14. $\frac{1}{\pi/2}\int_0^{\pi/2}\frac{\sin v\,\cos v}{\sqrt{1+\cos^2 v}}dv$

$=\frac{2}{\pi}\left[-\sqrt{1+\cos^2 v}\right]_0^{\pi/2}$

$=\frac{2}{\pi}\left(-1+\sqrt{2}\right)$

15. $\int_0^3 \sqrt{x+1}\,dx = \sqrt{c+1}(3-0)$

$\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = 3\sqrt{c+1}$

$14/3 = 3\sqrt{c+1};\ c = \frac{115}{81} \approx 1.42$

16. $\int_{-1}^1 x^2\,dx = c^2(1-(-1))$

$\left[\frac{1}{3}x^3\right]_{-1}^1 = 2c^2;\ c = \pm\frac{\sqrt{3}}{3} \approx \pm 0.58$

17. $\int_{-4}^3 (1-x^2)\,dx = (1-c^2)(3+4)$

$\left[x - \frac{1}{3}x^3\right]_{-4}^3 = 7 - 7c^2$

$c = \pm\frac{\sqrt{39}}{3} \approx \pm 2.08$

18. $\int_0^1 x(1-x)\,dx = c(1-c)(1-0)$

$\left[\frac{-x^2(2x-3)}{6}\right]_0^1 = c - c^2$

$c = \frac{3 \pm \sqrt{3}}{6} \approx 0.21 \text{ or } 0.79$

19. $\int_0^2 |x|\,dx = |c|(2-0);\ \left[\frac{x|x|}{2}\right]_0^2 = 2|c|;\ c = 1$

20. $\int_{-2}^2 |x|\,dx = |c|(2+2);\ \left[\frac{x|x|}{2}\right]_{-2}^2 = 4|c|;\ c = -1, 1$

21. $\int_{-\pi}^{\pi} \sin z\,dz = \sin c(\pi + \pi)$

$[-\cos z]_{-\pi}^{\pi} = 2\pi \sin c; \quad c = 0$

22. $\int_0^{\pi} \cos 2y\,dy = (\cos 2c)(\pi - 0)$

$\left[\frac{\sin 2y}{2}\right]_0^{\pi} = \pi \cos 2c; \quad c = \frac{\pi}{4}, \frac{3\pi}{4}$

23. $\int_0^2 (v^2 - v)\,dv = (c^2 - c)(2-0)$

$\left[\frac{1}{3}v^3 - \frac{1}{2}v^2\right]_0^2 = 2c^2 - 2c$

$c = \frac{\sqrt{21}+3}{6} \approx 1.26$

24. $\int_0^2 x^3 dx = c^3(2-0);\ \left[\frac{1}{4}x^4\right]_0^2 = 2c^3$

$c = \sqrt[3]{2} \approx 1.26$

25. $\int_1^4 (ax+b)\,dx = (ac+b)(4-1)$

$\left[\frac{a}{2}x^2 + bx\right]_1^4 = 3ac + 3b;\ c = \frac{5}{2}$

26. $\int_0^b y^2 dy = c^2(b-0);\ \left[\frac{1}{3}y^3\right]_0^b = bc^2$

$c = \frac{b}{\sqrt{3}}$

27. $\frac{\int_A^B (ax+b)\,dx}{B-A} = f(c)$

$\frac{\left[\frac{a}{2}x^2 + bx\right]_A^B}{B-A} = ac + b$

$\frac{\frac{a}{2}(B-A)(B+A) + b(B-A)}{B-A} = ac + b$

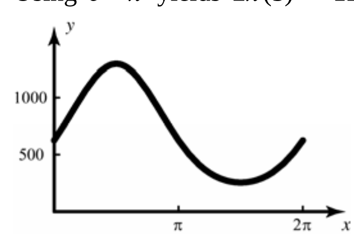
$\frac{a}{2}B + \frac{a}{2}A + b = ac + b;$

$c = \frac{1}{2}B + \frac{1}{2}A = (A+B)/2$

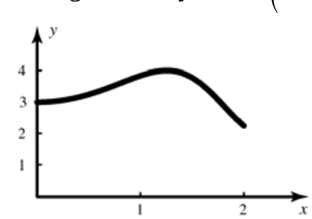
28. $\int_0^b ay^2 dy = ac^2(b-0);\ \left[\frac{1}{3}ay^3\right]_0^b = abc^2$

$c = \frac{b\sqrt{3}}{3}$

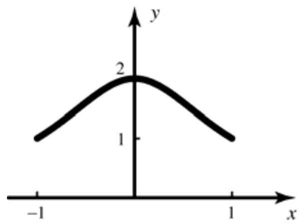
29.. Using $c = \pi$ yields $2\pi(5)^4 = 1250\pi \approx 3927$



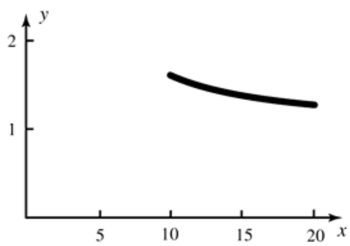
30. Using $c = 0.8$ yields $2\left(3 + \sin 0.8^2\right) \approx 7.19$



31. Using $c = 0.5$ yields $2\dfrac{2}{1+0.5^2} = 3.2$



32. Using c = 15 yields $\left(\dfrac{16}{15}\right)^5 (20-10) \approx 13.8$.



33. A rectangle with height 25 and width 7 has approximately the same area as that under the curve. Thus

$$\frac{1}{7}\int_0^7 H(t)\,dt \approx 25$$

34. **a.** A rectangle with height 28 and width 24 has approximately the same area as that under the curve. Thus,

$$\frac{1}{24-0}\int_0^{24} T(t)\,dt \approx 28$$

b. Yes. The Mean Value Theorem for Integrals guarantees the existence of a c such that

$$\frac{1}{24-0}\int_0^{24} T(t)\,dt = T(c)$$

The figure indicates that there are actually two such values of c, roughly, $c = 11$ and $c = 16$.

35. $\int_{-\pi}^{\pi} (\sin x + \cos x)\,dx = \int_{-\pi}^{\pi} \sin x\,dx + 2\int_0^{\pi} \cos x\,dx$

$= 0 + 2[\sin x]_0^{\pi} = 0$

36. $\int_{-1}^{1} \dfrac{x^3}{(1+x^2)^4}\,dx = 0$, since the integrand is odd.

37. $\int_{-\pi/2}^{\pi/2} \dfrac{\sin x}{1+\cos x}\,dx = 0$, since the integrand is odd.

38. $\int_{-\sqrt{3}\pi}^{\sqrt{3}\pi} x^2 \cos(x^3)\,dx = 2\int_0^{\sqrt{3}\pi} x^2 \cos(x^3)\,dx$

$= \dfrac{2}{3}\left[\sin(x^3)\right]_0^{\sqrt{3}\pi} = \dfrac{2}{3}\sin\left(3\sqrt{3}\,\pi^3\right)$

39. $\int_{-\pi}^{\pi} (\sin x + \cos x)^2\,dx$

$= \int_{-\pi}^{\pi} (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx$

$= \int_{-\pi}^{\pi} (1 + 2\sin x\cos x)\,dx = \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \sin 2x\,dx$

$= 2\int_0^{\pi} dx + 0 = 2[x]_0^{\pi} = 2\pi$

40. $\int_{-\pi/2}^{\pi/2} z\sin^2(z^3)\cos(z^3)dz = 0$, since

$(-z)\sin^2[(-z)^3]\cos[(-z)^3]$

$= -z\sin^2(-z^3)\cos(-z^3)$

$= -z[-\sin(z^3)]^2\cos(z^3)$

$= -z\sin^2(z^3)\cos(z^3)$

41. $\int_{-1}^{1} (1 + x + x^2 + x^3)\,dx$

$= \int_{-1}^{1} dx + \int_{-1}^{1} x\,dx + \int_{-1}^{1} x^2 dx + \int_{-1}^{1} x^3 dx$

$= 2[x]_0^1 + 0 + 2\left[\dfrac{x^3}{3}\right]_0^1 + 0 = \dfrac{8}{3}$

42. $\int_{-100}^{100} (v + \sin v + v\cos v + \sin^3 v)^5\,dv = 0$

since $(-v + \sin(-v) - v\cos(-v) + \sin^3(-v))^5$

$= (-v - \sin v - v\cos v - \sin^3 v)^5$

$= -(v + \sin v + v\cos v + \sin^3 v)^5$

43. $\int_{-1}^{1} \left(\left|x^3\right| + x^3\right)dx = 2\int_0^1 \left|x^3\right| dx + \int_{-1}^{1} x^3 dx$

$= 2\left[\dfrac{x^4}{4}\right]_0^1 + 0 = \dfrac{1}{2}$

44. $\int_{-\pi/4}^{\pi/4} \left(|x|\sin^5 x + |x|^2 \tan x\right)dx = 0$

since $|-x|\sin^5(-x) + |-x|^2 \tan(-x)$

$= -|x|\sin^5 x - |x|^2 \tan x$

45. $\int_{-b}^{-a} f(x)\,dx = \int_a^b f(x)\,dx$ when f is even.

$\int_{-b}^{-a} f(x)\,dx = -\int_a^b f(x)\,dx$ when f is odd.

46. $u = -x, du = -dx$

$$\int_a^b f(-x)\,dx = -\int_{-a}^{-b} f(u)\,du$$

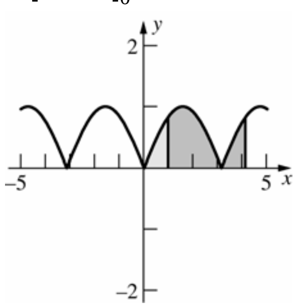
$= \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$ since the variable used in the integration is not important.

47. $$\int_0^{4\pi} |\cos x|\,dx = 8\int_0^{\pi/2} |\cos x|\,dx$$
$$= 8[\sin x]_0^{\pi/2} = 8$$

48. Since $\sin x$ is periodic with period 2π, $\sin 2x$ is periodic with period π.

$$\int_0^{4\pi} |\sin 2x|\,dx = 8\int_0^{\pi/2} \sin 2x\,dx$$
$$= 8\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = -4(-1-1) = 8$$

49. $$\int_1^{1+\pi} |\sin x|\,dx = \int_0^{\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx$$
$$= [-\cos x]_0^{\pi} = 2$$



50. $$\int_2^{2+\pi/2} |\sin 2x|\,dx = \int_0^{\pi/2} |\sin 2x|\,dx$$
$$= \frac{1}{2}[-\cos 2x]_0^{\pi/2} = 1$$

51. $$\int_1^{1+\pi} |\cos x|\,dx = \int_0^{\pi} |\cos x|\,dx = 2\int_0^{\pi/2} \cos x\,dx$$
$$= 2[\sin x]_0^{\pi/2} = 2(1-0) = 2$$

52. The statement is true. Recall that

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\int_a^b \overline{f}dx = \overline{f}\int_a^b dx = \frac{1}{b-a}\int_a^b f(x)dx \cdot \int_a^b dx$$
$$= \frac{1}{b-a}\int_a^b f(x)dx \cdot (b-a) = \int_a^b f(x)\,dx$$

53. All the statements are true.

a. $$\overline{u} + \overline{v} = \frac{1}{b-a}\int_a^b u\,dx + \frac{1}{b-a}\int_a^b v\,dx$$
$$= \frac{1}{b-a}\int_a^b (u+v)\,dx = \overline{u+v}$$

b. $$k\overline{u} = \frac{k}{b-a}\int_a^b u\,dx = \frac{1}{b-a}\int_a^b ku\,dx = \overline{ku}$$

c. Note that

$\overline{u} = \frac{1}{b-a}\int_a^b u(x)\,dx = \frac{1}{a-b}\int_b^a u(x)\,dx$, so we can assume $a < b$.

$$\overline{u} = \frac{1}{b-a}\int_a^b u\,dx \le \frac{1}{b-a}\int_a^b v\,dx = \overline{v}$$

54. **a.** $\overline{V} = 0$ by periodicity.

b. $\overline{V} = 0$ by periodicity.

c. $$V_{rms}^2 = \int_{\phi}^{\phi+1} \hat{V}^2 \sin^2(120\pi t + \phi)\,dt$$
$$= \int_0^1 \hat{V}^2 \sin^2(120\pi t)\,dt$$

by periodicity.

$u = 120\pi t, \quad du = 120\pi\,dt$

$$V_{rms}^2 = \frac{1}{120\pi}\int_0^{120\pi} \hat{V}^2 \sin^2 u\,du$$
$$= \frac{\hat{V}^2}{120\pi}\left[-\frac{1}{2}\cos u \sin u + \frac{1}{2}u\right]_0^{120\pi}$$
$$= \frac{1}{2}\hat{V}^2$$

d. $$120 = \frac{\hat{V}\sqrt{2}}{2}$$
$\hat{V} = 120\sqrt{2} \approx 169.71$ Volts

55. Since f is continuous on a closed interval $[a,b]$ there exist (by the Min-Max Existence Theorem) an m and M in $[a,b]$ such that

$f(m) \le f(x) \le f(M)$ for all x in $[a,b]$. Thus

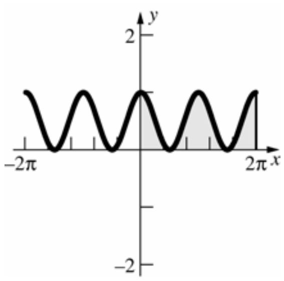
$$\int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx$$
$$(b-a)\,f(m) \le \int_a^b f(x)\,dx \le (b-a)\,f(M)$$
$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M)$$

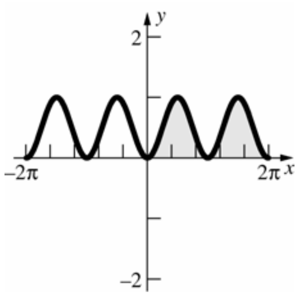
Since f is continuous, we can apply the Intermediate Value Theorem and say that f takes on every value between $f(m)$ and $f(M)$. Since $\frac{1}{b-a}\int_a^b f(x)\,dx$ is between $f(m)$ and $f(M)$, there exists a c in $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

56. **a.** $$\int_0^{2\pi} (\sin^2 x + \cos^2 x)\,dx = \int_0^{2\pi} dx = [x]_0^{2\pi} = 2\pi$$

b.





c. $2\pi = \int_0^{2\pi} \cos^2 x\,dx + \int_0^{2\pi} \sin^2 x\,dx$

$= 2\int_0^{2\pi} \cos^2 x\,dx$, thus $\int_0^{2\pi} \cos^2 x\,dx$

$= \int_0^{2\pi} \sin^2 x\,dx = \pi$

57. **a.** Even

b. 2π

c. On $[0,\pi]$, $|\sin x| = \sin x$.

$u = \cos x$, $du = -\sin x\,dx$

$$\int f(x)dx = \int \sin x \cdot \sin(\cos x)dx$$

$$= -\int \sin u\,du = \cos u + C$$

$$= \cos(\cos x) + C$$

Likewise, on $[\pi, 2\pi]$,

$$\int f(x)dx = -\cos(\cos x) + C$$

$$\int_0^{\pi/2} f(x)dx = 1 - \cos 1 \approx 0.46$$

$$\int_{-\pi/2}^{\pi/2} f(x)dx = 2\int_0^{\pi/2} f(x)dx$$

$$= 2(1 - \cos 1) \approx 0.92$$

$$\int_0^{3\pi/2} f(x)dx = \int_0^{\pi} f(x)dx + \int_{\pi}^{3\pi/2} f(x)dx$$

$$= \cos 1 - 1 \approx -0.46$$

$$\int_{-3\pi/2}^{3\pi/2} f(x)dx = 2\int_0^{3\pi/2} f(x)dx$$

$$= 2(\cos 1 - 1) \approx -0.92$$

$$\int_0^{2\pi} f(x)dx = 0$$

$$\int_{\pi/6}^{4\pi/3} f(x)dx = 2\cos 1 - \cos\left(\frac{\sqrt{3}}{2}\right) + \cos\left(\frac{1}{2}\right)$$

$$\approx -0.44$$

$$\int_{13\pi/6}^{10\pi/3} f(x)dx = \int_{\pi/6}^{4\pi/3} f(x)dx \approx -0.44$$

58. **a.** Odd

b. 2π

c. This function cannot be integrated in closed form. We can only simplify the integrals using symmetry and periodicity, and approximate them numerically.

Note that $\int_{-a}^{a} f(x)dx = 0$ since f is odd, and

$\int_{\pi-a}^{\pi+a} f(x)dx = 0$ since

$f(\pi + x) = -f(\pi - x)$.

$\int_0^{\pi/2} f(x)dx = \frac{\pi}{2} J_1(1) \approx 0.69$ (Bessel function)

$$\int_{-\pi/2}^{\pi/2} f(x)dx = 0$$

$$\int_0^{3\pi/2} f(x)dx = \int_0^{\pi/2} f(x)dx \approx 0.69$$

$$\int_{-3\pi/2}^{3\pi/2} f(x)dx = 0\ ; \int_0^{2\pi} f(x)dx = 0$$

$$\int_{\pi/6}^{13\pi/6} f(x)dx = \int_0^{2\pi} f(x)dx = 0$$

$\int_{\pi/6}^{4\pi/3} f(x)dx \approx 1.055$ (numeric integration)

$$\int_{13\pi/6}^{10\pi/3} f(x)dx = \int_{\pi/6}^{4\pi/3} f(x)dx \approx 1.055$$

59. **a.** Written response.

b. $A = \int_0^a g(x)\,dx = \int_0^a \frac{a}{c} f\left(\frac{c}{a}x\right)dx$

$$= \int_0^c \frac{a}{c} f(x)\frac{a}{c}dx = \frac{a^2}{c^2}\int_0^c f(x)\,dx$$

$$B = \int_0^b h(x)\,dx = \int_0^b \frac{b}{c} f\left(\frac{c}{b}x\right)dx$$

$$= \int_0^c \frac{b}{c} f(x)\frac{b}{c}dx = \frac{b^2}{c^2}\int_0^c f(x)\,dx$$

Thus, $\int_0^a g(x)\,dx + \int_0^b h(x)\,dx$

$$= \frac{a^2}{c^2}\int_0^c f(x)\,dx + \frac{b^2}{c^2}\int_0^c f(x)\,dx$$

$= \frac{a^2 + b^2}{c^2}\int_0^c f(x)\,dx = \int_0^c f(x)\,dx$ since

$a^2 + b^2 = c^2$ from the triangle.

60. If f is odd, then $f(-x) = -f(x)$ and we can write

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0}[-f(-x)]\,dx = \int_a^0 f(u)\,du$$

$$= -\int_0^a f(u)\,du = -\int_0^a f(x)\,dx$$

On the second line, we have made the substitution $u = -x$.

4.6 Concepts Review

1. 1, 2, 2, 2, …, 2, 1

2. 1, 4, 2, 4, 2, …, 4, 1

3. n^4

4. large

Problem Set 4.6

1. $f(x)=\frac{1}{x^2}; h=\frac{3-1}{8}=0.25$

$x_0=1.00$	$f(x_0)=1$	$x_5=2.25$	$f(x_5)\approx 0.1975$
$x_1=1.25$	$f(x_1)=0.64$	$x_6=2.50$	$f(x_6)=0.16$
$x_2=1.50$	$f(x_2)\approx 0.4444$	$x_7=2.75$	$f(x_7)\approx 0.1322$
$x_3=1.75$	$f(x_3)\approx 0.3265$	$x_8=3.00$	$f(x_8)\approx 0.1111$
$x_4=2.00$	$f(x_4)=0.25$		

Left Riemann Sum: $\int_1^3 \frac{1}{x^2}dx \approx 0.25[f(x_0)+f(x_1)+\ldots+f(x_7)]\approx 0.7877$

Right Riemann Sum: $\int_1^3 \frac{1}{x^2}dx \approx 0.25[f(x_1)+f\left(x_2\right)+...+f(x_8)]\approx 0.5655$

Trapezoidal Rule: $\int_1^3 \frac{1}{x^2}dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)]\approx 0.6766$

Parabolic Rule: $\int_1^3 \frac{1}{x^2}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_7)+f(x_8)]\approx 0.6671$

Fundamental Theorem of Calculus: $\int_1^3 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^3 = -\frac{1}{3}+1=\frac{2}{3}\approx 0.6667$

2. $f(x)=\frac{1}{x^3}; h=\frac{3-1}{8}=0.25$

$x_0=1.00$	$f(x_0)=1$	$x_5=2.25$	$f(x_5)\approx 0.0878$
$x_1=1.25$	$f(x_1)=0.5120$	$x_6=2.50$	$f(x_6)=0.0640$
$x_2=1.50$	$f(x_2)\approx 0.2963$	$x_7=2.75$	$f(x_7)\approx 0.0481$
$x_3=1.75$	$f(x_3)\approx 0.1866$	$x_8=3.00$	$f(x_8)\approx 0.0370$
$x_4=2.00$	$f(x_4)=0.1250$		

Left Riemann Sum: $\int_1^3 \frac{1}{x^3}dx \approx 0.25[f(x_0)+f(x_1)+\ldots+f(x_7)]\approx 0.5799$

Right Riemann Sum: $\int_1^3 \frac{1}{x^3}dx \approx 0.25[f(x_1)+f\left(x_2\right)+...+f(x_8)]\approx 0.3392$

Trapezoidal Rule: $\int_1^3 \frac{1}{x^3}dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)]\approx 0.4596$

Parabolic Rule: $\int_1^3 \frac{1}{x^3}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_7)+f(x_8)]\approx 0.4455$

Fundamental Theorem of Calculus: $\int_1^3 \frac{1}{x^3}dx = \left[-\frac{1}{2x^2}\right]_1^3 = \frac{4}{9}\approx 0.4444$

3. $f(x)=\sqrt{x}; h=\frac{2-0}{8}=0.25$

$x_0=0.00$	$f(x_0)=0$	$x_5=1.25$	$f(x_5)\approx 1.1180$
$x_1=0.25$	$f(x_1)=0.5$	$x_6=1.50$	$f(x_6)\approx 1.2247$
$x_2=0.50$	$f(x_2)\approx 0.7071$	$x_7=1.75$	$f(x_7)\approx 1.3229$
$x_3=0.75$	$f(x_3)\approx 0.8660$	$x_8=2.00$	$f(x_8)\approx 1.4142$
$x_4=1.00$	$f(x_4)=1$		

Left Riemann Sum: $\int_0^2 \sqrt{x}\,dx \approx 0.25[f(x_0)+f(x_1)+\ldots+f(x_7)]\approx 1.6847$

Right Riemann Sum: $\int_0^2 \sqrt{x}\,dx \approx 0.25[f(x_1)+f\left(x_2\right)+\ldots+f(x_8)]\approx 2.0383$

Trapezoidal Rule: $\int_0^2 \sqrt{x}dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)]\approx 1.8615$

Parabolic Rule: $\int_0^2 \sqrt{x}dx \approx \frac{0.25}{3}[f(x_1)+4f(x_2)+2f(x_3)+\ldots+4f(x_7)+f(x_8)]\approx 1.8755$

Fundamental Theorem of Calculus: $\int_0^2 \sqrt{x}dx=\left[\frac{2}{3}x^{3/2}\right]_0^2=\frac{4\sqrt{2}}{3}\approx 1.8856$

4. $f(x)=x\sqrt{x^2+1}; h=\frac{3-1}{8}=0.25$

$x_0=1.00$	$f(x_0)\approx 1.4142$	$x_5=2.25$	$f(x_5)\approx 5.5400$
$x_1=1.25$	$f(x_1)\approx 2.0010$	$x_6=2.50$	$f(x_6)\approx 6.7315$
$x_2=1.50$	$f(x_2)\approx 2.7042$	$x_7=2.75$	$f(x_7)\approx 8.0470$
$x_3=1.75$	$f(x_3)\approx 3.5272$	$x_8=3.00$	$f(x_8)\approx 9.4868$
$x_4=2.00$	$f(x_4)\approx 4.4721$		

Left Riemann Sum: $\int_1^3 x\sqrt{x^2+1}\,dx \approx 0.25[f(x_0)+f(x_1)+\cdots+f(x_7)]\approx 8.6093$

Right Riemann Sum: $\int_1^3 x\sqrt{x^2+1}\,dx \approx 0.25[f(x_1)+f\left(x_2\right)+\ldots+f(x_8)]\approx 10.6274$

Trapezoidal Rule: $\int_1^3 x\sqrt{x^2+1}\,dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)]\approx 9.6184$

Parabolic Rule: $\int_1^3 x\sqrt{x^2+1}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_7)+f(x_8)]\approx 9.5981$

Fundamental Theorem of Calculus: $\int_1^3 x\sqrt{x^2+1}dx=\left[\frac{1}{3}(x^2+1)^{3/2}\right]_1^3=\frac{1}{3}\left(10\sqrt{10}-2\sqrt{2}\right)\approx 9.5981$

5. $f(x)=x\left(x^2+1\right)^5; h=\frac{1-0}{8}=0.125$

$x_0=0.00$	$f(x_0)=0$	$x_5=0.625$	$f(x_5)\approx 3.2504$
$x_1=0.125$	$f(x_1)\approx 0.1351$	$x_6=0.750$	$f(x_6)\approx 6.9849$
$x_2=0.250$	$f(x_2)\approx 0.3385$	$x_7=0.875$	$f(x_7)\approx 15.0414$
$x_3=0.375$	$f(x_3)\approx 0.7240$	$x_8=1.000$	$f(x_8)=32$
$x_4=0.500$	$f(x_4)\approx 1.5259$		

Left Riemann Sum: $\int_0^1 x\left(x^2+1\right)^5 dx \approx 0.125[f(x_0)+f(x_1)+\cdots+f(x_7)] \approx 3.4966$

Right Riemann Sum: $\int_0^1 x\left(x^2+1\right)^5 dx \approx 0.125[f(x_1)+f\left(x_2\right)+...+f(x_8)] \approx 7.4966$

Trapezoidal Rule: $\int_0^1 x\left(x^2+1\right)^5 dx \approx \frac{0.125}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)] \approx 5.4966$

Parabolic Rule: $\int_0^1 x\left(x^2+1\right)^5 dx \approx \frac{0.125}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_7)+f(x_8)] \approx 5.2580$

Fundamental Theorem of Calculus: $\int_0^1 x\left(x^2+1\right)^5 dx = \left[\frac{1}{12}\left(x^2+1\right)^6\right]_0^1 = 5.25$

6. $f(x)=\left(x+1\right)^{3/2}; h=\frac{4-1}{8}=0.375$

$x_0=1.000$ $f(x_0)\approx 2.8284$
$x_1=1.375$ $f(x_1)\approx 3.6601$
$x_2=1.750$ $f(x_2)\approx 4.5604$
$x_3=2.125$ $f(x_3)\approx 5.5243$
$x_4=2.500$ $f(x_4)\approx 6.5479$
$x_5=2.875$ $f(x_5)\approx 7.6279$
$x_6=3.250$ $f(x_6)\approx 8.7616$
$x_7=3.625$ $f(x_7)\approx 9.9464$
$x_8=4.000$ $f(x_8)\approx 11.1803$

Left Riemann Sum: $\int_1^4 \left(x+1\right)^{3/2} dx \approx 0.375[f(x_0)+f(x_1)+\cdots+f(x_7)] \approx 18.5464$

Right Riemann Sum: $\int_1^4 \left(x+1\right)^{3/2} dx \approx 0.375[f(x_1)+f\left(x_2\right)+...+f(x_8)] \approx 21.6784$

Trapezoidal Rule: $\int_1^4 \left(x+1\right)^{3/2} dx \approx \frac{0.375}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)] \approx 20.1124$

Parabolic Rule: $\int_1^4 \left(x+1\right)^{3/2} dx \approx \frac{0.375}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_7)+f(x_8)] \approx 20.0979$

Fundamental Theorem of Calculus: $\int_1^4 \left(x+1\right)^{3/2} dx = \left[\frac{2}{5}\left(x+1\right)^{5/2}\right]_1^4 \approx 20.0979$

7.

	LRS	RRS	MRS	Trap	Parabolic
n = 4	0.5728	0.3728	0.4590	0.4728	0.4637
n = 8	0.5159	0.4159	0.4625	0.4659	0.4636
n = 16	0.4892	0.4392	0.4634	0.4642	0.4636

8.

	LRS	RRS	MRS	Trap	Parabolic
n = 4	1.2833	0.9500	1.0898	1.1167	1.1000
n = 8	1.1865	1.0199	1.0963	1.1032	1.0987
n = 16	1.1414	1.0581	1.0980	1.0998	1.0986

9.

	LRS	RRS	MRS	Trap	Parabolic
$n = 4$	2.6675	3.2855	2.9486	2.9765	2.9580
$n = 8$	2.8080	3.1171	2.9556	2.9625	2.9579
$n = 16$	2.8818	3.0363	2.9573	2.9591	2.9579

10.

	LRS	RRS	MRS	Trap	Parabolic
$n = 4$	10.3726	17.6027	13.6601	13.9876	13.7687
$n = 8$	12.0163	15.6314	13.7421	13.8239	13.7693
$n = 16$	12.8792	14.6867	13.7625	13.7830	13.7693

11. $f'(x) = -\frac{1}{x^2};\ f''(x) = \frac{2}{x^3}$

The largest that $\left|f''(c)\right|$ can be on $[1,3]$ occurs when $c = 1$, and $\left|f''(1)\right| = 2$

$\frac{(3-1)^3}{12n^2}(2) \le 0.01;\ \ n \ge \sqrt{\frac{400}{3}}$ Round up: $n = 12$

$\int_1^3 \frac{1}{x}\,dx \approx \frac{0.167}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{11}) + f(x_{12})]$

≈ 1.1007

12. $f'(x) = -\frac{1}{(1+x)^2};\ f''(x) = \frac{2}{(1+x)^3}$

The largest that $\left|f''(c)\right|$ can be on $[1,3]$ occurs when $c = 1$, and $\left|f''(1)\right| = \frac{1}{4}$.

$\frac{(3-1)^3}{12n^2}\left(\frac{1}{4}\right) \le 0.01;\ \ n \ge \sqrt{\frac{100}{6}}$ Round up: $n = 5$

$\int_1^3 \frac{1}{1+x}\,dx \approx \frac{0.4}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_4) + f(x_5)]$

≈ 0.6956

13. $f'(x) = \frac{1}{2\sqrt{x}};\ f''(x) = -\frac{1}{4x^{3/2}}$

The largest that $\left|f''(c)\right|$ can be on $[1,4]$ occurs when $c = 1$, and $\left|f''(1)\right| = \frac{1}{4}$.

$\frac{(4-1)^3}{12n^2}\left(\frac{1}{4}\right) \le 0.01;\ \ n \ge \sqrt{\frac{900}{16}}$ Round up: $n = 8$

$\int_1^4 \sqrt{x}\,dx \approx \frac{0.375}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_7) + f(x_8)]$

≈ 4.6637

14. $f'(x)=\frac{1}{2\sqrt{x+1}}$; $f''(x)=-\frac{1}{4(x+1)^{3/2}}$

The largest that $|f''(c)|$ can be on $[1,3]$ occurs when $c=1$, and $|f''(1)|=\frac{1}{4\times 2^{3/2}}$.

$\frac{(3-1)^3}{12n^2}\left(\frac{1}{4\times 2^{3/2}}\right)\le 0.01$; $n\ge\sqrt{\frac{100}{12\sqrt{2}}}$ Round up: $n = 3$

$$\int_1^3 \sqrt{x+1}\,dx \approx \frac{0.667}{2}[f(x_0)+2f(x_1)+2f(x_2)+f(x_3)]$$
$$\approx 3.4439$$

15. $f'(x)=-\frac{1}{x^2}$; $f''(x)=\frac{2}{x^3}$; $f'''(x)=-\frac{6}{x^4}$;
$f^{(4)}(x)=\frac{24}{x^5}$

The largest that $\left|f^{(4)}(c)\right|$ can be on $[1,3]$ occurs when $c=1$, and $\left|f^{(4)}(1)\right|=24$.

$\frac{(4-1)^5}{180n^4}(24)\le 0.01$; $n\approx 4.545$ Round up to even: $n = 6$

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{0.333}{3}[f(x_0)+4f(x_1)+\ldots+4f(x_5)+f(x_6)]$$
$$\approx 1.0989$$

16. $f'(x)=\frac{1}{2\sqrt{x+1}}$; $f''(x)=-\frac{1}{4(x+1)^{3/2}}$;

$f'''(x)=\frac{3}{8(x+1)^{5/2}}$; $f^{(4)}(x)=-\frac{15}{16(x+1)^{7/2}}$

The largest that $\left|f^{(4)}(c)\right|$ can be on $[4,8]$ occurs when $c=4$, and $\left|f^{(4)}(4)\right|=\frac{3}{400\sqrt{5}}$.

$\frac{(8-4)^5}{180n^4}\left(\frac{3}{400\sqrt{5}}\right)\le 0.01$; $n\approx 1.1753$ Round up to even: $n = 2$

$$\int_4^8 \sqrt{x+1}\,dx \approx \frac{2}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]\approx 10.5464$$

17. $\int_{m-h}^{m+h}(ax^2+bx+c)dx = \left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_{m-h}^{m+h}$

$= \frac{a}{3}(m+h)^3+\frac{b}{2}(m+h)^2+c(m+h)-\frac{a}{3}(m-h)^3-\frac{b}{2}(m-h)^2-c(m-h)$

$= \frac{a}{3}(6m^2h+2h^3)+\frac{b}{2}(4mh)+c(2h) = \frac{h}{3}[a(6m^2+2h^2)+b(6m)+6c]$

$\frac{h}{3}[f(m-h)+4f(m)+f(m+h)]$

$= \frac{h}{3}[a(m-h)^2+b(m-h)+c+4am^2+4bm+4c+a(m+h)^2+b(m+h)+c]$

$= \frac{h}{3}[a(6m^2+2h^2)+b(6m)+6c]$

18. a. To show that the Parabolic Rule is exact, examine it on the interval $[m-h, m+h]$.

Let $f(x)=ax^3+bx^2+cx+d$, then

$\int_{m-h}^{m+h} f(x)\,dx$

$= \frac{a}{4}\left[(m+h)^4-(m-h)^4\right]+\frac{b}{3}\left[(m+h)^3-(m-h)^3\right]+\frac{c}{2}\left[(m+h)^2-(m-h)^2\right]+d[(m+h)-(m-h)]$

$= \frac{a}{4}(8m^3h+8h^3m)+\frac{b}{3}(6m^2h+2h^3)+\frac{c}{2}(4mh)+d(2h).$

The Parabolic Rule with $n = 2$ gives

$\int_{m-h}^{m+h} f(x)\,dx = \frac{h}{3}[f(m-h)+4f(m)+f(m+h)] = 2am^3h+2amh^3+2bm^2h+\frac{2}{3}bh^3+2chm+2dh$

$= \frac{a}{4}(8m^3h+8mh^3)+\frac{b}{3}(6m^2h+2h^3)+\frac{c}{2}(4mh)+d(2h)$

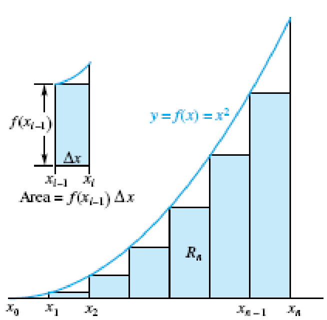
which agrees with the direct computation. Thus, the Parabolic Rule is exact for any cubic polynomial.

b. The error in using the Parabolic Rule is given by $E_n = -\frac{(l-k)^5}{180n^4}f^{(4)}(m)$ for some m between l and k.

However, $f'(x)=3ax^2+2bx+c$, $f''(x)=6ax+2b$, $f^{(3)}(x)=6a$, and $f^{(4)}(x)=0$, so $E_n = 0$.

19. The left Riemann sum will be smaller than $\int_a^b f(x)\,dx$.

If the function is increasing, then $f(x_i) < f(x_{i+1})$ on the interval $[x_i, x_{i+1}]$. Therefore, the left Riemann sum will underestimate the value of the definite integral. The following example illustrates this behavior:

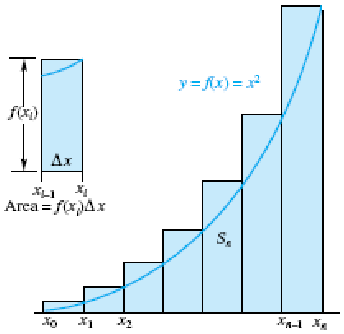


Inscribed polygon

If f is increasing, then $f'(c) > 0$ for any $c \in (a,b)$. Thus, the error $E_n = \frac{(b-a)^2}{2n}f'(c) > 0$. Since the error is positive, then the Riemann sum must be less than the integral.

20. The right Riemann sum will be larger than $\int_a^b f(x)\,dx$.

If the function is increasing, then $f(x_i) < f(x_{i+1})$ on the interval $[x_i, x_{i+1}]$. Therefore, the right Riemann sum will overestimate the value of the definite integral. The following example illustrates this behavior:



Circumscribed polygon

If f is increasing, then $f'(c) > 0$ for any $c \in (a,b)$. Thus, the error $E_n = -\frac{(b-a)^2}{2n} f'(c) < 0$. Since the error is negative, then the Riemann sum must be greater than the integral.

21. The midpoint Riemann sum will be larger than $\int_a^b f(x)\,dx$.

If f is concave down, then $f''(c) < 0$ for any $c \in (a,b)$. Thus, the error $E_n = \frac{(b-a)^3}{24n^2} f''(c) < 0$. Since the error is negative, then the Riemann sum must be greater than the integral.

22. The Trapezoidal Rule approximation will be smaller than $\int_a^b f(x)\,dx$.

If f is concave down, then $f''(c) < 0$ for any $c \in (a,b)$. Thus, the error $E_n = -\frac{(b-a)^3}{12n^2} f''(c) > 0$. Since the error is positive, then the Trapezoidal Rule approximation must be less than the integral.

23. Let $n = 2$.

$f(x) = x^k;\ h = a$

$x_0 = -a \qquad f(x_0) = -a^k$

$x_1 = 0 \qquad f(x_1) = 0$

$x_2 = a \qquad f(x_2) = a^k$

$$\int_{-a}^{a} x^k\,dx \approx \frac{a}{2}[-a^k + 2\cdot 0 + a^k] = 0$$

$$\int_{-a}^{a} x^k\,dx = \left[\frac{1}{k+1}x^{k+1}\right]_{-a}^{a} = \frac{1}{k+1}[a^{k+1} - (-a)^{k+1}] = \frac{1}{k+1}[a^{k+1} - a^{k+1}] = 0$$

A corresponding argument works for all n.

24. **a.** $T \approx 48.9414;\ f'(x) = 4x^3$

$$T - \frac{[4(3)^3 - 4(1)^3](0.25)^2}{12} \approx 48.9414 - 0.5417 = 48.3997$$

The correct value is 48.4.

b. $T \approx 1.9886;\ f'(x) = \cos x$

$$T - \frac{[\cos\pi - \cos 0]\left(\frac{\pi}{12}\right)^2}{12} \approx 1.999987$$

The correct value is 2.

25. The integrand is increasing and concave down. By problems 19-22, LRS < TRAP < MRS < RRS.

26. The integrand is increasing and concave up. By problems 19-22, LRS < MRS < TRAP < RRS

27. $A \approx \frac{10}{2}[75 + 2\cdot 71 + 2\cdot 60 + 2\cdot 45 + 2\cdot 45 + 2\cdot 52 + 2\cdot 57 + 2\cdot 60 + 59] = 4570 \text{ ft}^2$

28. $A \approx \frac{3}{3}[23 + 4\cdot 24 + 2\cdot 23 + 4\cdot 21 + 2\cdot 18 + 4\cdot 15 + 2\cdot 12 + 4\cdot 11 + 2\cdot 10 + 4\cdot 8 + 0] = 465 \text{ ft}^2$

$V = A\cdot 6 \approx 2790 \text{ ft}^3$

29. $A \approx \frac{20}{3}[0 + 4\cdot 7 + 2\cdot 12 + 4\cdot 18 + 2\cdot 20 + 4\cdot 20 + 2\cdot 17 + 4\cdot 10 + 0] = 2120 \text{ ft}^2$

4 mi/h = 21,120 ft/h

(2120)(21,120)(24) = 1,074,585,600 ft^3

30. Using a right-Riemann sum,

$$\text{Distance} = \int_0^{24} v(t)\,dt \approx \sum_{i=1}^{8} v(t_i)\,\Delta t$$

$$= (31 + 54 + 53 + 52 + 35 + 31 + 28)\frac{3}{60}$$

$$= \frac{852}{60} = 14.2 \text{ miles}$$

31. Using a right-Riemann sum,

$$\text{Water Usage} = \int_0^{120} F(t)\,dt$$

$$\approx \sum_{i=1}^{10} F(t_i)\,\Delta t = 12(71 + 68 + \cdots + 148)$$

$$= 13,740 \text{ gallons}$$

4.7 Chapter Review

Concepts Test

1. True: Theorem 4.3.D

2. True: Obtained by integrating both sides of the Product Rule

3. True: If $F(x) = \int f(x)\,dx,\ f(x)$ is a derivative of $F(x)$.

4. False: $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 7x - 5$ are a counterexample.

5. False: The two sides will in general differ by a constant term.

6. True: At any given height, speed on the downward trip is the negative of speed on the upward.

7. True:

$$a_1 + a_0 + a_2 + a_1 + a_3 + a_2 + \cdots + a_{n-1} + a_{n-2} + a_n + a_{n-1} = a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n$$

8. True:

$$\sum_{i=1}^{100}(2i-1) = 2\sum_{i=1}^{100} i - \sum_{i=1}^{100} 1 = \frac{2(100)(100+1)}{2} - 100 = 10,000$$

9. True:

$$\sum_{i=1}^{10}(a_i + 1)^2 = \sum_{i=1}^{10} a_i^2 + 2\sum_{i=1}^{10} a_i + \sum_{i=1}^{100} 1$$

= 100 + 2(20) + 10 = 150

10. False: f must also be continuous except at a finite number of points on $[a, b]$.

11. True: The area of a vertical line segment is 0.

12. False: $\int_{-1}^{1} x\,dx$ is a counterexample.

13. False: A counterexample is $f(x)=\begin{cases}0, & x\neq 0\\ 1, & x=0\end{cases}$ with $\int_{-1}^{1}\left[f(x)\right]^2 dx=0$.
If $f(x)$ is continuous, then $[f(x)]^2 \geq 0$, and if $[f(x)]^2$ is greater than 0 on [a, b], the integral will be also.

14. False: $D_x\left[\int_a^x f(z)\,dz\right]=f(x)$

15. True: $\sin x+\cos x$ has period 2π, so $\int_x^{x+2\pi}(\sin x+\cos x)\,dx$ is independent of x.

16. True: $\lim_{x\to a} kf(x)=k\lim_{x\to a} f(x)$ and $\lim_{x\to a}\left[f(x)+g(x)\right]$ $=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$ when all the limits exist.

17. True: $\sin^{13} x$ is an odd function.

18. True: Theorem 4.2.B

19. False: The statement is not true if $c > d$.

20. False: $D_x\left[\int_0^{x^2}\frac{1}{1+t^2}\,dt\right]=\frac{2x}{1+x^2}$

21. True: Both sides equal 4.

22. True: Both sides equal 4.

23. True: If f is odd, then the accumulation function $F(x)=\int_0^x f(t)\,dt$ is even, and so is $F(x)+C$ for any C.

24. False: $f(x)=x^2$ is a counterexample.

25. False: $f(x)=x^2$ is a counterexample.

26. False: $f(x)=x^2$ is a counterexample.

27. False: $f(x)=x^2$, $v(x) = 2x + 1$ is a counterexample.

28. False: $f(x)=x^3$ is a counterexample.

29. False: $f(x)=\sqrt{x}$ is a counterexample.

30. True: All rectangles have height 4, regardless of $\overline{x}_i$.

31. True: $F(b)-F(a)=\int_a^b F'(x)\,dx$ $=\int_a^b G'(x)\,dx=G(b)-G(a)$

32. False: $\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$ because f is even.

33. False: $z(t)=t^2$ is a counterexample.

34. False: $\int_0^b f(x)\,dx=F(b)-F(0)$

35. True: Odd-exponent terms cancel themselves out over the interval, since they are odd.

36. False: $a = 0$, $b = 1$, $f(x) = -1$, $g(x) = 0$ is a counterexample.

37. False: $a = 0$, $b = 1$, $f(x) = -1$, $g(x) = 0$ is a counterexample.

38. True: $\left|a_1+a_2+a_3+\cdots+a_n\right|$ $\leq\left|a_1\right|+\left|a_2\right|+\left|a_3\right|+\cdots+\left|a_n\right|$ because any negative values of a_i make the left side smaller than the right side.

39. True: Note that $-\left|f(x)\right|\leq f(x)\leq\left|f(x)\right|$ and use Theorem 4.3.B.

40. True: Definition of Definite Integral

41. True: Definition of Definite Integral

42. False: Consider $\int \cos\left(x^2\right)dx$

43. True. Right Riemann sum always bigger.

44. True. Midpoint of x coordinate is midpoint of y coordinate.

45. False. Trapeziod rule overestimates integral.

46. True. Parabolic Rule gives exact value for quadratic and cubic functions.

Sample Test Problems

1. $\left[\frac{1}{4}x^4 - x^3 + 2x^{3/2}\right]_0^1 = \frac{5}{4}$

2. $\left[\frac{2}{3}x^3 - 3x - \frac{1}{x}\right]_1^2 = \frac{13}{6}$

3. $\left[\frac{1}{3}y^3 + 9\cos y - \frac{26}{y}\right]_1^{\pi} = \frac{50}{3} - \frac{26}{\pi} + \frac{\pi^3}{3} - 9\cos 1$

4. $\left[\frac{1}{3}(y^2-4)^{3/2}\right]_4^9 = -8\sqrt{3} + \frac{77\sqrt{77}}{3}$

5. $\left[\frac{3}{16}(2z^2-3)^{4/3}\right]_2^8 = \frac{-15\left(-125+\sqrt[3]{5}\right)}{16}$

6. $\left[-\frac{1}{5}\cos^5 x\right]_0^{\pi/2} = \frac{1}{5}$

7. $u = \tan(3x^2+6x),\ du = (6x+6)\sec^2(3x^2+6x)$

$\frac{1}{6}\int u^2 du = \frac{1}{18}u^3 + C$

$\frac{1}{18}\left[\tan^3(3x^2+6x)\right]_0^{\pi} = \frac{1}{18}\tan^3(3\pi^2+6\pi)$

8. $u = t^4 + 9,\ du = 4t^3 dt$

$\frac{1}{4}\int_9^{25} u^{-1/2} du = \frac{1}{2}\left[u^{1/2}\right]_9^{25} = 1$

9. $\frac{1}{5}\left[\frac{3}{5}(t^5+5)^{5/3}\right]_1^2 = \frac{3}{25}\left[37^{5/3} - 6^{5/3}\right] \approx 46.9$

10. $\left[\frac{1}{9y-3y^3}\right]_2^3 = \frac{4}{27}$

11. $\int (x+1)\sin\left(x^2+2x+3\right)dx$

$= \frac{1}{2}\int \sin\left(x^2+2x+3\right)\left(2x+2\right)dx$

$= \frac{1}{2}\int \sin u\, du$

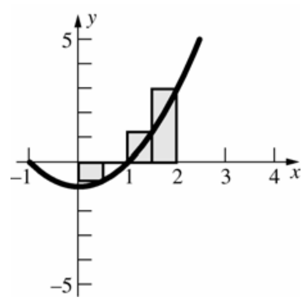
$= -\frac{1}{2}\cos\left(x^2+2x+3\right) + C$

12. $u = 2y^3 + 3y^2 + 6y,\ du = (6y^2+6y+6)\,dy$

$\int_1^5 \frac{\left(y^2+y+1\right)}{\sqrt[5]{2y^3+3y^2+6y}}dy = \frac{1}{6}\int_{11}^{355} u^{-1/5}du$

$= \frac{1}{6}\left[\frac{5}{4}u^{4/5}\right]_{11}^{355} = \frac{5}{24}\left(355^{4/5} - 11^{4/5}\right)$

13. $\sum_{i=1}^{4}\left[\left(\frac{i}{2}\right)^2 - 1\right]\left(\frac{1}{2}\right) = \frac{7}{4}$



14. $f'(x) = \frac{1}{x+3},\ f'(7) = \frac{1}{10}$

15. $\int_0^3 (2-\sqrt{x+1})^2 dx$

$= \int_0^3 \left(x+5-4\sqrt{x+1}\right)dx$

$= \left[\frac{1}{2}x^2 + 5x - \frac{8}{3}(x+1)^{3/2}\right]_0^3 = \frac{5}{6}$

16. $\frac{1}{5-2}\int_2^5 3x^2\sqrt{x^3-4}\,dx = \frac{1}{3}\left[\frac{2}{3}(x^3-4)^{3/2}\right]_2^5$

$= 294$

17. $\int_2^4 \left(5 - \frac{1}{x^2}\right)dx = \left[5x + \frac{1}{x}\right]_2^4 = \frac{39}{4}$

18. $\sum_{i=1}^{n}(3^i - 3^{i-1})$

$= (3-1) + (3^2-3) + (3^3-3^2) + \cdots + (3^n - 3^{n-1})$

$= 3^n - 1$

19. $\sum_{i=1}^{10}(6i^2 - 8i) = 6\sum_{i=1}^{10} i^2 - 8\sum_{i=1}^{10} i$

$= 6\left[\frac{10(11)(21)}{6}\right] - 8\left[\frac{10(11)}{2}\right] = 1870$

20. a. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$

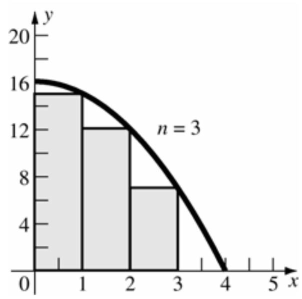
b. 1 + 0 + (–1) + (–2) + (–3) + (–4) = –9

c. $1+\frac{\sqrt{2}}{2}+0-\frac{\sqrt{2}}{2}-1=0$

21. a. $\sum_{n=2}^{78}\frac{1}{n}$

b. $\sum_{n=1}^{50} nx^{2n}$

22.



$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left[16-\left(\frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$
$$=\lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[\frac{48}{n}-\frac{27}{n^3}i^2\right]\right\}$$
$$=\lim_{n\to\infty}\left\{\frac{48}{n}\sum_{i=1}^{n}1-\frac{27}{n^3}\sum_{i=1}^{n}i^2\right\}$$
$$=\lim_{n\to\infty}\left\{48-\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right\}$$
$$=\lim_{n\to\infty}\left\{48-\frac{9}{2}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]\right\}$$
= 48 – 9 = 39

23. a. $\int_1^2 f(x)\,dx=\int_1^0 f(x)\,dx+\int_0^2 f(x)\,dx$
= –4 + 2 = –2

b. $\int_1^0 f(x)\,dx=-\int_0^1 f(x)\,dx=-4$

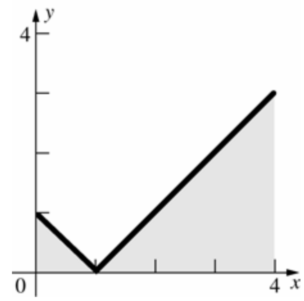
c. $\int_0^2 3f(u)\,du=3\int_0^2 f(u)\,du=3(2)=6$

d. $\int_0^2 [2g(x)-3f(x)]\,dx$

$=2\int_0^2 g(x)-3\int_0^2 f(x)\,dx$
= 2(–3) – 3(2) = –12

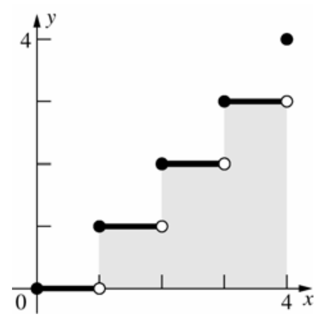
e. $\int_0^{-2} f(-x)\,dx=-\int_0^2 f(x)\,dx=-2$

24. a.



$\int_0^4 |x-1|\,dx=\frac{1}{2}(1)(1)+\frac{1}{2}(3)(3)=5$

b.



$\int_0^4 [\![x]\!]\,dx=1+2+3=6$

c. $\int_0^4 (x-[\![x]\!])\,dx=\int_0^4 x\,dx-\int_0^4 [\![x]\!]\,dx$

$\left[\frac{1}{2}x^2\right]_0^4-6=8-6=2$

25. a. $\int_{-2}^{2} f(x)\,dx=2\int_0^2 f(x)\,dx=2(-4)=-8$

b. Since $f(x)\le 0$, $|f(x)|=-f(x)$ and

$\int_{-2}^{2}|f(x)|\,dx=-\int_{-2}^{2} f(x)\,dx$

$=-2\int_0^2 f(x)\,dx=8$

c. $\int_{-2}^{2} g(x)\,dx=0$

d. $\int_{-2}^{2}[f(x)+f(-x)]\,dx$

$=2\int_0^2 f(x)\,dx+2\int_0^2 f(x)\,dx$
= 4(–4) = –16

e. $\int_0^2 [2g(x)+3f(x)]\,dx$

$= 2\int_0^2 g(x)\,dx + 3\int_0^2 f(x)\,dx$

$= 2(5) + 3(-4) = -2$

f. $\int_{-2}^0 g(x)\,dx = -\int_0^2 g(x)\,dx = -5$

26. $\int_{-100}^{100} (x^3 + \sin^5 x)\,dx = 0$

27. $\int_{-4}^{-1} 3x^2 dx = 3c^2(-1+4)$

$\left[x^3\right]_{-4}^{-1} = 9c^2$

$c^2 = 7$

$c = -\sqrt{7} \approx -2.65$

28. **a.** $G'(x) = \dfrac{1}{x^2+1}$

b. $G'(x) = \dfrac{2x}{x^4+1}$

c. $G'(x) = \dfrac{3x^2}{x^6+1} - \dfrac{1}{x^2+1}$

29. **a.** $G'(x) = \sin^2 x$

b. $G'(x) = f(x+1) - f(x)$

c. $G'(x) = -\dfrac{1}{x^2}\int_0^x f(z)\,dz + \dfrac{1}{x} f(x)$

d. $G'(x) = \int_0^x f(t)\,dt$

e. $G(x) = \int_0^{g(x)} \dfrac{dg(u)}{du}\,du = [g(u)]_0^{g(x)}$

$= g(g(x)) - g(0)$

$G'(x) = g'(g(x))g'(x)$

f. $G(x) = \int_0^{-x} f(-t)\,dt = \int_0^x f(u)(-du)$

$= -\int_0^x f(u)\,du$

$G'(x) = -f(x)$

30. **a.** $\int_0^4 \sqrt{x}\,dx = \dfrac{2}{3}\left[x^{3/2}\right]_0^4 = \dfrac{16}{3}$

b. $\int_1^3 x^2 dx = \dfrac{1}{3}\left[x^3\right]_1^3 = \dfrac{26}{3}$

31. $f(x) = \int_{2x}^{5x} \dfrac{1}{t}\,dt = \int_1^{5x} \dfrac{1}{t}\,dt - \int_1^{2x} \dfrac{1}{t}\,dt$

$f'(x) = \dfrac{1}{5x}\cdot 5 - \dfrac{1}{2x}\cdot 2 = 0$

32. Left Riemann Sum: $\int_1^2 \dfrac{1}{1+x^4}\,dx \approx 0.125[f(x_0) + f(x_1) + \ldots + f(x_7)] \approx 0.2319$

Right Riemann Sum: $\int_1^2 \dfrac{1}{1+x^4}\,dx \approx 0.125[f(x_1) + f(x_2) + \ldots + f(x_8)] \approx 0.1767$

Midpoint Riemann Sum: $\int_1^2 \dfrac{1}{1+x^4}\,dx \approx 0.125[f(x_{0.5}) + f(x_{1.5}) + \ldots + f(x_{7.5})] \approx 0.2026$

33. $\int_1^2 \dfrac{1}{1+x^4}\,dx \approx \dfrac{0.125}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_7) + f(x_8)] \approx 0.2043$

$$|f''(c)| = \left|\frac{4c^2(5c^4-3)}{(1+c^4)^3}\right| \le \frac{(4)(2^2)\left((5)(2^4)-3\right)}{\left(1+1^4\right)^3} = 154$$

$$|E_n| = \left|-\frac{(2-1)^3}{(12)8^2} f''(c)\right| = \frac{1}{(12)(64)}|f''(c)| \le \frac{154}{768} \approx 0.2005$$

Remark: A plot of f'' shows that in fact $|f''(c)| < 1.5$, so $|E_n| < 0.002$.

34. $\int_0^4 \frac{1}{1+2x}dx \approx \frac{0.5}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_7)+f(x_8)] \approx 1.1050$

$\left|f^{(4)}(c)\right| = \left|\frac{384}{(1+2c)^5}\right| \le 384$

$|E_n| = \left|-\frac{(4-0)^5}{180\cdot 8^4}\cdot f^{(4)}(c)\right| \le \frac{4^5\cdot 384}{180\cdot 8^4} = \frac{8}{15}$

35. $|f''(c)| = \left|\frac{4c^2(5c^4-3)}{(1+c^4)^3}\right| \le \frac{(4)(2^2)\left((5)(2^4)+3\right)}{\left(1+1^4\right)^3} = 166$

$|E_n| = \left|-\frac{(2-1)^3}{12n^2}f''(c)\right| = \frac{1}{12n^2}|f''(c)| \le \frac{166}{12n^2} < 0.0001$

$n^2 > \frac{166}{(12)(0.0001)} \approx 138{,}333$ so $n > \sqrt{138{,}333} \approx 371.9$ Round up to $n = 372$.

Remark: A plot of f'' shows that in fact $|f''(c)| < 1.5$ which leads to $n = 36$.

36. $\left|f^{(4)}(c)\right| = \left|\frac{384}{(1+2c)^5}\right| \le 384$

$|E_n| = \left|-\frac{(4-0)^5}{180\cdot n^4}\cdot f^{(4)}(c)\right| \le \frac{4^5\cdot 384}{180\cdot n^4} < 0.0001$

$n^4 > \frac{4^5\cdot 384}{180(0.0001)} \approx 21{,}845{,}333$, so $n \approx 68.4$. Round up to $n = 69$.

37. The integrand is decreasing and concave up. Therefore, we get:
Midpoint Rule, Trapezoidal rule, Left Riemann Sum

Review and Preview Problems

1. $\frac{1}{2}-\left(\frac{1}{2}\right)^2 = \frac{1}{2}-\frac{1}{4} = \frac{1}{4}$

2. $x - x^2$

3. the distance between $(1,4)$ and $(\sqrt[3]{4},4)$ is $\sqrt[3]{4}$ -1

4. the distance between $\left(\frac{y}{4}, y\right)$ and $\left(\sqrt[3]{y}, y\right)$ is $\sqrt[3]{y} - \frac{y}{4}$

5. the distance between (2,4) and (1,1) is $\sqrt{(2-1)^2+(4-1)^2} = \sqrt{10}$

6. $\sqrt{(x+h-x)^2+\left((x+h)^2-x^2\right)^2}$
$= \sqrt{h^2+\left(2xh+h^2\right)^2}$

7. $V = (\pi\cdot 2^2)0.4 = 1.6\pi$

8. $V = [\pi(4^2-1^2)]1 = 15\pi$

9. $V = [\pi(r_2^2 - r_1^2)]\Delta x$

10. $V = [\pi(5^2-4.5^2)]6 = 28.5\pi$

11. $\int_{-1}^{2}\left(x^4-2x^3+2\right)dx=\left[\frac{x^5}{5}-\frac{x^4}{2}+2x\right]_{-1}^{2}$

$=\frac{12}{5}-\left(-\frac{27}{10}\right)=\frac{51}{10}$

12. $\int_0^3 y^{2/3}dy=\frac{3}{5}\cdot y^{5/3}\Big|_0^3=\frac{3}{5}\cdot 3^{5/3}\approx 3.74$

13. $\int_0^2\left(1-\frac{x^2}{2}+\frac{x^4}{16}\right)dx=\left[x-\frac{x^3}{6}+\frac{x^5}{80}\right]_0^2=\frac{16}{15}$

14. Let $u=1+\frac{9}{4}x;$ then $du=\frac{9}{4}dx$ and

$\int\sqrt{1+\frac{9}{4}x}\,dx=\frac{4}{9}\int\sqrt{u}\,du=\frac{4}{9}\frac{2}{3}u^{3/2}+C$

$=\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}+C$

Thus, $\int_1^4\sqrt{1+\frac{9}{4}x}\,dx=\left[\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\right]_1^4$

$=\frac{8}{27}\left(10^{3/2}-\frac{13^{3/2}}{8}\right)\approx 7.63$

CHAPTER 5 Applications of the Integral

5.1 Concepts Review

1. $\int_a^b f(x)dx; -\int_a^b f(x)dx$
2. slice, approximate, integrate
3. $g(x) - f(x);\ f(x) = g(x)$
4. $\int_c^d [q(y) - p(y)]dy$

Problem Set 5.1

1. Slice vertically.
$\Delta A \approx (x^2 + 1)\Delta x$
$A = \int_{-1}^2 (x^2 + 1)dx = \left[\frac{1}{3}x^3 + x\right]_{-1}^2 = 6$

2. Slice vertically.
$\Delta A \approx (x^3 - x + 2)\Delta x$
$A = \int_{-1}^2 (x^3 - x + 2)dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 + 2x\right]_{-1}^2 = \frac{33}{4}$

3. Slice vertically.
$\Delta A \approx \left[(x^2 + 2) - (-x)\right]\Delta x = (x^2 + x + 2)\Delta x$
$A = \int_{-2}^2 (x^2 + x + 2)dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right]_{-2}^2$
$= \left(\frac{8}{3} + 2 + 4\right) - \left(-\frac{8}{3} + 2 - 4\right) = \frac{40}{3}$

4. Slice vertically.
$\Delta A \approx -(x^2 + 2x - 3)\Delta x = (-x^2 - 2x + 3)\Delta x$
$A = \int_{-3}^1 (-x^2 - 2x + 3)dx = \left[-\frac{1}{3}x^3 - x^2 + 3x\right]_{-3}^1 = \frac{32}{3}$

5. To find the intersection points, solve $2 - x^2 = x$.
$x^2 + x - 2 = 0$
$(x + 2)(x - 1) = 0$
$x = -2, 1$
Slice vertically.
$\Delta A \approx \left[(2 - x^2) - x\right]\Delta x = (-x^2 - x - 2)\Delta x$
$A = \int_{-2}^1 (-x^2 - x + 2)dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^1$
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$

6. To find the intersection points, solve
$x + 4 = x^2 - 2$.
$x^2 - x - 6 = 0$
$(x + 2)(x - 3) = 0$
$x = -2, 3$
Slice vertically.
$\Delta A \approx \left[(x + 4) - (x^2 - 2)\right]\Delta x = (-x^2 + x + 6)\Delta x$
$A = \int_{-2}^3 (-x^2 + x + 6)\,dx = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right]_{-2}^3$
$\left(-9 + \frac{9}{2} + 18\right) - \left(\frac{8}{3} + 2 - 12\right) = \frac{125}{6}$

7. Solve $x^3 - x^2 - 6x = 0$.
$x(x^2 - x - 6) = 0$
$x(x + 2)(x - 3) = 0$
$x = -2, 0, 3$
Slice vertically.
$\Delta A_1 \approx (x^3 - x^2 - 6x)\Delta x$
$\Delta A_2 \approx -(x^3 - x^2 - 6x)\Delta x = (-x^3 + x^2 + 6x)\Delta x$
$A = A_1 + A_2$
$= \int_{-2}^0 (x^3 - x^2 - 6x)dx + \int_0^3 (-x^3 + x^2 + 6x)dx$
$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2\right]_{-2}^0 + \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + 3x^2\right]_0^3$
$= \left[0 - \left(4 + \frac{8}{3} - 12\right)\right] + \left[-\frac{81}{4} + 9 + 27 - 0\right]$
$= \frac{16}{3} + \frac{63}{4} = \frac{253}{12}$

8. To find the intersection points, solve
$-x + 2 = x^2$.
$x^2 + x - 2 = 0$
$(x + 2)(x - 1) = 0$
$x = -2, 1$
Slice vertically.
$\Delta A \approx \left[(-x + 2) - x^2\right]\Delta x = (-x^2 - x + 2)\Delta x$
$A = \int_{-2}^1 (-x^2 - x + 2)dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^1$
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$

9. To find the intersection points, solve

$y+1=3-y^2$.

$y^2+y-2=0$

$(y + 2)(y - 1) = 0$

$y = -2, 1$

Slice horizontally.

$$\Delta A \approx \left[(3-y^2)-(y+1)\right]\Delta y = (-y^2-y+2)\Delta y$$

$$A=\int_{-2}^{1}(-y^2-y+2)dy = \left[-\frac{1}{3}y^3-\frac{1}{2}y^2+2y\right]_{-2}^{1}$$

$$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)=\frac{9}{2}$$

10. To find the intersection point, solve $y^2=6-y$.

$y^2+y-6=0$

$(y + 3)(y - 2) = 0$

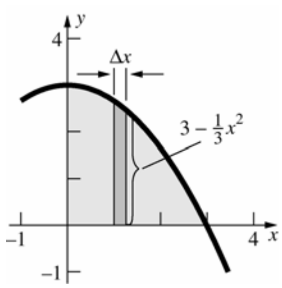
$y = -3, 2$

Slice horizontally.

$$\Delta A \approx \left[(6-y)-y^2\right]\Delta y = (-y^2-y+6)\Delta y$$

$$A=\int_{0}^{2}(-y^2-y+6)dy = \left[-\frac{1}{3}y^3-\frac{1}{2}y^2+6y\right]_{0}^{2} = \frac{22}{3}$$

11.

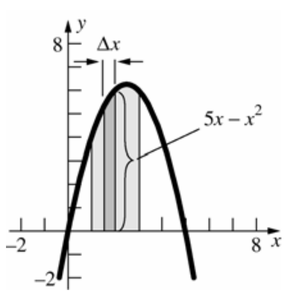


$$\Delta A \approx \left(3-\frac{1}{3}x^2\right)\Delta x$$

$$A=\int_0^3\left(3-\frac{1}{3}x^2\right)dx=\left[3x-\frac{1}{9}x^3\right]_0^3=9-3=6$$

Estimate the area to be (3)(2) = 6.

12.

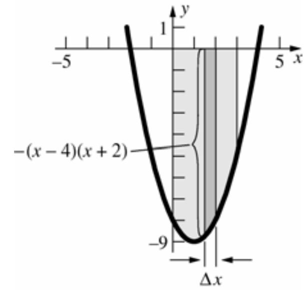


$$\Delta A \approx (5x-x^2)\Delta x$$

$$A=\int_1^3(5x-x^2)dx=\left[\frac{5}{2}x^2-\frac{1}{3}x^3\right]_1^3\approx 11.33$$

Estimate the area to be $(2)\left(5\frac{1}{2}\right)=11$.

13.



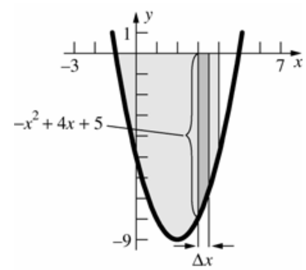
$$\Delta A \approx -(x-4)(x+2)\Delta x = (-x^2+2x+8)\Delta x$$

$$A=\int_0^3(-x^2+2x+8)dx=\left[-\frac{1}{3}x^3+x^2+8x\right]_0^3$$

$= -9 + 9 + 24 = 24$

Estimate the area to be (3)(8) = 24.

14.



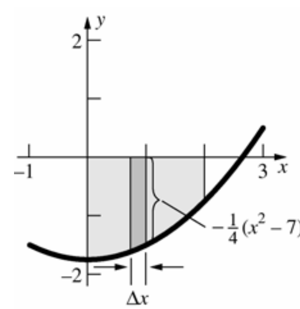
$$\Delta A \approx -(x^2-4x-5)\Delta x = (-x^2+4x+5)\Delta x$$

$$A=\int_{-1}^4(-x^2+4x+5)dx = \left[-\frac{1}{3}x^3+2x^2+5x\right]_{-1}^4$$

$$=\left(-\frac{64}{3}+32+20\right)-\left(\frac{1}{3}+2-5\right)=\frac{100}{3}\approx 33.33$$

Estimate the area to be $(5)\left(6\frac{1}{2}\right)=32\frac{1}{2}$.

15.



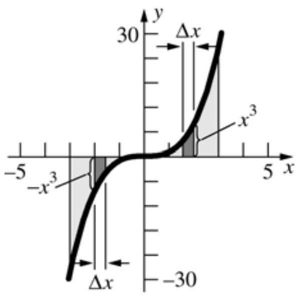
$$\Delta A \approx -\frac{1}{4}(x^2-7)\Delta x$$

$$A=\int_0^2-\frac{1}{4}(x^2-7)dx=-\frac{1}{4}\left[\frac{1}{3}x^3-7x\right]_0^2$$

$$=-\frac{1}{4}\left(\frac{8}{3}-14\right)=\frac{17}{6}\approx 2.83$$

Estimate the area to be $(2)\left(1\frac{1}{2}\right)=3$.

16.



$\Delta A_1 \approx -x^3 \Delta x$

$\Delta A_2 \approx x^3 \Delta x$

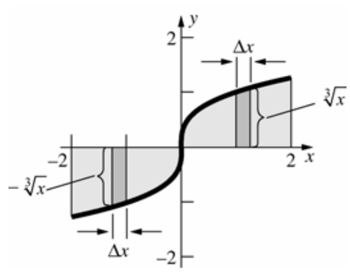
$A = A_1 + A_2 = \int_{-3}^{0} -x^3 dx + \int_0^3 x^3 dx$

$= \left[-\frac{1}{4}x^4\right]_{-3}^{0} + \left[\frac{1}{4}x^4\right]_0^3 = \left(\frac{81}{4}\right) + \left(\frac{81}{4}\right) = \frac{81}{2}$

= 40.5

Estimate the area to be (3)(7) + (3)(7) = 42.

17.



$\Delta A_1 \approx -\sqrt[3]{x}\, \Delta x$

$\Delta A_2 \approx \sqrt[3]{x}\, \Delta x$

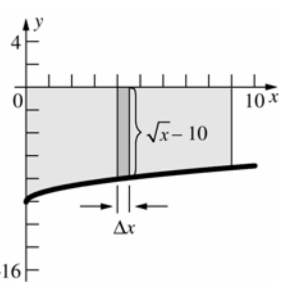
$A = A_1 + A_2 = \int_{-2}^{0} -\sqrt[3]{x}\, dx + \int_0^2 \sqrt[3]{x}\, dx$

$= \left[-\frac{3}{4}x^{4/3}\right]_{-2}^{0} + \left[\frac{3}{4}x^{4/3}\right]_0^2 = \left(\frac{3\sqrt[3]{2}}{2}\right) + \left(\frac{3\sqrt[3]{2}}{2}\right)$

$= 3\sqrt[3]{2} \approx 3.78$

Estimate the area to be (2)(1) + (2)(1) = 4.

18.



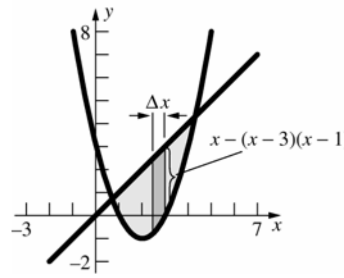
$\Delta A \approx -(\sqrt{x} - 10)\Delta x = (10 - \sqrt{x})\Delta x$

$A = \int_0^9 (10 - \sqrt{x})\, dx = \left[10x - \frac{2}{3}x^{3/2}\right]_0^9$

= 90 – 18 = 72

Estimate the area to be 9 · 8 =72.

19.



$\Delta A \approx \left[x - (x-3)(x-1)\right]\Delta x$

$= \left[x - (x^2 - 4x + 3)\right]\Delta x = (-x^2 + 5x - 3)\Delta x$

To find the intersection points, solve
$x = (x - 3)(x - 1)$.

$x^2 - 5x + 3 = 0$

$x = \frac{5 \pm \sqrt{25 - 12}}{2}$

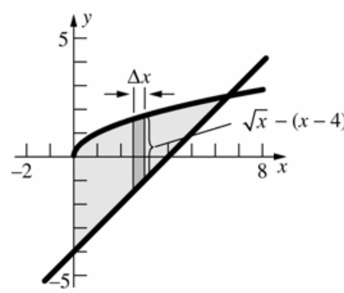
$x = \frac{5 \pm \sqrt{13}}{2}$

$A = \int_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} (-x^2 + 5x - 3)dx$

$= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 3x\right]_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} = \frac{13\sqrt{13}}{6} \approx 7.81$

Estimate the area to be $\frac{1}{2}(4)(4) = 8$.

20.



$\Delta A \approx \left[\sqrt{x} - (x - 4)\right]\Delta x = \left(\sqrt{x} - x + 4\right)\Delta x$

To find the intersection point, solve
$\sqrt{x} = (x - 4)$.

$x = (x - 4)^2$

$x^2 - 9x + 16 = 0$

$x = \frac{9 \pm \sqrt{81 - 64}}{2}$

$x = \frac{9 \pm \sqrt{17}}{2}$

$\left(x = \frac{9 - \sqrt{17}}{2} \text{ is extraneous so } x = \frac{9 + \sqrt{17}}{2}.\right)$

$$A = \int_0^{\frac{9+\sqrt{17}}{2}} \left(\sqrt{x} - x + 4\right) dx$$

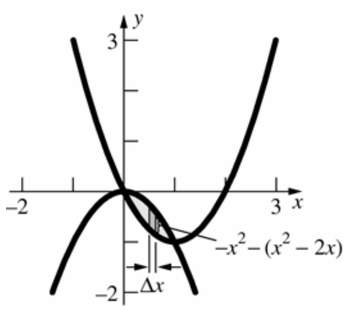
$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 4x\right]_0^{\frac{9+\sqrt{17}}{2}}$$

$$= \frac{2}{3}\left(\frac{9+\sqrt{17}}{2}\right)^{3/2} - \frac{1}{2}\left(\frac{9+\sqrt{17}}{2}\right)^2 + 4\left(\frac{9+\sqrt{17}}{2}\right)$$

$$= \frac{2}{3}\left(\frac{9+\sqrt{17}}{2}\right)^{3/2} + \frac{23}{4} - \frac{\sqrt{17}}{4} \approx 15.92$$

Estimate the area to be $\frac{1}{2}\left(5\frac{1}{2}\right)\left(5\frac{1}{2}\right) = 15\frac{1}{8}$.

21.



$$\Delta A \approx \left[-x^2 - (x^2 - 2x)\right]\Delta x = (-2x^2 + 2x)\Delta x$$

To find the intersection points, solve

$-x^2 = x^2 - 2x$.

$2x^2 - 2x = 0$

$2x(x - 1) = 0$

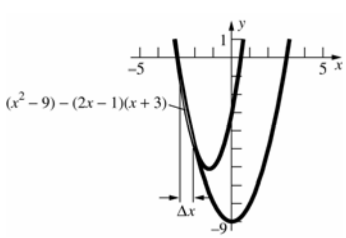
$x = 0, x = 1$

$$A = \int_0^1 (-2x^2 + 2x)dx = \left[-\frac{2}{3}x^3 + x^2\right]_0^1$$

$$= -\frac{2}{3} + 1 = \frac{1}{3} \approx 0.33$$

Estimate the area to be $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$.

22.



$$\Delta A \approx \left[(x^2 - 9) - (2x - 1)(x + 3)\right]\Delta x$$

$$= \left[(x^2 - 9) - (2x^2 + 5x - 3)\right]\Delta x$$

$$= (-x^2 - 5x - 6)\Delta x$$

To find the intersection points, solve

$(2x - 1)(x + 3) = x^2 - 9$.

$x^2 + 5x + 6 = 0$

$(x + 3)(x + 2) = 0$

$x = -3, -2$

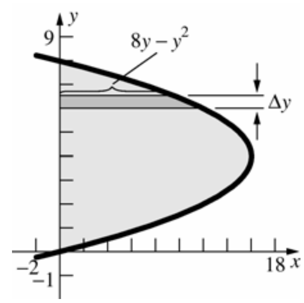
$$A = \int_{-3}^{-2} (-x^2 - 5x - 6)dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{5}{2}x^2 - 6x\right]_{-3}^{-2}$$

$$= \left(\frac{8}{3} - 10 + 12\right) - \left(9 - \frac{45}{2} + 18\right) = \frac{1}{6} \approx 0.17$$

Estimate the area to be $\frac{1}{2}(1)\left(5 - 4\frac{2}{3}\right) = \frac{1}{6}$.

23.



$$\Delta A \approx (8y - y^2)\Delta y$$

To find the intersection points, solve

$8y - y^2 = 0$.

$y(8 - y) = 0$

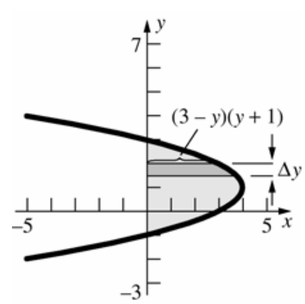
$y = 0, 8$

$$A = \int_0^8 (8y - y^2)\,dy = \left[4y^2 - \frac{1}{3}y^3\right]_0^8$$

$$= 256 - \frac{512}{3} = \frac{256}{3} \approx 85.33$$

Estimate the area to be (16)(5) = 80.

24.



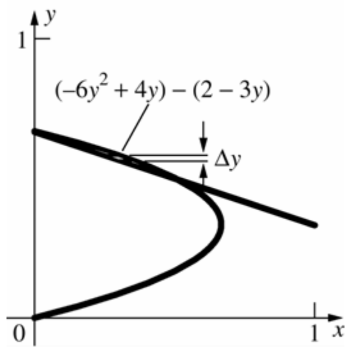
$$\Delta A \approx (3 - y)(y + 1)\Delta y = (-y^2 + 2y + 3)\Delta y$$

$$A = \int_{-1}^{3} (-y^2 + 2y + 3)dy = \left[-\frac{1}{3}y^3 + y^2 + 3y\right]_{-1}^{3}$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) = \frac{32}{3} \approx 10.67$$

Estimate the area to be $(4)\left(2\frac{1}{2}\right) = 10$.

25.



$\Delta A \approx \left[(-6y^2+4y)-(2-3y)\right]\Delta y$

$= (-6y^2+7y-2)\Delta y$

To find the intersection points, solve

$-6y^2+4y = 2-3y.$

$6y^2-7y+2=0$

$(2y-1)(3y-2)=0$

$y = \frac{1}{2}, \frac{2}{3}$

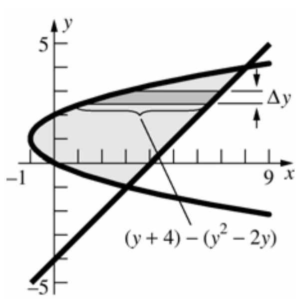
$A = \int_{1/2}^{2/3}(-6y^2+7y-2)dy = \left[-2y^3+\frac{7}{2}y^2-2y\right]_{1/2}^{2/3}$

$= \left(-\frac{16}{27}+\frac{14}{9}-\frac{4}{3}\right)-\left(-\frac{1}{4}+\frac{7}{8}-1\right) = \frac{1}{216} \approx 0.0046$

Estimate the area to be

$\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)-\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)=\frac{1}{120}.$

26.



$\Delta A \approx \left[(y+4)-(y^2-2y)\right]\Delta y = (-y^2+3y+4)\Delta y$

To find the intersection points, solve

$y^2-2y = y+4.$

$y^2-3y-4=0$

$(y+1)(y-4)=0$

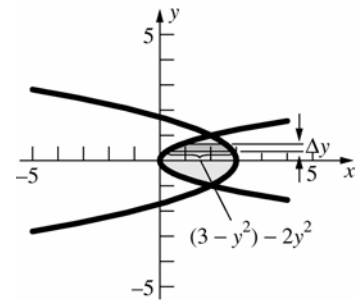
$y=-1, 4$

$A = \int_{-1}^{4}(-y^2+3y+4)dy = \left[-\frac{1}{3}y^3+\frac{3}{2}y^2+4y\right]_{-1}^{4}$

$= \left(-\frac{64}{3}+24+16\right)-\left(\frac{1}{3}+\frac{3}{2}-4\right)=\frac{125}{6}\approx 20.83$

Estimate the area to be (7)(3) = 21.

27.



$\Delta A \approx \left[(3-y^2)-2y^2\right]\Delta y = (-3y^2+3)\Delta y$

To find the intersection points, solve

$2y^2 = 3-y^2.$

$3y^2-3=0$

$3(y+1)(y-1)=0$

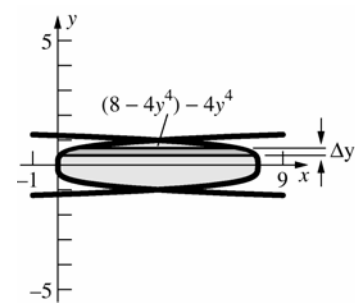
$y=-1, 1$

$A = \int_{-1}^{1}(-3y^2+3)dy = \left[-y^3+3y\right]_{-1}^{1}$

$= (-1+3)-(1-3) = 4$

Estimate the value to be (2)(2) = 4.

28.



$\Delta A \approx \left[(8-4y^4)-(4y^4)\right]\Delta y = (8-8y^4)\Delta y$

To find the intersection points, solve

$4y^4 = 8-4y^4.$

$8y^4=8$

$y^4=1$

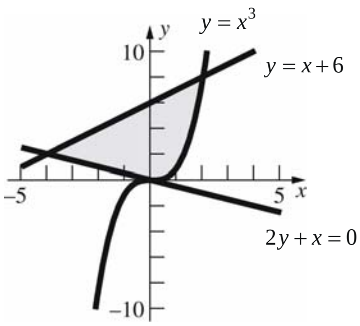
$y=\pm 1$

$A = \int_{-1}^{1}(8-8y^4)dy = \left[8y-\frac{8}{5}y^5\right]_{-1}^{1}$

$= \left(8-\frac{8}{5}\right)-\left(-8+\frac{8}{5}\right)=\frac{64}{5}=12.8$

Estimate the area to be $(8)\left(1\frac{1}{2}\right)=12.$

29.



Let R_1 be the region bounded by $2y + x = 0$, $y = x + 6$, and $x = 0$.

$$A(R_1) = \int_{-4}^{0}\left[(x+6) - \left(-\frac{1}{2}x\right)\right]dx$$

$$= \int_{-4}^{0}\left(\frac{3}{2}x + 6\right)dx$$

Let R_2 be the region bounded by $y = x + 6$, $y = x^3$, and $x = 0$.

$$A(R_2) = \int_0^2\left[(x+6) - x^3\right]dx = \int_0^2 (-x^3 + x + 6)dx$$

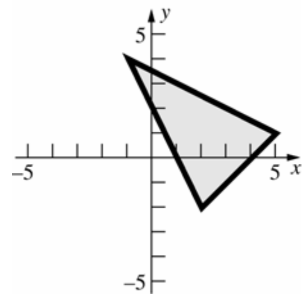
$$A(R) = A(R_1) + A(R_2)$$

$$= \int_{-4}^{0}\left(\frac{3}{2}x + 6\right)dx + \int_0^2 (-x^3 + x + 6)dx$$

$$= \left[\frac{3}{4}x^2 + 6x\right]_{-4}^{0} + \left[-\frac{1}{4}x^4 + \frac{1}{2}x^2 + 6x\right]_0^2$$

$= 12 + 10 = 22$

30.



An equation of the line through (–1, 4) and (5, 1) is $y = -\frac{1}{2}x + \frac{7}{2}$. An equation of the line through (–1, 4) and (2, –2) is $y = -2x + 2$. An equation of the line through (2, –2) and (5, 1) is $y = x - 4$. Two integrals must be used. The left-hand part of the triangle has area

$$\int_{-1}^{2}\left[-\frac{1}{2}x + \frac{7}{2} - (-2x+2)\right]dx = \int_{-1}^{2}\left(\frac{3}{2}x + \frac{3}{2}\right)dx.$$

The right-hand part of the triangle has area

$$\int_2^5\left[-\frac{1}{2}x + \frac{7}{2} - (x-4)\right]dx = \int_2^5\left(-\frac{3}{2}x + \frac{15}{2}\right)dx.$$

The triangle has area

$$\int_{-1}^{2}\left(\frac{3}{2}x + \frac{3}{2}\right)dx + \int_2^5\left(-\frac{3}{2}x + \frac{15}{2}\right)dx$$

$$= \left[\frac{3}{4}x^2 + \frac{3}{2}x\right]_{-1}^{2} + \left[-\frac{3}{4}x^2 + \frac{15}{2}x\right]_2^5$$

$$= \frac{27}{4} + \frac{27}{4} = \frac{27}{2} = 13.5$$

31. $\int_{-1}^{9}(3t^2 - 24t + 36)dt = \left[t^3 - 12t^2 + 36t\right]_{-1}^{9}$ = (729 – 972 + 324) – (–1 – 12 – 36) = 130

The displacement is 130 ft. Solve $3t^2 - 24t + 36 = 0$.

$3(t-2)(t-6) = 0$

$t = 2, 6$

$$|V(t)| = \begin{cases} 3t^2 - 24t + 36 & t \le 2, t \ge 6 \\ -3t^2 + 24t - 36 & 2 < t < 6 \end{cases}$$

$$\int_{-1}^{9}\left|3t^2 - 24t + 36\right|dt = \int_{-1}^{2}(3t^2 - 24t + 36)\,dt + \int_2^6(-3t^2 + 24t - 36)\,dt + \int_6^9(3t^2 - 24t + 36)\,dt$$

$$= \left[t^3 - 12t^2 + 36t\right]_{-1}^{2} + \left[-t^3 + 12t^2 - 36t\right]_2^6 + \left[t^3 - 12t^2 + 36t\right]_6^9 = 81 + 32 + 81 = 194$$

The total distance traveled is 194 feet.

32. $\int_0^{3\pi/2}\left(\frac{1}{2} + \sin 2t\right)dt = \left[\frac{1}{2}t - \frac{1}{2}\cos 2t\right]_0^{3\pi/2} = \left(\frac{3\pi}{4} + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right) = \frac{3\pi}{4} + 1$

The displacement is $\frac{3\pi}{4} + 1 \approx 3.36$ feet. Solve $\frac{1}{2} + \sin 2t = 0$ for $0 \le t \le \frac{3\pi}{2}$.

$$\sin 2t = -\frac{1}{2} \Rightarrow 2t = \frac{7\pi}{6}, \frac{11\pi}{6} \Rightarrow t = \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$\left|\frac{1}{2}+\sin 2t\right| = \begin{cases} \frac{1}{2}+\sin 2t & 0 \le t \le \frac{7\pi}{12}, \frac{11\pi}{12} \le t \le \frac{3\pi}{2} \\ -\frac{1}{2}-\sin 2t & \frac{7\pi}{12} < t < \frac{11\pi}{12} \end{cases}$$

$$\int_0^{3\pi/2}\left|\frac{1}{2}+\sin 2t\right|dt = \int_0^{7\pi/12}\left(\frac{1}{2}+\sin 2t\right)dt + \int_{7\pi/12}^{11\pi/12}\left(-\frac{1}{2}-\sin 2t\right)dt + \int_{11\pi/12}^{3\pi/2}\left(\frac{1}{2}+\sin 2t\right)dt$$

$$= \left[\frac{1}{2}t-\frac{1}{2}\cos 2t\right]_0^{7\pi/12} + \left[-\frac{1}{2}t+\frac{1}{2}\cos 2t\right]_{7\pi/12}^{11\pi/12} + \left[\frac{1}{2}t-\frac{1}{2}\cos 2t\right]_{11\pi/12}^{3\pi/2}$$

$$= \left(\frac{7\pi}{24}+\frac{\sqrt{3}}{4}+\frac{1}{2}\right)+\left(-\frac{\pi}{6}+\frac{\sqrt{3}}{2}\right)+\left(\frac{7\pi}{24}+\frac{\sqrt{3}}{4}+\frac{1}{2}\right) = \frac{5\pi}{12}+\sqrt{3}+1$$

The total distance traveled is $\frac{5\pi}{12}+\sqrt{3}+1 \approx 4.04$ feet.

33. $s(t) = \int v(t)dt = \int (2t-4)dt = t^2-4t+C$

Since $s(0) = 0$, $C = 0$ and $s(t) = t^2-4t$. $s = 12$ when $t = 6$, so it takes the object 6 seconds to get $s = 12$.

$$|2t-4| = \begin{cases} 4-2t & 0 \le t < 2 \\ 2t-4 & 2 \le t \end{cases}$$

$\int_0^2 |2t-4|\,dt = \left[-t^2+4t\right]_0^2 = 4,$ so the object travels a distance of 4 cm in the first two seconds.

$$\int_2^x |2t-4|\,dt = \left[t^2-4t\right]_2^x = x^2-4x+4$$

$x^2-4x+4 = 8$ when $x = 2+2\sqrt{2}$, so the object takes $2+2\sqrt{2} \approx 4.83$ seconds to travel a total distance of 12 centimeters.

34. **a.** $A = \int_1^6 x^{-2}dx = \left[-\frac{1}{x}\right]_1^6 = -\frac{1}{6}+1 = \frac{5}{6}$

b. Find c so that $\int_1^c x^{-2}dx = \frac{5}{12}$.

$$\int_1^c x^{-2}dx = \left[-\frac{1}{x}\right]_1^c = 1-\frac{1}{c}$$

$$1-\frac{1}{c} = \frac{5}{12}, c = \frac{12}{7}$$

The line $x = \frac{12}{7}$ bisects the area.

c. Slicing the region horizontally, the area is $\int_{1/36}^1 \frac{1}{\sqrt{y}}dy + \left(\frac{1}{36}\right)(5)$. Since $\frac{5}{36} < \frac{5}{12}$ the line that bisects the area is between $y = \frac{1}{36}$ and $y = 1$, so we find d such that

$$\int_d^1 \frac{1}{\sqrt{y}}dy = \frac{5}{12}; \int_d^1 \frac{1}{\sqrt{y}}dy = \left[2\sqrt{y}\right]_d^1$$

$$= 2-2\sqrt{d}\ ;\ 2-2\sqrt{d} = \frac{5}{12};$$

$$d = \frac{361}{576} \approx 0.627.$$

The line $y = 0.627$ approximately bisects the area.

35. Equation of line through (–2, 4) and (3, 9): $y = x + 6$
Equation of line through (2, 4) and (–3, 9): $y = –x + 6$

$$A(A) = \int_{-3}^0 [9-(-x+6)]dx + \int_0^3 [9-(x+6)]dx$$

$$= \int_{-3}^0 (3+x)dx + \int_0^3 (3-x)dx$$

$$= \left[3x+\frac{1}{2}x^2\right]_{-3}^0 + \left[3x-\frac{1}{2}x^2\right]_0^3 = \frac{9}{2}+\frac{9}{2} = 9$$

$$A(B) = \int_{-3}^{-2} [(-x+6)-x^2]dx + \int_{-2}^0 [(-x+6)-(x+6)]dx$$

$$= \int_{-3}^{-2} (-x^2-x+6)dx + \int_{-2}^0 (-2x)dx$$

$$= \left[-\frac{1}{3}x^3-\frac{1}{2}x^2+6x\right]_{-3}^{-2} + \left[-x^2\right]_{-2}^0 = \frac{37}{6}$$

$$A(C) = A(B) = \frac{37}{6} \text{ (by symmetry)}$$

$$A(D) = \int_{-2}^{0}[(x+6) - x^2]dx + \int_{0}^{2}[(-x+6) - x^2]dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right]_{-2}^{0} + \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_{0}^{2}$$

$$= \frac{44}{3}$$

$A(A) + A(B) + A(C) + A(D) = 36$

$$A(A+B+C+D) = \int_{-3}^{3}(9 - x^2)dx = \left[9x - \frac{1}{3}x^3\right]_{-3}^{3}$$

$= 36$

36. Let $f(x)$ be the width of region 1 at every x.

$\Delta A_1 \approx f(x)\Delta x$, so $A_1 = \int_a^b f(x)dx$.

Let $g(x)$ be the width of region 2 at every x.

$\Delta A_2 \approx g(x)\Delta x$, so $A_2 = \int_a^b g(x)dx$.

Since $f(x) = g(x)$ at every x in $[a, b]$,

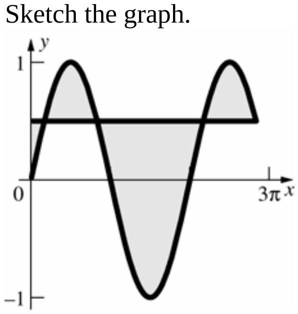
$A_1 = \int_a^b f(x)dx = \int_a^b g(x)dx = A_2$.

37. The height of the triangular region is given by for $0 \le x \le 1$. We need only show that the height of the second region is the same in order to apply Cavalieri''s Principle. The height of the second region is

$$h_2 = (x^2 - 2x + 1) - (x^2 - 3x + 1)$$
$$= x^2 - 2x + 1 - x^2 + 3x - 1$$
$$= x \text{ for } 0 \le x \le 1.$$

Since $h_1 = h_2$ over the same closed interval, we can conclude that their areas are equal.

38. Sketch the graph.



Solve $\sin x = \frac{1}{2}$ for $0 \le x \le \frac{17\pi}{6}$.

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

The area of the trapped region is

$$\int_0^{\pi/6}\left(\frac{1}{2} - \sin x\right)dx + \int_{\pi/6}^{5\pi/6}\left(\sin x - \frac{1}{2}\right)dx$$

$$+\int_{5\pi/6}^{13\pi/6}\left(\frac{1}{2} - \sin x\right)dx + \int_{13\pi/6}^{17\pi/6}\left(\sin x - \frac{1}{2}\right)dx$$

$$= \left[\frac{1}{2}x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \frac{1}{2}x\right]_{\pi/6}^{5\pi/6}$$

$$+ \left[\frac{1}{2}x + \cos x\right]_{5\pi/6}^{13\pi/6} + \left[-\cos x - \frac{1}{2}x\right]_{13\pi/6}^{17\pi/6}$$

$$= \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\right) + \left(\sqrt{3} - \frac{\pi}{3}\right) + \left(\sqrt{3} + \frac{2\pi}{3}\right)$$

$$+ \left(\sqrt{3} - \frac{\pi}{3}\right) = \frac{\pi}{12} + \frac{7\sqrt{3}}{2} - 1 \approx 5.32$$

5.2 Concepts Review

1. $\pi r^2 h$

2. $\pi(R^2 - r^2)h$

3. $\pi x^4 \Delta x$

4. $\pi[(x^2 + 2)^2 - 4]\Delta x$

Problem Set 5.2

1. Slice vertically.

$$\Delta V \approx \pi(x^2 + 1)^2 \Delta x = \pi(x^4 + 2x^2 + 1)\Delta x$$

$$V = \pi\int_0^2 (x^4 + 2x^2 + 1)dx$$

$$= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^2 = \pi\left(\frac{32}{5} + \frac{16}{3} + 2\right) = \frac{206\pi}{15}$$

≈ 43.14

2. Slice vertically.

$$\Delta V \approx \pi(-x^2 + 4x)^2 \Delta x = \pi(x^4 - 8x^3 + 16x^2)\Delta x$$

$$V = \pi\int_0^3 (x^4 - 8x^3 + 16x^2)dx$$

$$= \pi\left[\frac{1}{5}x^5 - 2x^4 + \frac{16}{3}x^3\right]_0^3$$

$$= \pi\left(\frac{243}{5} - 162 + 144\right)$$

$$= \frac{153\pi}{5} \approx 96.13$$

3. a. Slice vertically.

$$\Delta V \approx \pi(4-x^2)^2\Delta x = \pi(16-8x^2+x^4)\Delta x$$

$$V = \pi\int_0^2 (16-8x^2+x^4)dx$$

$$= \frac{256\pi}{15} \approx 53.62$$

b. Slice horizontally.

$$x = \sqrt{4-y}$$

Note that when $x = 0$, $y = 4$.

$$\Delta V \approx \pi\left(\sqrt{4-y}\right)^2 \Delta y = \pi(4-y)\Delta y$$

$$V = \pi\int_0^4 (4-y)dy = \pi\left[4y - \frac{1}{2}y^2\right]_0^4$$

$$= \pi(16-8) = 8\pi \approx 25.13$$

4. a. Slice vertically.

$$\Delta V \approx \pi(4-2x)^2\Delta x$$

$$0 \le x \le 2$$

$$V = \pi\int_0^2 (4-2x)^2 dx = \pi\left[-\frac{1}{6}(4-2x)^3\right]_0^2$$

$$= \frac{32\pi}{3} \approx 33.51$$

b. Slice vertically.

$$x = 2 - \frac{y}{2}$$

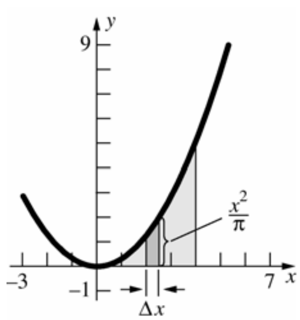
$$\Delta V \approx \pi\left(2-\frac{y}{2}\right)^2 \Delta y$$

$$0 \le y \le 4$$

$$V = \pi\int_0^4 \left(2-\frac{y}{2}\right)^2 dy = \pi\left[-\frac{2}{3}\left(2-\frac{y}{2}\right)^3\right]_0^4$$

$$= \frac{16\pi}{3} \approx 16.76$$

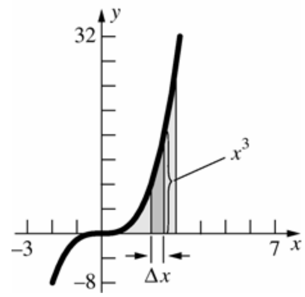
5.



$$\Delta V \approx \pi\left(\frac{x^2}{\pi}\right)^2 \Delta x = \frac{x^4}{\pi}\Delta x$$

$$V = \int_0^4 \frac{x^4}{\pi}dx = \frac{1}{\pi}\left[\frac{1}{5}x^5\right]_0^4 = \frac{1024}{5\pi} \approx 65.19$$

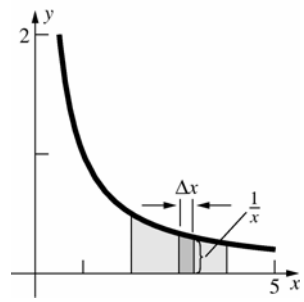
6.



$$\Delta V \approx \pi(x^3)^2\Delta x = \pi x^6 \Delta x$$

$$V = \pi\int_0^3 x^6 dx = \pi\left[\frac{1}{7}x^7\right]_0^3 = \frac{2187\pi}{7} \approx 981.52$$

7.

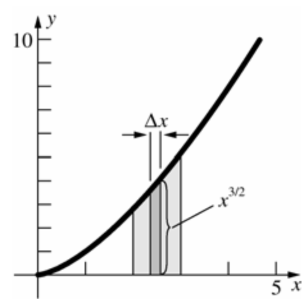


$$\Delta V \approx \pi\left(\frac{1}{x}\right)^2 \Delta x = \pi\left(\frac{1}{x^2}\right)\Delta x$$

$$V = \pi\int_2^4 \frac{1}{x^2}dx = \pi\left[-\frac{1}{x}\right]_2^4 = \pi\left(-\frac{1}{4}+\frac{1}{2}\right) = \frac{\pi}{4}$$

$$\approx 0.79$$

8.

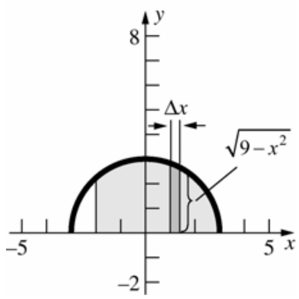


$$\Delta V \approx \pi(x^{3/2})^2\Delta x = \pi x^3 \Delta x$$

$$V = \pi\int_2^3 x^3 dx = \pi\left[\frac{1}{4}x^4\right]_2^3 = \pi\left(\frac{81}{4}-\frac{16}{4}\right)$$

$$= \frac{65\pi}{4} \approx 51.05$$

9.

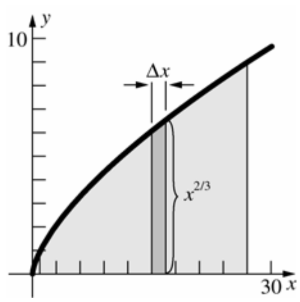


$$\Delta V \approx \pi\left(\sqrt{9-x^2}\right)^2 \Delta x = \pi(9-x^2)\Delta x$$

$$V = \pi\int_{-2}^{3} (9-x^2)dx = \pi\left[9x - \frac{1}{3}x^3\right]_{-2}^{3}$$

$$= \pi\left[(27-9) - \left(-18+\frac{8}{3}\right)\right] = \frac{100\pi}{3} \approx 104.72$$

10.

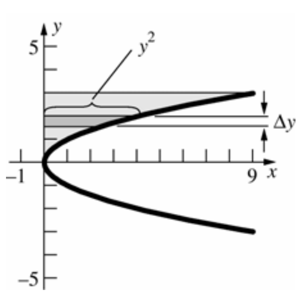


$$\Delta V \approx \pi(x^{2/3})^2 \Delta x = \pi x^{4/3}\Delta x$$

$$V = \pi\int_{1}^{27} x^{4/3}dx = \pi\left[\frac{3}{7}x^{7/3}\right]_{1}^{27} = \pi\left(\frac{6561}{7} - \frac{3}{7}\right)$$

$$= \frac{6558\pi}{7} \approx 2943.22$$

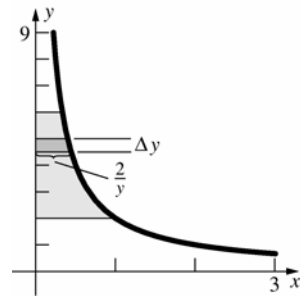
11.



$$\Delta V \approx \pi(y^2)^2 \Delta y = \pi y^4 \Delta y$$

$$V = \pi\int_{0}^{3} y^4 dy = \pi\left[\frac{1}{5}y^5\right]_{0}^{3} = \frac{243\pi}{5} \approx 152.68$$

12.

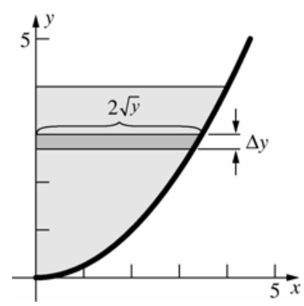


$$\Delta V \approx \pi\left(\frac{2}{y}\right)^2 \Delta y = 4\pi\left(\frac{1}{y^2}\right)\Delta y$$

$$V = 4\pi\int_{2}^{6} \frac{1}{y^2}dy = 4\pi\left[-\frac{1}{y}\right]_{2}^{6} = 4\pi\left(-\frac{1}{6}+\frac{1}{2}\right)$$

$$= \frac{4\pi}{3} \approx 4.19$$

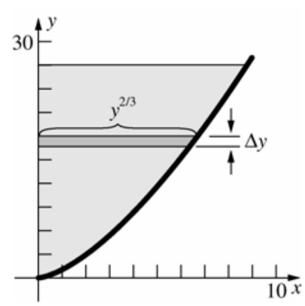
13.



$$\Delta V \approx \pi\left(2\sqrt{y}\right)^2 \Delta y = 4\pi y \Delta y$$

$$V = 4\pi\int_{0}^{4} y\,dy = 4\pi\left[\frac{1}{2}y^2\right]_{0}^{4} = 32\pi \approx 100.53$$

14.

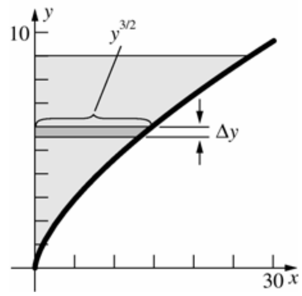


$$\Delta V \approx \pi(y^{2/3})^2 \Delta y = \pi y^{4/3}\Delta y$$

$$V = \pi\int_{0}^{27} y^{4/3}\,dy = \pi\left[\frac{3}{7}y^{7/3}\right]_{0}^{27} = \frac{6561\pi}{7}$$

$$\approx 2944.57$$

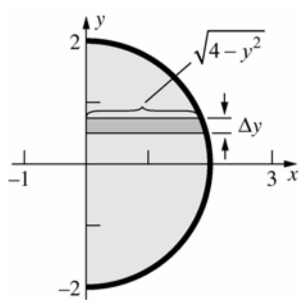
15.



$$\Delta V \approx \pi (y^{3/2})^2 \Delta y = \pi y^3 \Delta y$$

$$V = \pi \int_0^9 y^3 dy = \pi \left[\frac{1}{4} y^4 \right]_0^9 = \frac{6561\pi}{4} \approx 5153.00$$

16.



$$\Delta V \approx \pi \left(\sqrt{4 - y^2} \right)^2 \Delta y = \pi (4 - y^2) \Delta y$$

$$V = \pi \int_{-2}^{2} (4 - y^2) dy = \pi \left[4y - \frac{1}{3} y^3 \right]_{-2}^{2}$$

$$= \pi \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{32\pi}{3} \approx 33.51$$

17. The equation of the upper half of the ellipse is

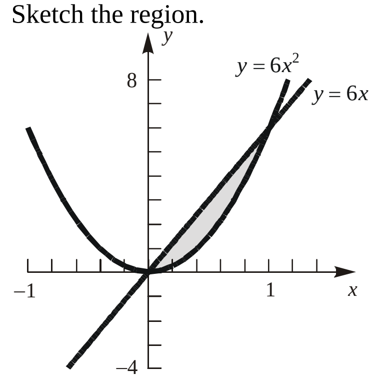
$y = b\sqrt{1 - \frac{x^2}{a^2}}$ or $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

$$V = \pi \int_{-a}^{a} \frac{b^2}{a^2} (a^2 - x^2) dx$$

$$= \frac{b^2 \pi}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a}$$

$$= \frac{b^2 \pi}{a^2} \left[\left(a^3 - \frac{a^3}{3} \right) - \left(-a^3 + \frac{a^3}{3} \right) \right] = \frac{4}{3} ab^2 \pi$$

18. Sketch the region.



To find the intersection points, solve $6x = 6x^2$.

$6(x^2 - x) = 0$

$6x(x - 1) = 0$

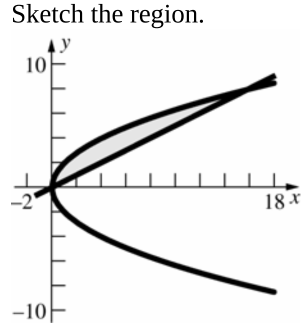
$x = 0, 1$

$$\Delta V \approx \pi \left[(6x)^2 - (6x^2)^2 \right] \Delta x = 36\pi (x^2 - x^4) \Delta x$$

$$V = 36\pi \int_0^1 (x^2 - x^4) dx = 36\pi \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_0^1$$

$$= 36\pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{24\pi}{5} \approx 15.08$$

19. Sketch the region.



To find the intersection points, solve $\frac{x}{2} = 2\sqrt{x}$.

$\frac{x^2}{4} = 4x$

$x^2 - 16x = 0$

$x(x - 16) = 0$

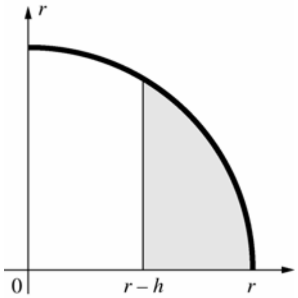
$x = 0, 16$

$$\Delta V \approx \pi \left[\left(2\sqrt{x} \right)^2 - \left(\frac{x}{2} \right)^2 \right] \Delta x = \pi \left(4x - \frac{x^2}{4} \right) \Delta x$$

$$V = \pi \int_0^{16} \left(4x - \frac{x^2}{4} \right) dx = \pi \left[2x^2 - \frac{x^3}{12} \right]_0^{16}$$

$$= \pi \left(512 - \frac{1024}{3} \right) = \frac{512\pi}{3} \approx 536.17$$

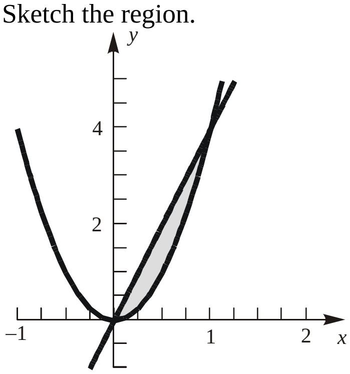
20. Sketch the region.



$$V = \pi\int_{r-h}^{r}(r^2 - x^2)dx = \pi\left[r^2x - \frac{1}{3}x^3\right]_{r-h}^{r}$$

$$= \frac{1}{3}\pi h^2(3r - h)$$

21. Sketch the region.



To find the intersection points, solve $\frac{y}{4} = \frac{\sqrt{y}}{2}$.

$$\frac{y^2}{16} = \frac{y}{4}$$

$$y^2 - 4y = 0$$

$y(y - 4) = 0$

$y = 0, 4$

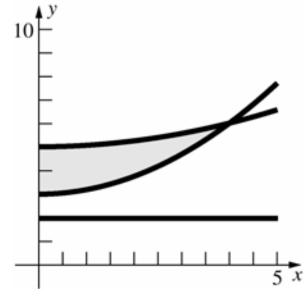
$$\Delta V \approx \pi\left[\left(\frac{\sqrt{y}}{2}\right)^2 - \left(\frac{y}{4}\right)^2\right]\Delta y = \pi\left(\frac{y}{4} - \frac{y^2}{16}\right)\Delta y$$

$$V = \pi\int_0^4\left(\frac{y}{4} - \frac{y^2}{16}\right)dy = \pi\left[\frac{y^2}{8} - \frac{y^3}{48}\right]_0^4$$

$$= \frac{2\pi}{3} \approx 2.0944$$

22. $y = \frac{3}{16}x^2 + 3,\ y = \frac{1}{16}x^2 + 5$

Sketch the region.



To find the intersection point, solve

$$\frac{3}{16}x^2 + 3 = \frac{1}{16}x^2 + 5.$$

$$\frac{1}{8}x^2 - 2 = 0$$

$$x^2 - 16 = 0$$

$(x + 4)(x - 4) = 0$

$x = -4, 4$

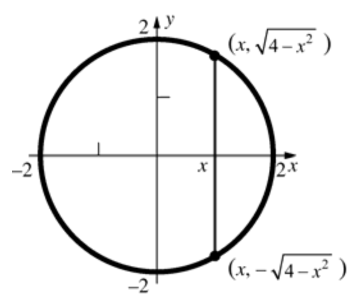
$$V = \pi\int_0^4\left[\left(\frac{1}{16}x^2 + 5 - 2\right)^2 - \left(\frac{3}{16}x^2 + 3 - 2\right)^2\right]dx$$

$$= \pi\int_0^4\left[\left(\frac{1}{256}x^4 - \frac{3}{8}x^2 + 9\right) - \left(\frac{9}{256}x^4 - \frac{3}{8}x^2 + 1\right)\right]dx$$

$$= \pi\int_0^4\left(8 - \frac{1}{32}x^4\right)dx = \pi\left[8x - \frac{1}{160}x^5\right]_0^4$$

$$= \pi\left(32 - \frac{32}{5}\right) = \frac{128\pi}{5} \approx 80.42$$

23.



The square at x has sides of length $2\sqrt{4 - x^2}$, as shown.

$$V = \int_{-2}^{2}\left(2\sqrt{4 - x^2}\right)^2 dx = \int_{-2}^{2}4(4 - x^2)dx$$

$$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 4\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] = \frac{128}{3}$$

≈ 42.67

24. The area of each cross section perpendicular to the x-axis is $\frac{1}{2}(4)\left(2\sqrt{4-x^2}\right) = 4\sqrt{4-x^2}$.

The area of a semicircle with radius 2 is $\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi$.

$V = \int_{-2}^{2} 4\sqrt{4-x^2}\,dx = 4(2\pi) = 8\pi \approx 25.13$

25. The square at x has sides of length $\sqrt{\cos x}$.

$V = \int_{-\pi/2}^{\pi/2}\cos x\,dx = [\sin x]_{-\pi/2}^{\pi/2} = 2$

26. The area of each cross section perpendicular to the x-axis is $[(1-x^2)-(1-x^4)]^2 = x^8 - 2x^6 + x^4$.

$V = \int_{-1}^{1}(x^8 - 2x^6 + x^4)dx$

$= \left[\frac{1}{9}x^9 - \frac{2}{7}x^7 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{16}{315} \approx 0.051$

27. The square at x has sides of length $\sqrt{1-x^2}$.

$V = \int_0^1 (1-x^2)dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} \approx 0.67$

28. From Problem 27 we see that horizontal cross sections of one octant of the common region are squares. The length of a side at height y is $\sqrt{r^2-y^2}$ where r is the common radius of the cylinders. The volume of the "+" can be found by adding the volumes of each cylinder and subtracting off the volume of the common region (which is counted twice). The volume of one octant of the common region is

$\int_0^r (r^2-y^2)dy = r^2y - \frac{1}{3}y^2 \Big|_0^r$

$= r^3 - \frac{1}{3}r^3 = \frac{2}{3}r^3$

Thus, the volume of the "+" is

$V =$ vol. of cylinders - vol. of common region

$=2(\pi \mathrm{r}^2 l) - 8\left(\frac{2}{3}r^3\right)$

$= 2\pi(2^2)(12) - 8\left(\frac{2}{3}(2)^3\right) = 96\pi - \frac{128}{3}$

$\approx 258.93 \text{ in}^2$

29. Using the result from Problem 28, the volume of one octant of the common region in the "+" is

$\int_0^r (r^2-y^2)dy = r^2y - \frac{1}{3}y^2 \Big|_0^r$

$= r^3 - \frac{1}{3}r^3 = \frac{2}{3}r^3$

Thus, the volume inside the "+" for two cylinders of radius r and length L is

$V =$ vol. of cylinders - vol. of common region

$= 2(\pi r^2 L) - 8\left(\frac{2}{3}r^3\right)$

$= 2\pi r^2 L - \frac{16}{3}r^3$

30. From Problem 28, the volume of one octant of the common region is $\frac{2}{3}r^3$. We can find the volume of the "T" similarly. Since the "T" has one-half the common region of the "+" in Problem 28, the volume of the "T" is given by

$V =$ vol. of cylinders - vol. of common region

$=(\pi r^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$

With $r = 2, L_1 = 12,$ and $L_2 = 8$ (inches), the volume of the "T" is

$V =$ vol. of cylinders - vol. of common region

$=(\pi \mathrm{r}^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$

$= (\pi 2^2)(12+8) - 4\left(\frac{2}{3}2^3\right)$

$= 80\pi - \frac{64}{3} \text{ in}^3$

$\approx 229.99 \text{ in}^3$

31. From Problem 30, the general form for the volume of a "T" formed by two cylinders with the same radius is

$V =$ vol. of cylinders - vol. of common region

$=(\pi \mathrm{r}^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$

$= \pi r^2(L_1 + L_2) - \frac{8}{3}r^3$

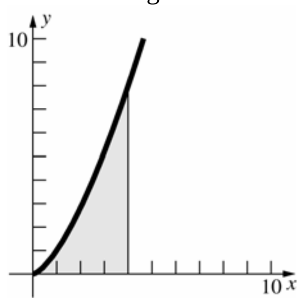
32. The area of each cross section perpendicular to the x-axis is $\frac{1}{2}\pi\left[\frac{1}{2}\left(\sqrt{x} - x^2\right)\right]^2$

$= \frac{\pi}{8}(x^4 - 2x^{5/2} + x).$

$V = \frac{\pi}{8}\int_0^1 (x^4 - 2x^{5/2} + x)dx$

$= \frac{\pi}{8}\left[\frac{1}{5}x^5 - \frac{4}{7}x^{7/2} + \frac{1}{2}x^2\right]_0^1 = \frac{9\pi}{560} \approx 0.050$

33. Sketch the region.



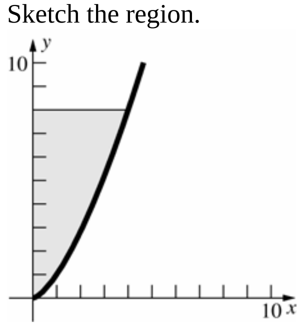
a. Revolving about the line $x = 4$, the radius of the disk at y is $4-\sqrt[3]{y^2} = 4 - y^{2/3}$.

$$V = \pi\int_0^8 (4-y^{2/3})^2\,dy$$
$$= \pi\int_0^8 (16-8y^{2/3}+y^{4/3})dy$$
$$= \pi\left[16y - \frac{24}{5}y^{5/3} + \frac{3}{7}y^{7/3}\right]_0^8$$
$$= \pi\left(128 - \frac{768}{5} + \frac{384}{7}\right)$$
$$= \frac{1024\pi}{35} \approx 91.91$$

b. Revolving about the line $y = 8$, the inner radius of the disk at x is $8-\sqrt{x^3} = 8 - x^{3/2}$.

$$V = \pi\int_0^4 \left[8^2 - (8-x^{3/2})^2\right]dx$$
$$= \pi\int_0^4 (16x^{3/2} - x^3)dx$$
$$= \pi\left[\frac{32}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^4 = \pi\left(\frac{1024}{5} - 64\right)$$
$$= \frac{704\pi}{5} \approx 442.34$$

34. Sketch the region.



a. Revolving about the line $x = 4$, the inner radius of the disk at y is $4-\sqrt[3]{y^2} = 4 - y^{2/3}$.

$$V = \pi\int_0^8 \left[4^2 - \left(4-y^{2/3}\right)^2\right]dy$$
$$= \pi\int_0^8 (8y^{2/3} - y^{4/3})dy$$
$$= \pi\left[\frac{24}{5}y^{5/3} - \frac{3}{7}y^{7/3}\right]_0^8$$
$$= \pi\left(\frac{768}{5} - \frac{384}{7}\right) = \frac{3456\pi}{35} \approx 310.21$$

b. Revolving about the line $y = 8$, the radius of the disk at x is $8-\sqrt{x^3} = 8 - x^{3/2}$.

$$V = \pi\int_0^4 (8-x^{3/2})^2\,dx$$
$$= \pi\int_0^4 (64 - 16x^{3/2} + x^3)dx$$
$$= \pi\left[64x - \frac{32}{5}x^{5/2} + \frac{1}{4}x^4\right]_0^4$$
$$= \pi\left[256 - \frac{1024}{5} + 64\right] = \frac{576\pi}{5} \approx 361.91$$

35. The area of a quarter circle with radius 2 is $\int_0^2 \sqrt{4-y^2}\,dy = \pi$.

$$\int_0^2 \left[2\sqrt{4-y^2} + 4 - y^2\right]dy$$
$$= 2\int_0^2 \sqrt{4-y^2}\,dy + \int_0^2 (4-y^2)dy$$
$$= 2\pi + \left[4y - \frac{1}{3}y^3\right]_0^2 = 2\pi + \left(8 - \frac{8}{3}\right)$$
$$= 2\pi + \frac{16}{3} \approx 11.62$$

36. Let the x-axis lie along the diameter at the base perpendicular to the water level and slice perpendicular to the x-axis. Let $x = 0$ be at the center. The slice has base length $2\sqrt{r^2-x^2}$ and height $\frac{hx}{r}$.

$$V = \frac{2h}{r}\int_0^r x\sqrt{r^2-x^2}\,dx$$
$$= \frac{2h}{r}\left[-\frac{1}{3}\left(r^2-x^2\right)^{3/2}\right]_0^r = \frac{2h}{r}\left(\frac{1}{3}r^3\right) = \frac{2}{3}r^2h$$

37. Let the x-axis lie on the base perpendicular to the diameter through the center of the base. The slice at x is a rectangle with base of length $2\sqrt{r^2-x^2}$ and height $x\tan\theta$.

$$V = \int_0^r 2x\tan\theta\sqrt{r^2-x^2}\,dx$$
$$= \left[-\frac{2}{3}\tan\theta(r^2-x^2)^{3/2}\right]_0^r$$
$$= \frac{2}{3}r^3\tan\theta$$

38. **a.** $x = \sqrt[4]{\frac{y}{k}}$

Slice horizontally.

$$\Delta V \approx \pi\left(\sqrt[4]{\frac{y}{k}}\right)^2 \Delta y = \pi\left(\sqrt{\frac{y}{k}}\right)\Delta y$$

If the depth of the tank is h, then

$$V = \pi\int_0^h \sqrt{\frac{y}{k}}dy = \frac{\pi}{\sqrt{k}}\left[\frac{2}{3}y^{3/2}\right]_0^h$$

$$= \frac{2\pi}{3\sqrt{k}}h^{3/2}.$$

The volume as a function of the depth of the tank is $V(y) = \frac{2\pi}{3\sqrt{k}}y^{3/2}$

b. It is given that $\frac{dV}{dt} = -m\sqrt{y}$.

From part **a**, $\frac{dV}{dt} = \frac{\pi}{\sqrt{k}}y^{1/2}\frac{dy}{dt}$.

Thus, $\frac{\pi}{\sqrt{k}}\sqrt{y}\frac{dy}{dt} = -m\sqrt{y}$ and $\frac{dy}{dt} = \frac{-m\sqrt{k}}{\pi}$ which is constant.

39. Let A lie on the xy-plane. Suppose $\Delta A = f(x)\Delta x$ where $f(x)$ is the length at x, so $A = \int f(x)dx$.

Slice the general cone at height z parallel to A. The slice of the resulting region is A_z and ΔA_z is a region related to $f(x)$ and Δx by similar triangles:

$$\Delta A_z = \left(1-\frac{z}{h}\right)f(x)\cdot\left(1-\frac{z}{h}\right)\Delta x$$

$$= \left(1-\frac{z}{h}\right)^2 f(x)\Delta x$$

Therefore, $A_z = \left(1-\frac{z}{h}\right)^2\int f(x)dx = \left(1-\frac{z}{h}\right)^2 A.$

$$\Delta V \approx A_z\Delta z = A\left(1-\frac{z}{h}\right)^2\Delta z \quad V = A\int_0^h\left(1-\frac{z}{h}\right)^2 dz$$

$$= A\left[-\frac{h}{3}\left(1-\frac{z}{h}\right)^3\right]_0^h = \frac{1}{3}Ah.$$

a. $A = \pi r^2$

$$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h$$

b. A face of a regular tetrahedron is an equilateral triangle. If the side of an equilateral triangle has length r, then the area is $A = \frac{1}{2}r\cdot\frac{\sqrt{3}}{2}r = \frac{\sqrt{3}}{4}r^2$.

The center of an equilateral triangle is $\frac{2}{3}\cdot\frac{\sqrt{3}}{2}r = \frac{1}{\sqrt{3}}r$ from a vertex. Then the height of a regular tetrahedron is

$$h = \sqrt{r^2 - \left(\frac{1}{\sqrt{3}}r\right)^2} = \sqrt{\frac{2}{3}r^2} = \frac{\sqrt{2}}{\sqrt{3}}r.$$

$$V = \frac{1}{3}Ah = \frac{\sqrt{2}}{12}r^3$$

40. If two solids have the same cross sectional area at every x in $[a, b]$, then they have the same volume.

41. First we examine the cross-sectional areas of each shape.

Hemisphere: cross-sectional shape is a circle. The radius of the circle at height y is $\sqrt{r^2 - y^2}$. Therefore, the cross-sectional area for the hemisphere is

$$A_h = \pi(\sqrt{r^2-y^2})^2 = \pi(r^2 - y^2)$$

Cylinder w/o cone: cross-sectional shape is a washer. The outer radius is a constant , r. The inner radius at height y is equal to y. Therefore, the cross-sectional area is

$$A_2 = \pi r^2 - \pi y^2 = \pi(r^2 - y^2).$$

Since both cross-sectional areas are the same, we can apply Cavaleri’s Principle. The volume of the hemisphere of radius r is

$V =$ vol. of cylinder - vol. of cone

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h$$

$$= \frac{2}{3}\pi r^2 h$$

With the height of the cylinder and cone equal to r, the volume of the hemisphere is

$$V = \frac{2}{3}\pi r^2(r) = \frac{2}{3}\pi r^3$$

5.3 Concepts Review

1. $2\pi x\, f(x)\Delta x$

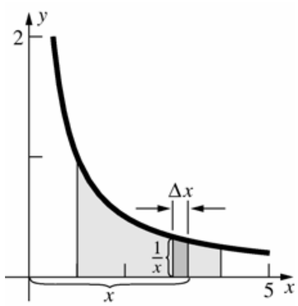
2. $2\pi\int_0^2 x^2 dx;\ \pi\int_0^2 (4-y^2)dy$

3. $2\pi\int_0^2 (1+x)x\, dx$

4. $2\pi\int_0^2 (1+y)(2-y)dy$

Problem Set 5.3

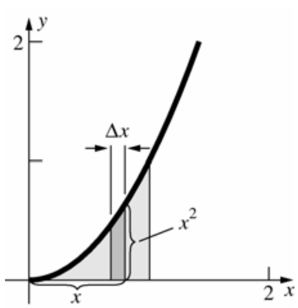
1. a, b.



c. $\Delta V \approx 2\pi x\left(\frac{1}{x}\right)\Delta x = 2\pi\Delta x$

d,e. $V = 2\pi\int_1^4 dx = 2\pi[x]_1^4 = 6\pi \approx 18.85$

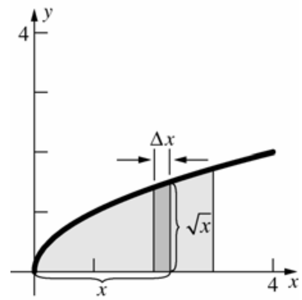
2. a, b.



c. $\Delta V \approx 2\pi x(x^2)\Delta x = 2\pi x^3\Delta x$

d, e. $V = 2\pi\int_0^1 x^3 dx = 2\pi\left[\frac{1}{4}x^4\right]_0^1 = \frac{\pi}{2} \approx 1.57$

3. a, b.

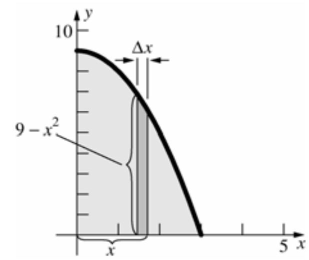


c. $\Delta V \approx 2\pi x\sqrt{x}\,\Delta x = 2\pi x^{3/2}\Delta x$

d, e. $V = 2\pi\int_0^3 x^{3/2}dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^3$

$= \frac{36\sqrt{3}}{5}\pi \approx 39.18$

4. a,b.

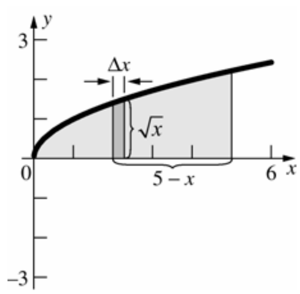


c. $\Delta V \approx 2\pi x(9-x^2)\Delta x = 2\pi(9x-x^3)\Delta x$

d, e. $V = 2\pi\int_0^3 (9x-x^3)dx = 2\pi\left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(\frac{81}{2} - \frac{81}{4}\right) = \frac{81\pi}{2} \approx 127.23$

5. a, b.



c. $\Delta V \approx 2\pi(5-x)\sqrt{x}\,\Delta x$

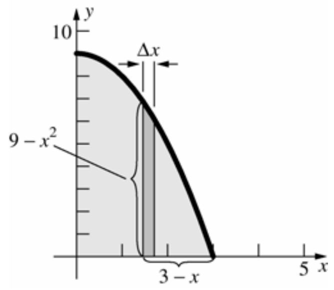
$= 2\pi(5x^{1/2} - x^{3/2})\Delta x$

d, e. $V = 2\pi\int_0^5 (5x^{1/2} - x^{3/2})dx$

$= 2\pi\left[\frac{10}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^5$

$= 2\pi\left(\frac{50\sqrt{5}}{3} - 10\sqrt{5}\right) = \frac{40\sqrt{5}}{3}\pi \approx 93.66$

6. a, b.



c. $\Delta V \approx 2\pi(3-x)(9-x^2)\Delta x$

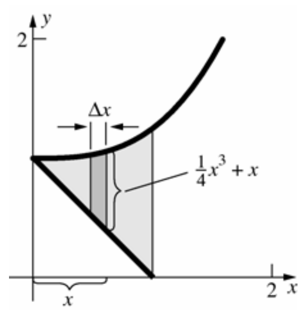
$= 2\pi(27-9x-3x^2+x^3)\Delta x$

d, e. $V = 2\pi\int_0^3 (27-9x-3x^2+x^3)dx$

$= 2\pi\left[27x-\frac{9}{2}x^2-x^3+\frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(81-\frac{81}{2}-27+\frac{81}{4}\right) = \frac{135\pi}{2} \approx 212.06$

7. a, b.



c. $\Delta V \approx 2\pi x\left[\left(\frac{1}{4}x^3+1\right)-(1-x)\right]\Delta x$

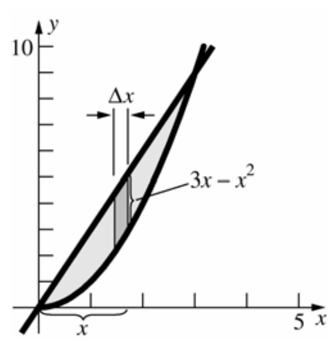
$= 2\pi\left(\frac{1}{4}x^4+x^2\right)\Delta x$

d, e. $V = 2\pi\int_0^1 \left(\frac{1}{4}x^4+x^2\right)dx$

$= 2\pi\left[\frac{1}{20}x^5+\frac{1}{3}x^3\right]_0^1 = 2\pi\left(\frac{1}{20}+\frac{1}{3}\right)$

$= \frac{23\pi}{30} \approx 2.41$

8. a, b.

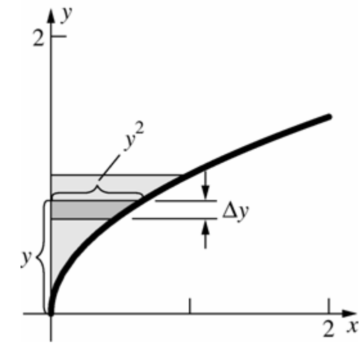


c. $\Delta V \approx 2\pi x(3x-x^2)\Delta x = 2\pi(3x^2-x^3)\Delta x$

d, e. $V = 2\pi\int_0^3 (3x^2-x^3)dx = 2\pi\left[x^3-\frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(27-\frac{81}{4}\right) = \frac{27\pi}{2} \approx 42.41$

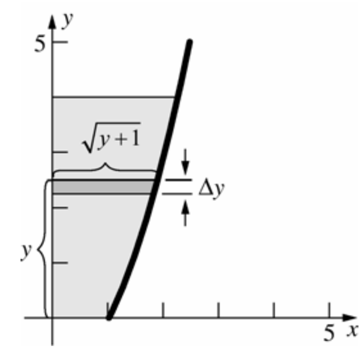
9. a, b.



c. $\Delta V \approx 2\pi y(y^2)\Delta y = 2\pi y^3\Delta y$

d, e. $V = 2\pi\int_0^1 y^3 dy = 2\pi\left[\frac{1}{4}y^4\right]_0^1 = \frac{\pi}{2} \approx 1.57$

10. a, b.



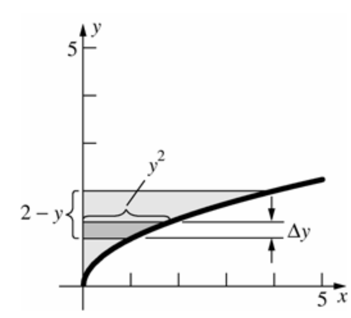
c. $\Delta V \approx 2\pi y\left(\sqrt{y}+1\right)\Delta y = 2\pi(y^{3/2}+y)\Delta y$

d, e. $V = 2\pi\int_0^4 (y^{3/2}+y)\,dy$

$= 2\pi\left[\frac{2}{5}y^{5/2}+\frac{1}{2}y^2\right]_0^4 = 2\pi\left(\frac{64}{5}+8\right)$

$= \frac{208\pi}{5} \approx 130.69$

11. a, b.

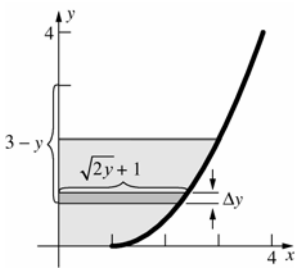


c. $\Delta V \approx 2\pi(2-y)y^2\Delta y = 2\pi(2y^2-y^3)\Delta y$

d, e. $V = 2\pi\int_0^2 (2y^2-y^3)dy = 2\pi\left[\frac{2}{3}y^3-\frac{1}{4}y^4\right]_0^2$

$= 2\pi\left(\frac{16}{3}-4\right) = \frac{8\pi}{3} \approx 8.38$

12. a, b.



c. $\Delta V \approx 2\pi(3-y)\left(\sqrt{2y}+1\right)\Delta y$

$= 2\pi\left(3+3\sqrt{2}y^{1/2}-y-\sqrt{2}y^{3/2}\right)\Delta y$

d, e. $V = 2\pi\int_0^2 \left(3+3\sqrt{2}y^{1/2}-y-\sqrt{2}\,y^{3/2}\right)dy$

$= 2\pi\left[3y+2\sqrt{2}y^{3/2}-\frac{1}{2}y^2-\frac{2\sqrt{2}}{5}y^{5/2}\right]_0^2$

$= 2\pi\left(6+8-2-\frac{16}{5}\right) = \frac{88\pi}{5} \approx 55.29$

13. a. $\pi\int_a^b \left[f(x)^2-g(x)^2\right]dx$

b. $2\pi\int_a^b x\left[f(x)-g(x)\right]dx$

c. $2\pi\int_a^b (x-a)\left[f(x)-g(x)\right]dx$

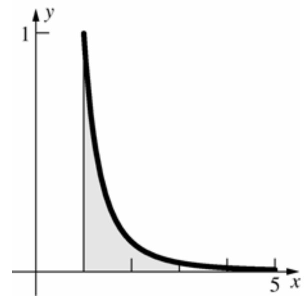
d. $2\pi\int_a^b (b-x)\left[f(x)-g(x)\right]dx$

14. a. $\pi\int_c^d \left[f(y)^2-g(y)^2\right]dy$

b. $2\pi\int_c^d y\left[f(y)-g(y)\right]dy$

c. $2\pi\int_c^d (3-y)\left[f(y)-g(y)\right]dy$

15.



a. $A = \int_1^3 \frac{1}{x^3}dx$

b. $V = 2\pi\int_1^3 x\left(\frac{1}{x^3}\right)dx = 2\pi\int_1^3 \frac{1}{x^2}dx$

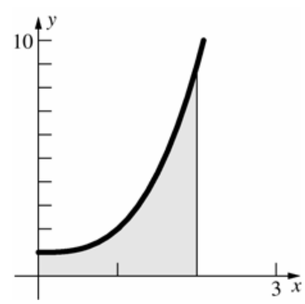
c. $V = \pi\int_1^3 \left[\left(\frac{1}{x^3}+1\right)^2-(-1)^2\right]dx$

$= \pi\int_1^3 \left(\frac{1}{x^6}+\frac{2}{x^3}\right)dx$

d. $V = 2\pi\int_1^3 (4-x)\left(\frac{1}{x^3}\right)dx$

$= 2\pi\int_1^3 \left(\frac{4}{x^3}-\frac{1}{x^2}\right)dx$

16.



a. $A = \int_0^2 (x^3+1)\,dx$

b. $V = 2\pi\int_0^2 x(x^3+1)dx = 2\pi\int_0^2 (x^4+x)dx$

c. $V = \pi\int_0^2 \left[(x^3+2)^2-(-1)^2\right]$

$= \pi\int_0^2 (x^6+4x^3+3)dx$

d. $V = 2\pi\int_0^2 (4-x)(x^3+1)dx$

$= 2\pi\int_0^2 (-x^4+4x^3-x+4)dx$

17. To find the intersection point, solve $\sqrt{y}=\frac{y^3}{32}$.

$$y=\frac{y^6}{1024}$$
$$y^6-1024y=0$$
$$y(y^5-1024)=0$$
$$y=0,\ 4$$
$$V=2\pi\int_0^4 y\left(\sqrt{y}-\frac{y^3}{32}\right)dy$$
$$=2\pi\int_0^4\left(y^{3/2}-\frac{y^4}{32}\right)dy$$
$$=2\pi\left[\frac{2}{5}y^{5/2}-\frac{y^5}{160}\right]_0^4=2\pi\left(\frac{64}{5}-\frac{32}{5}\right)=\frac{64\pi}{5}$$
$$\approx 40.21$$

18. $V=2\pi\int_0^4(4-y)\left(\sqrt{y}-\frac{y^3}{32}\right)dy$

$$=2\pi\int_0^4\left(4y^{1/2}-y^{3/2}-\frac{y^3}{8}+\frac{y^4}{32}\right)dy$$
$$=2\pi\left[\frac{8}{3}y^{3/2}-\frac{2}{5}y^{5/2}-\frac{y^4}{32}+\frac{y^5}{160}\right]_0^4$$
$$=2\pi\left(\frac{64}{3}-\frac{64}{5}-8+\frac{32}{5}\right)=\frac{208\pi}{15}\approx 43.56$$

19. Let R be the region bounded by $y=\sqrt{b^2-x^2}$, $y=-\sqrt{b^2-x^2}$, and $x=a$. When R is revolved about the y-axis, it produces the desired solid.

$$V=2\pi\int_a^b x\left(\sqrt{b^2-x^2}+\sqrt{b^2-x^2}\right)dx$$
$$=4\pi\int_a^b x\sqrt{b^2-x^2}\,dx=4\pi\left[-\frac{1}{3}(b^2-x^2)^{3/2}\right]_a^b$$
$$=4\pi\left[\frac{1}{3}(b^2-a^2)^{3/2}\right]=\frac{4\pi}{3}(b^2-a^2)^{3/2}$$

20. $y=\pm\sqrt{a^2-x^2},\ -a\le x\le a$

$$V=2\pi\int_{-a}^{a}(b-x)\left(2\sqrt{a^2-x^2}\right)dx$$
$$=4\pi b\int_{-a}^{a}\sqrt{a^2-x^2}\,dx-4\pi\int_{-a}^{a}x\sqrt{a^2-x^2}\,dx$$
$$=4\pi b\left(\frac{1}{2}\pi a^2\right)-4\pi\left[-\frac{1}{3}(a^2-x^2)^{3/2}\right]_{-a}^{a}=2\pi^2a^2b$$

(Note that the area of a semicircle with radius a is $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=\frac{1}{2}\pi a^2$.)

21. To find the intersection point, solve $\sin(x^2)=\cos(x^2)$.

$$\tan(x^2)=1$$
$$x^2=\frac{\pi}{4}$$
$$x=\frac{\sqrt{\pi}}{2}$$
$$V=2\pi\int_0^{\sqrt{\pi}/2}x\left[\cos(x^2)-\sin(x^2)\right]dx$$
$$=2\pi\int_0^{\sqrt{\pi}/2}\left[x\cos(x^2)-x\sin(x^2)\right]dx$$
$$=2\pi\left[\frac{1}{2}\sin(x^2)+\frac{1}{2}\cos(x^2)\right]_0^{\sqrt{\pi}/2}$$
$$=2\pi\left[\left(\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\right)-\frac{1}{2}\right]=\pi\left(\sqrt{2}-1\right)\approx 1.30$$

22. $V=2\pi\int_0^{2\pi}x(2+\sin x)dx$

$$=2\pi\int_0^{2\pi}(2x+x\sin x)dx$$
$$=2\pi\int_0^{2\pi}2x\,dx+2\pi\int_0^{2\pi}x\sin x\,dx$$
$$=2\pi\left[x^2\right]_0^{2\pi}+2\pi\left[\sin x-x\cos x\right]_0^{2\pi}$$
$$=2\pi(4\pi^2)+2\pi(-2\pi)=4\pi^2(2\pi-1)\approx 208.57$$

23. a. The curves intersect when $x=0$ and $x=1$.

$$V=\pi\int_0^1[x^2-(x^2)^2]dx=\pi\int_0^1(x^2-x^4)dx$$
$$=\pi\left[\frac{1}{3}x^3-\frac{1}{5}x^5\right]_0^1=\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{2\pi}{15}\approx 0.42$$

b. $V=2\pi\int_0^1 x(x-x^2)dx=2\pi\int_0^1(x^2-x^3)dx$

$$=2\pi\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$$
$$\approx 0.52$$

c. Slice perpendicular to the line $y=x$. At (a,a), the perpendicular line has equation $y=-(x-a)+a=-x+2a$. Substitute $y=-x+2a$ into $y=x^2$ and solve for $x\ge 0$.

$$x^2+x-2a=0$$
$$x=\frac{-1\pm\sqrt{1+8a}}{2}$$
$$x=\frac{-1+\sqrt{1+8a}}{2}$$

Substitute into $y = -x + 2a$, so

$y = \frac{1+4a-\sqrt{1+8a}}{2}$. Find an expression for r^2, the square of the distance from (a, a) to $\left(\frac{-1+\sqrt{1+8a}}{2}, \frac{1+4a-\sqrt{1+8a}}{2}\right)$.

$$r^2 = \left[a - \frac{-1+\sqrt{1+8a}}{2}\right]^2 + \left[a - \frac{1+4a-\sqrt{1+8a}}{2}\right]^2$$

$$= \left[\frac{2a+1-\sqrt{1+8a}}{2}\right]^2 + \left[-\frac{2a+1-\sqrt{1+8a}}{2}\right]^2$$

$$= 2\left[\frac{2a+1-\sqrt{1+8a}}{2}\right]^2$$

$$= 2a^2 + 6a + 1 - 2a\sqrt{1+8a} - \sqrt{1+8a}$$

$$\Delta V \approx \pi r^2 \Delta a$$

$$V = \pi \int_0^1 (2a^2 + 6a + 1 - 2a\sqrt{1+8a} - \sqrt{1+8a})\, da$$

$$= \pi \left[\frac{2}{3}a^3 + 3a^2 + a - \frac{1}{12}(1+8a)^{3/2}\right]_0^1 - \pi \int_0^1 2a\sqrt{1+8a}\, da$$

$$= \pi\left[\left(\frac{2}{3} + 3 + 1 - \frac{9}{4}\right) - \left(-\frac{1}{12}\right)\right] - \pi \int_0^1 2a\sqrt{1+8a}\, da$$

$$= \frac{5\pi}{2} - \pi \int_0^1 2a\sqrt{1+8a}\, da$$

To integrate $\int_0^1 2a\sqrt{1+8a}\, da$, use the substitution $u = 1 + 8a$.

$$\int_0^1 2a\sqrt{1+8a}\, da = \int_1^9 \frac{1}{4}(u-1)\sqrt{u}\,\frac{1}{8}du$$

$$= \frac{1}{32}\int_1^9 (u^{3/2} - u^{1/2})du$$

$$= \frac{1}{32}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9$$

$$= \frac{1}{32}\left[\left(\frac{486}{5} - 18\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right] = \frac{149}{60}$$

$$V = \frac{5\pi}{2} - \frac{149\pi}{60} = \frac{\pi}{60} \approx 0.052$$

24. $\Delta V \approx 4\pi x^2 \Delta x$

$$V = 4\pi \int_0^r x^2\, dx = 4\pi\left[\frac{1}{3}x^3\right]_0^r = \frac{4}{3}\pi r^3$$

25. $\Delta V \approx \frac{x^2}{r^2} S \Delta x$

$$V = \frac{S}{r^2}\int_0^r x^2 dx = \frac{S}{r^2}\left[\frac{1}{3}x^3\right]_0^r = \frac{1}{3}rS$$

5.4 Concepts Review

1. Circle

$x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$

2. x; $x^2 + 1$

3. $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$

4. Mean Value Theorem (for derivatives)

Problem Set 5.4

1. $f(x) = 4x^{3/2}$, $f'(x) = 6x^{1/2}$

$$L = \int_{1/3}^5 \sqrt{1+(6x^{1/2})^2}\, dx = \int_{1/3}^5 \sqrt{1+36x}\, dx$$

$$= \left[\frac{1}{36}\cdot\frac{2}{3}(1+36x)^{3/2}\right]_{1/3}^5$$

$$= \frac{1}{54}\left(181\sqrt{181} - 13\sqrt{13}\right) \approx 44.23$$

2. $f(x) = \frac{2}{3}(x^2+1)^{3/2}$, $f'(x) = 2x(x^2+1)^{1/2}$

$$L = \int_1^2 \sqrt{1 + \left[2x(x^2+1)^{1/2}\right]^2}\, dx$$

$$= \int_1^2 \sqrt{4x^4 + 4x^2 + 1}\, dx = \int_1^2 (2x^2+1)dx$$

$$= \left[\frac{2}{3}x^3 + x\right]_1^2 = \left(\frac{16}{3} + 2\right) - \left(\frac{2}{3} + 1\right) = \frac{17}{3} \approx 5.67$$

3. $f(x) = (4 - x^{2/3})^{3/2}$,

$$f'(x) = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$$

$$= -x^{-1/3}(4 - x^{2/3})^{1/2}$$

$$L = \int_1^8 \sqrt{1 + \left[-x^{-1/3}(4 - x^{2/3})^{1/2}\right]^2}\, dx$$

$$= \int_1^8 \sqrt{4x^{-2/3}}\, dx = \int_1^8 2x^{-1/3} dx$$

$$= 2\left[\frac{3}{2}x^{2/3}\right]_1^8 = 3(4 - 1) = 9$$

4. $f(x) = \dfrac{x^4 + 3}{6x} = \dfrac{x^3}{6} + \dfrac{1}{2x}$

$$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$L = \int_1^3 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx$$

$$= \int_1^3 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\, dx = \int_1^3 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx$$

$$= \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^3$$

$$= \left(\frac{9}{2} - \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{14}{3} \approx 4.67$$

5. $g(y) = \dfrac{y^4}{16} + \dfrac{1}{2y^2}$, $g'(y) = \dfrac{y^3}{4} - \dfrac{1}{y^3}$

$$L = \int_{-3}^{-2} \sqrt{1 + \left(\frac{y^3}{4} - \frac{1}{y^3}\right)^2}\, dy$$

$$= \int_{-3}^{-2} \sqrt{\frac{y^6}{16} + \frac{1}{2} + \frac{1}{y^6}}\, dy = \int_{-3}^{-2} \sqrt{\left(\frac{y^3}{4} + \frac{1}{y^3}\right)^2}\, dy$$

$$= \int_{-3}^{-2} -\left(\frac{y^3}{4} + \frac{1}{y^3}\right) dy = -\left[\frac{y^4}{16} - \frac{1}{2y^2}\right]_{-3}^{-2}$$

$$= -\left[\left(1 - \frac{1}{8}\right) - \left(\frac{81}{16} - \frac{1}{18}\right)\right] = \frac{595}{144} \approx 4.13$$

6. $x = \dfrac{y^5}{30} + \dfrac{1}{2y^3}$

$$g(y) = \frac{y^5}{30} + \frac{1}{2y^3},\ g'(y) = \frac{y^4}{6} - \frac{3}{2y^4}$$

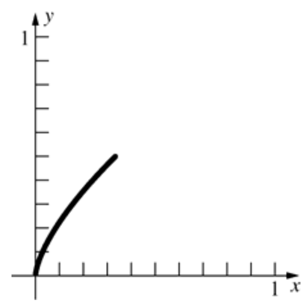
$$L = \int_1^3 \sqrt{1 + \left(\frac{y^4}{6} - \frac{3}{2y^4}\right)^2}\, dy$$

$$= \int_1^3 \sqrt{\frac{y^8}{36} + \frac{1}{2} + \frac{9}{4y^8}}\, dy = \int_1^3 \sqrt{\left(\frac{y^4}{6} + \frac{3}{2y^4}\right)^2}\, dy$$

$$= \int_1^3 \left(\frac{y^4}{6} + \frac{3}{2y^4}\right) dy = \left[\frac{y^5}{30} - \frac{1}{2y^3}\right]_1^3$$

$$= \left(\frac{81}{10} - \frac{1}{54}\right) - \left(\frac{1}{30} - \frac{1}{2}\right) = \frac{1154}{135} \approx 8.55$$

7.



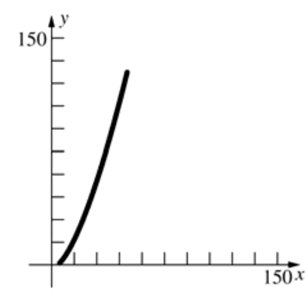
$$\frac{dx}{dt} = t^2, \frac{dy}{dt} = t$$

$$L = \int_0^1 \sqrt{(t^2)^2 + (t)^2}\, dt = \int_0^1 \sqrt{t^4 + t^2}\, dt$$

$$= \int_0^1 t\sqrt{t^2 + 1}\, dt = \left[\frac{1}{3}(t^2 + 1)^{3/2}\right]_0^1 = \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

$$\approx 0.61$$

8.



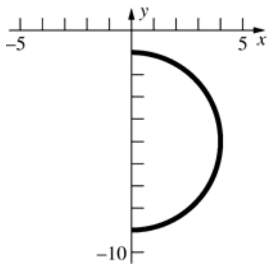
$$\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$$

$$L = \int_1^4 \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_1^4 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_1^4 6t\sqrt{1 + t^2}\, dt = \left[2(1 + t^2)^{3/2}\right]_1^4$$

$$= 2\left(17\sqrt{17} - 2\sqrt{2}\right) \approx 134.53$$

9.



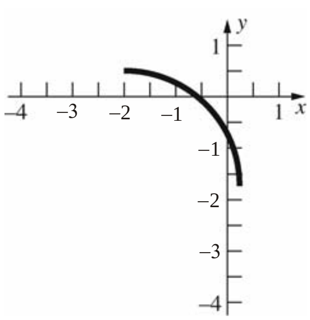
$$\frac{dx}{dt} = 4\cos t, \frac{dy}{dt} = -4\sin t$$

$$L = \int_0^{\pi} \sqrt{(4\cos t)^2 + (-4\sin t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{16\cos^2 t + 16\sin^2 t}\, dt = \int_0^{\pi} 4dt$$

$$= 4\pi \approx 12.57$$

10.



$$\frac{dx}{dt} = 2\sqrt{5}\cos 2t, \frac{dy}{dt} = -2\sqrt{5}\sin 2t$$

$$L = \int_0^{\pi/4} \sqrt{\left(2\sqrt{5}\cos 2t\right)^2 + \left(-2\sqrt{5}\sin 2t\right)^2}\, dt$$

$$= \int_0^{\pi/4} \sqrt{20\cos^2 2t + 20\sin^2 2t}\, dt = \int_0^{\pi/4} 2\sqrt{5}\, dt$$

$$= \frac{\sqrt{5}\,\pi}{2} \approx 3.51$$

11. $f(x) = 2x + 3,\ f'(x) = 2$

$$L = \int_1^3 \sqrt{1 + (2)^2}\, dx = \sqrt{5}\int_1^3 dx = 2\sqrt{5}$$

At $x = 1$, $y = 2(1) + 3 = 5$.
At $x = 3$, $y = 2(3) + 3 = 9$.

$$d = \sqrt{(3-1)^2 + (9-5)^2} = \sqrt{20} = 2\sqrt{5}$$

12. $x = y + \frac{3}{2}$

$$g(y) = y + \frac{3}{2}, g'(y) = 1$$

$$L = \int_1^3 \sqrt{1 + (1)^2} = \sqrt{2}\int_1^3 dy = 2\sqrt{2}$$

At $y = 1$, $x = 1 + \frac{3}{2} = \frac{5}{2}$.

At $y = 3$, $x = 3 + \frac{3}{2} = \frac{9}{2}$.

$$d = \sqrt{\left(\frac{9}{2} - \frac{5}{2}\right)^2 + (3-1)^2} = \sqrt{8} = 2\sqrt{2}$$

13. $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t$

$$L = \int_0^2 \sqrt{1^2 + (2t)^2}\, dt = \int_0^2 \sqrt{1 + 4t^2}\, dt$$

Let $f(t) = \sqrt{1 + 4t^2}$. Using the Parabolic Rule

with n = 8,

$$L \approx \frac{2-0}{3\times 8}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right)\right.$$

$$\left. +2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right]$$

$$\approx \frac{1}{12}[1 + 4\times 1.118 + 2\times 1.4142$$
$$+ 4\times 1.8028 + 2\times 2.2361$$
$$+ 4\times 2.6926 + 2\times 3.1623$$
$$+ 4\times 3.6401 + 4.1231] \approx 4.6468$$

14. $\frac{dx}{dt} = 2t, \frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$$L \approx \int_1^4 \sqrt{(2t)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2}\, dt = \int_1^4 \sqrt{4t^2 + \frac{1}{4t}}\, dt$$

Let $f(t) = \sqrt{4t^2 + \frac{1}{4t}}$. Using the Parabolic Rule

with n = 8,

$$L \approx \frac{4-1}{3\times 8}\left[f(1) + 4f\left(\frac{11}{8}\right) + 2f\left(\frac{14}{8}\right)\right.$$

$$+4f\left(\frac{17}{8}\right) + 2f\left(\frac{20}{8}\right) + 4f\left(\frac{23}{8}\right) + 2f\left(\frac{26}{8}\right)$$

$$\left. +4f\left(\frac{29}{8}\right) + f(4)\right] \approx \frac{1}{8}\big(2.0616 + 4\times 2.8118$$

$$+2\times 3.562 + 4\times 4.312 + 2\times 5.0621 + 4\times 5.8122$$

$$2\times 6.5622 + 4\times 7.3122 + 8.0623\big) \approx 15.0467$$

15. $\frac{dx}{dt} = \cos t, \frac{dy}{dt} = -2\sin 2t$

$L = \int_0^{\pi/2} \sqrt{(\cos t)^2 + (-2\sin 2t)^2}\, dt$

$= \int_0^{\pi/2} \sqrt{\cos^2 t + 4\sin^2 2t}\, dt$

Let $f(t) = \sqrt{\cos^2 t + 4\sin^2 2t}$. Using the Parabolic Rule with n = 8,

$L \approx \frac{\pi/2 - 0}{3 \times 8}\left[f(0) + 4f\left(\frac{\pi}{16}\right) + 2f\left(\frac{2\pi}{16}\right)\right.$

$+4f\left(\frac{3\pi}{16}\right) + 2f\left(\frac{4\pi}{16}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{6\pi}{16}\right)$

$\left. +4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx \frac{\pi}{48}[1 + 4\times 1.2441$

$+ 2\times 1.6892 + 4\times 2.0262 + 2\times 2.1213 + 4\times 1.9295$

$+2\times 1.4651 + 4\times 0.7898 + 0) \approx 2.3241$

16. $\frac{dx}{dt} = 1, \frac{dy}{dt} = \sec^2 t$

$L = \int_0^{\pi/4} \sqrt{1^2 + (\sec^2 t)^2}\, dt = \int_0^{\pi/4} \sqrt{1 + \sec^4 t}\, dt$

Let $f(t) = \sqrt{1 + \sec^4 t}$. Using the Parabolic Rule with n = 8, $L \approx \frac{\pi/4 - 0}{3\times 8}\left[f(0) + 4f\left(\frac{\pi}{32}\right)\right.$

$+2f\left(\frac{2\pi}{32}\right) + 4f\left(\frac{3\pi}{32}\right) + 2f\left(\frac{4\pi}{32}\right) + 4f\left(\frac{5\pi}{32}\right)$

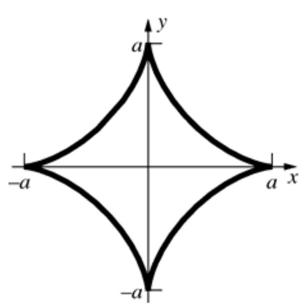
$\left. +2f\left(\frac{6\pi}{32}\right) + 4f\left(\frac{7\pi}{32}\right) + f\left(\frac{\pi}{4}\right)\right]$

$\approx \frac{\pi}{96}[1.4142 + 4\times 1.4211 + 2\times 1.4425 + 4\times 1.4807$

$+2\times 1.5403 + 4\times 1.6288 + 2\times 1.7585$

$+4\times 1.9495 + 2.2361] \approx 1.278$

17.



$\frac{dx}{dt} = 3a\cos t \sin^2 t, \frac{dy}{dt} = -3a\sin t\cos^2 t$

The first quadrant length is L

$= \int_0^{\pi/2} \sqrt{(3a\cos t\sin^2 t)^2 + (-3a\sin t\cos^2 t)^2}\, dt$

$= \int_0^{\pi/2} \sqrt{9a^2\cos^2 t\sin^4 t + 9a^2\sin^2 t\cos^4 t}\, dt$

$= \int_0^{\pi/2} \sqrt{9a^2\cos^2 t\sin^2 t(\sin^2 t + \cos^2 t)}\, dt$

$= \int_0^{\pi/2} 3a\cos t\sin t dt = 3a\left[-\frac{1}{2}\cos^2 t\right]_0^{\pi/2} = \frac{3a}{2}$

(The integral can also be evaluated as $3a\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2}$ with the same result.)

The total length is $6a$.

18. **a.** $\overline{OT} = \text{length}\left(\widehat{PT}\right) = a\theta$

b. From Figure 18 of the text,

$\sin\theta = \frac{\overline{PQ}}{\overline{PC}} = \frac{\overline{PQ}}{a}$ and $\cos\theta = \frac{\overline{QC}}{\overline{PC}} = \frac{\overline{QC}}{a}$.

Therefore $\overline{PQ} = a\sin\theta$ and $\overline{QC} = a\cos\theta$.

c. $x = \overline{OT} - \overline{PQ} = a\theta - a\sin\theta = a(\theta - \sin\theta)$

$y = \overline{CT} - \overline{CQ} = a - a\cos\theta = a(1 - \cos\theta)$

19. From Problem 18,

$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 - \cos\theta), \frac{dy}{d\theta} = a\sin\theta$ so

$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [a(1 - \cos\theta)]^2 + [a\sin\theta]^2$

$= a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta$

$= 2a^2 - 2a^2\cos\theta = 2a^2(1 - \cos\theta)$

$= 4a^2\frac{1 - \cos\theta}{2} = 4a^2\sin^2\left(\frac{\theta}{2}\right)$.

The length of one arch of the cycloid is

$\int_0^{2\pi} \sqrt{4a^2\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = \int_0^{2\pi} 2a\sin\left(\frac{\theta}{2}\right) d\theta$

$= 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 2a(2 + 2) = 8a$

20. a. Using $\theta = \omega t$, the point P is at $x = a\omega t - a\sin(\omega t),\quad y = a - a\cos(\omega t)$ at time t.

$$\frac{dx}{dt} = a\omega - a\omega\cos(\omega t) = a\omega(1-\cos(\omega t))$$

$$\frac{dy}{dt} = a\omega\sin(\omega t)$$

$$\frac{ds}{dt} = \sqrt{\left[\frac{dy}{dt}\right]^2 + \left[\frac{dx}{dt}\right]^2}$$

$$= \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2 - 2a^2\omega^2\cos(\omega t) + a^2\omega^2\cos^2(\omega t)} = \sqrt{2a^2\omega^2 - 2a^2\omega^2\cos(\omega t)}$$

$$= 2a\omega\sqrt{\frac{1}{2}(1-\cos(\omega t))} = 2a\omega\sqrt{\sin^2\frac{\omega t}{2}} = 2a\omega\left|\sin\frac{\omega t}{2}\right|$$

b. The speed is a maximum when $\left|\sin\frac{\omega t}{2}\right| = 1,$ which occurs when $t = \frac{\pi}{\omega}(2k+1).$ The speed is a minimum when $\left|\sin\frac{\omega t}{2}\right| = 0,$ which occurs when $t = \frac{2k\pi}{\omega}$.

c. From Problem 18a, the distance traveled by the wheel is $a\theta$, so at time t, the wheel has gone $a\theta = a\omega t$ miles. Since the car is going 60 miles per hour, the wheel has gone $60t$ miles at time t. Thus, $a\omega = 60$ and the maximum speed of the bug on the wheel is $2a\omega = 2(60) = 120$ miles per hour.

21. a. $\frac{dy}{dx} = \sqrt{x^3-1}$

$$L = \int_1^2 \sqrt{1+x^3-1}\,dx = \int_1^2 x^{3/2}dx$$

$$= \left[\frac{2}{5}x^{5/2}\right]_1^2 = \frac{2}{5}\left(4\sqrt{2}-1\right) \approx 1.86$$

b. $f'(t) = 1-\cos t,\ g'(t) = \sin t$

$$L = \int_0^{4\pi}\sqrt{2-2\cos t}\,dt = \int_0^{4\pi} 2\left|\sin\left(\frac{t}{2}\right)\right|dt$$

$\sin\left(\frac{t}{2}\right)$ is positive for $0 < t < 2\pi$, and by symmetry, we can double the integral from 0 to 2π.

$$L = 4\int_0^{2\pi}\sin\left(\frac{t}{2}\right)dt = \left[-8\cos\frac{t}{2}\right]_0^{2\pi}$$

$= 8 + 8 = 16$

22. a. $\frac{dy}{dx} = \sqrt{64\sin^2 x\cos^4 x - 1}$

$$L = \int_{\pi/6}^{\pi/3}\sqrt{1+64\sin^2\cos^4 x - 1}\,dx$$

$$= \int_{\pi/6}^{\pi/3} 8\sin x\cos^2 x\,dx = \left[-\frac{8}{3}\cos^3 x\right]_{\pi/6}^{\pi/3}$$

$$= -\frac{1}{3}+\sqrt{3} \approx 1.40$$

b. $\frac{dx}{dt} = -a\sin t + a\sin t + at\cos t = at\cos t$

$$\frac{dy}{dt} = a\cos t - a\cos t + at\sin t = at\sin t$$

$$L = \int_{-1}^{1}\sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}\,dt$$

$$= \int_{-1}^{1}|at|\,dt = \int_0^1 at\,dt - \int_{-1}^{0} at\,dt$$

$$= \left[\frac{a}{2}t^2\right]_0^1 - \left[\frac{a}{2}t^2\right]_{-1}^0 = \frac{a}{2}+\frac{a}{2} = a$$

23. $f(x) = 6x,\ f'(x) = 6$

$$A = 2\pi\int_0^1 6x\sqrt{1+36}\,dx = 12\sqrt{37}\pi\int_0^1 x\,dx$$

$$= 12\sqrt{37}\pi\left[\frac{1}{2}x^2\right]_0^1 = 6\sqrt{37}\pi \approx 114.66$$

24. $f(x) = \sqrt{25-x^2},\ f'(x) = -\frac{x}{\sqrt{25-x^2}}$

$$A = 2\pi\int_{-2}^{3}\sqrt{25-x^2}\sqrt{1+\frac{x^2}{25-x^2}}dx$$

$$= 2\pi\int_{-2}^{3}\sqrt{25-x^2+x^2}\,dx$$

$$= 2\pi\int_{-2}^{3}5dx = 10\pi[x]_{-2}^{3} = 50\pi \approx 157.08$$

25. $f(x)=\frac{x^3}{3},\ f'(x)=x^2$

$$A=2\pi\int_1^{\sqrt{7}}\frac{x^3}{3}\sqrt{1+x^4}\,dx$$

$$=2\pi\left[\frac{1}{18}(1+x^4)^{3/2}\right]_1^{\sqrt{7}}=\frac{\pi}{9}\left(250\sqrt{2}-2\sqrt{2}\right)$$

$$=\frac{248\pi\sqrt{2}}{9}\approx 122.43$$

26. $f(x)=\frac{x^6+2}{8x^2}=\frac{x^4}{8}+\frac{1}{4x^2},\ f'(x)=\frac{x^3}{2}-\frac{1}{2x^3}$

$$A=2\pi\int_1^3\left(\frac{x^4}{8}+\frac{1}{4x^2}\right)\sqrt{1+\left(\frac{x^3}{2}-\frac{1}{2x^3}\right)^2}$$

$$=2\pi\int_1^3\left(\frac{x^4}{8}+\frac{1}{4x^2}\right)\sqrt{\frac{x^6}{4}+\frac{1}{2}+\frac{1}{4x^6}}\,dx$$

$$=2\pi\int_1^3\left(\frac{x^4}{8}+\frac{1}{4x^2}\right)\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx$$

$$=2\pi\int_1^3\left(\frac{x^7}{16}+\frac{3x}{16}+\frac{1}{8x^5}\right)dx$$

$$=2\pi\left[\frac{x^8}{128}+\frac{3x^2}{32}-\frac{1}{32x^4}\right]_1^3$$

$$=2\pi\left[\left(\frac{6561}{128}+\frac{27}{32}-\frac{1}{2592}\right)-\left(\frac{1}{128}+\frac{3}{32}-\frac{1}{32}\right)\right]$$

$$=\frac{8429\pi}{81}\approx 326.92$$

27. $\frac{dx}{dt}=1,\frac{dy}{dt}=3t^2$

$$A=2\pi\int_0^1 t^3\sqrt{1+9t^4}\,dt$$

$$=2\pi\left[\frac{1}{54}(1+9t^4)^{3/2}\right]_0^1=\frac{\pi}{27}\left(10\sqrt{10}-1\right)$$

≈ 3.56

28. $\frac{dx}{dt}=-2t,\frac{dy}{dt}=2$

$$A=2\pi\int_0^1 2t\sqrt{4t^2+4}\,dt=8\pi\int_0^1 t\sqrt{t^2+1}\,dt$$

$$=8\pi\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^1=\frac{8\pi}{3}(2\sqrt{2}-1)\approx 15.32$$

29. $y=f(x)=\sqrt{r^2-x^2}$

$f'(x)=-x(r^2-x^2)^{-1/2}$

$$A=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left[-x(r^2-x^2)^{-1/2}\right]^2}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+x^2(r^2-x^2)^{-1}}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{\left(r^2-x^2\right)\left(1+x^2(r^2-x^2)^{-1}\right)}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2+x^2}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2}\,dx=2\pi\int_{-r}^{r}r\,dx=2\pi rx\,\Big|_{-r}^{r}=4\pi r^2$$

30. $x=f(t)=r\cos t$

$y=g(t)=r\sin t$

$f'(t)=-r\sin t$

$g'(t)=r\cos t$

$$A=2\pi\int_0^{\pi}r\sin t\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt$$

$$=2\pi\int_0^{\pi}r\sin t\sqrt{r^2\sin^2 t+r^2\cos^2 t}\,dt$$

$$=2\pi\int_0^{\pi}r\sin t\sqrt{r^2}\,dt$$

$$=2\pi\int_0^{\pi}r^2\sin t\,dt=-2\pi r^2\cos t\,\Big|_0^{\pi}$$

$$=-2r^2(-1-1)=4\pi r^2$$

31. **a.** The base circumference is equal to the arc length of the sector, so $2\pi r=\theta l$. Therefore, $\theta=\frac{2\pi r}{l}$.

b. The area of the sector is equal to the lateral surface area. Therefore, the lateral surface area is $\frac{1}{2}l^2\theta=\frac{1}{2}l^2\left(\frac{2\pi r}{l}\right)=\pi rl$.

c. Assume $r_2>r_1$. Let l_1 and l_2 be the slant heights for r_1 and r_2, respectively. Then $A=\pi r_2 l_2-\pi r_1 l_1=\pi r_2(l_1+l)-\pi r_1 l_1$.

From part a, $\theta=\frac{2\pi r_2}{l_2}=\frac{2\pi r_2}{l_1+l}=\frac{2\pi r_1}{l_1}$.

Solve for $l_1: l_1 r_2=l_1 r_1+l r_1$

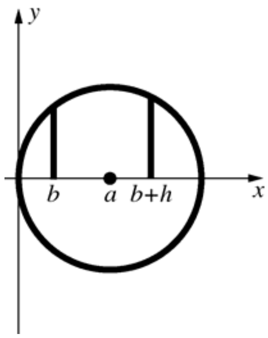
$l_1(r_2-r_1)=l r_1$

$l_1=\frac{l r_1}{r_2-r_1}$

$$A=\pi r_2\left(\frac{l r_1}{r_2-r_1}+l\right)-\pi r_1\left(\frac{l r_1}{r_2-r_1}\right)$$

$$=\pi(l r_1+l r_2)=2\pi\left[\frac{r_1+r_2}{2}\right]l$$

32. Put the center of a circle of radius a at $(a, 0)$. Revolving the portion of the circle from $x = b$ to $x = b + h$ about the x-axis results in the surface in question. (See figure.)



The equation of the top half of the circle is

$y = \sqrt{a^2 - (x-a)^2}$.

$$\frac{dy}{dx} = \frac{-(x-a)}{\sqrt{a^2-(x-a)^2}}$$

$$A = 2\pi\int_b^{b+h}\sqrt{a^2-(x-a)^2}\sqrt{1+\frac{(x-a)^2}{a^2-(x-a)^2}}dx$$

$$= 2\pi\int_b^{b+h}\sqrt{a^2-(x-a)^2+(x-a)^2}\,dx$$

$$= 2\pi\int_b^{b+h} a\,dx = 2\pi a[x]_b^{b+h} = 2\pi ah$$

A right circular cylinder of radius a and height h has surface area $2\pi ah$.

33. a. $\frac{dx}{dt} = a(1-\cos t), \frac{dy}{dt} = a\sin t$

$$A = 2\pi\int_0^{2\pi} a(1-\cos t)\cdot \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\;dt$$

$$= 2\pi a\int_0^{2\pi}(1-\cos t)\sqrt{2a^2-2a^2\cos t}\,dt$$

$$= 2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos t)^{3/2}dt$$

b. $1-\cos t = 2\sin^2\left(\frac{t}{2}\right)$, so

$$A = 2\sqrt{2}\pi a^2\int_0^{2\pi}2^{3/2}\sin^3\left(\frac{t}{2}\right)dt$$

$$= 8\pi a^2\int_0^{2\pi}\sin\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)dt$$

$$= 8\pi a^2\int_0^{2\pi}\sin\left(\frac{t}{2}\right)\left[1-\cos^2\left(\frac{t}{2}\right)\right]dt$$

$$= 8\pi a^2\left[-2\cos\left(\frac{t}{2}\right)+\frac{2}{3}\cos^3\left(\frac{t}{2}\right)\right]_0^{2\pi}$$

$$= 8\pi a^2\left[\left(2-\frac{2}{3}\right)-\left(-2+\frac{2}{3}\right)\right] = \frac{64}{3}\pi a^2$$

34. $\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t$

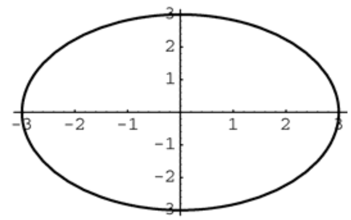
Since the circle is being revolved about the line $x = b$, the surface area is

$$A = 2\pi\int_0^{2\pi}(b-a\cos t)\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt$$

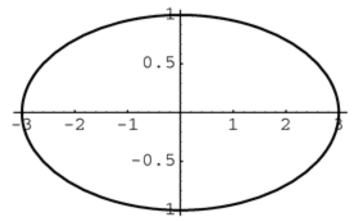
$$= 2\pi a\int_0^{2\pi}(b-a\cos t)dt$$

$$= 2\pi a[bt - a\sin t]_0^{2\pi} = 4\pi^2 ab$$

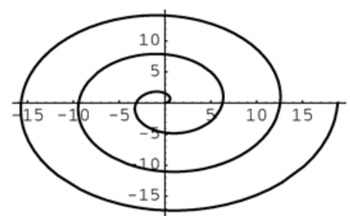
35. a.



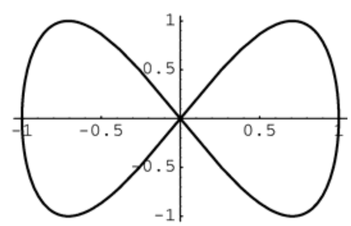
b.



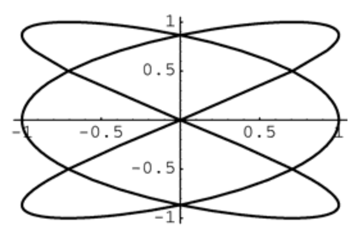
c.



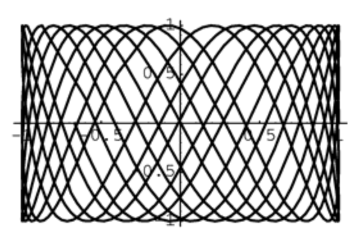
d.



e.



f.



36. a. $f'(t) = -3\sin t,\ g'(t) = 3\cos t$

$L = \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\,dt$

$= \int_0^{2\pi} 3dt = 3[t]_0^{2\pi} = 6\pi \approx 18.850$

b. $f'(t) = -3\sin t,\ g'(t) = \cos t$

$L = \int_0^{2\pi} \sqrt{9\sin^2 t + \cos^2 t}\,dt \approx 13.365$

c. $f'(t) = \cos t - t\sin t,\ g'(t) = t\cos t + \sin t$

$L = \int_0^{6\pi} \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}\,dt$

$= \int_0^{6\pi} \sqrt{1+t^2}\,dt \approx 179.718$

d. $f'(t) = -\sin t,\ g'(t) = 2\cos 2t$

$L = \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 2t}\,dt \approx 9.429$

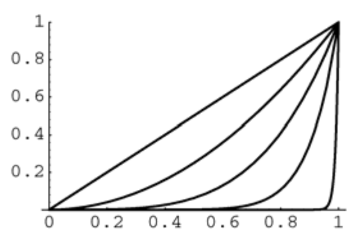
e. $f'(t) = -3\sin 3t,\ g'(t) = 2\cos 2t$

$L = \int_0^{2\pi} \sqrt{9\sin^2 3t + 4\cos^2 2t}\,dt \approx 15.289$

f. $f'(t) = -\sin t,\ g'(t) = \pi\cos \pi t$

$L = \int_0^{40} \sqrt{\sin^2 t + \pi^2\cos^2 \pi t}\,dt \approx 86.58$

37.



$y = x,\ y' = 1,$

$L = \int_0^1 \sqrt{2}dx = \left[\sqrt{2}\,x\right]_0^1 = \sqrt{2} \approx 1.41421$

$y = x^2,\ y' = 2x,\ L = \int_0^1 \sqrt{1+4x^2}\,dx \approx 1.47894$

$y = x^4,\ y' = 4x^3,\ \ L = \int_0^1 \sqrt{1+16x^6}\,dx \approx 1.60023$

$y = x^{10},\ y' = 10x^9,$

$L = \int_0^1 \sqrt{1+100^{18}}\,dx \approx 1.75441$

$y = x^{100},\ y' = 100x^{99},$

$L = \int_0^1 \sqrt{1+10{,}000x^{198}}\,dx \approx 1.95167$

When n = 10,000 the length will be close to 2.

5.5 Concepts Review

1. $F\cdot(b-a);\ \int_a^b F(x)\,dx$

2. $30 \cdot 10 = 300$

3. the depth of that part of the surface

4. δhA

Problem Set 5.5

1. $F\left(\frac{1}{2}\right) = 6;\ k\cdot\frac{1}{2} = 6,\ k = 12$

$F(x) = 12x$

$W = \int_0^{1/2} 12x\,dx = \left[6x^2\right]_0^{1/2} = \frac{3}{2} = 1.5$ ft-lb

2. From Problem 1, $F(x) = 12x$.

$W = \int_0^2 12x\,dx = \left[6x^2\right]_0^2 = 24$ ft-lb

3. $F(0.01) = 0.6;\ k = 60$

$F(x) = 60x$

$W = \int_0^{0.02} 60x\,dx = \left[30x^2\right]_0^{0.02} = 0.012$ Joules

4. $F(x) = kx$ and let l be the natural length of the spring.

$W = \int_{8-l}^{9-l} kx\,dx = \left[\frac{1}{2}kx^2\right]_{8-l}^{9-l}$

$= \frac{1}{2}k\left[(81-18l+l^2)-(64-16l+l^2)\right]$

$= \frac{1}{2}k(17-2l) = 0.05$

Thus, $k = \frac{0.1}{17-2l}$.

$W = \int_{9-l}^{10-l} kx\,dx = \left[\frac{1}{2}kx^2\right]_{9-l}^{10-l}$

$= \frac{1}{2}k\left[(100-20l+l^2)-(81-18l+l^2)\right]$

$= \frac{1}{2}k(19-2l) = 0.1$

Thus, $k = \frac{0.2}{19-2l}$.

Solving $\frac{0.1}{17-2l} = \frac{0.2}{19-2l},\ l = \frac{15}{2}$.

Thus $k = 0.05$, and the natural length is 7.5 cm.

5. $W=\int_0^d kx\,dx=\left[\frac{1}{2}kx^2\right]_0^d$

$=\frac{1}{2}k(d^2-0)=\frac{1}{2}kd^2$

6. $F(8)=2; k16=2, k=\frac{1}{8}$

$W=\int_0^{27}\frac{1}{8}s^{4/3}\,ds=\frac{1}{8}\left[\frac{3}{7}s^{7/3}\right]_0^{27}=\frac{6561}{56}$

$\approx$ 117.16 inch-pounds

7. $W=\int_0^2 9s\,ds=9\left[\frac{1}{2}s^2\right]_0^2=18$ ft-lb

8. One spring will move from 2 feet beyond its natural length to 3 feet beyond its natural length. The other will move from 2 feet beyond its natural length to 1 foot beyond its natural length.

$W=\int_2^3 6s\,ds+\int_2^1 6s\,ds=\left[3s^2\right]_2^3+\left[3s^2\right]_2^1$

$= 3(9 – 4) + 3(1 – 4) = 6$ ft-lb

9. A slab of thickness Δy at height y has width $4-\frac{4}{5}y$ and length 10. The slab will be lifted a distance $10 – y$.

$\Delta W\approx\delta\cdot 10\cdot\left(4-\frac{4}{5}y\right)\Delta y(10-y)$

$=8\delta(y^2-15y+50)\Delta y$

$W=\int_0^5 8\delta(y^2-15y+50)\,dy$

$=8(62.4)\left[\frac{1}{3}y^3-\frac{15}{2}y^2+50y\right]_0^5$

$=8(62.4)\left(\frac{125}{3}-\frac{375}{2}+250\right)=52{,}000$ ft-lb

10. A slab of thickness Δy at height y has width $4-\frac{4}{3}y$ and length 10. The slab will be lifted a distance $8 – y$.

$\Delta W\approx\delta\cdot 10\cdot\left(4-\frac{4}{3}y\right)\Delta y(8-y)$

$=\frac{40}{3}\delta(24-11y+y^2)\Delta y$

$W=\int_0^3\frac{40}{3}\delta(24-11y+y^2)dy$

$=\frac{40}{3}(62.4)\left[24y-\frac{11}{2}y^2+\frac{1}{3}y^3\right]_0^3$

$=\frac{40}{3}(62.4)\left(72-\frac{99}{2}+9\right)=26{,}208$ ft-lb

11. A slab of thickness Δy at height y has width $\frac{3}{4}y+3$ and length 10. The slab will be lifted a distance $9 – y$. $\Delta W\approx\delta\cdot 10\cdot\left(\frac{3}{4}y+3\right)\Delta y(9-y)$

$=\frac{15}{2}\delta(36+5y-y^2)\Delta y$

$W=\int_0^4\frac{15}{2}\delta(36+5y-y^2)dy$

$=\frac{15}{2}(62.4)\left[36y+\frac{5}{2}y^2-\frac{1}{3}y^3\right]_0^4$

$=\frac{15}{2}(62.4)\left(144+40-\frac{64}{3}\right)$

$= 76{,}128$ ft-lb

12. A slab of thickness Δy at height y has width $2\sqrt{6y-y^2}$ and length 10. The slab will be lifted a distance $8 – y$.

$\Delta W\approx\delta\cdot 10\cdot 2\sqrt{6y-y^2}\,\Delta y(8-y)$

$=20\delta\sqrt{6y-y^2}\,(8-y)\Delta y$

$W=\int_0^3 20\delta\sqrt{6y-y^2}\,(8-y)dy$

$=20\delta\int_0^3\sqrt{6y-y^2}\,(3-y)\,dy$

$+20\delta\int_0^3\sqrt{6y-y^2}\,(5)dy$

$=20\delta\left[\frac{1}{3}(6y-y^2)^{3/2}\right]_0^3+100\delta\int_0^3\sqrt{6y-y^2}\,dy$

Notice that $\int_0^3\sqrt{6y-y^2}\,dy$ is the area of a quarter of a circle with radius 3.

$W=20\delta(9)+100\delta\left(\frac{1}{4}\pi 9\right)$

$= (62.4)(180 + 225\pi) \approx 55{,}340$ ft-lb

13. The volume of a disk with thickness Δy is $16\pi\Delta y$. If it is at height y, it will be lifted a distance $10 – y$.

$\Delta W\approx\delta 16\pi\Delta y(10-y)=16\pi\delta(10-y)\Delta y$

$W=\int_0^{10}16\pi\delta(10-y)dy=16\pi(50)\left[10y-\frac{1}{2}y^2\right]_0^{10}$

$= 16\pi(50)(100 – 50) \approx 125{,}664$ ft-lb

14. The volume of a disk with thickness Δx at height x is $\pi(4+x)^2 \Delta x$. It will be lifted a distance of $10 - x$.

$$\Delta W \approx \delta\pi(4+x)^2 \Delta x(10-x)$$
$$= \pi\delta(160+64x+2x^2-x^3)\Delta x$$
$$W = \int_0^{10} \pi\delta(160+64x+2x^2-x^3)dx$$
$$= \pi(50)\left[160x+32x^2+\frac{2}{3}x^3-\frac{1}{4}x^4\right]_0^{10}$$
$$= \pi(50)\left(1600+3200+\frac{2000}{3}-2500\right)$$
$$\approx 466{,}003 \text{ ft-lb}$$

15. The total force on the face of the piston is $A \cdot f(x)$ if the piston is x inches from the cylinder head. The work done by moving the piston from x_1 to x_2 is $W = \int_{x_1}^{x_2} A \cdot f(x)dx = A\int_{x_1}^{x_2} f(x)dx$. This is the work done by the gas in moving the piston. The work done by the piston to compress the gas is the opposite of this or $A\int_{x_2}^{x_1} f(x)dx$.

16. $c = 40(16)^{1.4}$

$A = 1;\ p(v) = cv^{-1.4}$

$f(x) = cx^{-1.4}$

$x_1 = \frac{16}{1} = 16,\ x_2 = \frac{2}{1} = 2$

$$W = \int_2^{16} cx^{-1.4}dx = c\left[-2.5x^{-0.4}\right]_2^{16}$$
$$= 40(16)^{1.4}(-2.5)(16^{-0.4}-2^{-0.4})$$
$$\approx 2075.83 \text{ in.-lb}$$

17. $c = 40(16)^{1.4}$

$A = 2;\ p(v) = cv^{-1.4}$

$f(x) = c(2x)^{-1.4}$

$x_1 = \frac{16}{2} = 8,\ x_2 = \frac{2}{2} = 1$

$$W = 2\int_1^{8} c(2x)^{-1.4}dx = 2c\left[-1.25(2x)^{-0.4}\right]_1^{8}$$
$$= 80(16)^{1.4}(-1.25)(16^{-0.4}-2^{-0.4})$$
$$\approx 2075.83 \text{ in.-lb}$$

18. 80 lb/in.^{2} = 11,520 lb/ft^2

$c = 11{,}520(1)^{1.4} = 11{,}520$

$$\Delta W \approx p(v)\Delta v = 11{,}520v^{-1.4}\Delta v$$
$$W = \int_1^{4} 11{,}520v^{-1.4}dv = \left[-28{,}800v^{-0.4}\right]_1^{4}$$
$$= -28{,}800(4^{-0.4}-1^{-0.4}) \approx 12{,}259 \text{ ft-lb}$$

19. The total work is equal to the work W_1 to haul the load by itself and the work W_2 to haul the rope by itself.

$W_1 = 200 \cdot 500 = 100{,}000$ ft-lb

Let $y = 0$ be the bottom of the shaft. When the rope is at y, $\Delta W_2 \approx 2\Delta y(500 - y)$.

$$W_2 = \int_0^{500} 2(500-y)dy = 2\left[500y - \frac{1}{2}y^2\right]_0^{500}$$
$= 2(250{,}000 - 125{,}000) = 250{,}000$ ft-lb

$W = W_1 + W_2 = 100{,}000 + 250{,}000$

$= 350{,}000$ ft-lb

20. The total work is equal to the work W_1 to lift the monkey plus the work W_2 to lift the chain.

$W_1 = 10 \cdot 20 = 200$ ft-lb

Let $y = 20$ represent the top. As the monkey climbs the chain, the piece of chain at height y $(0 \le y \le 10)$ will be lifted $20 - 2y$ ft.

$$\Delta W_2 \approx \frac{1}{2}\Delta y(20-2y) = (10-y)\Delta y$$
$$W_2 = \int_0^{10}(10-y)dy = \left[10y - \frac{1}{2}y^2\right]_0^{10}$$
$= 100 - 50 = 50$ ft-lb

$W = W_1 + W_2 = 250$ ft-lb

21. $f(x) = \frac{k}{x^2};\ f(4000) = 5000$

$\frac{k}{4000^2} = 5000$, $k = 80{,}000{,}000{,}000$

$$W = \int_{4000}^{4200} \frac{80{,}000{,}000{,}000}{x^2}dx$$
$$= 80{,}000{,}000{,}000\left[-\frac{1}{x}\right]_{4000}^{4200}$$
$$= \frac{20{,}000{,}000}{21} \approx 952{,}381 \text{ mi-lb}$$

22. $F(x) = \frac{k}{x^2}$ where x is the distance between the charges. $F(2) = 10;\ \frac{k}{4} = 10,\ k = 40$

$$W = \int_1^5 \frac{40}{x^2}dx = \left[-\frac{40}{x}\right]_1^5 = 32 \text{ ergs}$$

23. The relationship between the height of the bucket and time is $y = 2t$, so $t = \frac{1}{2}y$. When the bucket is a height y, the sand has been leaking out of the bucket for $\frac{1}{2}y$ seconds. The weight of the bucket and sand is $100 + 500 - 3\left(\frac{1}{2}y\right) = 600 - \frac{3}{2}y.$

$$\Delta W \approx \left(600 - \frac{3}{2}y\right)\Delta y$$

$$W = \int_0^{80}\left(600 - \frac{3}{2}y\right)dy = \left[600y - \frac{3}{4}y^2\right]_0^{80}$$

$= 48{,}000 - 4800 = 43{,}200$ ft-lb

24. The total work is equal to the work W_1 needed to fill the pipe plus the work W_2 needed to fill the tank.

$$\Delta W_1 = \delta\pi\left(\frac{1}{2}\right)^2 \Delta y(y) = \frac{\delta\pi y}{4}\Delta y$$

$$W_1 = \int_0^{30}\frac{\delta\pi y}{4}dy = \frac{(62.4)\pi}{4}\left[\frac{1}{2}y^2\right]_0^{30}$$

$\approx 22{,}054$ ft-lb

The cross sectional area at height y feet $(30 \le y \le 50)$ is πr^2 where

$r = \sqrt{10^2 - (40-y)^2} = \sqrt{-y^2 + 80y - 1500}$.

$$\Delta W_2 = \delta\pi r^2 \Delta y\, y = \delta\pi(-y^3 + 80y^2 - 1500y)\Delta y$$

$$W_2 = \int_{30}^{50}\delta\pi(-y^3 + 80y^2 - 1500y)dy$$

$$= (62.4)\pi\left[-\frac{1}{4}y^4 + \frac{80}{3}y^3 - 750y^2\right]_{30}^{50}$$

$$= (62.4)\pi\left[\left(-1{,}562{,}500 + \frac{10{,}000{,}000}{3} - 1{,}875{,}000\right) - (-202{,}500 + 720{,}000 - 675{,}000)\right]$$

$\approx$ 10,455,220 ft-lb

$W = W_1 + W_2 \approx 10{,}477{,}274$ ft-lb

25. Let y measure the height of a narrow rectangle with $0 \le y \le 3$. The force against this rectangle at depth $3 - y$ is $\Delta F \approx \delta(3-y)(6)\Delta y$. Thus,

$$F = \int_0^3 \delta(3-y)(6)\,dy = 6\delta\left[3y - \frac{y^2}{2}\right]_0^3$$

$= 6 \cdot 62.4(4.5) = 1684.8$ pounds

26. Let y measure the height of a narrow rectangle with $0 \le y \le 3$. The force against this rectangle at depth $5 - y$ is $\Delta F \approx \delta(5-y)(6)\Delta y$. Thus,

$$F = \int_0^3 \delta(5-y)(6)\,dy = 6\delta\left[5y - \frac{y^2}{2}\right]_0^3$$

$= 6 \cdot 62.4 \cdot 10.5 = 3931.2$ pounds

27. Place the equilateral triangle in the coordinate system such that the vertices are $(-3,0), (3,0)$ and $\left(0, -3\sqrt{3}\right)$.

The equation of the line in Quadrant I is

$y = \sqrt{3}\cdot x - 3\sqrt{3}$ or $x = \frac{y}{\sqrt{3}} + 3.$

$\Delta F \approx \delta(-y)\left(2\left(\frac{y}{\sqrt{3}} + 3\right)\right)\Delta y$ and

$$F = \int_{-3\sqrt{3}}^{0}\delta(-y)\left(2\left(\frac{y}{\sqrt{3}} + 3\right)\right)dy$$

$$= -2\delta\int_{-3\sqrt{3}}^{0}\left(\frac{y^2}{\sqrt{3}} + 3y\right)dy$$

$$= -2\delta\left[\frac{y^3}{3\sqrt{3}} + \frac{3y^2}{2}\right]_{-3\sqrt{3}}^{0} = -2\cdot 62.4(0 - 13.5)$$

$= 1684.8$ pounds

28. Place the right triangle in the coordinate system such that the vertices are (0,0), (3,0) and (0,-4). The equation of the line in Quadrant IV is

$y = \frac{4}{3}x - 4$ or $x = \frac{3}{4}y + 3.$

$\Delta F \approx \delta(3-y)\left(\frac{3}{4}y + 3\right)\Delta y$ and

$$F = \int_{-4}^{0}\delta\left(9 - \frac{3}{4}y - \frac{3}{4}y^2\right)dy$$

$$= \delta\left[9y - \frac{3y^2}{8} - \frac{y^3}{4}\right]_{-4}^{0} = 62.4\cdot 26$$

$= 1622.4$ pounds

29. $\Delta F \approx \delta(1-y)\left(\sqrt{y}\right)\Delta y;\ F = \int_0^1 \delta(1-y)\left(\sqrt{y}\right)dy$

$$= \delta\int_0^1\left(y^{1/2} - y^{3/2}\right)dy$$

$$= \delta\left[\frac{2}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = 62.4\left(\frac{4}{15}\right)$$

$= 16.64$ pounds

30. Place the circle in the coordinate system so that the center is (0.0). The equation of the circle is $x^2 + y^2 = 16$ and in Quadrants I and IV, $x = \sqrt{16 - y^2}$. $\Delta F \approx \delta(6-y)\left(2\sqrt{16-y^2}\right)\Delta y$

$$F = \int_{-4}^{4} \delta(6-y)\left(2\sqrt{16-y^2}\right)dy$$

Using a CAS, $F \approx 18{,}819$ pounds.

31. Place a rectangle in the coordinate system such that the vertices are (0,0), (0,b), (a,0) and (a,b). The equation of the diagonal from (0,0) to (a,b) is $y = \frac{b}{a}x$ or $x = \frac{a}{b}y$. For the upper left triangle I,

$$\Delta F \approx \delta(b-y)\left(\frac{a}{b}y\right)\Delta y \text{ and}$$

$$F = \int_0^b \delta(b-y)\left(\frac{a}{b}y\right)dy$$

$$= \delta\int_0^b \left(y - \frac{a}{b}y^2\right)dy = \delta\left[\frac{ay^2}{2} - \frac{ay^3}{3b}\right]_0^b$$

$$= \delta\left(\frac{ab^2}{2} - \frac{ab^2}{3}\right) = \delta\frac{ab^2}{6}$$

For the lower right triangle II,

$$\Delta F \approx \delta(b-y)\left(a - \frac{a}{b}y\right)dy \text{ and}$$

$$F = \int_0^b \delta(b-y)\left(a - \frac{a}{b}y\right)dy$$

$$= \int_0^b \delta\left(ab - 2ay + \frac{a}{b}y^2\right)dy$$

$$= \delta\left[aby - ay^2 + \frac{ay^3}{3b}\right]_0^b = \delta\left(ab^2 - ab^2 + \frac{ab^2}{3}\right)$$

$$= \delta\frac{ab^2}{3}$$

The total force on one half of the dam is twice the total force on the other half since $\dfrac{\delta\frac{ab^2}{3}}{\delta\frac{ab^2}{6}} = 2.$

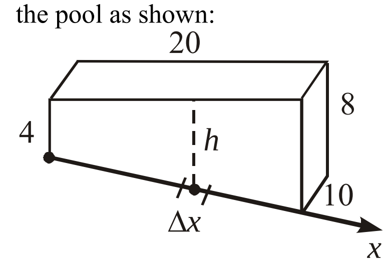
32. Consider one side of the cube and place the vertices of this square on (0,0), (0,2), (2,0) and (2,2).

$$\Delta F \approx \delta(102-y)(2)\Delta y;\ F = \int_0^2 2\delta(102-y)\,dy$$

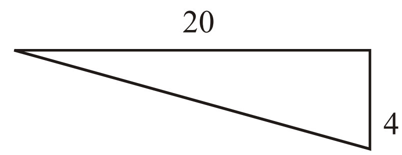
$$= 2\delta\left[102y - \frac{y^2}{2}\right]_0^2 = 2\cdot 62.4\cdot 202 = 25{,}209.6$$

The force on all six sides would be 6(25,209.6) = 151,257.6 pounds.

33. We can position the x-axis along the bottom of the pool as shown:



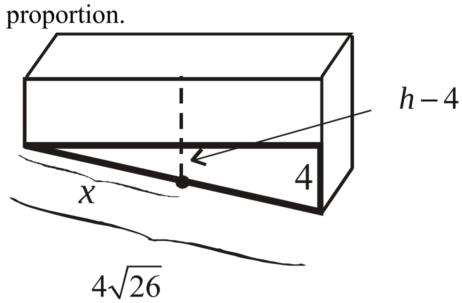
From the diagram, we let h = the depth of an arbitrary slice along the width of the bottom of the pool.



Using the Pythagorean Theorem, we can find that the length of the bottom of the pool is

$\sqrt{20^2 + 4^2} = \sqrt{416} = 4\sqrt{26}$

Next, we need to get h in terms of x. This can be done by using similar triangles to set up a proportion.



$$\frac{h-4}{4} = \frac{x}{4\sqrt{26}} \quad \rightarrow \quad h = 4 + \frac{x}{\sqrt{26}}$$

$$\Delta F = \delta \cdot h \cdot \Delta A$$

$$F = \int_0^{4\sqrt{26}} \delta\left(4 + \frac{x}{\sqrt{26}}\right)(10)\,dx$$

$$= \int_0^{4\sqrt{26}} 62.4\left(4 + \frac{x}{\sqrt{26}}\right)(10)\,dx$$

$$= 624\int_0^{4\sqrt{26}} \left(4 + \frac{x}{\sqrt{26}}\right)dx$$

$$= 624\left[4x + \frac{x^2}{2\sqrt{26}}\right]_0^{4\sqrt{26}}$$

$$= 624\left(16\sqrt{26} + 8\sqrt{26}\right) = 624\left(24\sqrt{26}\right)$$

$$= 14{,}976\sqrt{26}\ \text{ lb} \quad \left(\approx 76{,}362.92\text{ lb}\right)$$

34. If we imagine unrolling the cylinder so we have a flat sheet, then we need to find the total force against one side of a rectangular plate as if it had been submerged in the oil. The rectangle would be $2\pi(5) = 10\pi$ feet wide and 6 feet high.

Thus, the total lateral force is given by

$$F = \int_0^6 50 \cdot y \cdot 10\pi \, dy$$

$$= 500\pi \int_0^6 y \, dy = \left[250\pi y^2\right]_0^6$$

$$= 250\pi(36) = 9000\pi \text{ lbs} \quad (\approx 28,274.33 \text{ lb})$$

35. Let W_1 be the work to lift V to the surface and W_2 be the work to lift V from the surface to 15 feet above the surface. The volume displaced by the buoy y feet above its original position is

$$\frac{1}{3}\pi\left(a - \frac{a}{h}y\right)^2 (h - y) = \frac{1}{3}\pi a^2 h\left(1 - \frac{y}{h}\right)^3.$$

The weight displaced is $\frac{\delta}{3}\pi a^2 h\left(1 - \frac{y}{h}\right)^3$.

Note by Archimede's Principle $m = \frac{\delta}{3}\pi a^2 h$ or

$a^2 h = \frac{3m}{\delta\pi}$, so the displaced weight is

$$m\left(1 - \frac{y}{h}\right)^3.$$

$$\Delta W_1 \approx \left(m - m\left(1 - \frac{y}{h}\right)^3\right)\Delta y = m\left(1 - \left(1 - \frac{y}{h}\right)^3\right)\Delta y$$

$$W_1 = m\int_0^h \left(1 - \left(1 - \frac{y}{h}\right)^3\right) dy$$

$$= m\left[y + \frac{h}{4}\left(1 - \frac{y}{h}\right)^4\right]_0^h = \frac{3mh}{4}$$

$$W_2 = m \cdot 15 = 15m$$

$$W = W_1 + W_2 = \frac{3mh}{4} + 15m$$

36. First calculate the work W_1 needed to lift the contents of the bottom tank to 10 feet.

$$\Delta W_1 \approx \delta 40 \Delta y(10 - y)$$

$$W_1 = \int_0^4 \delta 40(10 - y) dy$$

$$= (62.4)(40)\left[-\frac{1}{2}(10 - y)^2\right]_0^4$$

$= (62.4)(40)(-18 + 50) = 79,872$ ft-lb

Next calculate the work W_2 needed to fill the top tank. Let y be the distance from the bottom of the top tank.

$$\Delta W_2 \approx \delta(36\pi)\Delta y \, y$$

Solve for the height of the top tank:

$$36\pi h = 160; h = \frac{160}{36\pi} = \frac{40}{9\pi}$$

$$W_2 = \int_0^{40/9\pi} \delta 36\pi y \, dy$$

$$= (62.4)(36\pi)\left[\frac{1}{2}y^2\right]_0^{40/9\pi}$$

$$= (62.4)(36\pi)\left(\frac{800}{81\pi^2}\right) \approx 7062 \text{ ft-lbs}$$

$$W = W_1 + W_2 \approx 86,934 \text{ ft-lbs}$$

37. Since $\delta\left(\frac{1}{3}\pi a^2\right)(8) = 300, a = \sqrt{\frac{225}{2\pi\delta}}$.

When the buoy is at z feet ($0 \le z \le 2$) below floating position, the radius r at the water level is

$$r = \left(\frac{8 + z}{8}\right)a = \sqrt{\frac{225}{2\pi\delta}}\left(\frac{8 + z}{8}\right).$$

$$F = \delta\left(\frac{1}{3}\pi r^2\right)(8 + z) - 300$$

$$= \frac{75}{128}(8 + z)^3 - 300$$

$$W = \int_0^2 \left[\frac{75}{128}(8 + z)^3 - 300\right] dz$$

$$= \left[\frac{75}{512}(8 + z)^4 - 300z\right]_0^2$$

$$= \left(\frac{46,875}{32} - 600\right) - (600 - 0)$$

$$= \frac{8475}{32} \approx 264.84 \text{ ft-lb}$$

5.6 Concepts Review

1. right; $\frac{4\cdot 1+6\cdot 3}{4+6}=2.2$

2. 2.5; right; $x(1+x)$; $1+x$

3. 1; 3

4. $\frac{24}{16};\frac{40}{16}$
The second lamina balances at $\bar{x}=3,\ \bar{y}=1$.
The first lamina has area 12 and the second lamina has area 4.
$\bar{x}=\frac{12\cdot 1+4\cdot 3}{12+4}=\frac{24}{16},\ \bar{y}=\frac{12\cdot 3+4\cdot 1}{12+4}=\frac{40}{16}$

Problem Set 5.6

1. $\bar{x}=\frac{2\cdot 5+(-2)\cdot 7+1\cdot 9}{5+7+9}=\frac{5}{21}$

2. Let x measure the distance from the end where John sits.
$\frac{180\cdot 0+80\cdot x+110\cdot 12}{180+80+110}=6$
$80x + 1320 = 6 \cdot 370$
$80x = 900$
$x = 11.25$
Tom should be 11.25 feet from John, or, equivalently, 0.75 feet from Mary.

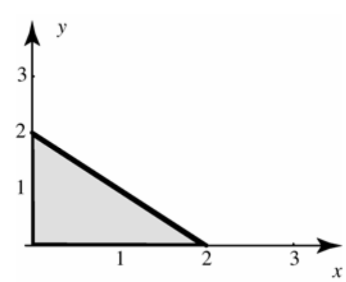
3. $\bar{x}=\frac{\int_0^7 x\sqrt{x}\,dx}{\int_0^7 \sqrt{x}\,dx}=\frac{\left[\frac{2}{5}x^{5/2}\right]_0^7}{\left[\frac{2}{3}x^{3/2}\right]_0^7}=\frac{\frac{2}{5}\left(49\sqrt{7}\right)}{\frac{2}{3}\left(7\sqrt{7}\right)}=\frac{21}{5}$

4. $\bar{x}=\frac{\int_0^7 x(1+x^3)dx}{\int_0^7 (1+x^3)dx}=\frac{\left[\frac{1}{2}x^2+\frac{1}{5}x^5\right]_0^7}{\left[x+\frac{1}{4}x^4\right]_0^7}$
$=\frac{\left(\frac{49}{2}+\frac{16,807}{5}\right)}{\left(7+\frac{2401}{4}\right)}=\frac{\frac{33,859}{10}}{\frac{2429}{4}}=\frac{9674}{1735}\approx 5.58$

5. $M_y=1\cdot 2+7\cdot 3+(-2)\cdot 4+(-1)\cdot 6+4\cdot 2=17$
$M_x=1\cdot 2+1\cdot 3+(-5)\cdot 4+0\cdot 6+6\cdot 2=-3$
$m = 2 + 3 + 4 + 6 + 2 = 17$
$\bar{x}=\frac{M_y}{m}=1,\ \bar{y}=\frac{M_x}{m}=-\frac{3}{17}$

6. $M_y=(-3)\cdot 5+(-2)\cdot 6+3\cdot 2+4\cdot 7+7\cdot 1=14$
$M_x=2\cdot 5+(-2)\cdot 6+5\cdot 2+3\cdot 7+(-1)\cdot 1=28$
$m = 5 + 6 + 2 + 7 + 1 = 21$
$\bar{x}=\frac{M_y}{m}=\frac{2}{3},\ \bar{y}=\frac{M_x}{m}=\frac{4}{3}$

7. Consider two regions R_1 and R_2 such that R_1 is bounded by *f(x)* and the *x*-axis, and R_2 is bounded by *g(x)* and the *x*-axis. Let R_3 be the region formed by R_1-R_2. Make a regular partition of the homogeneous region R_3 such that each sub-region is of width , Δx and let *x* be the distance from the *y*-axis to the center of mass of a sub-region. The heights of R_1 and R_2 at *x* are approximately *f(x)* and *g(x)* respectively. The mass of R_3 is approximately
$$\begin{aligned}\Delta m&=\Delta m_1-\Delta m_2\\&\approx \delta f(x)\Delta x-\delta g(x)\Delta x\\&=\delta[f(x)-g(x)]\Delta x\end{aligned}$$
where δ is the density. The moments for R_3 are approximately
$$\begin{aligned}M_x&=M_x(R_1)-M_x(R_2)\\&\approx\frac{\delta}{2}[f(x)]^2\Delta x-\frac{\delta}{2}[g(x)]^2\Delta x\\&=\frac{\delta}{2}\left[(f(x))^2-(g(x))^2\right]\Delta x\end{aligned}$$
$$\begin{aligned}M_y&=M_y(R_1)-M_y(R_2)\\&\approx x\delta f(x)\Delta x-x\delta g(x)\Delta x\\&=x\delta[f(x)-g(x)]\Delta x\end{aligned}$$
Taking the limit of the regular partition as $\Delta x\to 0$ yields the resulting integrals in Figure 10.

8.


$f(x)=2-x;\ g(x)=0$
$\bar{x}=\frac{\int_0^2 x[(2-x)-0]dx}{\int_0^2[(2-x)-0]dx}$
$=\frac{\int_0^2[2x-x^2]dx}{\int_0^2[2-x]dx}$

$$= \frac{\left(x^2 - \frac{1}{3}x^3\right)_0^2}{\left(2x - \frac{1}{2}x^2\right)_0^2} = \frac{4 - \frac{8}{3}}{4 - 2}$$

$$= \frac{2}{3}$$

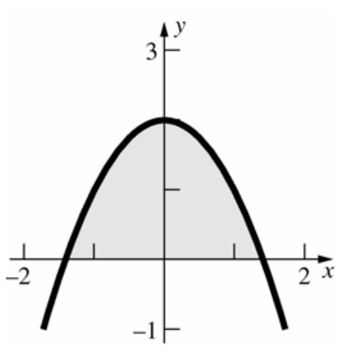
$$\overline{y} = \frac{\frac{1}{2}\int_0^2 [(2-x)^2 - 0^2]dx}{\int_0^2 [(2-x) - 0]dx}$$

$$= \frac{\int_0^2 [4 - 4x + x^2]dx}{4}$$

$$= \frac{\left(4x - 2x^2 + \frac{1}{3}x^3\right)_0^2}{4} = \frac{8 - 8 + \frac{8}{3}}{4}$$

$$= \frac{2}{3}$$

9.



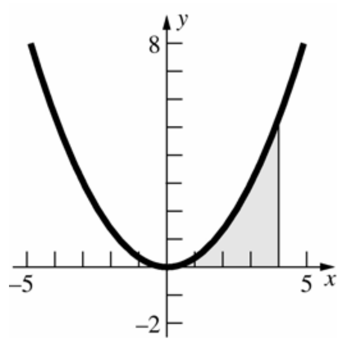
$\overline{x} = 0$ (by symmetry)

$$\overline{y} = \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^2\, dx}{\int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)dx}$$

$$= \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4x^2 + x^4)dx}{\left[2x - \frac{1}{3}x^3\right]_{-\sqrt{2}}^{\sqrt{2}}}$$

$$= \frac{\frac{1}{2}\left[4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}} = \frac{\frac{32\sqrt{2}}{15}}{\frac{8\sqrt{2}}{3}} = \frac{4}{5}$$

10.



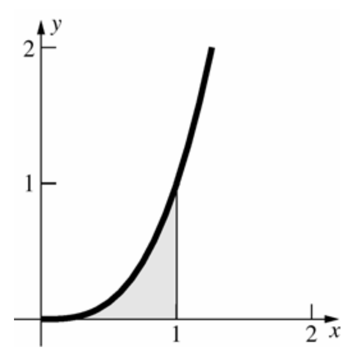
$$\overline{x} = \frac{\int_0^4 x\left(\frac{1}{3}x^2\right)dx}{\int_0^4 \frac{1}{3}x^2 dx} = \frac{\frac{1}{3}\int_0^4 x^3 dx}{\frac{1}{3}\int_0^4 x^2\, dx}$$

$$= \frac{\frac{1}{3}\left[\frac{1}{4}x^4\right]_0^4}{\frac{1}{3}\left[\frac{1}{3}x^3\right]_0^4} = \frac{\frac{64}{3}}{\frac{64}{9}} = 3$$

$$\overline{y} = \frac{\frac{1}{2}\int_0^4 \left(\frac{1}{3}x^2\right)^2 dx}{\int_0^4 \frac{1}{3}x^2\, dx} = \frac{\frac{1}{18}\int_0^4 x^4 dx}{\frac{64}{9}} = \frac{\frac{1}{18}\left[\frac{1}{5}x^5\right]_0^4}{\frac{64}{9}}$$

$$= \frac{\frac{512}{45}}{\frac{64}{9}} = \frac{8}{5}$$

11.

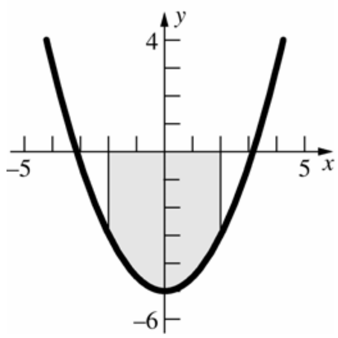


$$\overline{x} = \frac{\int_0^1 x(x^3)dx}{\int_0^1 x^3 dx} = \frac{\int_0^1 x^4\, dx}{\left[\frac{1}{4}x^4\right]_0^1} = \frac{\left[\frac{1}{5}x^5\right]_0^1}{\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}$$

$$\overline{y} = \frac{\frac{1}{2}\int_0^1 (x^3)^2 dx}{\int_0^1 x^3\, dx} = \frac{\frac{1}{2}\int_0^1 x^6 dx}{\frac{1}{4}} = \frac{\left[\frac{1}{14}x^7\right]_0^1}{\frac{1}{4}}$$

$$= \frac{\frac{1}{14}}{\frac{1}{4}} = \frac{2}{7}$$

12.



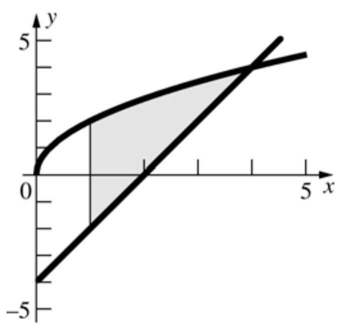
$\overline{x} = 0$ (by symmetry)

$$\overline{y} = \frac{\frac{1}{2}\int_{-2}^{2}\left[-\left(\frac{1}{2}(x^2-10)\right)^2\right]dx}{\int_{-2}^{2}\left[-\frac{1}{2}(x^2-10)\right]dx}$$

$$= \frac{-\frac{1}{8}\int_{-2}^{2}(x^4-20x^2+100)dx}{-\frac{1}{2}\int_{-2}^{2}(x^2-10)dx}$$

$$= \frac{-\frac{1}{8}\left[\frac{1}{5}x^5-\frac{20}{3}x^3+100x\right]_{-2}^{2}}{-\frac{1}{2}\left[\frac{1}{3}x^3-10x\right]_{-2}^{2}} = \frac{-\frac{574}{15}}{\frac{52}{3}} = -\frac{287}{130}$$

13.



To find the intersection point, solve

$2x-4 = 2\sqrt{x}$.

$x-2 = \sqrt{x}$

$x^2-4x+4 = x$

$x^2-5x+4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ ($x = 1$ is extraneous.)

$$\overline{x} = \frac{\int_1^4 x\left[2\sqrt{x}-(2x-4)\right]dx}{\int_1^4\left[2\sqrt{x}-(2x-4)\right]dx}$$

$$= \frac{2\int_1^4(x^{3/2}-x^2+2x)dx}{2\int_1^4(x^{1/2}-x+2)dx}$$

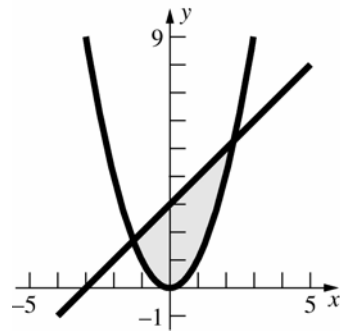
$$= \frac{2\left[\frac{2}{5}x^{5/2}-\frac{1}{3}x^3+x^2\right]_1^4}{2\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right]_1^4} = \frac{\frac{64}{5}}{\frac{19}{3}} = \frac{192}{95}$$

$$\overline{y} = \frac{\frac{1}{2}\int_1^4\left[\left(2\sqrt{x}\right)^2-(2x-4)^2\right]dx}{\int_1^4\left[2\sqrt{x}-(2x-4)\right]dx}$$

$$= \frac{2\int_1^4(-x^2+5x-4)dx}{\frac{19}{3}}$$

$$= \frac{2\left[-\frac{1}{3}x^3+\frac{5}{2}x^2-4x\right]_1^4}{\frac{19}{3}} = \frac{9}{\frac{19}{3}} = \frac{27}{19}$$

14.



To find the intersection points, $x^2 = x+3$.

$x^2-x-3 = 0$

$x = \frac{1\pm\sqrt{13}}{2}$

$$\overline{x} = \frac{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}} x(x+3-x^2)dx}{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}(x+3-x^2)dx}$$

$$= \frac{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}(x^2+3x-x^3)dx}{\left[\frac{1}{2}x^2+3x-\frac{1}{3}x^3\right]_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}}$$

$$= \frac{\left[\frac{1}{3}x^3+\frac{3}{2}x^2-\frac{1}{4}x^4\right]_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}}{\frac{13\sqrt{13}}{6}} = \frac{\frac{13\sqrt{3}}{12}}{\frac{13\sqrt{13}}{6}} = \frac{1}{2}$$

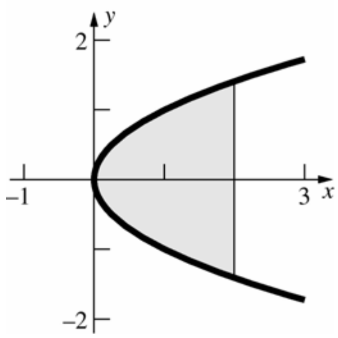
$$\overline{y} = \frac{\frac{1}{2}\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}\left[(x+3)^2-(x^2)^2\right]dx}{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}(x+3-x^2)dx}$$

$$= \frac{\frac{1}{2}\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}} (x^2+6x+9-x^4)}{\frac{13\sqrt{13}}{6}}$$

$$= \frac{\frac{1}{2}\left[\frac{1}{3}x^3+3x^2+9x-\frac{1}{5}x^5\right]_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}}{\frac{13\sqrt{13}}{6}}$$

$$= \frac{\frac{143\sqrt{13}}{30}}{\frac{13\sqrt{13}}{6}} = \frac{11}{5}$$

15.



To find the intersection points, solve $y^2 = 2$.

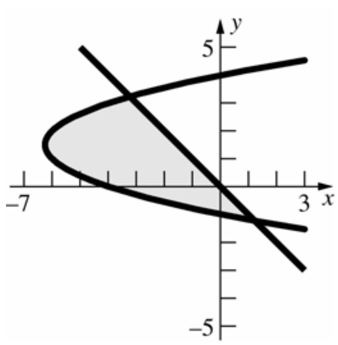
$y = \pm\sqrt{2}$

$$\bar{x} = \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} \left[2^2-(y^2)^2\right]dy}{\int_{-\sqrt{2}}^{\sqrt{2}} (2-y^2)dy} = \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (4-y^4)dy}{\left[2y-\frac{1}{3}y^3\right]_{-\sqrt{2}}^{\sqrt{2}}}$$

$$= \frac{\frac{1}{2}\left[4y-\frac{1}{5}y^5\right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}} = \frac{\frac{16\sqrt{2}}{5}}{\frac{8\sqrt{2}}{3}} = \frac{6}{5}$$

$\bar{y} = 0$ (by symmetry)

16.



To find the intersection points, solve

$y^2 - 3y - 4 = -y$.

$y^2 - 2y - 4 = 0$

$$y = \frac{2 \pm \sqrt{20}}{2}$$

$y = 1 \pm \sqrt{5}$

$$\bar{x} = \frac{\frac{1}{2}\int_{1-\sqrt{5}}^{1+\sqrt{5}} \left[(-y)^2-(y^2-3y-4)^2\right]dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}} \left[(-y)-(y^2-3y-4)\right]dy}$$

$$= \frac{\frac{1}{2}\int_{1-\sqrt{5}}^{1+\sqrt{5}} (-y^4+6y^3-24y-16)dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}} (-y^2+2y+4)dy}$$

$$= \frac{\frac{1}{2}\left[-\frac{1}{5}y^5+\frac{3}{2}y^4-12y^2-16y\right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\left[-\frac{1}{3}y^3+y^2+4y\right]_{1-\sqrt{5}}^{1+\sqrt{5}}} = \frac{-20\sqrt{5}}{\frac{20\sqrt{5}}{3}}$$

$= -3$

$$\bar{y} = \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}} y\left[(-y)-(y^2-3y-4)\right]dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}} \left[(-y)-(y^2-3y-4)\right]dy}$$

$$= \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}} (-y^3+2y^2+4y)dy}{\frac{20\sqrt{5}}{3}}$$

$$= \frac{\left[-\frac{1}{4}y^4+\frac{2}{3}y^3+2y^2\right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\frac{20\sqrt{5}}{3}} = \frac{\frac{20\sqrt{5}}{3}}{\frac{20\sqrt{5}}{3}} = 1$$

17. We let δ be the density of the regions and A_i be the area of region i.

Region R_1:

$$m(R_1) = \delta A_1 = \delta(1/2)(1)(1) = \frac{1}{2}\delta$$

$$\bar{x}_1 = \frac{\int_0^1 x(x)dx}{\int_0^1 xdx} = \frac{\frac{1}{3}x^3\Big|_0^1}{\frac{1}{2}x^2\Big|_0^1} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Since R_1 is symmetric about the line $y = 1 - x$, the centroid must lie on this line. Therefore,

$\bar{y}_1 = 1 - \bar{x}_1 = 1 - \frac{2}{3} = \frac{1}{3}$; and we have

$$M_y(R_1) = \bar{x}_2 \cdot m(R_1) = \frac{1}{3}\delta$$

$$M_x(R_1) = \bar{y}_2 \cdot m(R_1) = \frac{1}{6}\delta$$

Region R_2:

$m(R_2) = \delta A_2 = \delta(2)(1) = 2\delta$

By symmetry we get

$\bar{x}_2 = 2$ and $\bar{y}_2 = \frac{1}{2}$.

Thus,

$M_y(R_2) = \bar{x}_2 \cdot m(R_2) = 4\delta$

$M_x(R_2) = \bar{y}_2 \cdot m(R_2) = \delta$

18. We can obtain the mass and moments for the whole region by adding the individual regions. Using the results from Problem 17 we get that

$$m = m(R_1) + m(R_2) = \frac{1}{2}\delta + 2\delta = \frac{5}{2}\delta$$

$$M_y = M_y(R_1) + M_y(R_2) = \frac{1}{3}\delta + 4\delta = \frac{13}{3}\delta$$

$$M_x = M_x(R_1) + M_x(R_2) = \frac{1}{6}\delta + \delta = \frac{7}{6}\delta$$

Therefore, the centroid is given by

$$\overline{x} = \frac{M_y}{m} = \frac{\frac{13}{3}\delta}{\frac{5}{2}\delta} = \frac{26}{15}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\frac{7}{6}\delta}{\frac{5}{2}\delta} = \frac{7}{15}$$

19. $m(R_1) = \delta\int_a^b (g(x) - f(x))dx$

$$m(R_2) = \delta\int_b^c (g(x) - f(x))dx$$

$$M_x(R_1) = \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx$$

$$M_x(R_2) = \frac{\delta}{2}\int_b^c ((g(x))^2 - (f(x))^2)dx$$

$$M_y(R_1) = \delta\int_a^b x(g(x) - f(x))dx$$

$$M_y(R_2) = \delta\int_b^c x(g(x) - f(x))dx$$

Now,

$$\begin{aligned} m(R_3) &= \delta\int_a^c (g(x) - f(x))dx \\ &= \delta\int_a^b (g(x) - f(x))dx + \delta\int_b^c (g(x) - f(x))dx \\ &= m(R_1) + m(R_2) \end{aligned}$$

$$\begin{aligned} M_x(R_3) &= \frac{\delta}{2}\int_a^c ((g(x))^2 - (f(x))^2)dx \\ &= \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx \\ &\quad + \frac{\delta}{2}\int_b^c ((g(x))^2 - (f(x))^2)dx \\ &= M_x(R_1) + M_x(R_2) \end{aligned}$$

$$\begin{aligned} M_y(R_3) &= \delta\int_a^c x(g(x) - f(x))dx \\ &= \delta\int_a^b x(g(x) - f(x))dx \\ &\quad + \delta\int_b^c x(g(x) - f(x))dx \\ &= M_y(R_1) + M_y(R_2) \end{aligned}$$

20. $m(R_1) = \delta\int_a^b (h(x) - g(x))dx$

$$m(R_2) = \delta\int_a^b (g(x) - f(x))dx$$

$$M_x(R_1) = \frac{\delta}{2}\int_a^b ((h(x))^2 - (g(x))^2)dx$$

$$M_x(R_2) = \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx$$

$$M_y(R_1) = \delta\int_a^b x(h(x) - g(x))dx$$

$$M_y(R_2) = \delta\int_a^b x(g(x) - f(x))dx$$

Now,

$$\begin{aligned} m(R_3) &= \delta\int_a^b (h(x) - f(x))dx \\ &= \delta\int_a^b (h(x) - g(x) + g(x) - f(x))dx \\ &= \delta\int_a^b (h(x) - g(x))dx + \delta\int_a^b (g(x) - f(x))dx \\ &= m(R_1) + m(R_2) \end{aligned}$$

$$\begin{aligned} M_x(R_3) &= \frac{\delta}{2}\int_a^b ((h(x))^2 - (f(x))^2)dx \\ &= \frac{\delta}{2}\int_a^b ((h(x))^2 - (g(x))^2 + (g(x))^2 - (f(x))^2)dx \\ &= \frac{\delta}{2}\int_a^b ((h(x))^2 - (g(x))^2)dx \\ &\quad + \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx \\ &= M_x(R_1) + M_x(R_2) \end{aligned}$$

$$\begin{aligned} M_y(R_3) &= \delta\int_a^b x(h(x) - f(x))dx \\ &= \delta\int_a^b x(h(x) - g(x) + g(x) - f(x))dx \\ &= \delta\int_a^b x(h(x) - g(x))dx + \delta\int_a^b x(g(x) - f(x))dx \\ &= M_y(R_1) + M_y(R_2) \end{aligned}$$

21. Let region 1 be the region bounded by $x = -2$, $x = 2$, $y = 0$, and $y = 1$, so $m_1 = 4 \cdot 1 = 4$.

By symmetry, $\overline{x}_1 = 0$ and $\overline{y}_1 = \frac{1}{2}$. Therefore $M_{1y} = \overline{x}_1 m_1 = 0$ and $M_{1x} = \overline{y}_1 m_1 = 2$.

Let region 2 be the region bounded by $x = -2$, $x = 1$, $y = -1$, and $y = 0$, so $m_2 = 3 \cdot 1 = 3$.

By symmetry, $\overline{x}_2 = -\frac{1}{2}$ and $\overline{y}_2 = -\frac{1}{2}$. Therefore $M_{2y} = \overline{x}_2 m_2 = -\frac{3}{2}$ and $M_{2x} = \overline{y}_2 m_2 = -\frac{3}{2}$.

$$\overline{x} = \frac{M_{1y} + M_{2y}}{m_1 + m_2} = \frac{-\frac{3}{2}}{7} = -\frac{3}{14}$$

$$\overline{y} = \frac{M_{1x} + M_{2x}}{m_1 + m_2} = \frac{\frac{1}{2}}{7} = \frac{1}{14}$$

22. Let region 1 be the region bounded by $x = –3$, $x = 1$, $y = –1$, and $y = 4$, so $m_1 = 20$. By symmetry, $\bar{x} = -1$ and $\bar{y}_1 = \frac{3}{2}$. Therefore, $M_{1y} = \bar{x}_1 m_1 = -20$ and $M_{1x} = \bar{y}_1 m_1 = 30$. Let region 2 be the region bounded by $x = –3$, $x = –2$, $y = –3$, and $y = –1$, so $m_2 = 2$. By symmetry, $\bar{x}_2 = -\frac{5}{2}$ and $\bar{y}_2 = -2$. Therefore, $M_{2y} = \bar{x}_2 m_2 = -5$ and $M_{2x} = \bar{y}_2 m_2 = -4$. Let region 3 be the region bounded by $x = 0$, $x = 1$, $y = –2$, and $y = –1$, so $m_3 = 1$. By symmetry, $\bar{x}_3 = \frac{1}{2}$ and $\bar{y}_3 = -\frac{3}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = \frac{1}{2}$ and $M_{3x} = \bar{y}_3 m_3 = -\frac{3}{2}$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{-\frac{49}{2}}{23} = -\frac{49}{46}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{\frac{49}{2}}{23} = \frac{49}{46}$$

23. Let region 1 be the region bounded by $x = –2$, $x = 2$, $y = 2$, and $y = 4$, so $m_1 = 4 \cdot 2 = 8$. By symmetry, $\bar{x}_1 = 0$ and $\bar{y}_1 = 3$. Therefore, $M_{1y} = \bar{x}_1 m_1 = 0$ and $M_{1x} = \bar{y}_1 m_1 = 24$. Let region 2 be the region bounded by $x = –1$, $x = 2$, $y = 0$, and $y = 2$, so $m_2 = 3 \cdot 2 = 6$. By symmetry, $\bar{x}_2 = \frac{1}{2}$ and $\bar{y}_2 = 1$. Therefore, $M_{2y} = \bar{x}_2 m_2 = 3$ and $M_{2x} = \bar{y}_2 m_2 = 6$. Let region 3 be the region bounded by $x = 2$, $x = 4$, $y = 0$, and $y = 1$, so $m_3 = 2 \cdot 1 = 2$. By symmetry, $\bar{x}_3 = 3$ and $\bar{y}_2 = \frac{1}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = 6$ and $M_{3x} = \bar{y}_3 m_3 = 1$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{9}{16}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{31}{16}$$

24. Let region 1 be the region bounded by $x = –3$, $x = –1$, $y = –2$, and $y = 1$, so $m_1 = 6$. By symmetry, $\bar{x}_1 = -2$ and $\bar{y}_1 = -\frac{1}{2}$. Therefore, $M_{1y} = \bar{x}_1 m_1 = -12$ and $M_{1x} = \bar{y}_1 m_1 = -3$. Let region 2 be the region bounded by $x = –1$, $x = 0$, $y = –2$, and $y = 0$, so $m_2 = 2$. By symmetry, $\bar{x}_2 = -\frac{1}{2}$ and $\bar{y}_2 = -1$. Therefore, $M_{2y} = \bar{x}_2 m_2 = -1$ and $M_{2x} = \bar{y}_2 m_2 = -2$. Let region 3 be the remaining region, so $m_3 = 22$. By symmetry, $\bar{x}_3 = 2$ and $\bar{y}_3 = -\frac{1}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = 44$ and $M_{3x} = \bar{y}_3 m_3 = -11$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{31}{30}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = -\frac{16}{30} = -\frac{8}{15}$$

25. $A = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}$

From Problem 11, $\bar{x} = \frac{4}{5}$.

$$V = A(2\pi\bar{x}) = \frac{1}{4}\left(2\pi \cdot \frac{4}{5}\right) = \frac{2\pi}{5}$$

Using cylindrical shells:

$$V = 2\pi\int_0^1 x \cdot x^3 dx = 2\pi\int_0^1 x^4 dx = 2\pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{2\pi}{5}$$

26. The area of the region is πa^2. The centroid is the center (0, 0) of the circle. It travels a distance of $2\pi(2a) = 4\pi a$. $V = 4\pi^2 a^3$

27. The volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. If the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about the x-axis the result is a sphere of radius a. The centroid of the region travels a distance of $2\pi\bar{y}$. The area of the region is $\frac{1}{2}\pi a^2$. Pappus's Theorem says that

$$(2\pi\bar{y})\left(\frac{1}{2}\pi a^2\right) = \pi^2 a^2 \bar{y} = \frac{4}{3}\pi a^3.$$

$$\bar{y} = \frac{4a}{3\pi},\ \bar{x} = 0 \text{ (by symmetry)}$$

28. Consider a slice at x rotated about the y-axis.

$\Delta V = 2\pi x h(x)\Delta x$, so $V = 2\pi\int_a^b xh(x)dx$.

$\Delta m \approx h(x)\Delta x$, so $m = \int_a^b h(x)dx = A$.

$\Delta M_y \approx xh(x)\Delta x$, so $M_y = \int_a^b xh(x)dx$.

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b xh(x)dx}{A}$$

The distance traveled by the centroid is $2\pi\bar{x}$.

$$(2\pi\bar{x})A = 2\pi\int_a^b xh(x)dx$$

Therefore, $V = 2\pi\bar{x}A$.

29. a. $\Delta V \approx 2\pi(K-y)w(y)\Delta y$

$V = 2\pi\int_c^d (K-y)w(y)dy$

b. $\Delta m \approx w(y)\Delta y$, so $m = \int_c^d w(y)dy = A$.

$\Delta M_x \approx yw(y)\Delta y$, so $M_x = \int_c^d yw(y)dy$.

$\overline{y} = \dfrac{\int_c^d yw(y)dy}{A}$

The distance traveled by the centroid is $2\pi(K-\overline{y})$.

$2\pi(K-\overline{y})A = 2\pi(KA - M_x)$

$= 2\pi\left(\int_c^d Kw(y)dy - \int_c^d yw(y)dy\right)$

$= 2\pi\int_c^d (K-y)w(y)dy$

Therefore, $V = 2\pi(K-\overline{y})A$.

30. a. $m = \frac{1}{2}bh$

The length of a segment at y is $b - \frac{b}{h}y$.

$\Delta M_x \approx y\left(b - \frac{b}{h}y\right)\Delta y = \left(by - \frac{b}{h}y^2\right)\Delta y$

$M_x = \int_0^h \left(by - \frac{b}{h}y^2\right)dy$

$= \left[\frac{1}{2}by^2 - \frac{b}{3h}y^3\right]_0^h = \frac{1}{6}bh^2$

$\overline{y} = \frac{M_x}{m} = \frac{h}{3}$

b. $A = \frac{1}{2}bh$; the distance traveled by the centroid is $2\pi\left(k - \frac{h}{3}\right)$.

$V = 2\pi\left(k - \frac{h}{3}\right)\left(\frac{1}{2}bh\right) = \frac{\pi bh}{3}(3k-h)$

31. a. The area of a regular polygon P of $2n$ sides is $2r^2 n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}$. (To find this consider the isosceles triangles with one vertex at the center of the polygon and the other vertices on adjacent corners of the polygon. Each such triangle has base of length $2r\sin\frac{\pi}{2n}$ and height $r\cos\frac{\pi}{2n}$.) Since P is a regular polygon the centroid is at its center. The distance from the centroid to any side is $r\cos\frac{\pi}{2n}$, so the centroid travels a distance of $2\pi r\cos\frac{\pi}{2n}$.

Thus, by Pappus's Theorem, the volume of the resulting solid is

$\left(2\pi r\cos\frac{\pi}{2n}\right)\left(2r^2 n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}\right)$

$= 4\pi r^3 n\sin\frac{\pi}{2n}\cos^2\frac{\pi}{2n}$.

b. $\lim_{n\to\infty} 4\pi r^3 n\sin\frac{\pi}{2n}\cos^2\frac{\pi}{2n}$

$\lim_{n\to\infty} \frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} 2\pi^2 r^3\cos^2\frac{\pi}{2n} = 2\pi^2 r^3$

As $n\to\infty$, the regular polygon approaches a circle. Using Pappus's Theorem on the circle of area πr^2 whose centroid (= center) travels a distance of $2\pi r$, the volume of the solid is $(\pi r^2)(2\pi r) = 2\pi^2 r^3$ which agrees with the results from the polygon.

32. a. The graph of $f(\sin x)$ on $[0, \pi]$ is symmetric about the line $x = \frac{\pi}{2}$ since $f(\sin x) = f(\sin(\pi - x))$. Thus $\overline{x} = \frac{\pi}{2}$.

$\overline{x} = \dfrac{\int_0^\pi x f(\sin x)\,dx}{\int_0^\pi f(\sin x)dx} = \dfrac{\pi}{2}$

Therefore

$\int_0^\pi x f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$

b. $\sin x\cos^4 x = \sin x(1-\sin^2 x)^2$, so $f(x) = x(1-x^2)^2$.

$\int_0^\pi x\sin x\cos^4 x\,dx = \frac{\pi}{2}\int_0^\pi \sin x\cos^4 x\,dx$

$= \frac{\pi}{2}\left[-\frac{1}{5}\cos^5 x\right]_0^\pi = \frac{\pi}{5}$

33. Consider the region $S - R$.

$$\bar{y}_{S-R} = \frac{\frac{1}{2}\int_0^1 \left[g^2(x) - f^2(x)\right]dx}{S-R} \geq \bar{y}_R$$

$$= \frac{\frac{1}{2}\int_0^1 f^2(x)dx}{R}$$

$$\frac{1}{2}R\int_0^1 \left[g^2(x) - f^2(x)\right]dx \geq \frac{1}{2}(S-R)\int_0^1 f^2(x)dx$$

$$\frac{1}{2}R\int_0^1 \left[g^2(x) - f^2(x)\right]dx + \frac{1}{2}R\int_0^1 f^2(x)dx$$

$$\geq \frac{1}{2}(S-R)\int_0^1 f^2(x)dx + \frac{1}{2}R\int_0^1 f^2(x)dx$$

$$\frac{1}{2}R\int_0^1 g^2(x)dx \geq \frac{1}{2}S\int_0^1 f^2(x)dx$$

$$\frac{\frac{1}{2}\int_0^1 g^2(x)\,dx}{S} \geq \frac{\frac{1}{2}\int_0^1 f^2(x)dx}{R}$$

$$\bar{y}_S \geq \bar{y}_R$$

34. To approximate the centroid, we can lay the figure on the x-axis (flat side down) and put the shortest side against the y-axis. Next we can use the eight regions between measurements to approximate the centroid. We will let h_i ,the height of the *i*th region, be approximated by the height at the right end of the interval. Each interval is of width $\Delta x = 5$ cm. The centroid can be approximated as

$$\bar{x} \approx \frac{\sum_{i=1}^{8} x_i h_i}{\sum_{i=1}^{8} h_i} = \frac{(5)(6.5)+(10)(8)+\cdots+(35)(10)+(40)(8)}{6.5+8+\cdots+10+8}$$

$$= \frac{1695}{72.5} \approx 23.38$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{8}(h_i)^2}{\sum_{i=1}^{8} h_i} = \frac{(1/2)(6.5^2+8^2+\cdots+10^2+8^2)}{(6.5+8+\cdots+10+8)}$$

$$= \frac{335.875}{72.5} \approx 4.63$$

35. First we place the lamina so that the origin is centered inside the hole. We then recompute the centroid of Problem 34 (in this position) as

$$\bar{x} \approx \frac{\sum_{i=1}^{8} x_i h_i}{\sum_{i=1}^{8} h_i}$$

$$= \frac{(-25)(6.5)+(-15)(8)+\cdots+(5)(10)+(10)(8)}{6.5+8+\cdots+10+8}$$

$$= \frac{-480}{72.5} \approx -6.62$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{8}((h_i-4)^2-(-4)^2)}{\sum_{i=1}^{8} h_i}$$

$$= \frac{(1/2)((2.5^2-(-4)^2)+\cdots+(4^2-(-4)^2))}{6.5+8+\cdots+10+8}$$

$$= \frac{45.875}{72.5} \approx 0.633$$

A quick computation will show that these values agree with those in Problem 34 (using a different reference point).

Now consider the whole lamina as R_3 , the circular hole as R_2 , and the remaining lamina as R_1 . We can find the centroid of R_1 by noting that

$M_x(R_1) = M_x(R_3) - M_x(R_2)$

and similarly for $M_y(R_1)$.

From symmetry, we know that the centroid of a circle is at the center. Therefore, both $M_x(R_2)$ and $M_y(R_2)$ must be zero in our case.

This leads to the following equations

$$\bar{x} = \frac{M_y(R_3) - M_y(R_2)}{m(R_3) - m(R_2)}$$

$$= \frac{\delta\Delta x(-480)}{\delta\Delta x(72.5) - \delta\pi(2.5)^2}$$

$$= \frac{-2400}{342.87} \approx -7$$

$$\bar{y} = \frac{M_x(R_3) - M_x(R_2)}{m(R_3) - m(R_2)}$$

$$= \frac{\delta\Delta x(45.875)}{\delta\Delta x(72.5) - \delta\pi(2.5)^2}$$

$$= \frac{229.375}{342.87} \approx 0.669$$

Thus, the centroid is 7 cm above the center of the hole and 0.669 cm to the right of the center of the hole.

36. This problem is much like Problem 34 except we don't have one side that is completely flat. In this problem, it will be necessary, in some regions, to find the value of $g(x)$ instead of just $f(x) - g(x)$. We will use the 19 regions in the figure to approximate the centroid. Again we choose the height of a region to be approximately the value at the right end of that region. Each region has a width of 20 miles. We will place the north-east corner of the state at the origin.
The centroid is approximately

$$\overline{x} \approx \frac{\sum_{i=1}^{19} x_i (f(x_i) - g(x_i))}{\sum_{i=1}^{19} (f(x_i) - g(x_i))}$$

$$= \frac{(20)(145-13)+(40)(149-10)+\cdots(380)(85-85)}{(145-13)+(149-19)+\cdots(85-85)}$$

$$= \frac{482,860}{2780} \approx 173.69$$

$$\overline{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{19}[(f(x_i))^2 - (g(x_i))^2]}{\sum_{i=1}^{19} (f(x_i) - g(x_i))}$$

$$= \frac{\frac{1}{2}\left[(145^2-13^2)+(149^2-10^2)+\cdots+(85^2-85^2)\right]}{(145-13)+(149-19)+\cdots+(85-85)}$$

$$= \frac{230,805}{2780} \approx 83.02$$

This would put the geographic center of Illinois just south-east of Lincoln, IL.

5.7 Concepts Review

1. discrete, continuous

2. sum, integral

3. $\int_0^5 f(x)\,dx$

4. cumulative distribution function

Problem Set 5.7

1. **a.** $P(X \ge 2) = P(2) + P(3) = 0.05 + 0.05 = 0.1$

b. $E(X) = \sum_{i=1}^{4} x_i p_i$
$= 0(0.8) + 1(0.1) + 2(0.05) + 3(0.05)$
$= 0.35$

2. **a.** $P(X \ge 2) = P(2) + P(3) + P(4)$
$= 0.05 + 0.05 + 0.05 = 0.15$

b. $E(X) = \sum_{i=1}^{5} x_i p_i$
$= 0(0.7) + 1(0.15) + 2(0.05)$
$+ 3(0.05) + 4(0.5)$
$= 0.6$

3. **a.** $P(X \ge 2) = P(2) = 0.2$

b. $E(X) = -2(0.2) + (-1)(0.2) + 0(0.2)$
$+ 1(0.2) + 2(0.2)$
$= 0$

4. **a.** $P(X \ge 2) = P(2) = 0.1$

b. $E(X) = -2(0.1) + (-1)(0.2) + 0(0.4)$
$+1(0.2) + 2(0.1)$
$= 0$

5. **a.** $P(X \ge 2) = P(2) + P(3) + P(4)$
$= 0.2 + 0.2 + 0.2$
$= 0.6$

b. $E(X) = 1(0.4) + 2(0.2) + 3(0.2) + 4(0.2)$
$= 2.2$

6. **a.** $P(X \ge 2) = P(100) + P(1000)$
$= 0.018 + 0.002 = 0.02$

b. $E(X) = -0.1(0.98) + 100(0.018)$
$+ 1000(0.002)$
$= 3.702$

7. **a.** $P(X \ge 2) = P(2) + P(3) + P(4)$
$= \frac{3}{10} + \frac{2}{10} + \frac{1}{10} = \frac{6}{10} = 0.6$

b. $E(X) = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) = 2$

8. **a.** $P(X \ge 2) = P(2) + P(3) + P(4)$
$$= \frac{0^2}{10} + \frac{(-1)^2}{10} + \frac{(-2)^2}{10} = \frac{5}{10} = 0.5$$

b. $E(X) = 0(0.4) + 1(0.1) + 2(0) + 3(0.1) + 4(0.4)$
$= 2$

9. **a.** $P(X \ge 2) = \int_2^{20} \frac{1}{20}\,dx = \frac{1}{20} \cdot 18 = 0.9$

b. $E(X) = \int_0^{20} x \cdot \frac{1}{20}\,dx = \left[\frac{x^2}{40}\right]_0^{20} = 10$

c. For x between 0 and 20,
$$F(x) = \int_0^x \frac{1}{20}\,dt = \frac{1}{20} \cdot x = \frac{x}{20}$$

10. **a.** $P(X \ge 2) = \int_2^{20} \frac{1}{40}\,dx = \frac{1}{40} \cdot 18 = 0.45$

b. $E(X) = \int_{-20}^{20} x \cdot \frac{1}{40}\,dx = \left[\frac{x^2}{80}\right]_{-20}^{20} = 5 - 5 = 0$

c. For $-20 \le x \le 20$,
$$F(x) = \int_{-20}^x \frac{1}{40}\,dt = \frac{1}{40}(x + 20) = \frac{1}{40}x + \frac{1}{2}$$

11. **a.** $P(X \ge 2) = \int_2^8 \frac{3}{256} x(8 - x)\,dx$
$$= \frac{3}{256}\left[4x^2 - \frac{x^3}{3}\right]_2^8 = \frac{3}{256} \cdot 72 = \frac{27}{32}$$

b. $E(X) = \int_0^8 x \cdot \frac{3}{256} x(8 - x)\,dx$
$$= \frac{3}{256}\int_0^8 \left(8x^2 - x^3\right)dx$$
$$= \frac{3}{256}\left[\frac{8x^3}{3} - \frac{x^4}{4}\right]_0^8 = 4$$

c. For $0 \le x \le 8$
$$F(x) = \int_0^x \frac{3}{256} t(8 - t)\,dt = \frac{3}{256}\left[4t^2 - \frac{t^3}{3}\right]_0^x$$
$$= \frac{3}{64}x^2 - \frac{1}{256}x^3$$

12. **a.** $P(X \ge 2) = \int_2^{20} \frac{3}{4000} x(20 - x)\,dx$
$$= \frac{3}{4000}\left[10x^2 - \frac{x^3}{3}\right]_2^{20} = 0.972$$

b. $E(X) = \int_0^{20} x \cdot \frac{3}{4000} x(20 - x)\,dx$
$$= \frac{3}{4000}\int_0^{20}\left(20x^2 - x^3\right)dx$$
$$= \frac{3}{4000}\left[\frac{20x^3}{3} - \frac{x^4}{4}\right]_0^{20} = 10$$

c. For $0 \le x \le 20$
$$F(x) = \int_0^x \frac{3}{4000} t(20 - t)\,dt$$
$$= \frac{3}{4000}\left[10t^2 - \frac{t^3}{3}\right]_0^x = \frac{3}{400}x^2 - \frac{1}{4000}x^3$$

13. **a.** $P(X \ge 2) = \int_2^4 \frac{3}{64} x^2(4 - x)\,dx$
$$= \frac{3}{64}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_2^4 = 0.6875$$

b. $E(X) = \int_0^4 x \cdot \frac{3}{64} x^2(4 - x)\,dx$
$$= \frac{3}{64}\int_0^4\left(4x^3 - x^4\right)dx = \frac{3}{64}\left[x^4 - \frac{x^5}{5}\right]_0^4 = 2.4$$

c. For $0 \le x \le 4$
$$F(x) = \int_0^x \frac{3}{64} t^2(4 - t)\,dt = \frac{3}{64}\left[\frac{4t^3}{3} - \frac{t^4}{4}\right]_0^x$$
$$= \frac{1}{16}x^3 - \frac{3}{256}x^4$$

14. **a.** $P(X \ge 2) = \int_2^8 \frac{1}{32}(8 - x)\,dx$
$$= \frac{1}{32}\left[8x - \frac{x^2}{2}\right]_2^8 = \frac{9}{16}$$

b. $E(X) = \int_0^8 x \cdot \frac{1}{32}(8 - x)\,dx$
$$= \frac{1}{32}\left[4x^2 - \frac{x^3}{3}\right]_0^8 = \frac{8}{3}$$

c. For $0 \le x \le 8$
$$F(x) = \int_0^x \frac{1}{32}(8 - t)\,dt = \frac{1}{32}\left[8t - \frac{t^2}{2}\right]_0^x$$
$$= \frac{1}{4}x - \frac{1}{64}x^2$$

15. **a.** $P(X \ge 2) = \int_2^4 \frac{\pi}{8}\sin\left(\frac{\pi x}{4}\right)dx$

$= \frac{\pi}{8}\left[-\frac{4}{\pi}\cos\frac{\pi x}{4}\right]_2^4 = -\frac{1}{2}(-1-0) = \frac{1}{2}$

b. $E(X) = \int_0^4 x \cdot \frac{\pi}{8}\sin\left(\frac{\pi x}{4}\right)dx$

Using integration by parts or a CAS, $E(X) = 2$.

c. For $0 \le x \le 4$

$F(x) = \int_0^x \frac{\pi}{8}\sin\left(\frac{\pi t}{4}\right)dt = \frac{\pi}{8}\left[\frac{-4}{\pi}\cos\frac{\pi t}{4}\right]_0^x$

$= -\frac{1}{2}\left(\cos\frac{\pi x}{4} - 1\right) = -\frac{1}{2}\cos\frac{\pi x}{4} + \frac{1}{2}$

16. **a.** $P(X \ge 2) = \int_2^4 \frac{\pi}{8}\cos\left(\frac{\pi x}{8}\right)dx$

$= \left[\sin\left(\frac{\pi x}{8}\right)\right]_2^4 = \sin\frac{\pi}{2} - \sin\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}}$

b. $E(X) = \int_0^4 x \cdot \frac{\pi}{8}\cos\left(\frac{\pi x}{8}\right)dx$

Using a CAS, $E(X) \approx 1.4535$

c. For $0 \le x \le 4$

$F(x) = \int_0^x \frac{\pi}{8}\cos\left(\frac{\pi t}{8}\right)dt = \left[\sin\left(\frac{\pi t}{8}\right)\right]_0^x$

$= \sin\left(\frac{\pi x}{8}\right)$

17. **a.** $P(X \ge 2) = \int_2^4 \frac{4}{3x^2}dx = \left[-\frac{4}{3x}\right]_2^4 = \frac{1}{3}$

b. $E(X) = \int_1^4 x \cdot \frac{4}{3x^2}dx = \left[\frac{4}{3}\ln x\right]_1^4$

$= \frac{4}{3}\ln 4 \approx 1.85$

c. For $1 \le x \le 4$

$F(x) = \int_1^x \frac{4}{3t^2}dt = \left[-\frac{4}{3t}\right]_1^1 = \frac{-4}{3x} + \frac{4}{3}$

$= \frac{4x-4}{3x}$

18. **a.** $P(X \ge 2) = \int_2^9 \frac{81}{40x^3}dx = \left[-\frac{81}{80x^2}\right]_2^9$

$= \frac{77}{320} \approx 0.24$

b. $E(X) = \int_1^9 x \cdot \frac{81}{40x^3}dx = \left[-\frac{81}{40x}\right]_1^9 = 1.8$

c. For $1 \le x \le 9$

$F(x) = \int_1^x \frac{81}{40t^3}dt = \left[-\frac{81}{80t^2}\right]_1^x$

$= -\frac{81}{80x^2} + \frac{81}{80} = \frac{81x^2 - 81}{80x^2}$

19. Proof of $F'(x) = f(x)$:

By definition, $F(x) = \int_A^x f(t)\,dt$. By the First Fundamental Theorem of Calculus, $F'(x) = f(x)$.

Proof of $F(A) = 0$ and $F(B) = 1$:

$F(A) = \int_A^A f(x)\,dx = 0;$

$F(B) = \int_A^B f(x)\,dx = 1$

Proof of $P(a \le X \le b) = F(b) - F(a)$:

$P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$ due to the Second Fundamental Theorem of Calculus.

20. **a.** The midpoint of the interval [a,b] is $\frac{a+b}{2}$.

$P\left(X < \frac{a+b}{2}\right) = P\left(X \le \frac{a+b}{2}\right)$

$= \int_a^{\frac{a+b}{2}} \frac{1}{b-a}dx = \frac{1}{b-a}\left(\frac{a+b}{2} - a\right)$

$= \frac{1}{b-a} \cdot \frac{b-a}{2} = \frac{1}{2}$

b. $E(X) = \int_a^b x \cdot \frac{1}{b-a}dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$

$= \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$

c. $F(x) = \int_a^x \frac{1}{b-a}dt = \frac{1}{b-a}(x-a) = \frac{x-a}{b-a}$

21. The median will be the solution to the equation $\int_a^{x_0} \frac{1}{b-a}dx = 0.5$.

$\frac{1}{b-a}(x_0 - a) = 0.5$

$x_0 - a = \frac{b-a}{2}$

$x_0 = \frac{a+b}{2}$

22. The graph of $f(x)=\frac{15}{512}x^2(4-x)^2$ is symmetric about the line x = 2. Consequently, $P(X\le 2)=0.5$ and 2 must be the median of *X*.

23. Since the PDF must integrate to one, solve $\int_0^5 kx(5-x)\,dx=1$.

$$\left[\frac{5kx^2}{2}-\frac{kx^3}{3}\right]_0^5=1$$

$$\frac{125k}{2}-\frac{125k}{3}=1$$

$$375k-250k=6$$

$$k=\frac{6}{125}$$

24. Solve $\int_0^5 \mathrm{kx}^2(5-x)^2\,dx=1$

$$k\int_0^5\left(25x^2-10x^3+x^4\right)dx=1$$

$$k\left[\frac{25x^3}{3}-\frac{5x^4}{2}+\frac{x^5}{5}\right]_0^5=1$$

$$\frac{625}{6}k=1$$

$$k=\frac{6}{625}$$

25. a. Solve $\int_0^4 k\left(2-|x-2|\right)dx=1$

Due to the symmetry about the line x = 2, the solution can be found by solving

$$2\int_0^2 kx\,dx=1$$

$$k\cdot x^2\Big|_0^2=1$$

$$4k=1$$

$$k=\frac{1}{4}$$

b. $P(3\le X\le 4)=\int_3^4\frac{1}{4}\left(2-|x-2|\right)dx$

$$=\int_3^4\frac{1}{4}(2-(x-2))\,dx=\frac{1}{4}\int_3^4(4-x)\,dx$$

$$=\frac{1}{4}\left[4x-\frac{x^2}{2}\right]_3^4=\frac{1}{8}$$

c. $E(X)=\int_0^4 x\cdot\frac{1}{4}(2-|x-2|)\,dx$

$$=\int_0^2 x\cdot\frac{1}{4}(2+(x-2))\,dx+\int_2^4 x\cdot\frac{1}{4}(2-(x-2))\,dx$$

$$=\frac{1}{4}\int_0^2 x^2\,dx+\frac{1}{4}\int_2^4(4x-x^2)\,dx$$

$$=\frac{1}{12}x^3\Big|_0^2+\frac{1}{4}\left[2x^2-\frac{x^3}{3}\right]_2^4=\frac{2}{3}+\frac{4}{3}=2$$

d. If $0\le x\le 2$, $F(x)=\int_0^x\frac{1}{4}t\,dt=\left[\frac{t^2}{8}\right]_0^x=\frac{x^2}{8}$

If $2<x\le 4$, $F(x)=\int_0^2\frac{1}{4}x\,dx+\int_2^x\frac{1}{4}(4-t)\,dt$

$$=\frac{x^2}{8}\Big|_0^2+\frac{1}{4}\left[4t-\frac{t^2}{2}\right]_2^x=\frac{1}{2}+\left(x-\frac{x^2}{8}-\frac{3}{2}\right)$$

$$=-\frac{x^2}{8}+x-1$$

$$F(x)=\begin{cases}0 & \text{if } x<0\\ \frac{x^2}{8} & \text{if } 0\le x\le 2\\ -\frac{x^2}{8}+x-1 & \text{if } 2<x\le 4\\ 1 & \text{if } x>4\end{cases}$$

e. Using a similar procedure as shown in part (a), the PDF for *Y* is

$$f(y)=\frac{1}{14,400}\left(120-|y-120|\right)$$

If $0\le y<120$, $F(y)=\int_0^y\frac{1}{14,400}t\,dt$

$$=\left[\frac{t^2}{28,800}\right]_0^y=\frac{y^2}{28,800}$$

If $120<y\le 240$,

$$F(y)=\frac{1}{2}+\int_{120}^y\frac{1}{14,400}(240-t)\,dt$$

$$=\frac{1}{2}+\frac{1}{14,400}\left[240t-\frac{t^2}{2}\right]_{120}^y$$

$$=\frac{1}{2}+\frac{y}{60}-\frac{y^2}{28,800}-\frac{3}{2}=-\frac{y^2}{28,800}+\frac{y}{60}-1$$

$$F(x)=\begin{cases}0 & \text{if } y<0\\ \frac{y^2}{28,800} & \text{if } 0\le y\le 120\\ -\frac{y^2}{28,800}+\frac{y}{60}-1 & \text{if } 120<y\le 240\\ 1 & \text{if } y>240\end{cases}$$

26. **a.** Solve $\int_0^{180} kx^2(180-x)\,dx = 1.$

$$k\left[60x^3 - \frac{x^4}{4}\right]_0^{180} = 1$$

$$k = \frac{1}{87,480,000}$$

b. $P(100 \le X \le 150)$

$$= \int_{100}^{150} \frac{1}{87,480,000} x^2(180-x)\,dx$$

$$= \frac{1}{87,480,000}\left[60x^3 - \frac{x^4}{4}\right]_{100}^{150} \approx 0.468$$

c. $E(X) = \int_0^{180} x \cdot \frac{1}{87,480,000} x^2(180-x)\,dx$

$$= \frac{1}{87,480,000}\left[45x^4 - \frac{x^5}{5}\right]_0^{180} = 108$$

27. **a.** Solve $\int_0^{0.6} kx^6(0.6-x)^8\,dx = 1.$

$$k\int_0^{0.6} x^6(0.6-x)^8\,dx = 1$$

Using a CAS, k $\approx$ 95,802,719

b. The probability that a unit is scrapped is $1-P(0.35 \le X \le 0.45)$

$$= 1 - k\int_{0.35}^{0.45} x^6(0.6-x)^8\,dx$$

≈ 0.884 using a CAS

c. $E(X) = \int_0^{0.6} x \cdot kx^6(0.6-x)^8\,dx$

$$= k\int_0^{0.6} x^7(0.6-x)^8\,dx$$

≈ 0.2625

d. $F(x) = \int_0^x 95,802,719t^6(0.6-t)^8\,dt$

Using a CAS,

$$F(x) \approx 6,386,850x^7(x^8 - 5.14286x^7 + 11.6308x^6 - 15.12x^5 + 12.3709x^4 - 6.53184x^3 + 2.17728x^2 - 0.419904x + 0.36)$$

e. If X = measurement in mm, and Y = measurement in inches, then $Y = X/25.4$. Thus,

$$F_Y(y) = P(Y \le y) = P(X/25.4 \le y) = P(X \le 25.4y) = F(25.4y)$$

where $F(x)$ is given in part (d).

Alternatively, we can proceed as follows:

Solve $\int_0^{3/127} k \cdot y^6\left(\frac{3}{127} - y\right)^8 dy = 1$ using a CAS.

$k \approx 1.132096857 \times 10^{29}$

$$F_Y(y) = \int_0^y k \cdot t^6\left(\frac{3}{127} - t\right)^8 dt$$

Using a CAS,

$$F_Y(y) \approx (7.54731\times10^{27})y^7(y^8 - 0.202475y^7 + 0.01802y^6 - 0.000923y^5 + 0.00003y^4 - (6.17827\times10^{-7})y^3 + (8.108\times10^{-9})y^2 - (6.156\times10^{-11})y + 2.07746\times10^{-13})$$

28. **a.** Solve $\int_0^{200} kx^2(200-x)^8\,dx = 1.$

Using a CAS, $k \approx 2.417\times10^{-23}$

b. The probability that a batch is not accepted is

$$P(X \ge 100) = k\int_{100}^{200} x^2(200-x)^8\,dx$$

≈ 0.0327 using a CAS.

c. $E(X) = k\int_0^{200} x \cdot x^2(200-x)^8\,dx$

$= 50$ using a CAS

d. $F(x) = \int_0^x (2.417\times10^{-23})t^2(200-t)^8\,dt$

Using a CAS, F(x) $\approx (2.19727\times10^{-24})x^3 \cdot$

$$(x^8 - 1760x^7 + 136889x^6 - (6.16\times10^8)x^5 + (1.76\times10^{11})x^4 - (3.2853\times10^{13})x^3 + (3.942\times10^{15})x^2 - (2.816\times10^{17})x + 9.39\times10^{18})$$

e. Solve $\int_0^{100} kx^2(100-x)^8\,dx.$ Using a CAS,

$k = 4.95\times10^{-20}$

$$F(x) = \int_0^y \left(4.95\times10^{-20}\right)t^2(100-t)^8\,dt$$

Using a CAS,

$$F(x) \approx (4.5\times10^{-21})x^3 (x^8 - 880x^7 + 342,222x^6 - (7.7\times10^7)x^5 + (1.1\times10^{10})x^4 - (1.027\times10^{12})x^3 + (6.16\times10^{13})x^2 - (2.2\times10^{15})x + 3.667\times10^{16})$$

29. The PDF for the random variable X is

$$f(x)=\begin{cases}1 & if\ \ 0\le x\le 1\\ 0 & otherwise\end{cases}$$

From Problem 20, the CDF for X is $F(x)=x$

Y is the distance from $(1, X)$ to the origin, so

$$Y=\sqrt{(1-0)^2+(X-0)^2}=\sqrt{1+X^2}$$

Here we have a one-to-one transformation from the set $\{x: 0\le x\le 1\}$ to $\{y: 1\le y\le \sqrt{2}\}$. For every $1<a<b<\sqrt{2}$, the event $a<Y<b$ will occur when, and only when,

$\sqrt{a^2-1}<X<\sqrt{b^2-1}$.

If we let $a=1$ and $b=y$, we can obtain the CDF for Y.

$$P(1\le Y\le y)=P\left(\sqrt{1^2-1}\le X\le \sqrt{y^2-1}\right)$$
$$=P\left(0\le X\le\sqrt{y^2-1}\right)$$
$$=F\left(\sqrt{y^2-1}\right)=\sqrt{y^2-1}$$

To find the PDF, we differentiate the CDF with respect to y.

$$PDF=\frac{d}{dy}\sqrt{y^2-1}=\frac{1}{2}\cdot\frac{1}{\sqrt{y^2-1}}\cdot 2y=\frac{y}{\sqrt{y^2-1}}$$

Therefore, for $0\le y\le\sqrt{2}$ the PDF and CDF are respectively

$g(y)=\dfrac{y}{\sqrt{y^2-1}}$ and $G(y)=\sqrt{y^2-1}$.

30. $P(X=x)=\int_x^x f(t)\,dt=0.$ Consequently, $P(X<c)=P(X\le c).$ As a result, all four expressions, $P(a<X<b), P(a\le X\le b),$ $P(a<X\le b)$ and $P(a\le X<b),$ are equivalent.

31. By the defintion of a complement of a set, $A\cup A^c=S$, where S denotes the sample space. Since $P(S)=1$, $P(A\cup A^c)=1.$ Since $P(A\cup A^c)=P(A)+P(A^c),$ $P(A)+P(A^c)=1$ and $P(A^c)=1-P(A).$

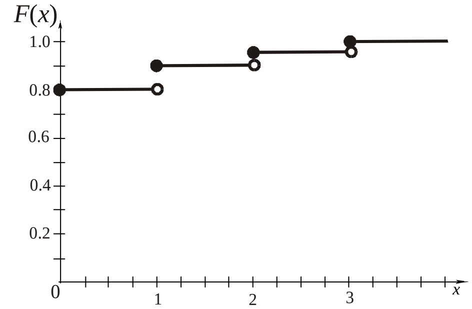
32. $P(X\ge 1)=1-P(X<1)$

For Problem 1, $1-P(X<1)=1-P(X=0)$
$=1-0.8=0.2$

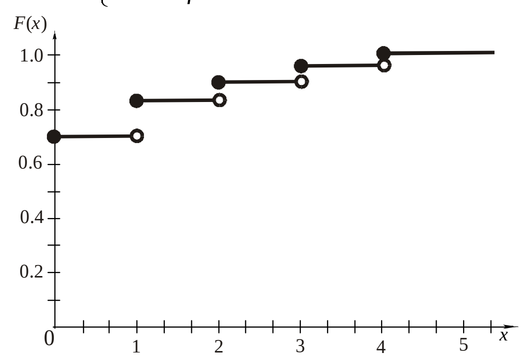
For Problem 2, $1-P(X<1)=1-P(X=0)$
$=1-0.7=0.3$

For Problem 5, $1-P(X<1)=1-0=1$

33. $$F(x)=\begin{cases}0 & if\ \ x<0\\ 0.8 & if\ \ 0\le x<1\\ 0.9 & if\ \ 1\le x<2\\ 0.95 & if\ \ 2\le x<3\\ 1 & if\ \ x>3\end{cases}$$



34. $$F(x)=\begin{cases}0 & if\ \ x<0\\ 0.7 & if\ \ 0\le x<1\\ 0.85 & if\ \ 1\le x<2\\ 0.9 & if\ \ 2\le x<3\\ 0.95 & if\ \ 3\le x<4\\ 1 & if\ \ x\ge 4\end{cases}$$



35. **a.** $P(Y<2)=P(Y\le 2)=F(2)=1$

b. $P(0.5<Y<0.6)=F(0.6)-F(0.5)$
$=\dfrac{1.2}{1.6}-\dfrac{1}{1.5}=\dfrac{1}{12}$

c. $f(y)=F'(y)=\dfrac{2}{(y+1)^2}, 0\le y\le 1$

d. $E(Y)=\int_0^1 y\cdot\dfrac{2}{(y+1)^2}\,dy\approx 0.38629$

36. **a.** $P(Z>1)=1-P(Z\le 1)=1-F(1)$
$=1-\frac{1}{9}=\frac{8}{9}$

b. $P(1<Z<2)=P(1\le Z\le 2)=F(2)-F(1)$
$=\frac{4}{9}-\frac{1}{9}=\frac{1}{3}$

c. $f(z)=F'(z)=\frac{2z}{9}, 0\le z\le 3$

d. $E(Z)=\int_0^3 z\cdot\frac{2z}{9}\,dz=\left[\frac{2z^3}{27}\right]_0^3=2$

37. $E(X)=\int_0^4 x\cdot\frac{15}{512}x^2(4-x)^2\,dx=2$

and $E(X^2)=\int_0^4 x^2\cdot\frac{15}{512}x^2(4-x)^2\,dx$

$=\frac{32}{7}\approx 4.57$ using a CAS

38. $E(X^2)=\int_0^8 x^2\cdot\frac{3}{256}x(8-x)\,dx=19.2$ and

$E(X^3)=\int_0^8 x^3\cdot\frac{3}{256}x(8-x)\,dx=102.4$

using a CAS

39. $V(X)=E\left[(X-\mu)^2\right]$, where $\mu=E(X)=2$

$V(X)=\int_0^4 (x-2)^2\cdot\frac{15}{512}x^2(4-x)^2\,dx=\frac{4}{7}$

40. $\mu=E(X)=\int_0^8 x\cdot\frac{3}{256}x(8-x)\,dx=4$

$V(X)=\int_0^8 (x-4)^2\cdot\frac{3}{256}x(8-x)\,dx=\frac{16}{5}$

41. $E\left[(X-\mu)^2\right]=E(X^2-2X\mu+\mu^2)$

$=E(X^2)-E(2X\mu)+E(\mu^2)$

$=E(X^2)-2\mu\cdot E(X)+\mu^2$

$=E(X^2)-2\mu^2+\mu^2$ since $E(X)=\mu$

$=E(X^2)-\mu^2$

For Problem 37, $V(X)=E(X^2)-\mu^2$ and using previous results, $V(X)=\frac{32}{7}-2^2=\frac{4}{7}$

5.8 Chapter Review

Concepts Test

1. False: $\int_0^\pi \cos x\,dx=0$ because half of the area lies above the x-axis and half below the x-axis.

2. True: The integral represents the area of the region in the first quadrant if the center of the circle is at the origin.

3. False: The statement would be true if either $f(x)\ge g(x)$ or $g(x)\ge f(x)$ for $a\le x\le b$. Consider Problem 1 with $f(x)=\cos x$ and $g(x)=0$.

4. True: The area of a cross section of a cylinder will be the same in any plane parallel to the base.

5. True: Since the cross sections in all planes parallel to the bases have the same area, the integrals used to compute the volumes will be equal.

6. False: The volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$. If the radius is doubled and the height halved the volume is $\frac{2}{3}\pi r^2 h$.

7. False: Using the method of shells, $V=2\pi\int_0^1 x(-x^2+x)dx$. To use the method of washers we need to solve $y=-x^2+x$ for x in terms of y.

8. True: The bounded region is symmetric about the line $x=\frac{1}{2}$. Thus the solids obtained by revolving about the lines $x=0$ and $x=1$ have the same volume.

9. False: Consider the curve given by $x=\frac{\cos t}{t}$, $y=\frac{\sin t}{t}, 2\le t<\infty$.

10. False: The work required to stretch a spring 2 inches beyond its natural length is $\int_0^2 kx\,dx=2k$, while the work required to stretch it 1 inch beyond its natural length is $\int_0^1 kx\,dx=\frac{1}{2}k$.

11. False: If the cone-shaped tank is placed with the point downward, then the amount of water that needs to be pumped from near the bottom of the tank is much less than the amount that needs to be pumped from near the bottom of the cylindrical tank.

12. False: The force depends on the depth, but the force is the same at all points on a surface as long as they are at the same depth.

13. True: This is the definition of the center of mass.

14. True: The region is symmetric about the point (π , 0).

15. True: By symmetry, the centroid is on the line $x=\frac{\pi}{2}$, so the centroid travels a distance of $2\pi\left(\frac{\pi}{2}\right)=\pi^2$.

16. True: At slice y, $\Delta A \approx (9-y^2)\Delta y$.

17. True: Since the density is proportional to the square of the distance from the midpoint, equal masses are on either side of the midpoint.

18. True: See Problem 30 in Section 5.6.

19. True: A discrete random variable takes on a finite number of possible values, or an infinite set of possible outcomes provided that these outcomes can be put in a list such as $\{x_1, x_2, \ldots\}$.

20. True: The computation of *E(X)* would be the same as the computation for the center of mass of the wire.

21. True: $E(X)=5\cdot 1=5$

22. True: If $F(x)=\int_A^x f(t)\,dt$, then $F'(x)=f(x)$ by the First Fundamental Theorem of Calculus.

23. True: $P(X=1)=P(1\le X\le 1)=\int_1^1 f(x)\,dx=0$

Sample Test Problems

1. $A=\int_0^1 (x-x^2)dx=\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1=\frac{1}{6}$

2. $$V=\pi\int_0^1 (x-x^2)^2\,dx$$
$$=\pi\int_0^1 (x^2-2x^3+x^4)dx$$
$$=\pi\left[\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_0^1=\frac{\pi}{30}$$

3. $$V=2\pi\int_0^1 x(x-x^2)dx=2\pi\int_0^1 (x^2-x^3)dx$$
$$=2\pi\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1=\frac{\pi}{6}$$

4. $$V=\pi\int_0^1\left[(x-x^2+2)^2-(2)^2\right]dx$$
$$=\pi\int_0^1 (x^4-2x^3-3x^2+4x)\,dx$$
$$=\pi\left[\frac{1}{5}x^5-\frac{1}{2}x^4-x^3+2x^2\right]_0^1=\frac{7\pi}{10}$$

5. $$V=2\pi\int_0^1 (3-x)(x-x^2)dx$$
$$=2\pi\int_0^1 (x^3-4x^2+3x)\,dx$$
$$=2\pi\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_0^1=\frac{5\pi}{6}$$

6. $$\overline{x}=\frac{\int_0^1 x(x-x^2)dx}{\int_0^1 (x-x^2)dx}=\frac{\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1}{\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1}=\frac{1}{2}$$
$$\overline{y}=\frac{\frac{1}{2}\int_0^1 (x-x^2)^2\,dx}{\int_0^1 (x-x^2)dx}=\frac{\frac{1}{2}\left[\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_0^1}{\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1}$$
$$=\frac{1}{10}$$

7. From Problem 1, $A=\frac{1}{6}$.

From Problem 6, $\overline{x}=\frac{1}{2}$ and $\overline{y}=\frac{1}{10}$.

$$V(S_1)=2\pi\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)=\frac{\pi}{30}$$
$$V(S_2)=2\pi\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)=\frac{\pi}{6}$$
$$V(S_3)=2\pi\left(\frac{1}{10}+2\right)\left(\frac{1}{6}\right)=\frac{7\pi}{10}$$
$$V(S_4)=2\pi\left(3-\frac{1}{2}\right)\left(\frac{1}{6}\right)=\frac{5\pi}{6}$$

8. $8 = F(8) = 8k$, $k = 1$

a. $W = \int_2^8 x\,dx = \left[\frac{1}{2}x^2\right]_2^8 = \frac{1}{2}(64-4)$

$= 30$ in.-lb

b. $W = \int_0^{-4} x\,dx = \left[\frac{1}{2}x^2\right]_0^4 = 8$ in.-lb

9. $W = \int_0^6 (62.4)(5^2)\pi(10-y)dy$

$= 1560\pi\int_0^6 (10-y)dy = 1560\pi\left[10y - \frac{1}{2}y^2\right]_0^6$

$= 65{,}520\pi \approx 205{,}837$ ft-lb

10. The total work is equal to the work W_1 to pull up the object to the top without the cable and the work W_2 to pull up the cable.

$W_1 = 200 \cdot 100 = 20{,}000$ ft-lb

The cable weighs $\frac{120}{100} = \frac{6}{5}$ lb/ft.

$\Delta W_2 = \frac{6}{5}\Delta y \cdot y = \frac{6}{5}y\Delta y$

$W_2 = \int_0^{100} \frac{6}{5}y\,dy = \frac{6}{5}\left[\frac{1}{2}y^2\right]_0^{100}$

$= 6000$ ft-lb

$W = W_1 + W_2 = 26{,}000$ ft-lb

11. a. To find the intersection points, solve

$4x = x^2$.

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$

$A = \int_0^4 (4x - x^2)dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \left(32 - \frac{64}{3}\right) = \frac{32}{3}$

b. To find the intersection points, solve

$\frac{y}{4} = \sqrt{y}$.

$\frac{y^2}{16} = y$

$y^2 - 16y = 0$

$y(y-16) = 0$

$y = 0, 16$

$A = \int_0^{16}\left(\sqrt{y} - \frac{y}{4}\right)dy = \left[\frac{2}{3}y^{3/2} - \frac{1}{8}y^2\right]_0^{16}$

$= \left(\frac{128}{3} - 32\right) = \frac{32}{3}$

12. $\bar{x} = \dfrac{\int_0^4 x(4x-x^2)dx}{\int_0^4 (4x-x^2)\,dx} = \dfrac{\int_0^4 (4x^2-x^3)\,dx}{\frac{32}{3}}$

$= \dfrac{\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4}{\frac{32}{3}} = \dfrac{\frac{64}{3}}{\frac{32}{3}} = 2$

$\bar{y} = \dfrac{\frac{1}{2}\int_0^4\left[(4x)^2 - (x^2)^2\right]dx}{\int_0^4 (4x-x^2)dx}$

$= \dfrac{\frac{1}{2}\int_0^4 (16x^2 - x^4)\,dx}{\frac{32}{3}}$

$= \dfrac{\frac{1}{2}\left[\frac{16}{3}x^3 - \frac{1}{5}x^5\right]_0^4}{\frac{32}{3}} = \dfrac{\frac{1024}{15}}{\frac{32}{3}} = \frac{32}{5}$

13. $V = \pi\int_0^4\left[(4x)^2 - (x^2)^2\right]dx$

$= \pi\int_0^4 (16x^2 - x^4)\,dx$

$= \pi\left[\frac{16}{3}x^3 - \frac{1}{5}x^5\right]_0^4 = \frac{2048\pi}{15}$

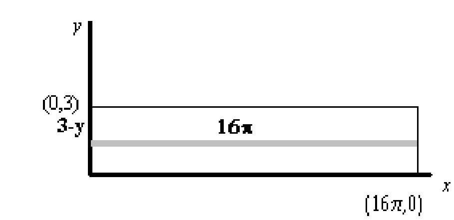
Using Pappus's Theorem:

From Problem 11, $A = \frac{32}{3}$.

From Problem 12, $\bar{y} = \frac{32}{5}$.

$V = 2\pi\bar{y}\cdot A = 2\pi\left(\frac{32}{5}\right)\left(\frac{32}{3}\right) = \frac{2048\pi}{15}$

14. a. (See example 4, section 5.5). Think of cutting the barrel vertically and opening the lateral surface into a rectangle as shown in the sketch below.



At depth $3 - y$, a narrow rectangle has width 16π, so the total force on the lateral surface is (δ = density of water = $62.4\ \text{lbs}/\text{ft}^3$)

$\int_0^3 \delta(3-y)(16\pi)\,dy = 16\pi\delta\int_0^3 (3-y)\,dy$

$= 16\pi\delta\left[3y - \frac{y^2}{2}\right]_0^3 = 16\pi\delta(4.5) \approx 14{,}114.55$ lbs.

b. All points on the bottom of the barrel are at the same depth; thus the total force on the bottom is simply the weight of the column of water in the barrel, namely

$F = \pi(8^2)(3)\delta \approx 37{,}638.8$ lbs.

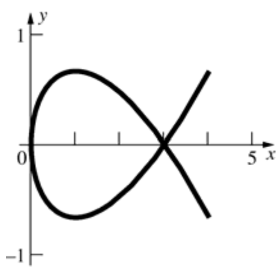
15. $\dfrac{dy}{dx} = x^2 - \dfrac{1}{4x^2}$

$$L = \int_1^3 \sqrt{1+\left(x^2 - \frac{1}{4x^2}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\,dx = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right)dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^3 = \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{53}{6}$$

16.



The loop is $-\sqrt{3} \le t \le \sqrt{3}$. By symmetry, we can double the length of the loop from $t = 0$ to $t = \sqrt{3}$, $\dfrac{dx}{dt} = 2t; \dfrac{dy}{dt} = t^2 - 1$

$$L = 2\int_0^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\,dt = 2\int_0^{\sqrt{3}} (t^2 + 1)dt$$

$$= 2\left[\frac{1}{3}t^3 + t\right]_0^{\sqrt{3}} = 4\sqrt{3}$$

17. $V = \int_{-3}^3 \left(\sqrt{9-x^2}\right)^2 dx = \int_{-3}^3 (9-x^2)\,dx$

$$= \left[9x - \frac{1}{3}x^3\right]_{-3}^3 = (27-9) - (-27+9) = 36$$

18. $A = \int_a^b [f(x) - g(x)]dx$

19. $V = \pi\int_a^b \left[f^2(x) - g^2(x)\right]dx$

20. $V = 2\pi\int_a^b (x-a)[f(x) - g(x)]dx$

21. $M_y = \delta\int_a^b x[f(x) - g(x)]dx$

$$M_x = \frac{\delta}{2}\int_a^b \left[f^2(x) - g^2(x)\right]dx$$

22. $L_1 = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$

$L_2 = \int_a^b \sqrt{1+[g'(x)]^2}\,dx$

$L_3 = f(a) - g(a)$

$L_4 = f(b) - g(b)$

Total length $= L_1 + L_2 + L_3 + L_4$

23. $A_1 = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$

$A_2 = 2\pi\int_a^b g(x)\sqrt{1+[g'(x)]^2}\,dx$

$A_3 = \pi\left[f^2(a) - g^2(a)\right]$

$A_4 = \pi\left[f^2(b) - g^2(b)\right]$

Total surface area $= A_1 + A_2 + A_3 + A_4$.

24. a. $P(X \ge 1) = P(1 \le X \le 2)$

$$= \int_1^2 \frac{1}{12}\left(8 - x^3\right)dx = \frac{1}{12}\left[8x - \frac{x^4}{4}\right]_1^2$$

$$= \frac{17}{48} \approx 0.354$$

b. $P(0 \le X < 0.5) = P(0 \le X \le 0.5)$

$$= \int_0^{0.5} \frac{1}{12}\left(8 - x^3\right)dx = \frac{1}{12}\left[8x - \frac{x^4}{4}\right]_1^2$$

$$= \frac{85}{256} \approx 0.332$$

c. $E(X) = \int_0^2 x \cdot \frac{1}{12}\left(8 - x^3\right)dx = \frac{1}{12}\left[4x^2 - \frac{x^5}{5}\right]_0^2$

$= 0.8$

d. $F(x) = \int_0^x \frac{1}{12}\left(8 - t^3\right)dt = \frac{1}{12}\left[8t - \frac{t^4}{4}\right]_0^x$

$$= \frac{1}{12}\left(8x - \frac{x^4}{4}\right) = \frac{2}{3}x - \frac{x^4}{48}$$

25. a. $P(X \le 3) = F(3) = 1 - \dfrac{(6-3)^2}{36} = \dfrac{3}{4}$

b. $f(x) = F'(x) = 2 \cdot \dfrac{1}{36}(6-x) = \dfrac{6-x}{18}$, $0 \le x \le 6$

c. $E(X) = \int_0^6 x \cdot \left(\dfrac{6-x}{18}\right)dx$

$$= \frac{1}{18}\left[3x^2 - \frac{x^3}{3}\right]_0^6 = 2$$

Review and Preview Problems

1. By the Power Rule

$$\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x} + C$$

2. By the Power Rule

$$\int \frac{1}{x^{1.5}}dx = \int x^{-1.5}dx = \frac{x^{-1.5+1}}{-1.5+1} = \frac{x^{-0.5}}{-0.5} = -\frac{2}{\sqrt{x}} + C$$

3. By the Power Rule

$$\int \frac{1}{x^{1.01}}dx = \int x^{-1.01}dx = \frac{x^{-1.01+1}}{-1.01+1} = \frac{x^{-0.01}}{-0.01} = -\frac{100}{x^{0.01}} + C$$

4. By the Power Rule

$$\int \frac{1}{x^{0.99}}dx = \int x^{-0.99}dx = \frac{x^{-0.99+1}}{-0.99+1} = \frac{x^{0.01}}{0.01} = 100x^{0.01} + C$$

5. $F(1) = \int_1^1 \frac{1}{t}dt = 0$

6. By the First Fundamental Theorem of Calculus

$$F'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}$$

7. Let $g(x) = x^2$; then by the Chain Rule and problem 6,

$$D_x F(x^2) = D_x F(g(x)) = F'(g(x))g'(x)$$
$$= \left(\frac{1}{x^2}\right)(2x) = \frac{2}{x}$$

8. Let $h(x) = x^3$; then by the Chain Rule and problem 6,

$$D_x \int_1^{x^3} \frac{1}{t}dt = D_x F(h(x)) = F'(h(x))h'(x)$$
$$= \left(\frac{1}{x^3}\right)(3x^2) = \frac{3}{x}$$

9. **a.** $(1+1)^{1/1} = 2^1 = 2$

b. $(1+\frac{1}{5})^{1/\frac{1}{5}} = \left(\frac{6}{5}\right)^5 = 2.48832$

c. $(1+\frac{1}{10})^{1/\frac{1}{10}} = \left(\frac{11}{10}\right)^{10} \approx 2.593742$

d. $(1+\frac{1}{50})^{1/\frac{1}{50}} = \left(\frac{51}{50}\right)^{50} \approx 2.691588$

e. $(1+\frac{1}{100})^{1/\frac{1}{100}} = \left(\frac{101}{100}\right)^{100} \approx 2.704814$

10. **a.** $(1+\frac{1}{1})^1 = 2^1 = 2$

b. $(1+\frac{1}{10})^{10} = \left(\frac{11}{10}\right)^{10} \approx 2.593742$

c. $(1+\frac{1}{100})^{100} = \left(\frac{101}{100}\right)^{100} \approx 2.704814$

d. $(1+\frac{1}{1000})^{1000} = \left(\frac{1001}{1000}\right)^{1000} \approx 2.7169239$

11. **a.** $(1+\frac{1}{2})^{2/1} = \left(\frac{3}{2}\right)^2 = 2.25$

b. $(1+\frac{\frac{1}{5}}{2})^{2/\frac{1}{5}} = \left(1+\frac{1}{10}\right)^{10} = \left(\frac{11}{10}\right)^{10} \approx 2.593742$

c. $(1+\frac{\frac{1}{10}}{2})^{2/\frac{1}{10}} = \left(1+\frac{1}{20}\right)^{20} = \left(\frac{21}{20}\right)^{20} \approx 2.6533$

d. $(1+\frac{\frac{1}{50}}{2})^{2/\frac{1}{50}} = \left(1+\frac{1}{100}\right)^{100} = \left(\frac{101}{100}\right)^{100}$
≈ 2.70481

e. $(1+\frac{\frac{1}{100}}{2})^{2/\frac{1}{100}} = \left(1+\frac{1}{200}\right)^{200} = \left(\frac{201}{200}\right)^{200}$
≈ 2.71152

12. **a.** $(1+\frac{2}{1})^{1/2} = \sqrt{3} \approx 1.732051$

b. $(1+\frac{2}{10})^{10/2} = (1.2)^5 \approx 2.48832$

c. $(1+\frac{2}{100})^{100/2} = (1.02)^{50} \approx 2.691588$

d. $(1+\frac{2}{1000})^{1000/2} = (1.002)^{500} \approx 2.715569$

13. We know from trigonometry that, for any x and any integer k, $\sin(x+2k\pi) = \sin(x)$. Since

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2},$$

$$\sin(x) = \frac{1}{2} \text{ if } x = \frac{\pi}{6} + 2k\pi = \frac{12k+1}{6}\pi$$
$$\text{or } x = \frac{5\pi}{6} + 2k\pi = \frac{12k+5}{6}\pi$$

where k is any integer.

14. We know from trigonometry that, for any x and any integer k, $\cos(x+2k\pi)=\cos(x)$. Since $\cos(\pi)=\text{-}1,\ \cos(x)=-1$ if $x=\pi+2k\pi=(2k+1)\pi$ where k is any integer.

15. We know from trigonometry that, for any x and any integer k, $\tan(x+k\pi)=\tan(x)$. Since $\tan\left(\frac{\pi}{4}\right)=1,\ \tan(x)=1$ if $x=\frac{\pi}{4}+k\pi=\frac{4k+1}{4}\pi$ where k is any integer.

16. Since $\sec(x)=\frac{1}{\cos(x)}$, $\sec(x)$ is never 0.

17. In the triangle, relative to θ,

$opp=\sqrt{x^2-1}$, $adj=1$, $hypot=x$ so that

$$\sin\theta=\frac{\sqrt{x^2-1}}{x}\quad \cos\theta=\frac{1}{x}\quad \tan\theta=\sqrt{x^2-1}$$

$$\cot\theta=\frac{1}{\sqrt{x^2-1}}\quad \sec\theta=x\quad \csc\theta=\frac{x}{\sqrt{x^2-1}}$$

18. In the triangle, relative to θ,

$opp=x$, $adj=\sqrt{1-x^2}$, $hypot=1$ so that

$$\sin\theta=x\quad \cos\theta=\sqrt{1-x^2}\quad \tan\theta=\frac{x}{\sqrt{1-x^2}}$$

$$\cot\theta=\frac{\sqrt{1-x^2}}{x}\quad \sec\theta=\frac{1}{\sqrt{1-x^2}}\quad \csc\theta=\frac{1}{x}$$

19. In the triangle, relative to θ,

$opp=1$, $adj=x$, $hypot=\sqrt{1+x^2}$ so that

$$\sin\theta=\frac{1}{\sqrt{1+x^2}}\quad \cos\theta=\frac{x}{\sqrt{1+x^2}}\quad \tan\theta=\frac{1}{x}$$

$$\cot\theta=x\quad \sec\theta=\frac{\sqrt{1+x^2}}{x}\quad \csc\theta=\sqrt{1+x^2}$$

20. In the triangle, relative to θ,

$opp=\sqrt{1-x^2}$, $adj=x$, $hypot=1$ so that

$$\sin\theta=\sqrt{1-x^2}\quad \cos\theta=x\quad \tan\theta=\frac{\sqrt{1-x^2}}{x}$$

$$\cot\theta=\frac{x}{\sqrt{1-x^2}}\quad \sec\theta=\frac{1}{x}\quad \csc\theta=\frac{1}{\sqrt{1-x^2}}$$

21. $y'=xy^2 \quad\rightarrow\quad dy=xy^2dx$

$$\frac{1}{y^2}dy=xdx$$

$$\int\frac{dy}{y^2}=\int xdx$$

$$-\frac{1}{y}=\frac{1}{2}x^2+C$$

When $x=0$ and $y=1$ we get $C=-1$. Thus,

$$-\frac{1}{y}=\frac{1}{2}x^2-1=\frac{x^2-2}{2}$$

$$y=-\frac{2}{x^2-2}$$

22. $y'=\frac{\cos x}{y} \quad\rightarrow\quad dy=\frac{\cos x}{y}dx$

$$y\,dy=\cos x\,dx$$

$$\int ydy=\int\cos x\,dx$$

$$\frac{1}{2}y^2=\sin x+C$$

When $x=0$ and $y=4$ we get $C=8$. Thus,

$$\frac{1}{2}y^2=\sin x+8$$

$$y^2=2\sin x+16$$

CHAPTER 6 Transcendental Functions

6.1 Concepts Review

1. $\int_1^x \frac{1}{t} dt; (0, \infty); (-\infty, \infty)$

2. $\frac{1}{x}$

3. $\frac{1}{x}; \ln|x| + C$

4. $\ln x + \ln y$; $\ln x - \ln y$; $r \ln x$

Problem Set 6.1

1. a. $\ln 6 = \ln (2 \cdot 3) = \ln 2 + \ln 3$
$= 0.693 + 1.099 = 1.792$

b. $\ln 1.5 = \ln\left(\frac{3}{2}\right) = \ln 3 - \ln 2 = 0.406$

c. $\ln 81 = \ln 3^4 = 4\ln 3 = 4(1.099) = 4.396$

d. $\ln\sqrt{2} = \ln 2^{1/2} = \frac{1}{2}\ln 2 = \frac{1}{2}(0.693) = 0.3465$

e. $\ln\left(\frac{1}{36}\right) = -\ln 36 = -\ln(2^2 \cdot 3^2)$
$= -2\ln 2 - 2\ln 3 = -3.584$

f. $\ln 48 = \ln(2^4 \cdot 3) = 4\ln 2 + \ln 3 = 3.871$

2. a. 1.792 b. 0.405
c. 4.394 d. 0.3466
e. –3.584 f. 3.871

3. $D_x \ln(x^2 + 3x + \pi)$
$= \frac{1}{x^2 + 3x + \pi} \cdot D_x(x^2 + 3x + \pi) = \frac{2x + 3}{x^2 + 3x + \pi}$

4. $D_x \ln(3x^3 + 2x) = \frac{1}{3x^3 + 2x} D_x(3x^3 + 2x)$
$= \frac{9x^2 + 2}{3x^3 + 2x}$

5. $D_x \ln(x-4)^3 = D_x 3\ln(x-4)$
$= 3 \cdot \frac{1}{x-4} D_x(x-4) = \frac{3}{x-4}$

6. $D_x \ln\sqrt{3x-2} = D_x \frac{1}{2}\ln(3x-2)$
$= \frac{1}{2} \cdot \frac{1}{3x-2} D_x(3x-2) = \frac{3}{2(3x-2)}$

7. $\frac{dy}{dx} = 3 \cdot \frac{1}{x} = \frac{3}{x}$

8. $\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + 2x \cdot \ln x = x(1 + 2\ln x)$

9. $z = x^2 \ln x^2 + (\ln x)^3 = x^2 \cdot 2\ln x + (\ln x)^3$
$\frac{dz}{dx} = x^2 \cdot \frac{2}{x} + 2x \cdot 2\ln x + 3(\ln x)^2 \cdot \frac{1}{x}$
$= 2x + 4x\ln x + \frac{3}{x}(\ln x)^2$

10. $r = \frac{\ln x}{x^2 \ln x^2} + \left(\ln\frac{1}{x}\right)^3 = \frac{\ln x}{x^2 \cdot 2\ln x} + (-\ln x)^3$
$= \frac{1}{2}x^{-2} - (\ln x)^3$
$\frac{dr}{dx} = \frac{-2}{2}x^{-3} - 3(\ln x)^2 \cdot \frac{1}{x} = -\frac{1}{x^3} - \frac{3(\ln x)^2}{x}$

11. $g'(x) = \frac{1}{x + \sqrt{x^2+1}}\left[1 + \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x\right]$
$= \frac{1}{\sqrt{x^2+1}}$

12. $h'(x) = \frac{1}{x + \sqrt{x^2-1}}\left[1 + \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x\right]$
$= \frac{1}{\sqrt{x^2-1}}$

13. $f(x) = \ln\sqrt[3]{x} = \frac{1}{3}\ln x$
$f'(x) = \frac{1}{3} \cdot \frac{1}{x} = \frac{1}{3x}$
$f'(81) = \frac{1}{3 \cdot 81} = \frac{1}{243}$

14. $f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$
$f'\left(\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$.

15. Let $u = 2x + 1$ so $du = 2\,dx$.

$$\int \frac{1}{2x+1}dx = \frac{1}{2}\int \frac{1}{u}du$$
$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x+1| + C$$

16. Let $u = 1 - 2x$ so $du = -2dx$.

$$\int \frac{1}{1-2x}dx = -\frac{1}{2}\int \frac{1}{u}du$$
$$= -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|1-2x| + C$$

17. Let $u = 3v^2 + 9v$ so $du = 6v + 9$.

$$\int \frac{6v+9}{3v^2+9v}dv = \int \frac{1}{u}du = \ln|u| + C$$
$$= \ln\left|3v^2 + 9v\right| + C$$

18. Let $u = 2z^2 + 8$ so $du = 4z\,dz$.

$$\int \frac{z}{2z^2+8}dz = \frac{1}{4}\int \frac{1}{u}du$$
$$= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left(2z^2 + 8\right) + C$$

19. Let $u = \ln x$ so $du = \frac{1}{x}dx$

$$\int \frac{2\ln x}{x}dx = 2\int u\,du$$
$$= u^2 + C = (\ln x)^2 + C$$

20. Let $u = \ln x$, so $du = \frac{1}{x}dx$.

$$\int \frac{-1}{x(\ln x)^2}dx = -\int u^{-2}du$$
$$= \frac{1}{u} + C = \frac{1}{\ln x} + C$$

21. Let $u = 2x^5 + \pi$ so $du = 10x^4 dx$.

$$\int \frac{x^4}{2x^5+\pi}dx = \frac{1}{10}\int \frac{1}{u}du$$
$$= \frac{1}{10}\ln|u| + C = \frac{1}{10}\ln\left|2x^5+\pi\right| + C$$
$$\int_0^3 \frac{x^4}{2x^5+\pi}dx = \left[\frac{1}{10}\ln\left|2x^5+\pi\right|\right]_0^3$$
$$= \frac{1}{10}[\ln(486+\pi) - \ln\pi] = \ln\sqrt[10]{\frac{486+\pi}{\pi}} \approx 0.5048$$

22. Let $u = 2t^2 + 4t + 3$ so $du = (4t+4)dt$.

$$\int \frac{t+1}{2t^2+4t+3}dt = \frac{1}{4}\int \frac{1}{u}du$$
$$= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left|2t^2+4t+3\right| + C$$
$$\int_0^1 \frac{t+1}{2t^2+4t+3}dt = \left[\frac{1}{4}\ln\left|2t^2+4t+3\right|\right]_0^1$$
$$= \frac{1}{4}\ln 9 - \frac{1}{4}\ln 3 = \ln\sqrt[4]{\frac{9}{3}} = \ln\sqrt[4]{3} = \frac{1}{4}\ln 3$$

23. By long division, $\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$

so $\int \frac{x^2}{x-1}dx = \int x\,dx + \int 1\,dx + \int \frac{1}{x-1}dx$

$$= \frac{x^2}{2} + x + \ln|x-1| + C$$

24. By long division, $\frac{x^2+x}{2x-1} = \frac{x}{2} + \frac{3}{4} + \frac{3}{4(2x-1)}$ so

$$\int \frac{x^2+x}{2x-1}dx = \int \frac{x}{2}dx + \int \frac{3}{4}dx + \int \frac{3}{4(2x-1)}dx$$
$$= \frac{x^2}{4} + \frac{3}{4}x + \frac{3}{4}\int \frac{1}{2x-1}dx$$

Let $u = 2x - 1$; then $du = 2dx$. Hence

$$\int \frac{1}{2x-1}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln|2x-1| + C$$

and $\int \frac{x^2+x}{2x-1}dx = \frac{x^2}{4} + \frac{3}{4}x + \frac{3}{8}\ln|2x-1| + C$

25. By long division,

$\frac{x^4}{x+4} = x^3 - 4x^2 + 16x - 64 + \frac{256}{x+4}$ so

$$\int \frac{x^4}{x+4}dx =$$
$$\int x^3dx - \int 4x^2dx + \int 16x\,dx - \int 64dx + 256\int \frac{1}{x+4}dx$$
$$= \frac{x^4}{4} - \frac{4x^3}{3} + 8x^2 - 64x + 256\ln|x+4| + C$$

26. By long division, $\frac{x^3+x^2}{x+2} = x^2 - x + 2 - \frac{4}{x+2}$ so

$$\int \frac{x^3+x^2}{x+2}dx = \int x^2dx - \int x\,dx + \int 2dx - 4\int \frac{1}{x+2}dx$$
$$= \frac{x^3}{3} - \frac{x^2}{2} + 2x - 4\ln|x+2| + C$$

27. $2\ln(x+1)-\ln x=\ln(x+1)^2-\ln x=\ln\dfrac{(x+1)^2}{x}$

28. $\dfrac{1}{2}\ln(x-9)+\dfrac{1}{2}\ln x=\ln\sqrt{x-9}-\ln\sqrt{x}$

$=\ln\dfrac{\sqrt{x-9}}{\sqrt{x}}=\ln\sqrt{\dfrac{x-9}{x}}$

29. $\ln(x-2)-\ln(x+2)+2\ln x$

$=\ln(x-2)-\ln(x+2)+\ln x^2=\ln\dfrac{x^2(x-2)}{x+2}$

30. $\ln(x^2-9)-2\ln(x-3)-\ln(x+3)$

$=\ln(x^2-9)-\ln(x-3)^2-\ln(x+3)$

$=\ln\dfrac{x^2-9}{(x-3)^2(x+3)}\;=\ln\dfrac{1}{x-3}$

31. $\ln y=\ln(x+11)-\dfrac{1}{2}\ln(x^3-4)$

$\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{1}{x+11}\cdot 1-\dfrac{1}{2}\cdot\dfrac{1}{x^3-4}\cdot 3x^2$

$=\dfrac{1}{x+11}-\dfrac{3x^2}{2(x^3-4)}$

$\dfrac{dy}{dx}=y\cdot\left[\dfrac{1}{x+11}-\dfrac{3x^2}{2(x^3-4)}\right]$

$=\dfrac{x+11}{\sqrt{x^3-4}}\left[\dfrac{1}{x+11}-\dfrac{3x^2}{2(x^3-4)}\right]$

$=-\dfrac{x^3+33x^2+8}{2(x^3-4)^{3/2}}$

32. $\ln y=\ln(x^2+3x)+\ln(x-2)+\ln(x^2+1)$

$\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{2x+3}{x^2+3x}+\dfrac{1}{x-2}+\dfrac{2x}{x^2+1}$

$\dfrac{dy}{dx}=(x^2+3x)(x-2)(x^2+1)\left(\dfrac{2x+3}{x^2+3x}+\dfrac{1}{x-2}+\dfrac{2x}{x^2+1}\right)=5x^4+4x^3-15x^2+2x-6$

33. $\ln y=\dfrac{1}{2}\ln(x+13)-\ln(x-4)-\dfrac{1}{3}\ln(2x+1)$

$\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{1}{2(x+13)}-\dfrac{1}{x-4}-\dfrac{2}{3(2x+1)}$

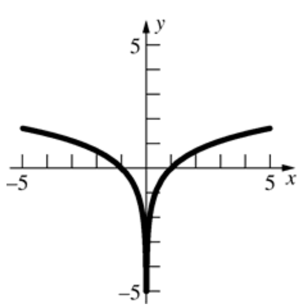
$\dfrac{dy}{dx}=\dfrac{\sqrt{x+13}}{(x-4)\sqrt[3]{2x+1}}\left[\dfrac{1}{2(x+13)}-\dfrac{1}{x-4}-\dfrac{2}{3(2x+1)}\right]=-\dfrac{10x^2+219x-118}{6(x-4)^2(x+13)^{1/2}(2x+1)^{4/3}}$

34. $\ln y=\dfrac{2}{3}\ln(x^2+3)+2\ln(3x+2)-\dfrac{1}{2}\ln(x+1)$

$\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{2}{3}\cdot\dfrac{2x}{x^2+3}+\dfrac{2\cdot 3}{3x+2}-\dfrac{1}{2(x+1)}$

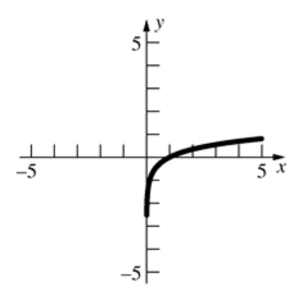
$\dfrac{dy}{dx}=\dfrac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\left[\dfrac{4x}{3(x^2+3)}+\dfrac{6}{3x+2}-\dfrac{1}{2(x+1)}\right]=\dfrac{(3x+2)(51x^3+70x^2+97x+90)}{6(x^2+3)^{1/3}(x+1)^{3/2}}$

35.



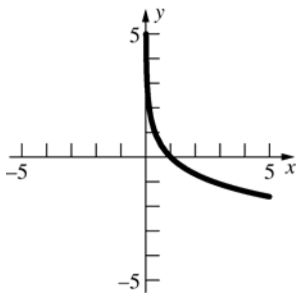
$y=\ln x$ is reflected across the y-axis.

36.



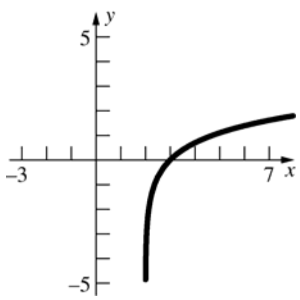
The y-values of $y=\ln x$ are multiplied by $\dfrac{1}{2}$, since $\ln\sqrt{x}=\dfrac{1}{2}\ln x$.

37.



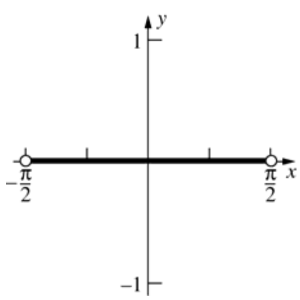
$y = \ln x$ is reflected across the x-axis since $\ln\left(\frac{1}{x}\right) = -\ln x.$

38.



$y = \ln x$ is shifted two units to the right.

39.



$$y = \ln\cos x + \ln\sec x$$
$$= \ln\cos x + \ln\frac{1}{\cos x}$$
$$= \ln\cos x - \ln\cos x = 0 \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

40. Since ln is continuous,

$$\lim_{x\to 0}\ln\frac{\sin x}{x} = \ln\lim_{x\to 0}\frac{\sin x}{x} = \ln 1 = 0$$

41. The domain is $(0,\infty)$.

$$f'(x) = 4x\ln x + 2x^2\left(\frac{1}{x}\right) - 2x = 4x\ln x$$

$f'(x) = 0$ if $\ln x = 0$, or $x = 1$.

$f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$

so $f(1) = -1$ is a minimum.

42. Let $r(x)$ = rate of transmission

$$= kx^2\ln\frac{1}{x} = -kx^2\ln x.$$

$$r'(x) = -2kx\ln x - kx^2\left(\frac{1}{x}\right) = -kx(2\ln x + 1)$$

$r'(x) = 0$ if $\ln x = -\frac{1}{2}$, or $-\ln x = \frac{1}{2}$, so

$$\ln\frac{1}{x} = \frac{1}{2}.$$

$\ln 1.65 \approx \frac{1}{2}$, so $x \approx \frac{1}{1.65} \approx 0.606.$

$$r''(x) = -k(2\ln x + 1) - kx\left(2\cdot\frac{1}{x}\right) = -k(2\ln x + 3)$$

$r''(0.606) \approx -2k < 0$ since $k > 0$, so $x \approx 0.606$ gives the maximum rate of transmission.

43. $\ln 4 > 1$

so $\ln 4^m = m\ln 4 > m\cdot 1 = m$

Thus $x > 4^m \Rightarrow \ln x > m$

so $\lim_{x\to\infty}\ln x = \infty$

44. Let $z = \frac{1}{x}$ so $z \to \infty$ as $x \to 0^+$

Then $\lim_{x\to 0^+}\ln x = \lim_{z\to\infty}\ln\left(\frac{1}{z}\right) = \lim_{z\to\infty}(-\ln z)$

$= -\lim_{z\to\infty}\ln z = -\infty$

45. $\int_{1/3}^{x}\frac{1}{t}dt = 2\int_1^x\frac{1}{t}dt$

$$\int_{1/3}^{1}\frac{1}{t}dt + \int_1^x\frac{1}{t}dt = 2\int_1^x\frac{1}{t}dt$$

$$\int_{1/3}^{1}\frac{1}{t}dt = \int_1^x\frac{1}{t}dt$$

$$-\int_1^{1/3}\frac{1}{t}dt = \int_1^x\frac{1}{t}dt$$

$$-\ln\frac{1}{3} = \ln x$$

$\ln 3 = \ln x$

$x = 3$

46. a. $\frac{1}{t} < \frac{1}{\sqrt{t}}$ for $t > 1$,

so $\ln x = \int_1^x\frac{1}{t}dt < \int_1^x\frac{1}{\sqrt{t}}dt = \int_1^x t^{-1/2}dt$

$$= \left[2\sqrt{t}\right]_1^x = 2(\sqrt{x} - 1)$$

so $\ln x < 2(\sqrt{x} - 1)$

b. If $x > 1$, $0 < \ln x < 2(\sqrt{x}-1)$,

so $0 < \frac{\ln x}{x} < \frac{2(\sqrt{x}-1)}{x}$.

Hence $0 \le \lim_{x\to\infty} \frac{\ln x}{x} \le \lim_{x\to\infty} \frac{2(\sqrt{x}+1)}{x} = 0$

and $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

47. $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$

$= \lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right]\cdot\frac{1}{n}$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\cdot\frac{1}{n} \ = \int_1^2 \frac{1}{x}dx = \ln 2 \approx 0.693$

48. $\frac{1,000,000}{\ln 1,000,000} \approx 72,382$

49. a. $f(x) = \ln\left(\frac{ax-b}{ax+b}\right)^c = c\ln\left(\frac{ax-b}{ax+b}\right)$

$= \frac{a^2-b^2}{2ab}[\ln(ax-b)-\ln(ax+b)]$

$f'(x) = \frac{a^2-b^2}{2ab}\left[\frac{a}{ax-b}-\frac{a}{ax+b}\right]$

$= \frac{a^2-b^2}{2ab}\left[\frac{2ab}{(ax-b)(ax+b)}\right] = \frac{a^2-b^2}{a^2x^2-b^2}$

$f'(1) = \frac{a^2-b^2}{a^2-b^2} = 1$

b. $f'(x) = \cos^2 u\cdot\frac{du}{dx}$

$= \cos^2[\ln(x^2+x-1)]\cdot\frac{2x+1}{x^2+x-1}$

$f'(1) = \cos^2[\ln(1^2+1-1)]\cdot\frac{2\cdot1+1}{1^2+1-1}$

$= 3\cos^2(0) = 3$

50. From Ex 9,

$\int_0^{\pi/3}\tan x\,dx = \left[-\ln\left|\cos x\right|\right]_0^{\pi/3}$

$= \ln\left|\cos 0\right| - \ln\left|\cos \pi/3\right|$

$= \ln(1)-\ln(0.5) = \ln\left(\frac{1}{0.5}\right)$

$= \ln 2 \approx 0.69315$

51. From Ex 10,

$\int_{\pi/4}^{\pi/3}\sec x\csc x\,dx = \left[-\ln\left|\cos x\right| + \ln\left|\sin x\right|\right]_{\pi/4}^{\pi/3}$

$= \left[\ln\left|\tan x\right|\right]_{\pi/4}^{\pi/3} = \ln\left|\tan\pi/3\right| - \ln\left|\tan\pi/4\right|$

$= \ln(\sqrt{3}) - \ln 1 = 0.5493 - 0 = 0.5493$

52. Let $u = 1+\sin x$; then $du = \cos x\,dx$ so that

$\int\frac{\cos x}{1+\sin x}dx = \int\frac{1}{u}du = \ln|u| + C$

$= \ln\left|1+\sin x\right| + C = \ln(1+\sin x) + C$

(since $1+\sin x \ge 0$ for all x).

53. $V = 2\pi\int_1^4 xf(x)dx = \int_1^4\frac{2\pi x}{x^2+4}dx$

Let $u = x^2+4$ so $du = 2x\,dx$.

$\int\frac{2\pi x}{x^2+4}dx = \pi\int\frac{1}{u}du = \pi\ln|u| + C$

$= \pi\ln\left|x^2+4\right| + C$

$\int_1^4\frac{2\pi x}{x^2+4}dx = \left[\pi\ln\left|x^2+4\right|\right]_1^4$

$= \pi\ln 20 - \pi\ln 5 = \pi\ln 4 \approx 4.355$

54. $y = \frac{x^2}{4} - \ln\sqrt{x} = \frac{x^2}{4} - \frac{1}{2}\ln x$

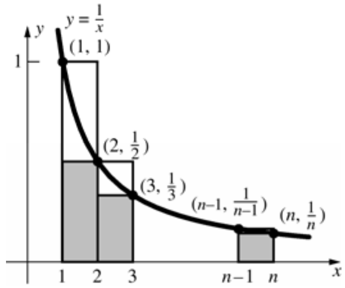
$\frac{dy}{dx} = \frac{2x}{4} - \frac{1}{2}\cdot\frac{1}{x} = \frac{x}{2} - \frac{1}{2x}$

$L = \int_1^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_1^2\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}dx$

$= \int_1^2\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}dx = \int_1^2\left(\frac{x}{2}+\frac{1}{2x}\right)dx$

$= \frac{1}{2}\left[\frac{x^2}{2}+\ln|x|\right]_1^2 = \frac{1}{2}\left[2+\ln 2 - \left(\frac{1}{2}+\ln 1\right)\right]$

$= \frac{3}{4}+\frac{1}{2}\ln 2 \approx 1.097$

55. 

$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=$ the lower approximate area

$1+\frac{1}{2}+\cdots+\frac{1}{n-1}=$ the upper approximate area

$\ln n$ = the exact area under the curve

Thus,
$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\ln n<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}$.

56. $\frac{\ln y-\ln x}{y-x}=\frac{\int_1^y \frac{1}{t}dt-\int_1^x \frac{1}{t}dt}{y-x}=\frac{\int_x^y \frac{1}{t}dt}{y-x}$

= the average value of $\frac{1}{t}$ on $[x, y]$.

Since $\frac{1}{t}$ is decreasing on the interval $[x, y]$, the average value is between the minimum value of $\frac{1}{y}$ and the maximum value of $\frac{1}{x}$.

57. a. $f'(x)=\frac{1}{1.5+\sin x}\cdot\cos x=\frac{\cos x}{1.5+\sin x}$

$f'(x)=0$ when $\cos x = 0$.

Critical points: $0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, 3\pi$

$f(0) \approx 0.405$,

$f\left(\frac{\pi}{2}\right)\approx 0.916$, $f\left(\frac{3\pi}{2}\right)\approx -0.693$,

$f\left(\frac{5\pi}{2}\right)\approx 0.916$, $f(3\pi)\approx 0.405$.

On $[0,3\pi]$, the maximum value points are $\left(\frac{\pi}{2},0.916\right),\left(\frac{5\pi}{2},0.916\right)$ and the minimum value point is $\left(\frac{3\pi}{2},-0.693\right)$.

b. $f''(x)=-\frac{1+1.5\sin x}{(1.5+\sin x)^2}$

On $[0,3\pi]$, $f''(x)=0$ when $x \approx 3.871$, 5.553.
Inflection points are (3.871, –0.182), (5.553, –0.182).

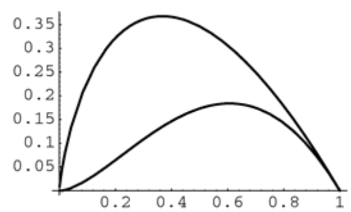
c. $\int_0^{3\pi}\ln(1.5+\sin x)dx \approx 4.042$

58. a. $f'(x)=-\frac{\sin(\ln x)}{x}$

On [0.1, 20], $f'(x)=0$ when $x = 1$.
Critical points: 0.1, 1, 20
$f(0.1) \approx -0.668$, $f(1) = 1$, $f(20) \approx -0.989$
On [0.1, 20], the maximum value point is (1, 1) and minimum value point is (20, –0.989).

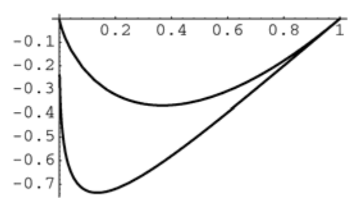
b. On [0.01, 0.1], $f'(x)=0$ when $x \approx 0.043$.
$f(0.01) \approx -0.107$, $f(0.043) \approx -1$
On [0.01, 20], the maximum value point is (1, 1) and the minimum value point is (0.043, –1).

c. $\int_{0.1}^{20}\cos(\ln x)dx \approx -8.37$

59. 

a. $\int_0^1\left[x\ln\left(\frac{1}{x}\right)-x^2\ln\left(\frac{1}{x}\right)\right]dx=\frac{5}{36}\approx 0.139$

b. Maximum of ≈ 0.260 at $x \approx 0.236$

60. 

a. $\int_0^1[x\ln x-\sqrt{x}\ln x]dx=\frac{7}{36}\approx 0.194$

b. Maximum of ≈ 0.521 at $x \approx 0.0555$

6.2 Concepts Review

1. $f(x_1) \neq f(x_2)$

2. x; $f^{-1}(y)$

3. monotonic; strictly increasing; strictly decreasing

4. $(f^{-1})'(y) = \dfrac{1}{f'(x)}$

Problem Set 6.2

1. $f(x)$ is one-to-one, so it has an inverse. Since $f(4) = 2$, $f^{-1}(2) = 4$.

2. $f(x)$ is one-to-one, so it has an inverse. Since $f(1) = 2$, $f^{-1}(2) = 1$.

3. $f(x)$ is not one-to-one, so it does not have an inverse.

4. $f(x)$ is not one-to-one, so it does not have an inverse.

5. $f(x)$ is one-to-one, so it has an inverse. Since $f(-1.3) \approx 2$, $f^{-1}(2) \approx -1.3$.

6. $f(x)$ is one-to-one, so it has an inverse. Since $f\left(\dfrac{1}{2}\right) = 2$, $f^{-1}(2) = \dfrac{1}{2}$.

7. $f'(x) = -5x^4 - 3x^2 = -(5x^4 + 3x^2) < 0$ for all $x \neq 0$. $f(x)$ is strictly decreasing at $x = 0$ because $f(x) > 0$ for $x < 0$ and $f(x) < 0$ for $x > 0$. Therefore $f(x)$ is strictly decreasing for x and so it has an inverse.

8. $f'(x) = 7x^6 + 5x^4 > 0$ for all $x \neq 0$. $f(x)$ is strictly increasing at $x = 0$ because $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Therefore $f(x)$ is strictly increasing for all x and so it has an inverse.

9. $f'(\theta) = -\sin\theta < 0$ for $0 < \theta < \pi$
$f(\theta)$ is decreasing at $\theta = 0$ because $f(0) = 1$ and $f(\theta) < 1$ for $0 < \theta < \pi$. $f(\theta)$ is decreasing at $\theta = \pi$ because $f(\pi) = -1$ and $f(\theta) > -1$ for $0 < \theta < \pi$. Therefore $f(\theta)$ is strictly decreasing on $0 \le \theta \le \pi$ and so it has an inverse.

10. $f'(x) = -\csc^2 x < 0$ for $0 < x < \dfrac{\pi}{2}$
$f(x)$ is decreasing on $0 < x < \dfrac{\pi}{2}$ and so it has an inverse.

11. $f'(z) = 2(z-1) > 0$ for $z > 1$
$f(z)$ is increasing at $z = 1$ because $f(1) = 0$ and $f(z) > 0$ for $z > 1$. Therefore, $f(z)$ is strictly increasing on $z \ge 1$ and so it has an inverse.

12. $f'(x) = 2x + 1 > 0$ for $x \ge 2$. $f(x)$ is strictly increasing on $x \ge 2$ and so it has an inverse.

13. $f'(x) = \sqrt{x^4 + x^2 + 10} > 0$ for all real x. $f(x)$ is strictly increasing and so it has an inverse.

14. $f(r) = \int_r^1 \cos^4 t\,dt = -\int_1^r \cos^4 t\,dt$
$f'(r) = -\cos^4 r < 0$ for all $r \neq k\pi + \dfrac{\pi}{2}$, k any integer.
$f(r)$ is decreasing at $r = k\pi + \dfrac{\pi}{2}$ since $f'(r) < 0$ on the deleted neighborhood $\left(k\pi + \dfrac{\pi}{2} - \varepsilon, k\pi + \dfrac{\pi}{2} + \varepsilon\right)$. Therefore, $f(r)$ is strictly decreasing for all r and so it has an inverse.

15. Step 1:
$y = x + 1$
$x = y - 1$
Step 2: $f^{-1}(y) = y - 1$
Step 3: $f^{-1}(x) = x - 1$
Check:
$f^{-1}(f(x)) = (x+1) - 1 = x$
$f(f^{-1}(x)) = (x-1) + 1 = x$

16. Step 1:
$y = -\dfrac{x}{3} + 1$
$-\dfrac{x}{3} = y - 1$
$x = -3(y - 1) = 3 - 3y$
Step 2: $f^{-1}(y) = 3 - 3y$
Step 3: $f^{-1}(x) = 3 - 3x$
Check:
$f^{-1}(f(x)) = 3 - 3\left(-\dfrac{x}{3} + 1\right) = 3 + (x - 3) = x$
$f(f^{-1}(x)) = \dfrac{-(3-3x)}{3} + 1 = (-1 + x) + 1 = x$

17. Step 1:

$y = \sqrt{x+1}$ (note that $y \ge 0$)

$x + 1 = y^2$

$x = y^2 - 1,\ y \ge 0$

Step 2: $f^{-1}(y) = y^2 - 1,\ y \ge 0$

Step 3: $f^{-1}(x) = x^2 - 1,\ x \ge 0$

Check:

$f^{-1}(f(x)) = (\sqrt{x+1})^2 - 1 = (x+1) - 1 = x$

$f(f^{-1}(x)) = \sqrt{(x^2-1)+1} = \sqrt{x^2} = |x| = x$

18. Step 1:

$y = -\sqrt{1-x}$ (note that $y \le 0$)

$\sqrt{1-x} = -y$

$1 - x = (-y)^2 = y^2$

$x = 1 - y^2,\ y \le 0$

Step 2: $f^{-1}(y) = 1 - y^2,\ y \le 0$

Step 3: $f^{-1}(x) = 1 - x^2,\ x \le 0$

Check:

$f^{-1}(f(x)) = 1 - (-\sqrt{1-x})^2 = 1 - (1-x) = x$

$f(f^{-1}(x)) = -\sqrt{1-(1-x^2)} = -\sqrt{x^2} = -|x|$

$= -(-x) = x$

19. Step 1:

$y = -\dfrac{1}{x-3}$

$x - 3 = -\dfrac{1}{y}$

$x = 3 - \dfrac{1}{y}$

Step 2: $f^{-1}(y) = 3 - \dfrac{1}{y}$

Step 3: $f^{-1}(x) = 3 - \dfrac{1}{x}$

Check:

$f^{-1}(f(x)) = 3 - \dfrac{1}{-\frac{1}{x-3}} = 3 + (x-3) = x$

$f(f^{-1}(x)) = -\dfrac{1}{\left(3 - \frac{1}{x}\right) - 3} = -\dfrac{1}{-\frac{1}{x}} = x$

20. Step 1:

$y = \sqrt{\dfrac{1}{x-2}}$ (note that $y > 0$)

$y^2 = \dfrac{1}{x-2}$

$x - 2 = \dfrac{1}{y^2}$

$x = 2 + \dfrac{1}{y^2},\ y > 0$

Step 2: $f^{-1}(y) = 2 + \dfrac{1}{y^2},\ y > 0$

Step 3: $f^{-1}(x) = 2 + \dfrac{1}{x^2},\ x > 0$

Check:

$f^{-1}(f(x)) = 2 + \dfrac{1}{\left(\sqrt{\frac{1}{x-2}}\right)^2} = 2 + \dfrac{1}{\left(\frac{1}{x-2}\right)}$

$= 2 + (x - 2) = x$

$f(f^{-1}(x)) = \sqrt{\dfrac{1}{\left(2 + \frac{1}{x^2}\right) - 2}} = \sqrt{\dfrac{1}{\left(\frac{1}{x^2}\right)}} = \sqrt{x^2}$

$= |x| = x$

21. Step 1:

$y = 4x^2,\ x \le 0$ (note that $y \ge 0$)

$x^2 = \dfrac{y}{4}$

$x = -\sqrt{\dfrac{y}{4}} = -\dfrac{\sqrt{y}}{2}$, negative since $x \le 0$

Step 2: $f^{-1}(y) = -\dfrac{\sqrt{y}}{2}$

Step 3: $f^{-1}(x) = -\dfrac{\sqrt{x}}{2}$

Check:

$f^{-1}(f(x)) = -\dfrac{\sqrt{4x^2}}{2} = -\sqrt{x^2} = -|x| = -(-x) = x$

$f(f^{-1}(x)) = 4\left(-\dfrac{\sqrt{x}}{2}\right)^2 = 4 \cdot \dfrac{x}{4} = x$

22. Step 1:

$y = (x-3)^2,\ x \ge 3$ (note that $y \ge 0$)

$x - 3 = \sqrt{y}$

$x = 3 + \sqrt{y}$

Step 2: $f^{-1}(y) = 3 + \sqrt{y}$

Step 3: $f^{-1}(x) = 3 + \sqrt{x}$

Check:

$f^{-1}(f(x)) = 3 + \sqrt{(x-3)^2} = 3 + |x-3|$

$= 3 + (x-3) = x$

$f(f^{-1}(x)) = [(3+\sqrt{x}) - 3]^2 = (\sqrt{x})^2 = x$

23. Step 1:

$y = (x-1)^3$

$x - 1 = \sqrt[3]{y}$

$x = 1 + \sqrt[3]{y}$

Step 2: $f^{-1}(y) = 1 + \sqrt[3]{y}$

Step 3: $f^{-1}(x) = 1 + \sqrt[3]{x}$

Check: $f^{-1}(f(x)) = 1 + \sqrt[3]{(x-1)^3} = 1 + (x-1) = x$

$f(f^{-1}(x)) = [(1+\sqrt[3]{x}) - 1]^3 = (\sqrt[3]{x})^3 = x$

24. Step 1:

$y = x^{5/2}, x \ge 0$

$x = y^{2/5}$

Step 2: $f^{-1}(y) = y^{2/5}$

Step 3: $f^{-1}(x) = x^{2/5}$

Check:

$f^{-1}(f(x)) = (x^{5/2})^{2/5} = x$

$f(f^{-1}(x)) = (x^{2/5})^{5/2} = x$

25. Step 1:

$y = \frac{x-1}{x+1}$

$xy + y = x - 1$

$x - xy = 1 + y$

$x = \frac{1+y}{1-y}$

Step 2: $f^{-1}(y) = \frac{1+y}{1-y}$

Step 3: $f^{-1}(x) = \frac{1+x}{1-x}$

Check:

$$f^{-1}(f(x)) = \frac{1 + \frac{x-1}{x+1}}{1 - \frac{x-1}{x+1}} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x$$

26. Step 1:

$y = \left(\frac{x-1}{x+1}\right)^3$

$y^{1/3} = \frac{x-1}{x+1}$

$xy^{1/3} + y^{1/3} = x - 1$

$x - xy^{1/3} = 1 + y^{1/3}$

$x = \frac{1+y^{1/3}}{1-y^{1/3}}$

Step 2: $f^{-1}(y) = \frac{1+y^{1/3}}{1-y^{1/3}}$

Step 3: $f^{-1}(x) = \frac{1+x^{1/3}}{1-x^{1/3}}$

Check:

$$f^{-1}(f(x)) = \frac{1 + \left[\left(\frac{x-1}{x+1}\right)^3\right]^{1/3}}{1 - \left[\left(\frac{x-1}{x+1}\right)^3\right]^{1/3}} = \frac{1 + \frac{x-1}{x+1}}{1 - \frac{x-1}{x+1}}$$

$$= \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = \left(\frac{\frac{1+x^{1/3}}{1-x^{1/3}} - 1}{\frac{1+x^{1/3}}{1-x^{1/3}} + 1}\right)^3 = \left(\frac{1+x^{1/3} - 1 + x^{1/3}}{1+x^{1/3} + 1 - x^{1/3}}\right)^3$$

$$= \left(\frac{2x^{1/3}}{2}\right)^3 = (x^{1/3})^3 = x$$

27. Step 1:

$y = \frac{x^3+2}{x^3+1}$

$x^3y + y = x^3 + 2$

$x^3y - x^3 = 2 - y$

$x^3 = \frac{2-y}{y-1}$

$x = \left(\frac{2-y}{y-1}\right)^{1/3}$

Step 2: $f^{-1}(y) = \left(\frac{2-y}{y-1}\right)^{1/3}$

Step 3: $f^{-1}(x) = \left(\frac{2-x}{x-1}\right)^{1/3}$

Check:

$$f^{-1}(f(x)) = \left(\frac{2 - \frac{x^3+2}{x^3+1}}{\frac{x^3+2}{x^3+1} - 1}\right)^{1/3} = \left(\frac{2x^3 + 2 - x^3 - 2}{x^3 + 2 - x^3 - 1}\right)^{1/3}$$

$$= \left(\frac{x^3}{1}\right)^{1/3} = x$$

$$f(f^{-1}(x)) = \frac{\left[\left(\frac{2-x}{x-1}\right)^{1/3}\right]^3 + 2}{\left[\left(\frac{2-x}{x-1}\right)^{1/3}\right]^3 + 1} = \frac{\frac{2-x}{x-1} + 2}{\frac{2-x}{x-1} + 1}$$

$$= \frac{2-x+2x-2}{2-x+x-1} = \frac{x}{1} = x$$

28. Step 1:

$$y=\left(\frac{x^3+2}{x^3+1}\right)^5$$

$$y^{1/5}=\frac{x^3+2}{x^3+1}$$

$$x^3y^{1/5}+y^{1/5}=x^3+2$$

$$x^3y^{1/5}-x^3=2-y^{1/5}$$

$$x^3=\frac{2-y^{1/5}}{y^{1/5}-1}$$

$$x=\left(\frac{2-y^{1/5}}{y^{1/5}-1}\right)^{1/3}$$

Step 2: $f^{-1}(y)=\left(\dfrac{2-y^{1/5}}{y^{1/5}-1}\right)^{1/3}$

Step 3: $f^{-1}(x)=\left(\dfrac{2-x^{1/5}}{x^{1/5}-1}\right)^{1/3}$

Check:

$$f^{-1}(f(x))=\left\{\frac{2-\left[\left(\frac{x^3+2}{x^3+1}\right)^5\right]^{1/5}}{\left[\left(\frac{x^3+2}{x^3+1}\right)^5\right]^{1/5}-1}\right\}^{1/3}$$

$$=\left(\frac{2-\frac{x^3+2}{x^3+1}}{\frac{x^3+2}{x^3+1}-1}\right)^{1/3}=\left(\frac{2x^3+2-x^3-2}{x^3+2-x^3-1}\right)^{1/3}$$

$$=\left(\frac{x^3}{1}\right)^{1/3}=x$$

$$f(f^{-1}(x))=\left\{\frac{\left[\left(\frac{2-x^{1/5}}{x^{1/5}-1}\right)^{1/3}\right]^3+2}{\left[\left(\frac{2-x^{1/5}}{x^{1/5}-1}\right)^{1/3}\right]^3+1}\right\}^5$$

$$=\left(\frac{\frac{2-x^{1/5}}{x^{1/5}-1}+2}{\frac{2-x^{1/5}}{x^{1/5}-1}+1}\right)^5=\left(\frac{2-x^{1/5}+2x^{1/5}-2}{2-x^{1/5}+x^{1/5}-1}\right)^5$$

$$=\left(\frac{x^{1/5}}{1}\right)^5=x$$

29. By similar triangles, $\frac{r}{h}=\frac{4}{6}$. Thus, $r=\frac{2h}{3}$

This gives

$$V=\frac{\pi r^2h}{3}=\frac{\pi\left(4h^2/9\right)h}{3}=\frac{4\pi h^3}{27}$$

$$h^3=\frac{27V}{4\pi}$$

$$h=3\sqrt[3]{\frac{V}{4\pi}}$$

30. $v=v_0-32t$

$v=0$ when $v_0=32t$, that is, when $t=\frac{v_0}{32}$. The position function is $s(t)=v_0t-16t^2$. The ball then reaches a height of

$$H=s(v_0/32)=v_0\frac{v_0}{32}-16\frac{v_0^2}{32^2}=\frac{v_0^2}{64}$$

$$v_0^2=64H$$

$$v_0=8\sqrt{H}$$

31. $f'(x)=4x+1$; $f'(x)>0$ when $x>-\frac{1}{4}$ and $f'(x)<0$ when $x<-\frac{1}{4}$.

The function is decreasing on $\left(-\infty,-\frac{1}{4}\right]$ and increasing on $\left[-\frac{1}{4},\infty\right)$. Restrict the domain to $\left(-\infty,-\frac{1}{4}\right]$ or restrict it to $\left[-\frac{1}{4},\infty\right)$.

Then $f^{-1}(x)=\frac{1}{4}(-1-\sqrt{8x+33})$ or $f^{-1}(x)=\frac{1}{4}(-1+\sqrt{8x+33})$.

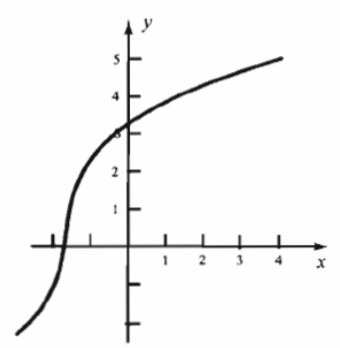
32. $f'(x) = 2x - 3$; $f'(x) > 0$ when $x > \frac{3}{2}$ and $f'(x) < 0$ when $x < \frac{3}{2}$.

The function is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and increasing on $\left[\frac{3}{2}, \infty\right)$. Restrict the domain to $\left(-\infty, \frac{3}{2}\right]$ or restrict it to $\left[\frac{3}{2}, \infty\right)$. Then

$f^{-1}(x) = \frac{1}{2}(3 - \sqrt{4x+5})$ or

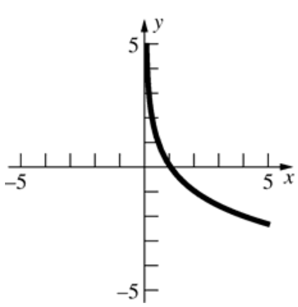
$f^{-1}(x) = \frac{1}{2}(3 + \sqrt{4x+5})$.

33.

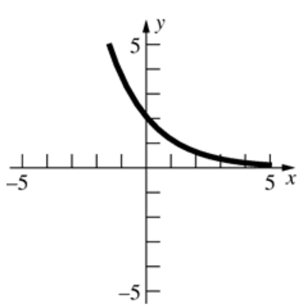


$(f^{-1})'(3) \approx \frac{1}{3}$

34. $(f^{-1})'(3) \approx -\frac{1}{2}$

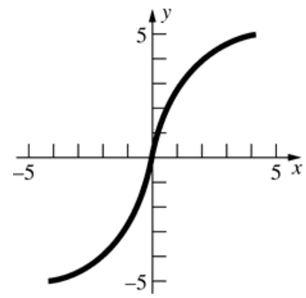


35.



$(f^{-1})'(3) \approx -\frac{1}{3}$

36. $(f^{-1})'(3) \approx \frac{1}{2}$



37. $f'(x) = 15x^4 + 1$ and $y = 2$ corresponds to $x = 1$, so $(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{15+1} = \frac{1}{16}$.

38. $f'(x) = 5x^4 + 5$ and $y = 2$ corresponds to $x = 1$, so $(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{5+5} = \frac{1}{10}$

39. $f'(x) = 2\sec^2 x$ and $y = 2$ corresponds to $x = \frac{\pi}{4}$, so $(f^{-1})'(2) = \frac{1}{f'\left(\frac{\pi}{4}\right)} = \frac{1}{2\sec^2\left(\frac{\pi}{4}\right)} = \frac{1}{2}\cos^2\left(\frac{\pi}{4}\right)$

$= \frac{1}{4}$.

40. $f'(x) = \frac{1}{2\sqrt{x+1}}$ and $y = 2$ corresponds to $x = 3$, so $(f^{-1})'(2) = \frac{1}{f'(3)} = 2\sqrt{3+1} = 4$.

41. $(g^{-1} \circ f^{-1})(h(x)) = (g^{-1} \circ f^{-1})(f(g(x)))$

$= g^{-1} \circ [f^{-1}(f(g(x)))] = g^{-1} \circ [g(x)] = x$

Similarly,

$h((g^{-1} \circ f^{-1})(x)) = f(g((g^{-1} \circ f^{-1})(x)))$

$= f(g(g^{-1}(f^{-1}(x)))) = f(f^{-1}(x)) = x$

Thus $h^{-1} = g^{-1} \circ f^{-1}$

42. Find $f^{-1}(x)$:

$y = \frac{1}{x}$, $x = \frac{1}{y}$

$f^{-1}(y) = \frac{1}{y}$

$f^{-1}(x) = \frac{1}{x}$

Find $g^{-1}(x)$:

$y = 3x + 2$

$x = \frac{y-2}{3}$

$g^{-1}(y) = \frac{y-2}{3}$

$g^{-1}(x) = \frac{x-2}{3}$

$$h(x) = f(g(x)) = f(3x+2) = \frac{1}{3x+2}$$

$$h^{-1}(x) = g^{-1}(f^{-1}(x)) = g^{-1}\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x}\right)-2}{3}$$

$$h^{-1}(h(x)) = h^{-1}\left(\frac{1}{3x+2}\right) = \frac{(3x+2)-2}{3} = \frac{3x}{3} = x$$

$$h(h^{-1}(x)) = h\left(\frac{\left(\frac{1}{x}\right)-2}{3}\right) = \frac{1}{\left[\left(\frac{1}{x}\right)-2\right]+2} = \frac{1}{\left(\frac{1}{x}\right)} = x$$

43. f has an inverse because it is monotonic (increasing):

$f'(x) = \sqrt{1+\cos^2 x} > 0$

a. $(f^{-1})'(A) = \frac{1}{f'\left(\frac{\pi}{2}\right)} = \frac{1}{\sqrt{1+\cos^2\left(\frac{\pi}{2}\right)}} = 1$

b. $(f^{-1})'(B) = \frac{1}{f'\left(\frac{5\pi}{6}\right)} = \frac{1}{\sqrt{1+\cos^2\left(\frac{5\pi}{6}\right)}} = \frac{1}{\sqrt{\frac{7}{4}}}$

$= \frac{2}{\sqrt{7}}$

c. $(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{\sqrt{1+\cos^2(0)}} = \frac{1}{\sqrt{2}}$

44. a. $y = \frac{ax+b}{cx+d}$

$cxy + dy = ax + b$

$(cy - a)x = b - dy$

$x = \frac{b-dy}{cy-a} = -\frac{dy-b}{cy-a}$

$f^{-1}(y) = -\frac{dy-b}{cy-a}$

$f^{-1}(x) = -\frac{dx-b}{cx-a}$

b. If $bc - ad = 0$, then $f(x)$ is either a constant function or undefined.

c. If $f = f^{-1}$, then for all x in the domain we have:

$\frac{ax+b}{cx+d} + \frac{dx-b}{cx-a} = 0$

$(ax + b)(cx - a) + (dx - b)(cx + d) = 0$

$acx^2 + (bc - a^2)x - ab + dcx^2$

$+ (d^2 - bc)x - bd = 0$

$(ac + dc)x^2 + (d^2 - a^2)x + (-ab - bd) = 0$

Setting the coefficients equal to 0 gives three requirements:

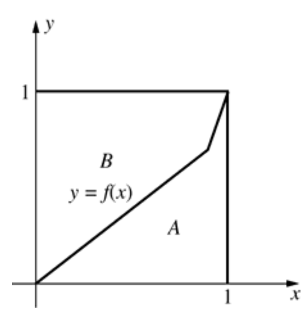
(1) $a = -d$ or $c = 0$

(2) $a = \pm d$

(3) $a = -d$ or $b = 0$

If $a = d$, then $f = f^{-1}$ requires $b = 0$ and $c = 0$, so $f(x) = \frac{ax}{d} = x$. If $a = -d$, there are no requirements on b and c (other than $bc - ad \neq 0$). Therefore, $f = f^{-1}$ if $a = -d$ or if f is the identity function.

45.



$\int_0^1 f^{-1}(y)\,dy =$ (Area of region B)

$= 1 -$ (Area of region A)

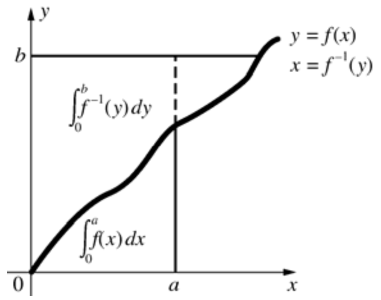
$= 1 - \int_0^1 f(x)\,dx = 1 - \frac{2}{5} = \frac{3}{5}$

46. $\int_0^a f(x)dx =$ the area bounded by $y = f(x)$, $y = 0$, and $x = a$ [the area under the curve].

$\int_0^b f^{-1}(y)dy =$ the area bounded by $x = f^{-1}(y)$ $x = 0$, and $y = b$.

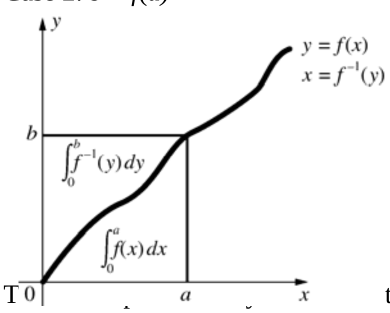
$ab =$ the area of the rectangle bounded by $x = 0$, $x = a$, $y = 0$, and $y = b$.

Case 1: $b > f(a)$



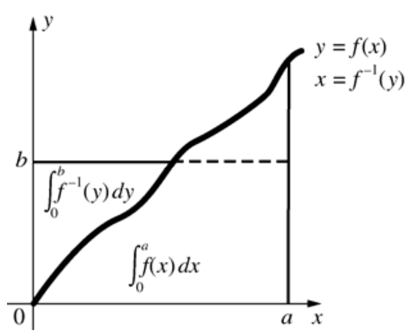
The area above the curve is greater than the area of the part of the rectangle above the curve, so the total area represented by the sum of the two integrals is greater than the area ab of the rectangle.

Case 2: $b = f(a)$



T the two integrals = the area ab of the rectangle.

Case 3: $b < f(a)$



The area below the curve is greater than the area of the part of the rectangle which is below the curve, so the total area represented by the sum of the two integrals is greater than the area ab of the rectangle.

$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$ with equality holding when $b = f(a)$.

47. Given $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $f(x) = x^{p-1}$, solving $\frac{1}{p} + \frac{1}{q} = 1$ for p gives $p = \frac{q}{q-1}$, so

$$\frac{1}{p-1} = \frac{1}{\frac{q}{q-1} - 1} = \frac{1}{\left[\frac{q-(q-1)}{q-1}\right]} = \frac{q-1}{1} = q-1.$$

Thus, if $y = x^{p-1}$ then $x = y^{\frac{1}{p-1}} = y^{q-1}$, so $f^{-1}(y) = y^{q-1}$.

By Problem 44, since $f(x) = x^{p-1}$ is strictly increasing for $p > 1$, $ab \le \int_0^a x^{p-1}dx + \int_0^b y^{q-1}dy$

$$ab \le \left[\frac{x^p}{p}\right]_0^a + \left[\frac{y^q}{q}\right]_0^b$$

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

6.3 Concepts Review

1. increasing; exp

2. $\ln e = 1$; 2.72

3. x; x

4. $e^x; e^x + C$

Problem Set 6.3

1. **a.** 20.086

b. 8.1662

c. $e^{\sqrt{2}} \approx e^{1.41} \approx 4.1$

d. $e^{\cos(\ln 4)} \approx e^{0.18} \approx 1.20$

2. **a.** $e^{3\ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$

b. $e^{\frac{\ln 64}{2}} = e^{\ln(64^{1/2})} = e^{\ln 8} = 8$

3. $e^{3\ln x} = e^{\ln x^3} = x^3$

4. $e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

5. $\ln e^{\cos x} = \cos x$

6. $\ln e^{-2x-3} = -2x-3$

7. $\ln(x^3e^{-3x}) = \ln x^3 + \ln e^{-3x} = 3\ln x - 3x$

8. $e^{x-\ln x} = \dfrac{e^x}{e^{\ln x}} = \dfrac{e^x}{x}$

9. $e^{\ln 3+2\ln x} = e^{\ln 3}\cdot e^{2\ln x} = 3\cdot e^{\ln x^2} = 3x^2$

10. $e^{\ln x^2 - y\ln x} = \dfrac{e^{\ln x^2}}{e^{y\ln x}} = \dfrac{x^2}{e^{\ln x^y}} = \dfrac{x^2}{x^y} = x^{2-y}$

11. $D_xe^{x+2} = e^{x+2}D_x(x+2) = e^{x+2}$

12. $D_xe^{2x^2-x} = e^{2x^2-x}D_x(2x^2-x)$
$= e^{2x^2-x}(4x-1)$

13. $D_xe^{\sqrt{x+2}} = e^{\sqrt{x+2}}D_x\sqrt{x+2} = \dfrac{e^{\sqrt{x+2}}}{2\sqrt{x+2}}$

14. $D_xe^{-\frac{1}{x^2}} = e^{-\frac{1}{x^2}}D_x\left(-\dfrac{1}{x^2}\right)$
$= e^{-\frac{1}{x^2}}\cdot 2x^{-3} = \dfrac{2e^{-\frac{1}{x^2}}}{x^3}$

15. $D_xe^{2\ln x} = D_xe^{\ln x^2} = D_xx^2 = 2x$

16. $D_xe^{\frac{x}{\ln x}} = e^{\frac{x}{\ln x}}D_x\dfrac{x}{\ln x} = e^{\frac{x}{\ln x}}\cdot\dfrac{(\ln x)\cdot 1 - x\cdot\frac{1}{x}}{(\ln x)^2}$
$= \dfrac{e^{\frac{x}{\ln x}}(\ln x - 1)}{(\ln x)^2}$

17. $D_x(x^3e^x) = x^3D_xe^x + e^xD_x(x^3)$
$= x^3e^x + e^x\cdot 3x^2 = x^2e^x(x+3)$

18. $D_xe^{x^3\ln x} = e^{x^3\ln x}D_x(x^3\ln x)$
$= e^{x^3\ln x}\left(x^3\cdot\dfrac{1}{x} + \ln x\cdot 3x^2\right)$
$= e^{x^3\ln x}(x^2 + 3x^2\ln x)$
$= x^2e^{x^3\ln x}(1+3\ln x)$

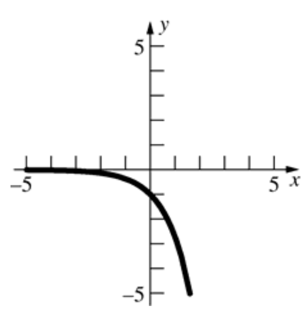
19. $D_x[\sqrt{e^{x^2}} + e^{\sqrt{x^2}}] = D_x(e^{x^2})^{1/2} + D_xe^{\sqrt{x^2}}$
$= \dfrac{1}{2}(e^{x^2})^{-1/2}D_xe^{x^2} + e^{\sqrt{x^2}}D_x\sqrt{x^2}$
$= \dfrac{1}{2}(e^{x^2})^{-1/2}e^{x^2}D_xx^2 + e^{\sqrt{x^2}}\cdot\dfrac{x}{\sqrt{x^2}}$
$= \dfrac{1}{2}(e^{x^2})^{1/2}2x + e^{\sqrt{x^2}}\cdot\dfrac{x}{|x|}$
$= x\sqrt{e^{x^2}} + \dfrac{xe^{\sqrt{x^2}}}{|x|}$

20. $D_x\left[e^{1/x^2} + \dfrac{1}{e^{x^2}}\right] = D_xe^{x^{-2}} + D_xe^{-x^2}$
$= e^{x^{-2}}D_xx^{-2} + e^{-x^2}D_x[-x^2]$
$= e^{x^{-2}}\cdot(-2x^{-3}) + e^{-x^2}\cdot(-2x)$
$= -\dfrac{2e^{1/x^2}}{x^3} - \dfrac{2x}{e^{x^2}}$

21. $D_x[e^{xy} + xy] = D_x[2]$
$e^{xy}(xD_xy + y) + (xD_xy + y) = 0$
$xe^{xy}D_xy + ye^{xy} + xD_xy + y = 0$
$xe^{xy}D_xy + xD_xy = -ye^{xy} - y$
$D_xy = \dfrac{-ye^{xy} - y}{xe^{xy} + x} = -\dfrac{y(e^{xy}+1)}{x(e^{xy}+1)} = -\dfrac{y}{x}$

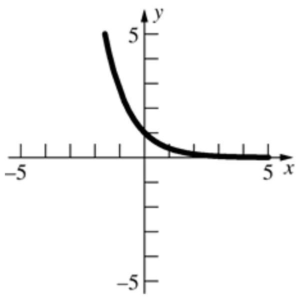
22. $D_x[e^{x+y}] = D_x[4+x+y]$
$e^{x+y}(1+D_xy) = 1 + D_xy$
$e^{x+y} + e^{x+y}D_xy = 1 + D_xy$
$e^{x+y}D_xy - D_xy = 1 - e^{x+y}$
$D_xy = \dfrac{1-e^{x+y}}{e^{x+y}-1} = -1$

23. a.



The graph of $y = e^x$ is reflected across the x-axis.

b.



The graph of $y = e^x$ is reflected across the y-axis.

24. $a < b \Rightarrow -a > -b \Rightarrow e^{-a} > e^{-b}$, since e^x is an increasing function.

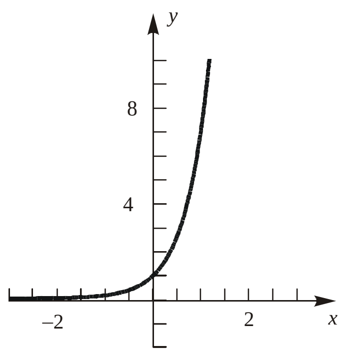
25. $f(x) = e^{2x}$ Domain = $(-\infty, \infty)$

$f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$

Since $f'(x) > 0$ for all x, f is increasing on $(-\infty, \infty)$.

Since $f''(x) > 0$ for all x, f is concave upward on $(-\infty, \infty)$.

Since f and f' are both monotonic, there are no extreme values or points of inflection.



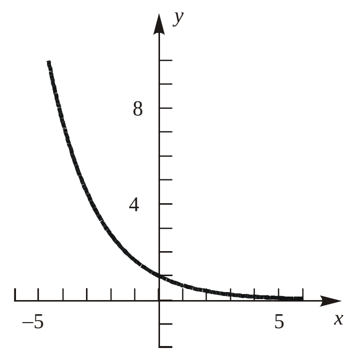
26. $f(x) = e^{-x/2}$ Domain = $(-\infty, \infty)$

$f'(x) = -\frac{1}{2}e^{-x/2}$, $f''(x) = \frac{1}{4}e^{-x/2}$

Since $f'(x) < 0$ for all x, f is decreasing on $(-\infty, \infty)$.

Since $f''(x) > 0$ for all x, f is concave upward on $(-\infty, \infty)$.

Since f and f' are both monotonic, there are no extreme values or points of inflection.



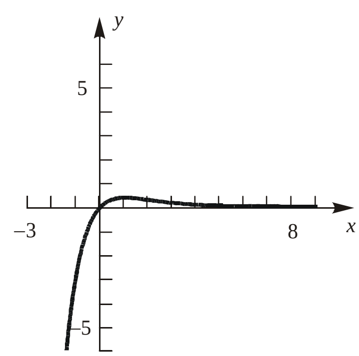
27. $f(x) = xe^{-x}$ Domain = $(-\infty, \infty)$

$f'(x) = (1-x)e^{-x}$, $f''(x) = (x-2)e^{-x}$

x	$(-\infty,1)$	1	$(1,2)$	2	$(2,\infty)$
f'	+	0	–	–	–
f''	–	–	–	0	+

f is increasing on $(-\infty, 1]$ and decreasing on $[1, \infty)$. f has a maximum at $(1, 1/e)$

f is concave up on $(2, \infty)$ and concave down on $(-\infty, 2)$. f has a point of inflection at $(2, 2/e^2)$



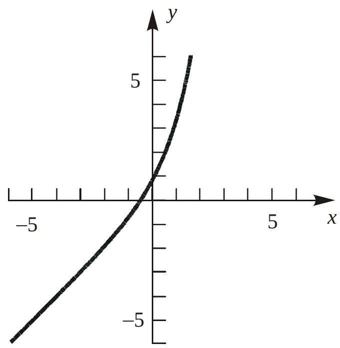
28. $f(x) = e^x + x$ Domain = $(-\infty, \infty)$

$f'(x) = e^x + 1$, $f''(x) = e^x$

Since $f'(x) > 0$ for all x, f is increasing on $(-\infty, \infty)$.

Since $f''(x) > 0$ for all x, f is concave upward on $(-\infty, \infty)$.

Since f and f' are both monotonic, there are no extreme values or points of inflection.



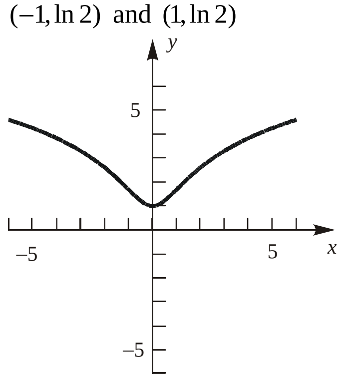
29. $f(x) = \ln(x^2 + 1)$ Since $x^2 + 1 > 0$ for all x, domain = $(-\infty, \infty)$

$f'(x) = \frac{2x}{x^2 + 1}$, $f''(x) = \frac{-2(x^2 - 1)}{(x^2 + 1)^2}$

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
f'	–	–	–	0	+	+	+
f''	–	0	+	+	+	0	–

f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. f has a minimum at $(0, 0)$

f is concave up on $(-1, 1)$ and concave down on $(-\infty, -1) \cup (1, \infty)$. f has points of inflection at $(-1, \ln 2)$ and $(1, \ln 2)$



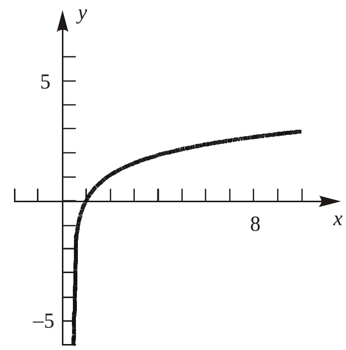
30. $f(x) = \ln(2x - 1)$. Since $2x - 1 > 0$ if and only if $x > \frac{1}{2}$, domain = $(\frac{1}{2}, \infty)$

$f'(x) = \frac{2}{2x - 1}$, $f''(x) = \frac{-4}{(2x - 1)^2}$

Since $f'(x) > 0$ for all domain values, f is increasing on $(\frac{1}{2}, \infty)$.

Since $f''(x) < 0$ for all domain values, f is concave downward on $(\frac{1}{2}, \infty)$.

Since f and f' are both monotonic, there are no extreme values or points of inflection.



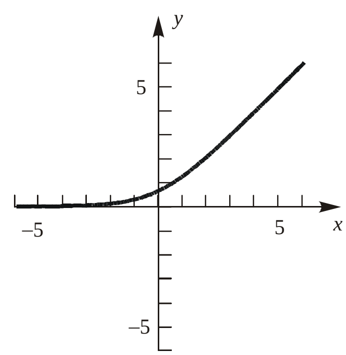
31. $f(x) = \ln(1 + e^x)$ Since $1 + e^x > 0$ for all x, domain = $(-\infty, \infty)$

$f'(x) = \frac{e^x}{1 + e^x}$, $f''(x) = \frac{e^x}{(1 + e^x)^2}$

Since $f'(x) > 0$ for all x, f is increasing on $(-\infty, \infty)$.

Since $f''(x) > 0$ for all x, f is concave upward on $(-\infty, \infty)$.

Since f and f' are both monotonic, there are no extreme values or points of inflection.



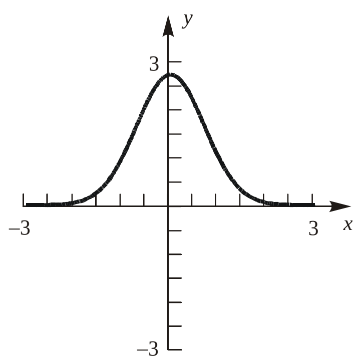
32. $f(x) = e^{1-x^2}$ Domain = $(-\infty, \infty)$

$f'(x) = -2xe^{1-x^2}$, $f''(x) = (4x^2 - 2)e^{1-x^2}$

x	$(-\infty, -\frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}$	$(-\frac{\sqrt{2}}{2}, 0)$	0	$(0, \frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}$	$(\frac{\sqrt{2}}{2}, \infty)$
f'	+	+	+	0	–	–	–
f''	+	0	–	–	–	0	+

f is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$. f has a maximum at $(0, e)$

f is concave up on $(-\infty, -\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, \infty)$ and concave down on $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. f has points of inflection at $(-\frac{\sqrt{2}}{2}, \sqrt{e})$ and $(\frac{\sqrt{2}}{2}, \sqrt{e})$



33. $f(x) = e^{-(x-2)^2}$ Domain = $(-\infty, \infty)$

$f'(x) = (4 - 2x)e^{-(x-2)^2}$,

$f''(x) = (4x^2 - 16x + 14)e^{-(x-2)^2}$

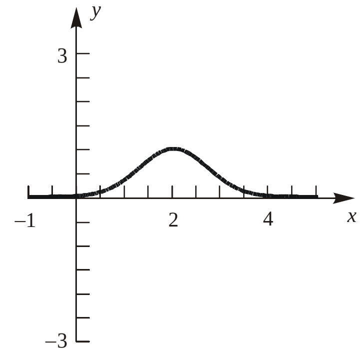
Note that $4x^2 - 16x + 14 = 0$ when

$$x = \frac{4 \pm \sqrt{2}}{2} \approx 2 \pm 0.707$$

x	$(-\infty, 1.293)$	≈ 1.293	$(1.293, 2)$	2	$(2, 2.707)$	≈ 2.707	$(2.707, \infty)$
f'	+	+	+	0	–	–	–
f''	+	0	–	–	–	0	+

f is increasing on $(-\infty, 2]$ and decreasing on $[2, \infty)$. f has a maximum at $(2, 1)$

f is concave up on $(-\infty, \frac{4-\sqrt{2}}{2}) \cup (\frac{4+\sqrt{2}}{2}, \infty)$ and concave down on $(\frac{4-\sqrt{2}}{2}, \frac{4+\sqrt{2}}{2})$. f has points of inflection at $(\frac{4-\sqrt{2}}{2}, \frac{1}{\sqrt{e}})$ and $(\frac{4+\sqrt{2}}{2}, \frac{1}{\sqrt{e}})$.

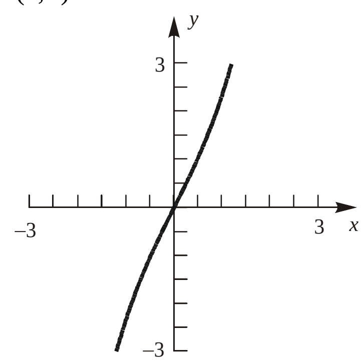


34. $f(x) = e^x - e^{-x}$ Domain = $(-\infty, \infty)$

$f'(x) = e^x + e^{-x}$, $f''(x) = e^x - e^{-x}$

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	+	+	+
f''	–	0	+

f is increasing on $(-\infty, \infty)$ and so has no extreme values. *f* is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. f has a point of inflection at $(0, 0)$

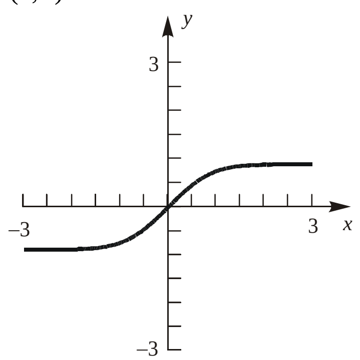


35. $f(x)=\int_0^x e^{-t^2}dt$ Domain $=(-\infty,\infty)$

$f'(x)=e^{-x^2}$, $f''(x)=-2xe^{-x^2}$

x	$(-\infty,0)$	0	$(0,\infty)$
f'	+	+	+
f''	+	0	−

f is increasing on $(-\infty,\infty)$ and so has no extreme values. f is concave up on $(-\infty,0)$ and concave down on $(0,\infty)$. f has a point of inflection at $(0,0)$



36. $f(x)=\int_0^x te^{-t}dt$ Domain $=(-\infty,\infty)$

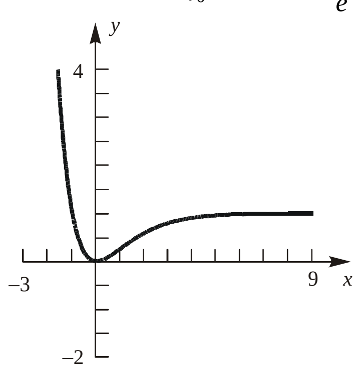
$f'(x)=xe^{-x}$, $f''(x)=(1-x)e^{-x}$

x	$(-\infty,0)$	0	$(0,1)$	1	$(1,\infty)$
f'	−	0	+	+	+
f''	+	+	+	0	−

f is increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$. f has a minimum at $(0,0)$

f is concave up on $(-\infty,1)$ and concave down on $(1,\infty)$. f has a point of inflection at $(1,\int_0^1 te^{-t}dt)$.

Note: It can be shown with techniques in Chapter 7 that $\int_0^1 te^{-t}dt=1-\frac{2}{e}\approx 0.264$



37. Let $u = 3x + 1$, so $du = 3dx$.

$$\int e^{3x+1}dx=\frac{1}{3}\int e^{3x+1}3dx=\frac{1}{3}\int e^u du=\frac{1}{3}e^u+C$$
$$=\frac{1}{3}e^{3x+1}+C$$

38. Let $u=x^2-3$, so $du = 2x\,dx$.

$$\int xe^{x^2-3}dx=\frac{1}{2}\int e^{x^2-3}2x\,dx=\frac{1}{2}\int e^u du$$
$$=\frac{1}{2}e^u+C=\frac{1}{2}e^{x^2-3}+C$$

39. Let $u=x^2+6x$, so $du = (2x + 6)dx$.

$$\int (x+3)e^{x^2+6x}dx=\frac{1}{2}\int e^u du=\frac{1}{2}e^u+C$$
$$=\frac{1}{2}e^{x^2+6x}+C$$

40. Let $u=e^x-1$, so $du=e^x dx$.

$$\int \frac{e^x}{e^x-1}dx=\int \frac{1}{u}du=\ln|u|+C=\ln\left|e^x-1\right|+C$$

41. Let $u=-\frac{1}{x}$, so $du=\frac{1}{x^2}dx$.

$$\int \frac{e^{-1/x}}{x^2}dx=\int e^u du=e^u+C=e^{-1/x}+C$$

42. $\int e^{x+e^x}dx=\int e^x\cdot e^{e^x}dx$

Let $u=e^x$, so $du=e^x dx$.

$$\int e^x\cdot e^{e^x}dx=\int e^u du=e^u+C=e^{e^x}+C$$

43. Let $u = 2x + 3$, so $du = 2dx$

$$\int e^{2x+3}dx=\frac{1}{2}\int e^u du=\frac{1}{2}e^u+C=\frac{1}{2}e^{2x+3}+C$$
$$\int_0^1 e^{2x+3}dx=\left[\frac{1}{2}e^{2x+3}\right]_0^1=\frac{1}{2}e^5-\frac{1}{2}e^3$$
$$=\frac{1}{2}e^3(e^2-1)\approx 64.2$$

44. Let $u=\frac{3}{x}$, so $du=-\frac{3}{x^2}dx$.

$$\int \frac{e^{3/x}}{x^2}dx=-\frac{1}{3}\int e^u du=-\frac{1}{3}e^u+C$$
$$=-\frac{1}{3}e^{3/x}+C$$
$$\int_1^2 \frac{e^{3/x}}{x^2}dx=\left[-\frac{1}{3}e^{3/x}\right]_1^2=-\frac{1}{3}e^{3/2}+\frac{1}{3}e^3\approx 5.2$$

45. $V=\pi\int_0^{\ln 3}(e^x)^2\,dx=\pi\int_0^{\ln 3}e^{2x}dx$

$=\pi\left[\frac{1}{2}e^{2x}\right]_0^{\ln 3}=\pi\left(\frac{1}{2}e^{2\ln 3}-\frac{1}{2}e^0\right)=4\pi\approx 12.57$

46. $V=\int_0^1 2\pi xe^{-x^2}\,dx$.

Let $u=-x^2$, so $du = -2x\,dx$.

$\int 2\pi xe^{-x^2}dx=-\pi\int e^{-x^2}(-2x)dx=-\pi\int e^u du$

$=-\pi e^u+C=-\pi e^{-x^2}+C$

$\int_0^1 2\pi xe^{-x^2}dx=-\pi\left[e^{-x^2}\right]_0^1=-\pi(e^{-1}-e^0)$

$=\pi(1-e^{-1})\ \approx 1.99$

47. The line through (0, 1) and $\left(1,\frac{1}{e}\right)$ has slope

$\frac{\frac{1}{e}-1}{1-0}=\frac{1}{e}-1=\frac{1-e}{e}\Rightarrow y-1=\frac{1-e}{e}(x-0);$

$y=\frac{1-e}{e}x+1$

$\int_0^1\left[\left(\frac{1-e}{e}x+1\right)-e^{-x}\right]dx=\left[\frac{1-e}{2e}x^2+x+e^{-x}\right]_0^1$

$=\frac{1-e}{2e}+1+\frac{1}{e}-1=\frac{3-e}{2e}\approx 0.052$

48. $f'(x)=\frac{(e^x-1)(1)-x(e^x)}{(e^x-1)^2}-\frac{1}{1-e^{-x}}(-e^{-x})(-1)$

$=\frac{e^x-1-xe^x}{(e^x-1)^2}-\frac{1}{1-e^{-x}}\left(\frac{1}{e^x}\right)$

$=\frac{e^x-1-xe^x}{(e^x-1)^2}-\frac{1}{e^x-1}=\frac{e^x-1-xe^x-(e^x-1)}{(e^x-1)^2}$

$=-\frac{xe^x}{(e^x-1)^2}$

When $x > 0$, $f'(x) < 0$, so $f(x)$ is decreasing for $x > 0$.

49. **a.** Exact:

$10!=10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

$=3,628,800$

Approximate:

$10!\approx\sqrt{20\pi}\left(\frac{10}{e}\right)^{10}\approx 3,598,696$

b. $60!\approx\sqrt{120\pi}\left(\frac{60}{e}\right)^{60}\approx 8.31\times 10^{81}$

50. $e^{0.3}\approx\left\{\left[\left(\frac{0.3}{4}+1\right)\frac{0.3}{3}+1\right]\frac{0.3}{2}+1\right\}(0.3)+1$

$=1.3498375$

$e^{0.3}\approx 1.3498588$ by direct calculation

51. $x=e^t\sin t$, so $dx=(e^t\sin t+e^t\cos t)dt$

$y=e^t\cos t$, so $dy=(e^t\cos t-e^t\sin t)dt$

$ds=\sqrt{dx^2+dy^2}$

$=e^t\sqrt{(\sin t+\cos t)^2+(\cos t-\sin t)^2}\,dt$

$=e^t\sqrt{2\sin^2 t+2\cos^2 t}\,dt=\sqrt{2}e^t\,dt$

The length of the curve is

$\int_0^{\pi}\sqrt{2}e^t\,dt=\sqrt{2}\left[e^t\right]_0^{\pi}=\sqrt{2}(e^{\pi}-1)\approx 31.312$

52. Use $x = 30$, $n = 8$, and $k = 0.25$.

$P_n(x)=\frac{(kx)^n e^{-kx}}{n!}=\frac{(0.25\cdot 30)^8 e^{-0.25\cdot 30}}{8!}\approx 0.14$

53. **a.** $\lim_{x\to 0^+}\frac{\ln x}{1+(\ln x)^2}$ is of the form $\frac{\infty}{\infty}$.

$=\lim_{x\to 0^+}\frac{D_x\ln x}{D_x[1+(\ln x)^2]}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{2\ln x\cdot\frac{1}{x}}$

$=\lim_{x\to 0^+}\frac{1}{2\ln x}=0$

$\lim_{x\to\infty}\frac{\ln x}{1+(\ln x)^2}=\lim_{x\to\infty}\frac{1}{2\ln x}=0$

b. $f'(x)=\frac{[1+(\ln x)^2]\cdot\frac{1}{x}-\ln x\cdot 2\ln x\cdot\frac{1}{x}}{[1+(\ln x)^2]^2}$

$=\frac{1-(\ln x)^2}{x[1+(\ln x)^2]^2}$

$f'(x)=0$ when $\ln x=\pm 1$ so $x=e^1=e$

or $x=e^{-1}=\frac{1}{e}$

$f(e)=\frac{\ln e}{1+(\ln e)^2}=\frac{1}{1+1^2}=\frac{1}{2}$

$f\left(\frac{1}{e}\right)=\frac{\ln\frac{1}{e}}{1+\left(\ln\frac{1}{e}\right)^2}=\frac{-1}{1+(-1)^2}=-\frac{1}{2}$

Maximum value of $\frac{1}{2}$ at $x = e$; minimum value of $-\frac{1}{2}$ at $x=e^{-1}$.

c. $F(x) = \int_1^{x^2} \frac{\ln t}{1+(\ln t)^2}\,dt$

$F'(x) = \frac{\ln x^2}{1+(\ln x^2)^2} \cdot 2x$

$F'(\sqrt{e}) = \frac{\ln(\sqrt{e})^2}{1+[\ln(\sqrt{e})^2]^2} \cdot 2\sqrt{e} = \frac{1}{1+1^2} \cdot 2\sqrt{e}$

$= \sqrt{e} \approx 1.65$

54. Let (x_0, e^{x_0}) be the point of tangency. Then

$\frac{e^{x_0} - 0}{x_0 - 0} = f'(x_0) = e^{x_0} \Rightarrow e^{x_0} = x_0 e^{x_0} \Rightarrow x_0 = 1$

so the line is $y = e^{x_0}x$ or $y = ex$.

a. $A = \int_0^1 (e^x - ex)dx = \left[e^x - \frac{ex^2}{2}\right]_0^1$

$= e - \frac{e}{2} - (e^0 - 0) = \frac{e}{2} - 1 \approx 0.36$

b. $V = \pi\int_0^1 [(e^x)^2 - (ex)^2]dx$

$= \pi\int_0^1 (e^{2x} - e^2x^2)dx = \pi\left[\frac{1}{2}e^{2x} - \frac{e^2x^3}{3}\right]_0^1$

$= \pi\left[\frac{1}{2}e^2 - \frac{e^2}{3} - \left(\frac{1}{2}e^0\right)\right] = \frac{\pi}{6}(e^2 - 3) \approx 2.30$

55. a. $\int_{-3}^{3} \exp\left(-\frac{1}{x^2}\right)dx = 2\int_0^3 \exp\left(-\frac{1}{x^2}\right)dx \approx 3.11$

b. $\int_0^{8\pi} e^{-0.1x}\sin x\,dx \approx 0.910$

56. a. $\lim_{x\to 0}(1+x)^{1/x} = e \approx 2.72$

b. $\lim_{x\to 0}(1+x)^{-1/x} = \frac{1}{e} \approx 0.368$

57. $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2e^{-x^2}(2x^2 - 1)$

$y = f(x)$ and $y = f''(x)$ intersect when

$e^{-x^2} = 2e^{-x^2}(2x^2 - 1); 1 = 4x^2 - 2;$

$4x^2 - 3 = 0, x = \pm\frac{\sqrt{3}}{2}$

Both graphs are symmetric with respect to the y-axis so the area is

$$2\left\{\int_0^{\frac{\sqrt{3}}{2}} [e^{-x^2} - 2e^{x^2}(2x^2 - 1)]dx + \int_{\frac{\sqrt{3}}{2}}^{3} [2e^{-x^2}(2x^2 - 1) - e^{-x^2}]dx\right\}$$

≈ 4.2614

58. a. $\lim_{x\to\infty} x^p e^{-x} = 0$

b. $f'(x) = x^p e^{-x}(-1) + e^{-x} \cdot px^{p-1}$

$= x^{p-1}e^{-x}(p - x)$

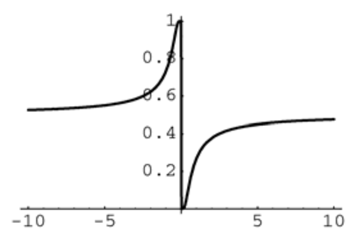
$f'(x) = 0$ when $x = p$

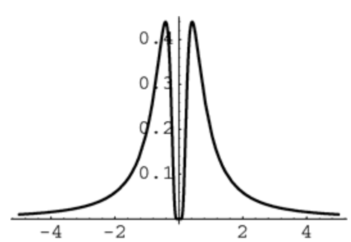
59. $\lim_{x\to-\infty} \ln(x^2 + e^{-x}) = \infty$ (behaves like $-x$)

$\lim_{x\to\infty} \ln(x^2 + e^{-x}) = \infty$ (behaves like $2\ln x$)

60. $f'(x) = -(1 + e^{x^{-1}})^{-2} \cdot e^{x^{-1}}(-x^{-2})$

$= \frac{e^{1/x}}{x^2(1 + e^{1/x})^2}$





a. $\lim_{x\to 0^+} f(x) = 0$

b. $\lim_{x\to 0^-} f(x) = 1$

c. $\lim_{x\to\pm\infty} f(x) = \frac{1}{2}$

d. $\lim_{x\to 0} f'(x) = 0$

e. f has no minimum or maximum values.

6.4 Concepts Review

1. $e^{\sqrt{3}\ln\pi}$; $e^{x\ln a}$

2. e

3. $\frac{\ln x}{\ln a}$

4. ax^{a-1}; $a^x \ln a$

Problem Set 6.4

1. $2^x = 8 = 2^3$; $x = 3$

2. $x = 5^2 = 25$

3. $x = 4^{3/2} = 8$

4. $x^4 = 64$

$x = \sqrt[4]{64} = 2\sqrt{2}$

5. $\log_9\left(\frac{x}{3}\right) = \frac{1}{2}$

$\frac{x}{3} = 9^{1/2} = 3$

$x = 9$

6. $4^3 = \frac{1}{2x}$

$x = \frac{1}{2\cdot 4^3} = \frac{1}{128}$

7. $\log_2(x+3) - \log_2 x = 2$

$\log_2 \frac{x+3}{x} = 2$

$\frac{x+3}{x} = 2^2 = 4$

$x + 3 = 4x$

$x = 1$

8. $\log_5(x+3) - \log_5 x = 1$

$\log_5 \frac{x+3}{x} = 1$

$\frac{x+3}{x} = 5^1 = 5$

$x + 3 = 5x$

$x = \frac{3}{4}$

9. $\log_5 12 = \frac{\ln 12}{\ln 5} \approx 1.544$

10. $\log_7 0.11 = \frac{\ln 0.11}{\ln 7} \approx -1.1343$

11. $\log_{11}(8.12)^{1/5} = \frac{1}{5}\frac{\ln 8.12}{\ln 11} \approx 0.1747$

12. $\log_{10}(8.57)^7 = 7\frac{\ln 8.57}{\ln 10} \approx 6.5309$

13. $x \ln 2 = \ln 17$

$x = \frac{\ln 17}{\ln 2} \approx 4.08746$

14. $x \ln 5 = \ln 13$

$x = \frac{\ln 13}{\ln 5} \approx 1.5937$

15. $(2s - 3) \ln 5 = \ln 4$

$2s - 3 = \frac{\ln 4}{\ln 5}$

$s = \frac{1}{2}\left(3 + \frac{\ln 4}{\ln 5}\right) \approx 1.9307$

16. $\frac{1}{\theta - 1}\ln 12 = \ln 4$

$\frac{\ln 12}{\ln 4} = \theta - 1$

$\theta = 1 + \frac{\ln 12}{\ln 4} \approx 2.7925$

17. $D_x(6^{2x}) = 6^{2x} \ln 6 \cdot D_x(2x) = 2\cdot 6^{2x} \ln 6$

18. $D_x(3^{2x^2-3x}) = 3^{2x^2-3x} \ln 3 \cdot D_x(2x^2 - 3x)$

$= (4x-3)\cdot 3^{2x^2-3x} \ln 3$

19. $D_x \log_3 e^x = \frac{1}{e^x \ln 3}\cdot D_x e^x$

$= \frac{e^x}{e^x \ln 3} = \frac{1}{\ln 3} \approx 0.9102$

Alternate method:

$D_x \log_3 e^x = D_x(x \log_3 e) = \log_3 e$

$= \frac{\ln e}{\ln 3} = \frac{1}{\ln 3} \approx 0.9102$

20. $D_x \log_{10}(x^3+9) = \dfrac{1}{(x^3+9)\ln 10} \cdot D_x(x^3+9)$

$= \dfrac{3x^2}{(x^3+9)\ln 10}$

21. $D_z[3^z \ln(z+5)]$

$= 3^z \cdot \dfrac{1}{z+5}(1) + \ln(z+5) \cdot 3^z \ln 3$

$= 3^z \left[\dfrac{1}{z+5} + \ln(z+5)\ln 3\right]$

22. $D_\theta \sqrt{\log_{10}(3^{\theta^2-\theta})} = D_\theta \sqrt{(\theta^2-\theta)\log_{10} 3}$

$= D_\theta \sqrt{\dfrac{(\theta^2-\theta)\ln 3}{\ln 10}} = \sqrt{\dfrac{\ln 3}{\ln 10}} \cdot D_\theta \sqrt{\theta^2-\theta}$

$= \sqrt{\dfrac{\ln 3}{\ln 10}} \cdot \dfrac{1}{2}(\theta^2-\theta)^{-1/2}(2\theta-1)$

$= \dfrac{2\theta-1}{2\sqrt{\theta^2-\theta}}\sqrt{\dfrac{\ln 3}{\ln 10}}$

23. Let $u = x^2$ so $du = 2x dx$.

$\int x \cdot 2^{x^2} dx = \dfrac{1}{2}\int 2^u du = \dfrac{1}{2} \cdot \dfrac{2^u}{\ln 2} + C$

$= \dfrac{2^{x^2}}{2\ln 2} + C = \dfrac{2^{x^2-1}}{\ln 2} + C$

24. Let $u = 5x - 1$, so $du = 5\,dx$.

$\int 10^{5x-1} dx = \dfrac{1}{5}\int 10^u du = \dfrac{1}{5} \cdot \dfrac{10^u}{\ln 10} + C$

$= \dfrac{10^{5x-1}}{5\ln 10} + C$

25. Let $u = \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}dx$.

$\int \dfrac{5^{\sqrt{x}}}{\sqrt{x}} dx = 2\int 5^u du = 2 \cdot \dfrac{5^u}{\ln 5} + C$

$= \dfrac{2 \cdot 5^{\sqrt{x}}}{\ln 5} + C$

$\int_1^4 \dfrac{5^{\sqrt{x}}}{\sqrt{x}} dx = 2\left[\dfrac{5^{\sqrt{x}}}{\ln 5}\right]_1^4 = 2\left(\dfrac{25}{\ln 5} - \dfrac{5}{\ln 5}\right)$

$= \dfrac{40}{\ln 5} \approx 24.85$

26. $\int_0^1 (10^{3x} + 10^{-3x})dx = \int_0^1 10^{3x} dx + \int_0^1 10^{-3x} dx$

Let $u = 3x$, so $du = 3dx$.

$\int 10^{3x} dx = \dfrac{1}{3}\int 10^u du = \dfrac{1}{3} \cdot \dfrac{10^u}{\ln 10} + C$

$= \dfrac{10^{3x}}{3\ln 10} + C$

Now let $u = -3x$, so $du = -3dx$.

$\int 10^{-3x} dx = -\dfrac{1}{3}\int 10^u du = -\dfrac{1}{3} \cdot \dfrac{10^u}{\ln 10} + C$

$= -\dfrac{10^{-3x}}{3\ln 10} + C$

Thus, $\int_0^1 (10^{3x} + 10^{-3x})dx = \left[\dfrac{10^{3x} - 10^{-3x}}{3\ln 10}\right]_0^1$

$= \dfrac{1}{3\ln 10}\left(1000 - \dfrac{1}{1000}\right) = \dfrac{999{,}999}{3000\ln 10}$

≈ 144.76

27. $\dfrac{d}{dx}10^{(x^2)} = 10^{(x^2)} \ln 10 \dfrac{d}{dx}x^2 = 10^{(x^2)} 2x \ln 10$

$\dfrac{d}{dx}(x^2)^{10} = \dfrac{d}{dx}x^{20} = 20x^{19}$

$\dfrac{dy}{dx} = \dfrac{d}{dx}[10^{(x^2)} + (x^2)^{10}]$

$= 10^{(x^2)} 2x \ln 10 + 20x^{19}$

28. $\dfrac{d}{dx}\sin^2 x = 2\sin x \dfrac{d}{dx}\sin x = 2\sin x \cos x$

$\dfrac{d}{dx}2^{\sin x} = 2^{\sin x} \ln 2 \dfrac{d}{dx}\sin x = 2^{\sin x} \ln 2 \cos x$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^2 x + 2^{\sin x})$

$= 2\sin x \cos x + 2^{\sin x} \cos x \ln 2$

29. $\dfrac{d}{dx}x^{\pi+1} = (\pi+1)x^\pi$

$\dfrac{d}{dx}(\pi+1)^x = (\pi+1)^x \ln(\pi+1)$

$\dfrac{dy}{dx} = \dfrac{d}{dx}[x^{\pi+1} + (\pi+1)^x]$

$= (\pi+1)x^\pi + (\pi+1)^x \ln(\pi+1)$

30. $\frac{d}{dx}2^{(e^x)} = 2^{(e^x)}\ln 2\frac{d}{dx}e^x = 2^{(e^x)}e^x\ln 2$

$\frac{d}{dx}(2^e)^x = (2^e)^x\ln 2^e = (2^e)^x e\ln 2$

$\frac{dy}{dx} = \frac{d}{dx}[2^{(e^x)} + (2^e)^x]$

$= 2^{(e^x)}e^x\ln 2 + (2^e)^x e\ln 2$

31. $y = (x^2+1)^{\ln x} = e^{(\ln x)\ln(x^2+1)}$

$\frac{dy}{dx} = e^{(\ln x)\ln(x^2+1)}\frac{d}{dx}[(\ln x)\ln(x^2+1)]$

$= e^{(\ln x)\ln(x^2+1)}\left[\frac{1}{x}\ln(x^2+1) + \ln x\frac{2x}{x^2+1}\right]$

$= (x^2+1)^{\ln x}\left(\frac{\ln(x^2+1)}{x} + \frac{2x\ln x}{x^2+1}\right)$

32. $y = (\ln x^2)^{2x+3} = e^{(2x+3)\ln(\ln x^2)}$

$\frac{dy}{dx} = e^{(2x+3)\ln(\ln x^2)}\frac{d}{dx}[(2x+3)\ln(\ln x^2)]$

$= e^{(2x+3)\ln(\ln x^2)}\left[2\ln(\ln x^2) + (2x+3)\frac{1}{\ln x^2}\frac{1}{x^2}(2x)\right]$

$= \underbrace{(2\ln x)}_{\ln x^2}{}^{2x+3}\left[2\ln\underbrace{(2\ln x)}_{\ln x^2} + \frac{2x+3}{x\ln x}\right]$

33. $f(x) = x^{\sin x} = e^{\sin x\ln x}$

$f'(x) = e^{\sin x\ln x}\frac{d}{dx}(\sin x\ln x)$

$= e^{\sin x\ln x}\left[(\sin x)\left(\frac{1}{x}\right) + (\cos x)(\ln x)\right]$

$= x^{\sin x}\left(\frac{\sin x}{x} + \cos x\ln x\right)$

$f'(1) = 1^{\sin 1}\left(\frac{\sin 1}{1} + \cos 1\ln 1\right) = \sin 1 \approx 0.8415$

34. $f(e) = \pi^e \approx 22.46$

$g(e) = e^\pi \approx 23.14$

$g(e)$ is larger than $f(e)$.

$f'(x) = \frac{d}{dx}\pi^x = \pi^x\ln\pi$

$f'(e) = \pi^e\ln\pi \approx 25.71$

$g'(x) = \frac{d}{dx}x^\pi = \pi x^{\pi-1}$

$g'(e) = \pi e^{\pi-1} \approx 26.74$

$g'(e)$ is larger than $f'(e)$.

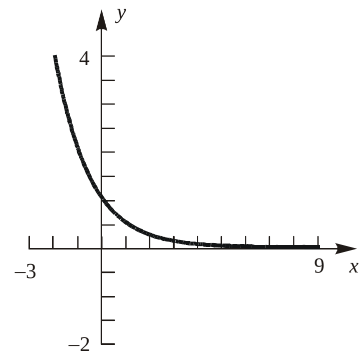
35. $f(x) = 2^{-x} = e^{(\ln 2)(-x)}$ Domain = $(-\infty,\infty)$

$f'(x) = (-\ln 2)2^{-x}$, $f''(x) = (\ln 2)^2 2^{-x}$

Since $f'(x) < 0$ for all x, f is decreasing on $(-\infty,\infty)$.

Since $f''(x) > 0$ for all x, f is concave upward on $(-\infty,\infty)$.

Since f and f' are both monotonic, there are no extreme values or points of inflection.



36. $f(x) = x2^{-x}$ Domain = $(-\infty,\infty)$

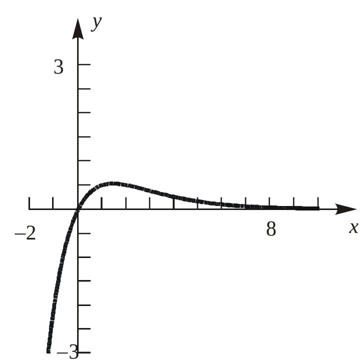
$f'(x) = [1-(\ln 2)x]2^{-x}$,

$f''(x) = (\ln 2)[(\ln 2)x - 2]2^{-x}$

x	$(-\infty,\frac{1}{\ln 2})$	$\frac{1}{\ln 2}$	$(\frac{1}{\ln 2},\frac{2}{\ln 2})$	$\frac{2}{\ln 2}$	$(\frac{2}{\ln 2},\infty)$
f'	+	0	−	−	−
f''	−	−	−	0	+

f is increasing on $\left(-\infty,\frac{1}{\ln 2}\right]$ and decreasing on $\left[\frac{1}{\ln 2},\infty\right)$. f has a maximum at $\left(\frac{1}{\ln 2}, 1/(e\ln 2)\right)$

f is concave up on $(\frac{2}{\ln 2},\infty)$ and concave down on $(-\infty,\frac{2}{\ln 2})$. f has a point of inflection at $\left(\frac{2}{\ln 2}, 2/(e^2\ln 2)\right)$



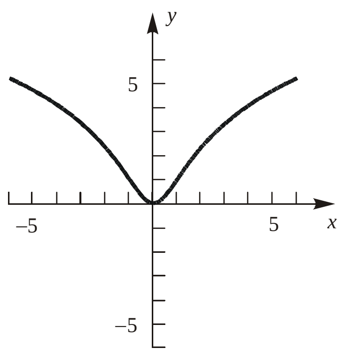
37. $f(x)=\log_2(x^2+1)=\dfrac{\ln(x^2+1)}{\ln 2}$. Since $x^2+1>0$ for all x, domain $=(-\infty,\infty)$

$$f'(x)=\left(\frac{2}{\ln 2}\right)\left(\frac{x}{x^2+1}\right),\quad f''(x)=\left(\frac{2}{\ln 2}\right)\left(\frac{1-x^2}{(x^2+1)^2}\right)$$

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
f'	$-$	$-$	$-$	0	$+$	$+$	$+$
f''	$-$	0	$+$	$+$	$+$	0	$-$

f is increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$. f has a minimum at $(0,0)$

f is concave up on $(-1,1)$ and concave down on $(-\infty,-1)\cup(1,\infty)$. f has points of inflection at $(-1,1)$ and $(1,1)$

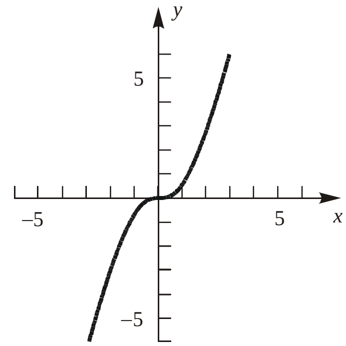


38. $f(x)=x\log_3(x^2+1)=\dfrac{x\ln(x^2+1)}{\ln 3}$. Since $x^2+1>0$ for all x, domain $=(-\infty,\infty)$

$$f'(x)=\frac{1}{\ln 3}\left[\frac{2x^2}{x^2+1}+\ln(x^2+1)\right],\quad f''(x)=\frac{2}{\ln 3}\left[\frac{x^3+3x}{x^2+1}\right]$$

x	$(-\infty,0)$	0	$(0,\infty)$
f'	$+$	0	$+$
f''	$-$	0	$+$

f is increasing on $(-\infty,\infty)$ and so has no extreme values. f is concave up on $(0,\infty)$ and concave down on $(-\infty,0)$. f has a point of inflection at $(0,0)$



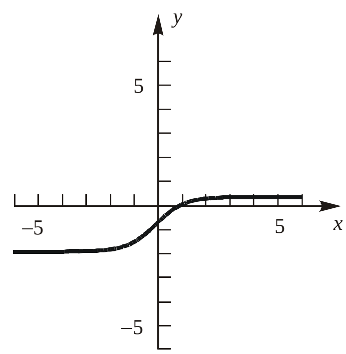
39. $f(x)=\int_1^x 2^{-t^2}\,dt$ Domain $=(-\infty,\infty)$

$f'(x)=2^{-x^2}$, $f''(x)=-2(\ln 2)x2^{-x^2}$

x	$(-\infty,0)$	0	$(0,\infty)$
f'	$+$	$+$	$+$
f''	$+$	0	$-$

f is increasing on $(-\infty,\infty)$ and so has no extreme values.

f is concave up on $(-\infty,0)$ and concave down on $(0,\infty)$. f has a point of inflection at $(0,\int_1^0 2^{-t^2}\,dt)\approx(0,-0.81)$



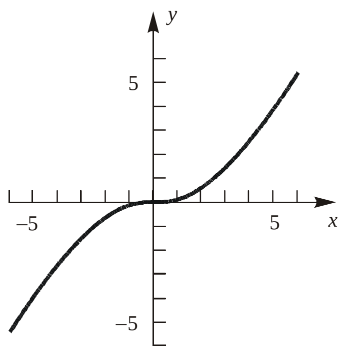
40. $f(x)=\int_0^x \log_{10}(t^2+1)dt$. Since $\log_{10}(t^2+1)$ has domain = $(-\infty,\infty)$, f also has domain = $(-\infty,\infty)$

$$f'(x)=\log_{10}(x^2+1)=\frac{\ln(x^2+1)}{\ln 10},$$

$$f''(x)=\left(\frac{1}{\ln 10}\right)\left(\frac{2x}{x^2+1}\right)$$

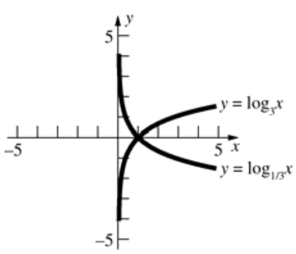
x	$(-\infty,0)$	0	$(0,\infty)$
f'	+	0	+
f''	–	0	+

f is increasing on $(-\infty,\infty)$ and so has no extreme values.
f is concave up on $(0,\infty)$ and concave down on $(-\infty,0)$. f has a point of inflection at $(0,0)$



41. $\log_{1/2} x=\frac{\ln x}{\ln\frac{1}{2}}=\frac{\ln x}{-\ln 2}=-\log_2 x$

42.



43. $M=0.67\log_{10}(0.37E)+1.46$

$$\log_{10}(0.37E)=\frac{M-1.46}{0.67}$$

$$E=\frac{10^{\frac{M-1.46}{0.67}}}{0.37}$$

Evaluating this expression for $M = 7$ and $M = 8$ gives $E\approx 5.017\times 10^8$ kW-h and $E\approx 1.560\times 10^{10}$ kW-h, respectively.

44. $115=20\log_{10}(121.3P)$

$\log_{10}(121.3P)=5.75$

$$P=\frac{10^{5.75}}{121.3}\approx 4636 \text{ lb/in.}^2$$

45. If r is the ratio between the frequencies of successive notes, then the frequency of $\overline{C}=r^{12}$ (the frequency of C). Since $\overline{C}$ has twice the frequency of C, $r=2^{1/12}\approx 1.0595$

Frequency of $\overline{\text{C}}=440(2^{1/12})^3=440\sqrt[4]{2}\approx 523.25$

46. Assume $\log_2 3=\frac{p}{q}$ where p and q are integers, $q\neq 0$. Then $2^{p/q}=3$ or $2^p=3^q$. But $2^p=2\cdot 2\ldots 2$ (p times) and has only powers of 2 as factors and $3^q=3\cdot 3\ldots 3$ (q times) and has only powers of 3 as factors.

$2^p=3^q$ only for $p=q=0$ which contradicts our assumption, so $\log_2 3$ cannot be rational.

47. If $y=A\cdot b^x$, then $\ln y = \ln A + x\ln b$, so the $\ln y$ vs. x plot will be linear.

If $y=C\cdot x^d$, then $\ln y = \ln C + d\ln x$, so the $\ln y$ vs. $\ln x$ plot will be linear.

48. WRONG 1:

$y=f(x)^{g(x)}$

$y'=g(x)f(x)^{g(x)-1}f'(x)$

WRONG 2:

$y=f(x)^{g(x)}$

$y'=f(x)^{g(x)}(\ln f(x))\cdot g'(x)=f(x)^{g(x)}g'(x)\ln f(x)$

RIGHT:

$y=f(x)^{g(x)}=e^{g(x)\ln f(x)}$

$$y'=e^{g(x)\ln f(x)}\frac{d}{dx}[g(x)\ln f(x)]$$

$$=f(x)^{g(x)}\left[g'(x)\ln f(x)+g(x)\frac{1}{f(x)}f'(x)\right]$$

$$=f(x)^{g(x)}g'(x)\ln f(x)+f(x)^{g(x)-1}g(x)f'(x)$$

Note that RIGHT = WRONG 2 + WRONG 1.

49. $f(x)=(x^x)^x = x^{(x^2)} \neq x^{(x^x)} = g(x)$

$f(x)=x^{(x^2)} = e^{x^2 \ln x}$

$f'(x) = e^{x^2 \ln x}\dfrac{d}{dx}(x^2 \ln x)$

$= e^{x^2 \ln x}\left(2x\ln x + x^2 \cdot \dfrac{1}{x}\right)$

$= x^{(x^2)}(2x \ln x + x)$

$g(x) = x^{(x^x)} = e^{x^x \ln x}$

Using the result from Example 5

$\left(\dfrac{d}{dx}x^x = x^x(1+\ln x)\right)$:

$g'(x) = e^{x^x \ln x}\dfrac{d}{dx}(x^x \ln x)$

$= e^{x^x \ln x}\left[x^x(1+\ln x)\ln x + x^x \cdot \dfrac{1}{x}\right]$

$= x^{(x^x)}x^x\left[(1+\ln x)\ln x + \dfrac{1}{x}\right]$

$= x^{x^x + x}\left[\ln x + (\ln x)^2 + \dfrac{1}{x}\right]$

50. $f(x) = \dfrac{a^x - 1}{a^x + 1}$

$f'(x) = \dfrac{(a^x+1)a^x \ln a - (a^x - 1)a^x \ln a}{(a^x+1)^2} = \dfrac{2a^x \ln a}{(a^x+1)^2}$

Since a is positive, a^x is always positive. $(a^x+1)^2$ is also always positive, thus $f'(x) > 0$ if $\ln a > 0$ and $f'(x) < 0$ if $\ln a < 0$. $f(x)$ is either always increasing or always decreasing, depending on a, so $f(x)$ has an inverse.

$y = \dfrac{a^x - 1}{a^x + 1}$

$y(a^x + 1) = a^x - 1$

$a^x(y-1) = -1 - y$

$a^x = \dfrac{1+y}{1-y}$

$x \ln a = \ln \dfrac{1+y}{1-y}$

$x = \dfrac{\ln \frac{1+y}{1-y}}{\ln a} = \log_a \dfrac{1+y}{1-y}$

$f^{-1}(y) = \log_a \dfrac{1+y}{1-y}$

$f^{-1}(x) = \log_a \dfrac{1+x}{1-x}$

51. a. Let $g(x) = \ln f(x) = \ln\left(\dfrac{x^a}{a^x}\right) = a\ln x - x \ln a$.

$g'(x) = \left(\dfrac{a}{x}\right) - \ln a$

$g'(x) < 0$ when $x > \dfrac{a}{\ln a}$, so as $x \to \infty$ $g(x)$ is decreasing. $g''(x) = -\dfrac{a}{x^2}$, so $g(x)$ is concave down. Thus, $\lim\limits_{x\to\infty} g(x) = -\infty$, so $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{g(x)} = 0$.

b. Again let $g(x) = \ln f(x) = a \ln x - x \ln a$. Since $y = \ln x$ is an increasing function, $f(x)$ is maximized when $g(x)$ is maximized.

$g'(x) = \left(\dfrac{a}{x}\right) - \ln a$, so $g'(x) > 0$ on $\left(0, \dfrac{a}{\ln a}\right)$ and $g'(x) < 0$ on $\left(\dfrac{a}{\ln a}, \infty\right)$.

Therefore, $g(x)$ (and hence $f(x)$) is maximized at $x_0 = \dfrac{a}{\ln a}$.

c. Note that $x^a = a^x$ is equivalent to $g(x) = 0$.

By part b., $g(x)$ is maximized at $x_0 = \dfrac{a}{\ln a}$.

If $a = e$, then

$g(x_0) = g\left(\dfrac{e}{\ln e}\right) = g(e) = e\ln e - e\ln e = 0.$

Since $g(x) < g(x_0) = 0$ for all $x \neq x_0$, the equation $g(x) = 0$ (and hence $x^a = a^x$) has just one positive solution. If $a \neq e$, then

$g(x_0) = g\left(\dfrac{a}{\ln a}\right) = a\ln\left(\dfrac{a}{\ln a}\right) - \dfrac{a}{\ln a}(\ln a)$

$= a\left[\ln\left(\dfrac{a}{\ln a}\right) - 1\right].$

Now $\dfrac{a}{\ln a} > e$ (justified below), so

$g(x_0) = a\left[\ln \dfrac{a}{\ln a} - 1\right] > a(\ln e - 1) = 0.$ Since $g'(x) > 0$ on $(0, x_0)$, $g(x_0) > 0$, and $\lim\limits_{x\to 0} g(x) = -\infty$, $g(x) = 0$ has exactly one solution on $(0, x_0)$.

Since $g'(x) < 0$ on (x_0, ∞), $g(x_0) > 0$, and $\lim\limits_{x\to\infty} g(x) = -\infty$, $g(x) = 0$ has exactly one solution on (x_0, ∞). Therefore,

the equation $g(x) = 0$ (and hence $x^a = a^x$) has exactly two positive solutions.

To show that $\frac{a}{\ln a} > e$ when $a \neq e$:

Consider the function $h(x) = \frac{x}{\ln x}$, for $x > 1$.

$$h'(x) = \frac{\ln(x)(1) - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

Note that $h'(x) < 0$ on $(1, e)$ and $h'(x) > 0$ on (e, ∞), so $h(x)$ has its minimum at (e, e).

Therefore $\frac{x}{\ln x} > e$ for all $x \neq e$, $x > 1$.

d. For the case $a = e$, part c. shows that $g(x) = e\ln x - x\ln e < 0$ for $x \neq e$.

Therefore, when $x \neq e$, $\ln x^e < \ln e^x$, which implies $x^e < e^x$. In particular, $\pi^e < e^\pi$.

52. a. $f_u(x) = x^u e^{-x}$

$f_u'(x) = ux^{u-1}e^{-x} - x^u e^{-x} = (u - x)x^{u-1}e^{-x}$

Since $f_u'(x) > 0$ on $(0, u)$ and $f_u'(x) < 0$ on (u, ∞), $f_u(x)$ attains its maximum at $x_0 = u$.

b. $f_u(u) > f_u(u+1)$ means

$u^u e^{-u} > (u+1)^u e^{-(u+1)}$.

Multiplying by $\frac{e^{u+1}}{u^u}$ gives $e > \left(\frac{u+1}{u}\right)^u$.

$f_{u+1}(u+1) > f_{u+1}(u)$ means

$(u+1)^{u+1}e^{-(u+1)} > u^{u+1}e^{-u}$.

Multiplying by $\frac{e^{u+1}}{u^{u+1}}$ gives $\left(\frac{u+1}{u}\right)^{u+1} > e$.

Combining the two inequalities,

$$\left(\frac{u+1}{u}\right)^u < e < \left(\frac{u+1}{u}\right)^{u+1}.$$

c. From part b., $e < \left(\frac{u+1}{u}\right)^{u+1}$.

Multiplying by $\frac{u}{u+1}$ gives

$$\frac{u}{u+1}e < \left(\frac{u+1}{u}\right)^u.$$

We showed $\left(\frac{u+1}{u}\right)^u < e$ in part b., so

$$\frac{u}{u+1}e < \left(\frac{u+1}{u}\right)^u < e.$$

Since $\lim_{u\to\infty} \frac{u}{u+1}e = e$, this implies that

$$\lim_{u\to\infty}\left(\frac{u+1}{u}\right)^u = e, \text{ i.e., } \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u = e.$$

53. $f(x) = x^x = e^{x\ln x}$

Let $g(x) = x\ln x$.

Using L'Hôpital's Rule,

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

Therefore, $\lim_{x\to 0^+} x^x = e^0 = 1$.

$g'(x) = 1 + \ln x$

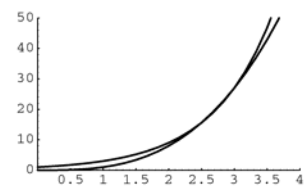
Since $g'(x) < 0$ on $(0, 1/e)$ and $g'(x) > 0$ on $(1/e, \infty)$, $g(x)$ has its minimum at $x = \frac{1}{e}$.

Therefore, $f(x)$ has its minimum at $(e^{-1}, e^{-1/e})$.

Note: this point could also be written as

$$\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{\frac{1}{e}}\right).$$

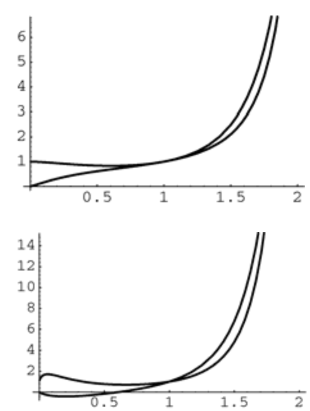
54.



(2.4781, 15.2171), (3, 27)

55. $\int_0^{4\pi} x^{\sin x}\,dx \approx 20.2259$

56.



57. **a.** In order of increasing slope, the graphs represent the curves $y = 2^x$, $y = 3^x$, and $y = 4^x$.

b. ln *y* is linear with respect to *x*, and at *x* = 0, *y* = 1 since *C* = 1.

c. The graph passes through the points (0.2, 4) and (0.6, 8). Thus, $4 = Cb^{0.2}$ and $8 = Cb^{0.6}$. Dividing the second equation by the first, gets $2 = b^{0.4}$ so $b = 2^{5/2}$.
Therefore $C = 2^{3/2}$.

58. The graph of the equation whose log-log plot has negative slope contains the points (2, 7) and (7, 2).

Thus, $7 = C2^r$ and $2 = C7^r$, so $\frac{7}{2} = \left(\frac{2}{7}\right)^r$.

$\ln\frac{7}{2} = r\ln\frac{2}{7} \Rightarrow r = \frac{\ln 7 - \ln 2}{\ln 2 - \ln 7} = -1$ and *C* = 14.

Hence, one equation is $y = 14x^{-1}$.

The graph of one equation contains the points (7, 30) and (10, 70). Thus, $30 = C7^r$ and $70 = C10^r$, so $\frac{3}{7} = \left(\frac{7}{10}\right)^r$

$\ln\frac{3}{7} = r\ln\frac{7}{10} \Rightarrow r = \frac{\ln 3 - \ln 7}{\ln 7 - \ln 10} \approx 2.38$ and $C \approx 30 \cdot 7^{-2.38} \approx 0.29$. Hence, another equation is $y = 0.29x^{2.38}$.

The graph of another equation contains the points (1, 2) and (7, 5). Thus, $2 = C1^r$ and $5 = C7^r$, so *C* = 2 and

$\ln 5 - \ln 2 = r\ln 7 \Rightarrow r = \frac{\ln 5 - \ln 2}{\ln 7} \approx 0.47.$

Hence, the last equation is $y = 2x^{0.47}$.

The given answers are only approximate. Student answers may also vary.

6.5 Concepts Review

1. ky; $ky(L - y)$

2. $2^3 = 8$

3. half-life

4. $(1+h)^{1/h}$

Problem Set 6.5

1. $k = -6$, $y_0 = 4$, so $y = 4e^{-6t}$

2. $k = 6$, $y_0 = 1$, so $y = e^{6t}$

3. $k = 0.005$, so $y = y_0e^{0.005t}$

$y(10) = y_0e^{0.005(10)} = y_0e^{0.05}$

$y(10) = 2 \Rightarrow y_0 = \frac{2}{e^{0.05}}$

$y = \frac{2}{e^{0.05}}e^{0.005t} = 2e^{0.005t-0.05} = 2e^{0.005(t-10)}$

4. $k = -0.003$, so $y = y_0e^{-0.003t}$

$y(-2) = y_0e^{(-0.003)(-2)} = y_0e^{0.006}$

$y(-2) = 3 \Rightarrow y_0 = \frac{3}{e^{0.006}}$

$y = \frac{3}{e^{0.006}}e^{-0.003t} = 3e^{-0.003t-0.006} = 3e^{-0.003(t+2)}$

5. $y_0 = 10{,}000$, $y(10) = 20{,}000$

$20{,}000 = 10{,}000e^{k(10)}$

$2 = e^{10k}$

ln 2 = 10*k*; $k = \frac{\ln 2}{10}$

$y = 10{,}000e^{((\ln 2)/10)t} = 10{,}000 \cdot 2^{t/10}$

After 25 days, $y = 10{,}000 \cdot 2^{2.5} \approx 56{,}568.$

6. Since the growth is exponential and it doubles in 10 days (from *t* = 0 to *t* = 10), it will always double in 10 days.

7. $3y_0 = y_0e^{((\ln 2)/10)t}$

$3 = e^{((\ln 2)/10)t}$

$\ln 3 = \frac{\ln 2}{10}t$

$t = \frac{10\ln 3}{\ln 2} \approx 15.8$ days

8. Let $P(t)$ = population (in millions) in year 1790 + t.
In 1960, t = 170.

$P(t) = P_0 e^{kt}$

$178 = 3.9e^{170k}$

$45.64 = e^{170k}$

$k = \frac{\ln 45.64}{170} \approx 0.02248$

In 2000, t = 210

$P(210) \approx 3.9e^{0.02248 \cdot 210} \approx 438$

The model predicts that the population will be about 438 million. The actual number, 275 million, is quite a bit smaller because the rate of growth has declined in recent decades.

9. 1 year: (4.5 million) $(1.032) \approx 4.64$ million

2 years: (4.5 million) $(1.032)^2 \approx 4.79$ million

10 years: (4.5 million) $(1.032)^{10} \approx 6.17$ million

100 years: (4.5 million) $(1.032)^{100} \approx 105$ million

10. $y = y_0 e^{kt}$

$1.032A = Ae^{k(1)}$

$k = \ln 1.032 \approx 0.03150$

At t = 100, $y = 4.5e^{(0.03150)(100)} \approx 105$.

After 100 years, the population will be about 105 million.

11. The formula to use is $y = y_0 e^{kt}$, where y = population after t years, y_0 =population at time t = 0, and k is the rate of growth. We are given

$235,000 = y_0 e^{k(12)}$ and

$164,000 = y_0 e^{k(5)}$

Dividing one equation by the other yields

$1.43293 = e^{12k-5k} = e^{7k}$ or

$k = \frac{\ln(1.43293)}{7} \approx 0.0513888$

Thus $y_0 = \frac{235,000}{e^{12(0.0513888)}} = 126,839.$

12. The formula to use is $y = y_0 e^{kt}$, where y = mass t months after initial measurement, y_0 = mass at time of initial measurement, and k is the rate of growth. We are given

$6.76 = 4e^{k(4)}$ so that

$k = \frac{1}{4}\ln\left(\frac{6.76}{4}\right) = \frac{0.5247}{4} \approx 0.1312$

Thus, 6 months before the initial measurement, the mass was $y = 4e^{(0.1312)(-6)} \approx 1.82$ grams. The tumor would have been detectable at that time.

13. $\frac{1}{2} = e^{k(700)}$ and $y_0 = 10$

$-\ln 2 = 700k$

$k = -\frac{\ln 2}{700} \approx -0.00099$

$y = 10e^{-0.00099t}$

At t = 300, $y = 10e^{-0.00099 \cdot 300} \approx 7.43$.

After 300 years there will be about 7.43 g.

14. $0.85 = e^{k(2)}$

$\ln 0.85 = 2k$

$k = \frac{\ln 0.85}{2} \approx -0.0813$

$\frac{1}{2} = e^{-0.0813t}$

$-\ln 2 = -0.0813t$

$t = \frac{\ln 2}{0.0813} \approx 8.53$

The half-life is about 8.53 days.

15. The basic formula is $y = y_0 e^{kt}$. If t_* denotes the half-life of the material, then (see Example 3)

$\frac{1}{2} = e^{kt_*}$ or $k = \frac{\ln(0.5)}{t_*}$. Thus

$k_C = \frac{-0.693}{30.22} = -0.0229$ and $k_S = \frac{-0.693}{28.8} = -0.0241$

To find when 1% of each material will remain, we use $0.01y_0 = y_0 e^{kt}$ or $t = \frac{\ln(0.01)}{k}$. Thus

$t_C = \frac{-4.6052}{-0.0229} \approx 201$ years (2187) and

$t_S = \frac{-4.6052}{-0.0241} \approx 191$ years (2177)

16. The basic formula is $y = y_0 e^{kt}$. We are given

$15.231 = y_0 e^{k(2)}$ and $9.086 = y_0 e^{k(8)}$

Dividing one equation by the other gives

$\frac{15.231}{9.086} = e^{k(2)-k(8)} = e^{k(-6)}$ so $k = -0.0861$

Thus $y_0 = \frac{15.231}{e^{(-.0861)(2)}} \approx 18.093$ grams.

To find the half-life:

$t_* = \frac{\ln(0.5)}{k} = \frac{-0.693}{-0.0861} \approx 8$ days

17. $\frac{1}{2} = e^{5730k}$

$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -1.210 \times 10^{-4}$

$0.7y_0 = y_0 e^{(-1.210\times 10^{-4})t}$

$t = \frac{\ln 0.7}{-1.210 \times 10^{-4}} \approx 2950$

The fort burned down about 2950 years ago.

18. $\frac{1}{2} = e^{5730k}$

$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -1.210 \times 10^{-4}$

$0.51y_0 = y_0 e^{(-1.210\times 10^{-4})t}$

$t = \frac{\ln 0.51}{-1.210 \times 10^{-4}} \approx 5565$

The body was buried about 5565 years ago.

19. From Example 4, $T(t) = T_1 + (T_0 - T_1)e^{kt}$. In this problem, $200 = T(0.5) = 75 + (300 - 75)e^{k(0.5)}$ so

$k = \frac{\ln\left(\frac{125}{225}\right)}{0.5} = -1.1756$ and

$T(3) = 75 + 225e^{(-1.1756)(3)} = 81.6^\circ$ F

20. From Example 4, $T(t) = T_1 + (T_0 - T_1)e^{kt}$. In this problem, $0 = T(5) = 24 + (-20 - 24)e^{k(5)}$ so

$k = \frac{\ln\left(\frac{-24}{-44}\right)}{5} = -0.1212$; the thermometer will register 20° C when $20 = 24 + (-44)e^{-0.1212\,t}$ or

$t = \frac{\ln\left(\frac{-4}{-44}\right)}{-0.1212} = 19.78$ min.

21. From Example 4, $T(t) = T_1 + (T_0 - T_1)e^{kt}$. In this problem, $70 = T(5) = 90 + (26 - 90)e^{k(5)}$ so

$k = \frac{\ln\left(\frac{-20}{-64}\right)}{5} = -0.2326$ and

$T(10) = 90 - 64e^{(-0.2326)(10)} = 90 - 64(0.0977) = 83.7^\circ$C

22. From Example 4, $T(t) = T_1 + (T_0 - T_1)e^{kt}$. In this problem, $250 = T(15) = 40 + (350 - 40)e^{k(15)}$ so

$k = \frac{\ln\left(\frac{210}{310}\right)}{15} = -0.026$; the brownies will be 110° F when $110 = 40 + (310)e^{-0.026\,t}$ or

$t = \frac{\ln\left(\frac{70}{310}\right)}{-0.026} = 57.2$ min.

23. From Example 4, $T(t) = T_1 + (T_0 - T_1)e^{kt}$.
Let w = the time of death; then

$82 = T(10 - w) = 70 + (98.6 - 70)e^{k(10-w)}$

$76 = T(11 - w) = 70 + (98.6 - 70)e^{k(11-w)}$

or $\quad 12 = 28.6e^{k(10-w)}$

$\quad 6 = 28.6e^{k(11-w)}$

Dividing: $2 = e^{k(-1)}$ or $k = \ln(0.5) = -0.693$

To find w:

$12 = 28.6e^{-0.693(10-w)}$ so $10 - w = \frac{\ln\left(\frac{12}{28.6}\right)}{-0.693} = 1.25$

Therefore $w = 10 - 1.25 = 8.75 = 8:45\,\text{pm}$.

24. a. From example 4 of this section,

$\frac{dT}{dt} = k(T - T_1)$ or

$\int \frac{dT}{T - T_1} = k\,dt$ or $\ln\left|\text{T(t)-T}_1\right| = kt + C$

This gives $\left|T(t) - T_1\right| = e^{kt}e^C$. Now, if T_0 is the temperature at $t = 0$, $\left|T_0 - T_1\right| = e^C$ and the Law of Cooling becomes

$\left|T(t) - T_1\right| = \left|T_0 - T_1\right|e^{kt}$. Note that $T(t)$ is always between T_0 and T_1 so that $\left|T(t) - T_1\right|$ and $\left|T_0 - T_1\right|$ always have the same sign; this simplifies the Law of Cooling to

$T(t) - T_1 = (T_0 - T_1)e^{kt}$

or

$T(t) = T_1 + (T_0 - T_1)e^{kt}$

b. Since $T(t)$ is always between T_0 and T_1, it follows that $e^{kt} = \frac{T(t) - T_1}{T_0 - T_1} < 1$ so that $k < 0$.

Hence

$\lim_{t\to\infty} T(t) = T_1 + (T_0 - T_1)\lim_{t\to\infty} e^{kt} = T_1 + 0 = T_1$

25. a. $(\$375)(1.035)^2 \approx \$401.71$

b. $(\$375)\left(1+\frac{0.035}{12}\right)^{24} \approx \$402.15$

c. $(\$375)\left(1+\frac{0.035}{365}\right)^{730} \approx \$402.19$

d. $(\$375)e^{0.035\cdot 2} \approx \$402.19$

26. a. $(\$375)(1.046)^2 = \$410.29$

b. $(\$375)\left(1+\frac{0.046}{12}\right)^{24} \approx \$411.06$

c. $(\$375)\left(1+\frac{0.046}{365}\right)^{730} \approx \$411.13$

d. $(\$375)e^{0.046\cdot 2} \approx \$411.14$

27. a. $\left(1+\frac{0.06}{12}\right)^{12t} = 2$

$1.005^{12t} = 2$

$12t = \frac{\ln 2}{\ln 1.005}$ so $t = \frac{\ln 2}{12\ln 1.005} \approx 11.58$

It will take about 11.58 years or 11 years, 6 months, 29 days.

b. $e^{0.06t} = 2 \Rightarrow t = \frac{\ln 2}{0.06} \approx 11.55$

It will take about 11.55 years or 11 years, 6 months, and 18 days.

28. $\$20,000(1.025)^5 \approx \$22,628.16$

29. 1626 to 2000 is 374 years.

$y = 24e^{0.06\cdot 374} \approx \$133.6$ billion

30. $\$100(1.04)^{969} \approx \$3.201\times 10^{18}$

31. $1000e^{(0.05)(1)} = \$1051.27$

32. $A_0 e^{(0.05)(1)} = 1000$

$A_0 = 1000e^{-0.05} \approx \$951.23$

33. If t is the doubling time, then

$\left(1+\frac{p}{100}\right)^t = 2$

$t\ln\left(1+\frac{p}{100}\right) = \ln 2$

$$t = \frac{\ln 2}{\ln\left(1+\frac{p}{100}\right)} \approx \frac{\ln 2}{\frac{p}{100}} = \frac{100\ln 2}{p} \approx \frac{70}{p}$$

34. $\frac{dy}{dt} = ky(L-y)$

$\frac{1}{y(L-y)}dy = k\,dt$

$\left[\frac{1}{Ly} + \frac{1}{L(L-y)}\right]dy = k\,dt$

$\frac{1}{L}\int\left(\frac{1}{y} + \frac{1}{L-y}\right)dy = \int k\,dt$

$\frac{1}{L}[\ln|y| - \ln|L-y|] = kt + C_1$

$\ln\left|\frac{y}{L-y}\right| = Lkt + LC_1$

$\left|\frac{y}{L-y}\right| = e^{Lkt+LC_1} = e^{LC_1}\cdot e^{Lkt}$, so $\frac{y}{L-y} = Ce^{Lkt}$

$\left(\text{Note that: } C = Ce^0 = Ce^{Lk\cdot 0} = \frac{y(0)}{L-y(0)} = \frac{y_0}{L-y_0}.\right)$

$y = LCe^{Lkt} - yCe^{Lkt}$

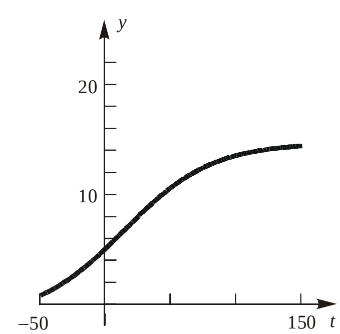
$y + yCe^{Lkt} = LCe^{Lkt}$

$$y = \frac{LCe^{Lkt}}{1+Ce^{Lkt}} = \frac{LC}{\frac{1}{e^{Lkt}} + C} = \frac{LC}{C + e^{-Lkt}}$$

$$= \frac{L\cdot\frac{y_0}{L-y_0}}{\frac{y_0}{L-y_0} + e^{-Lkt}} = \frac{Ly_0}{y_0 + (L-y_0)e^{-Lkt}}$$

35. $y = \frac{16(6.4)}{6.4 + (16-6.4)e^{-16(0.00186)t}}$

$= \frac{102.4}{6.4 + 9.6e^{-0.02976t}}$



36. a. $\lim_{x\to 0}(1+x)^{1000} = 1^{1000} = 1$

b. $\lim_{x\to 0} 1^{1/x} = \lim_{x\to 0} 1 = 1$

c. $\lim_{x\to 0^+}(1+\varepsilon)^{1/x} = \lim_{n\to\infty}(1+\varepsilon)^n = \infty$

d. $\lim_{x\to 0^-}(1+\varepsilon)^{1/x} = \lim_{n\to\infty}\frac{1}{(1+\varepsilon)^n} = 0$

e. $\lim_{x\to 0}(1+x)^{1/x} = e$

37. a. $\lim_{x\to 0}(1-x)^{1/x} = \lim_{x\to 0}\frac{1}{[1+(-x)]^{1/(-x)}} = \frac{1}{e}$

b. $\lim_{x\to 0}(1+3x)^{1/x} = \lim_{x\to 0}\left[(1+3x)^{\frac{1}{3x}}\right]^3 = e^3$

c. $\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

$= \lim_{x\to 0^+}(1+2x)^{1/x}$

$= \lim_{x\to 0^+}\left[(1+2x)^{\frac{1}{2x}}\right]^2 = e^2$

d. $\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{2n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$

$= \lim_{x\to 0^+}(1-x)^{2/x}$

$= \lim_{x\to 0^+}\left[(1-x)^{\frac{1}{-x}}\right]^{-2} = \frac{1}{e^2}$

38. $\frac{dy}{dt} = ay + b$

$\int \frac{dy}{y+\frac{b}{a}} = \int a\,dt$

$\ln\left|y+\frac{b}{a}\right| = at + C$

$\left|y+\frac{b}{a}\right| = e^{at+C}$; $y+\frac{b}{a} = Ae^{at}$

$y = Ae^{at} - \frac{b}{a}$

$y_0 = A - \frac{b}{a} \Rightarrow A = y_0 + \frac{b}{a}$

$y = \left(y_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}$

39. Let y = population in millions, t = 0 in 1985, a = 0.012, b = 0.06, $y_0 = 10$

$\frac{dy}{dt} = 0.012y + 0.06$

$y = \left(10 + \frac{0.06}{0.012}\right)e^{0.012t} - \frac{0.06}{0.012} = 15e^{0.012t} - 5$

From 1985 to 2010 is 25 years. At t = 25, $y = 15e^{0.012\cdot 25} - 5 \approx 15.25$. The population in 2010 will be about 15.25 million.

40. Let $N(t)$ be the number of people who have heard the news after t days. Then $\frac{dN}{dt} = k(L-N)$.

$\int \frac{1}{L-N}dN = \int k\,dt$

$-\ln(L-N) = kt + C$

$L - N = e^{-kt-C}$

$N = L - Ae^{-kt}$

$N(0) = 0, \Rightarrow A = L$

$N(t) = L(1-e^{-kt})$.

$N(5) = \frac{L}{2} \Rightarrow \frac{L}{2} = L(1-e^{-5k})$

$\frac{1}{2} = e^{-5k}$

$k = \frac{\ln\frac{1}{2}}{-5} \approx 0.1386$

$N(t) = L(1-e^{-0.1386t})$

$0.99L = L(1-e^{-0.1386t})$

$0.01 = e^{-0.1386t}$

$t = \frac{\ln 0.01}{-0.1386} \approx 33$

99% of the people will have heard about the scandal after 33 days.

41. If $f(t) = e^{kt}$, then $\frac{f'(t)}{f(t)} = \frac{ke^{kt}}{e^{kt}} = k$.

42. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$\lim_{x\to\infty}\frac{f'(x)}{f(x)}$

$= \lim_{x\to\infty}\frac{na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1}{a_n x^n + a_{n-1}x + \cdots + a_1 x + a_0}$

$= \lim_{x\to\infty}\frac{\frac{na_n}{x} + \frac{(n-1)a_{n-1}}{x^2} + \cdots + \frac{a_1}{x^n}}{a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}} = 0$

43. $\dfrac{f'(x)}{f(x)} = k > 0$ can be written as

$\dfrac{1}{y}\dfrac{dy}{dx} = k$ where $y = f(x)$.

$\dfrac{dy}{y} = k\,dx$ has the solution $y = Ce^{kx}$.

Thus, the equation $f(x) = Ce^{kx}$ represents exponential growth since $k > 0$.

44. $\dfrac{f'(x)}{f(x)} = k < 0$ can be written as $\dfrac{1}{y}\dfrac{dy}{dx} = k$ where $y = f(x)$. $\dfrac{dy}{y} = k\,dx$ has the solution $y = Ce^{kx}$.

Thus, $f(x) = Ce^{kx}$ which represents exponential decay since $k < 0$.

45. Maximum population:

$$13{,}500{,}000 \text{ mi}^2 \cdot \frac{640 \text{ acres}}{1 \text{ mi}^2} \cdot \frac{1 \text{ person}}{\frac{1}{2} \text{ acre}}$$

$= 1.728 \times 10^{10}$ people

Let $t = 0$ be in 2004.

$(6.4 \times 10^9)e^{0.0132t} = 1.728 \times 10^{10}$

$t = \dfrac{\ln\left(\dfrac{1.728 \cdot 10^{10}}{6.4 \cdot 10^9}\right)}{0.0132} \approx 75.2$ years from 2004, or sometime in the year 2079.

46. **a.** $k = 0.0132 - 0.0002t$

b. $y' = (0.0132 - 0.0002t)\,y$

c. $\dfrac{dy}{dt} = (0.0132 - 0.0002t)\,y$

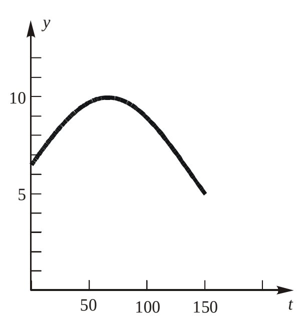
$\dfrac{dy}{y} = (0.0132 - 0.0002t)\,dt$

$\ln y = 0.0132t - 0.0001t^2 + C_0$

$y = C_1 e^{0.0132t - 0.0001t^2}$

The initial condition $y(0) = 6.4$ implies that $C_1 = 6.4$. Thus $y = 6.4e^{0.0132t - 0.0001t^2}$

d.



e. The maximum population will occur when

$\dfrac{d}{dt}\left(0.0132t - 0.0001t^2\right) = 0$

$0.0132 = 0.0002t$

$t = 0.0132/0.0002 = 66$

$t = 66$, which is year 2070.

The population will equal the 2004 value of 6.4 billion when $0.0132t - 0.0001t^2 = 0$

$t = 0$ or $t = 132$.

The model predicts that the population will return to the 2004 level in year 2136.

47. **a.** $k = 0.0132 - 0.0001t$

b. $y' = (0.0132 - 0.0001t)\,y$

c. $\dfrac{dy}{dt} = (0.0132 - 0.0001t)\,y$

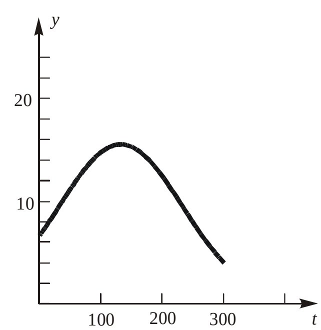
$\dfrac{dy}{y} = (0.0132 - 0.0001t)\,dt$

$\ln y = 0.0132t - 0.00005t^2 + C_0$

$y = C_1 e^{0.0132t - 0.00005t^2}$

The initial condition $y(0) = 6.4$ implies that $C_1 = 6.4$. Thus $y = 6.4e^{0.0132t - 0.00005t^2}$

d.



e. The maximum population will occur when

$\dfrac{d}{dt}\left(0.0132t - 0.00005t^2\right) = 0$

$0.0132 = 0.0001t$

$t = 0.0132/0.0001 = 132$

$t = 132$, which is year 2136

The population will equal the 2004 value of 6.4 billion when $0.0132t - 0.00005t^2 = 0$

$t = 0$ or $t = 264$.

The model predicts that the population will return to the 2004 level in year 2268.

48. $E'(x) = \lim_{h \to 0} \frac{E(x+h) - E(x)}{h}$

$= \lim_{h \to 0} \frac{E(x)E(h) - E(x)}{h}$

$= \lim_{h \to 0} E(x) \cdot \frac{E(h)-1}{h} = E(x) \lim_{h \to 0} \frac{E(h)-1}{h}$

$E(x) = E(x+0) = E(x) \cdot E(0)$

so $E(0) = 1$.

Thus, $E'(x) = E(x) \lim_{h \to 0} \frac{E(h) - E(0)}{h}$

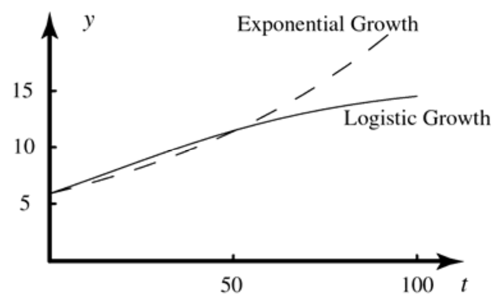
$= E(x) \lim_{h \to 0} \frac{E(0+h) - E(0)}{h} = E(x) \cdot E'(0)$

$= kE(x)$ where $k = E'(0)$.

Hence, $E(x) = E_0 e^{kx} = E(0)e^{kx} = 1 \cdot e^{kx} = e^{kx}$.

Check: $E(u+v) = e^{k(u+v)} = e^{ku+kv}$

$= e^{ku} \cdot e^{kv} = E(u) \cdot E(v)$

49. 

Exponential growth:
In 2010 (t = 6): 6.93 billion
In 2040 (t = 36): 10.29 billion
In 2090 (t = 86): 19.92 billion
Logistic growth:
In 2010 (t = 6): 7.13 billion
In 2040 (t = 36): 10.90 billion
In 2090 (t = 86): 15.15 billion

50. a. $\lim_{x \to 0} (1+x)^{1/x} = e$

b. $\lim_{x \to 0} (1-x)^{1/x} = \frac{1}{e}$

6.6 Concepts Review

1. $\exp\left(\int P(x)dx\right)$

2. $y \exp\left(\int P(x)dx\right)$

3. $\frac{1}{x}; \frac{d}{dx}\left(\frac{y}{x}\right) = 1; x^2 + Cx$

4. particular

Problem Set 6.6

1. Integrating factor is e^x.

$D(ye^x) = 1$

$y = e^{-x}(x + C)$

2. The left-hand side is already an exact derivative.

$D[y(x+1)] = x^2 - 1$

$y = \frac{x^3 - 3x + C}{3(x+1)}$

3. $y' + \frac{x}{1-x^2} y = \frac{ax}{1-x^2}$

Integrating factor:

$\exp \int \frac{x}{1-x^2} dx = \exp\left[\ln(1-x^2)^{-1/2}\right]$

$= (1-x^2)^{-1/2}$

$D[y(1-x^2)^{-1/2}] = ax(1-x^2)^{-3/2}$

Then $y(1-x^2)^{-1/2} = a(1-x^2)^{-1/2} + C$, so

$y = a + C(1-x^2)^{1/2}$.

4. Integrating factor is sec x.

$D[y \sec x] = \sec^2 x$

$y = \sin x + C \cos x$

5. Integrating factor is $\frac{1}{x}$.

$D\left[\frac{y}{x}\right] = e^x$

$y = xe^x + Cx$

6. $y' - ay = f(x)$

Integrating factor: $e^{\int -a\,dx} = e^{-ax}$

$D[ye^{-ax}] = e^{-ax} f(x)$

Then $ye^{-ax} = \int e^{-ax} f(x)dx,$ so

$y = e^{ax}\int e^{-ax} f(x)dx.$

7. Integrating factor is x. $D[yx] = 1$; $y = 1 + Cx^{-1}$

8. Integrating factor is $(x+1)^2$.

$D[y(x+1)^2] = (x+1)^5$

$y = \left(\frac{1}{6}\right)(x+1)^4 + C(x+1)^{-2}$

9. $y' + f(x)y = f(x)$

Integrating factor: $e^{\int f(x)dx}$

$D\left[ye^{\int f(x)dx}\right] = f(x)e^{\int f(x)dx}$

Then $ye^{\int f(x)dx} = e^{\int f(x)dx} + C,$ so

$y = 1 + Ce^{-\int f(x)dx}.$

10. Integrating factor is e^{2x}.

$D[ye^{2x}] = xe^{2x}$

$y = \left(\frac{1}{2}\right)x - \left(\frac{1}{4}\right) + Ce^{-2x}$

11. Integrating factor is $\frac{1}{x}$. $D\left[\frac{y}{x}\right] = 3x^2$; $y = x^4 + Cx$

$y = x^4 + 2x$ goes through (1, 3).

12. $y' + 3y = e^{2x}$

Integrating factor: $e^{\int 3dx} = e^{3x}$

$D[ye^{3x}] = e^{5x}$

Then $ye^{3x} = \frac{e^{5x}}{5} + C.$ $x = 0, y = 1 \Rightarrow C = \frac{4}{5},$ so

$ye^{3x} = \frac{e^{5x}}{5} + \frac{4}{5}.$

Therefore, $y = \frac{e^{2x} + 4e^{-3x}}{5}$ is the particular solution through (0, 1).

13. Integrating factor: xe^x

$d[yxe^x] = 1$; $y = e^{-x}(1 + Cx^{-1})$; $y = e^{-x}(1 - x^{-1})$ goes through (1, 0).

14. Integrating factor is $\sin^2 x$.

$D[y\sin^2 x] = 2\sin^2 x\cos x$

$y\sin^2 x = \frac{2}{3}\sin^3 x + C$

$y = \frac{2}{3}\sin x + \frac{C}{\sin^2 x}$

$y = \frac{2}{3}\sin x + \frac{5}{12}\csc^2 x$

goes through $\left(\frac{\pi}{6}, 2\right)$.

15. Let y denote the number of pounds of chemical A after t minutes.

$\frac{dy}{dt} = \left(2\frac{\text{lbs}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y\text{ lbs}}{20\text{ gal}}\right)\left(\frac{3\text{ gal}}{\text{min}}\right)$

$= 6 - \frac{3y}{20}$ lb/min

$y' + \frac{3}{20}y = 6$

Integrating factor: $e^{\int (3/20)dt} = e^{3t/20}$

$D[ye^{3t/20}] = 6e^{3t/20}$

Then $ye^{3t/20} = 40e^{3t/20} + C.$ $t = 0, y = 10$ $\Rightarrow C = -30.$

Therefore, $y(t) = 40 - 30e^{-3t/20},$ so

$y(20) = 40 - 30e^{-3} \approx 38.506$ lb.

16. $\frac{dy}{dt} = (2)(4) - \left(\frac{y}{200}\right)(4)$ or $y' + \frac{y}{50} = 8$

Integrating factor is $e^{t/50}$.

$D[ye^{t/50}] = 8e^{t/50}$

$y(t) = 400 + Ce^{-t/50}$

$y(t) = 400 - 350e^{-t/50}$ goes through (0, 50).

$y(40) = 400 - 350e^{-0.8} \approx 242.735$ lb of salt

17. $\frac{dy}{dt} = 4 - \left[\frac{y}{(120 - 2t)}\right](6)$ or $y' + \left[\frac{3}{(60 - t)}\right]y = 4$

Integrating factor is $(60 - t)^{-3}$.

$D[y(60 - t)^{-3}] = 4(60 - t)^{-3}$

$y(t) = 2(60 - t) + C(60 - t)^3$

$y(t) = 2(60 - t) - \left(\frac{1}{1800}\right)(60 - t)^3$ goes through (0, 0).

18. $\dfrac{dy}{dt}=\dfrac{-2y}{50+t}$ or $y'+\dfrac{2}{50+t}y=0.$

Integrating factor:

$\exp\left(\int \dfrac{2}{50+t}dt\right)=e^{2\ln(50+t)}=(50+t)^2$

$D[y(50+t)^2]=0$

Then $y(50+t)^2=C.$ $t=0, y=30 \Rightarrow C=75000$

Thus, $y(50+t)^2=75{,}000.$

If $y=25$, $25(50+t)^2=75{,}000$, so

$t=\sqrt{3000}-50\approx 4.772$ min.

19. $I'+10^6 I=1$

Integrating factor $=\exp(10^6 t)$

$D[I\exp(10^6 t)]=\exp(10^6 t)$

$I(t)=10^{-6}+C\exp(-10^6 t)$

$I(t)=10^{-6}[1-\exp(-10^6 t)]$ goes through (0, 0).

20. $3.5I'=120\sin 377t$

$I'=\left(\dfrac{240}{7}\right)\sin 377t$

$I=\left(-\dfrac{240}{2639}\right)\cos 377t+C$

$I(t)=\left(\dfrac{240}{2639}\right)(1-\cos 377t)$ through (0, 0).

21. $1000I=120\sin 377t$

$I(t)=0.12\sin 377t$

22. $\dfrac{dx}{dt}=-\dfrac{2x}{100}$

$x'+\left(\dfrac{1}{50}\right)x=0$

Integrating factor is $e^{t/50}$.

$D[xe^{t/50}]=0$

$x=Ce^{-t/50}$

$x(t)=50e^{-t/50}$ satisfies $t=0$, $x=50$.

$\dfrac{dy}{dt}=2\left(\dfrac{50e^{-t/50}}{100}\right)-2\left(\dfrac{y}{200}\right)$

$y'+\left(\dfrac{1}{100}\right)y=e^{-t/50}$

Integrating factor is $e^{t/100}$.

$D[ye^{t/100}]=e^{-t/100}$

$y(t)=e^{-t/100}(C-100e^{-t/100})$

$y(t)=e^{-t/100}(250-100e^{-t/100})$ satisfies $t=0$, $y=150$.

23. Let y be the number of gallons of pure alcohol in the tank at time t.

a. $y'=\dfrac{dy}{dt}=5(0.25)-\left(\dfrac{5}{100}\right)y=1.25-0.05y$

Integrating factor is $e^{0.05t}$.

$y(t)=25+Ce^{-0.05t}$; $y=100, t=0, C=75$

$y(t)=25+75e^{-0.05t}$; $y=50$, $t=T$,

$T=20(\ln 3)\approx 21.97$ min

b. Let A be the number of gallons of pure alcohol drained away.

$(100-A)+0.25A=50 \Rightarrow A=\dfrac{200}{3}$

It took $\dfrac{\frac{200}{3}}{5}$ minutes for the draining and the same amount of time to refill, so

$T=\dfrac{2\left(\frac{200}{3}\right)}{5}=\dfrac{80}{3}\approx 26.67$ min.

c. c would need to satisfy

$\dfrac{\frac{200}{3}}{5}+\dfrac{\frac{200}{3}}{c}<20(\ln 3).$

$c>\dfrac{10}{(3\ln 3-2)}\approx 7.7170$

d. $y'=4(0.25)-0.05y=1-0.05y$

Solving for y, as in part a, yields $y=20+80e^{-0.05t}$. The drain is closed when $t=0.8T$. We require that

$(20+80e^{-0.05\cdot 0.8T})+4\cdot 0.25\cdot 0.2T=50,$

or $400e^{-0.04T}+T=150.$

24. a. $v'+av=-g$

Integrating factor: e^{at}

$e^{at}(v'+av)=-ge^{at}; \dfrac{d}{dt}(ve^{at})=-ge^{at}$

$ve^{at}=\int -ge^{at}dt=\dfrac{-g}{a}e^{at}+C; v=\dfrac{-g}{a}+Ce^{-at}$

$v=v_0, t=0$

$v_0=\dfrac{-g}{a}+C\Rightarrow C=v_0+\dfrac{g}{a}$

Therefore, $v=\dfrac{-g}{a}+\left(v_0+\dfrac{g}{a}\right)e^{-at}$, so

$v(t)=v_\infty+(v_0-v_\infty)e^{-at}.$

b. $\frac{dy}{dt} = v_\infty + (v_0 - v_\infty)e^{-at}$, so

$$y = v_\infty \cdot t - \frac{(v_0 - v_\infty)e^{-at}}{a} + C.$$

$$y = y_0, t = 0 \Rightarrow y_0 = \frac{-(v_0 - v_\infty)}{a} + C$$

$$\Rightarrow C = y_0 + \frac{v_0 - v_\infty}{a}$$

$$y = v_\infty t - \frac{(v_0 - v_\infty)e^{-at}}{a} + \left(y_0 + \frac{v_0 - v_\infty}{a}\right)$$

$$= y_0 + v_\infty t + \frac{v_0 - v_\infty}{a}(1 - e^{-at})$$

25. a. $v_\infty = -\frac{32}{0.05} = -640$

$v(t) = [120 - (-640)]e^{-0.05t} + (-640) = 0$ if $t = 20\ln\left(\frac{19}{16}\right)$.

$$y(t) = 0 + (-640)t + \left(\frac{1}{0.05}\right)[120 - (-640)](1 - e^{-0.05t})$$

$$= -640t + 15,200(1 - e^{-0.05t})$$

Therefore, the maximum altitude is

$$y\left(20\ln\left(\frac{19}{16}\right)\right) = -12,800\ln\left(\frac{19}{16}\right) + \frac{45,600}{19} \approx 200.32 \text{ ft}$$

b. $-640T + 15,200(1 - e^{-0.05T}) = 0;$

$95 - 4T - 95e^{-0.05T} = 0$

26. For t in [0, 15],

$$v_\infty = \frac{-32}{0.10} = -320.$$

$v(t) = (0 + 320)e^{-0.1t} - 320 = 320(e^{-0.1t} - 1);$

$v(15) = 320(e^{-1.5} - 1) \approx -248.6$

$y(t) = 8000 - 320t + 10(320)(1 - e^{-0.1t});$

$y(15) = 3200(2 - e^{-1.5}) \approx 5686$

Let t be the number of seconds after the parachute opens that it takes Megan to reach the ground.

For t in [15, 15+T], $v_\infty = -\frac{32}{1.6} = -20.$

$0 = y(T + 15)$

$= [3200(2 - e^{-1.5})]$

$-20T + (0.625)[320(e^{-1.5} - 1) + 20](1 - e^{-1.6T})$

$\approx 5543 - 20T - 142.9e^{-1.6T} \approx 5543 - 20T$ [since $T > 50$, so $e^{-1.6T} < 10^{-35}$ (very small)]

Therefore, $T \approx 277$, so it takes Megan about 292 s (4 min, 52 s) to reach the ground.

27. a. $e^{-\ln x + C}\left(\frac{dy}{dx} - \frac{y}{x}\right) = x^2 e^{-\ln x + C}$

$$e^{-\ln x}e^C\left(\frac{dy}{dx} - \frac{y}{x}\right) = x^2 e^C e^{-\ln x}$$

$$\frac{1}{x}e^C \frac{dy}{dx} - ye^C \frac{1}{x^2} = x^2 e^C \frac{1}{x}$$

$$\frac{d}{dx}\left(e^C \frac{1}{x} y\right) = xe^C$$

b. $e^C \frac{y}{x} = e^C \int x\,dx$

$$\frac{y}{x} = \frac{x^2}{2} + C_1$$

$$y = \frac{x^3}{2} + C_1 x$$

28. $e^{\int P(x)\,dx + C} \frac{dy}{dx} + P(x)e^{\int P(x)\,dx + C} y$

$$= Q(x)e^{\int P(x)\,dx + C}$$

$$\frac{d}{dx}\left(e^{\int P(x)\,dx + C} y\right) = Q(x)e^{\int P(x)\,dx + C}$$

$$ye^{\int P(x)\,dx + C} = \int Q(x)e^{\int P(x)\,dx} e^C\,dx + C_1$$

$$y = e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx + C_2 e^{-\int P(x)\,dx}$$

6.7 Concepts Review

1. slope field

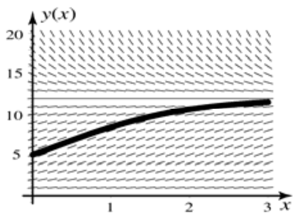
2. tangent line

3. $y_{n-1} + hf(x_{n-1}, y_{n-1})$

4. underestimate

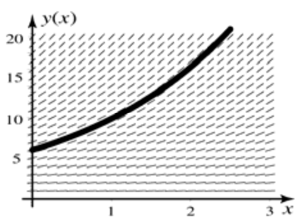
Problem Set 6.7

1.



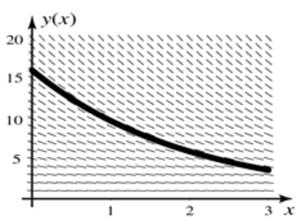
$\lim_{x\to\infty} y(x) = 12$ and $y(2) \approx 10.5$

2.



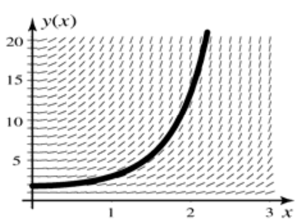
$\lim_{x\to\infty} y(x) = \infty$ and $y(2) \approx 16$

3.



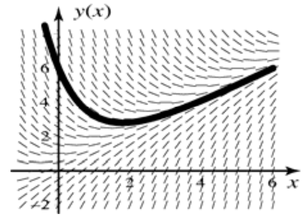
$\lim_{x\to\infty} y(x) = 0$ and $y(2) \approx 6$

4.



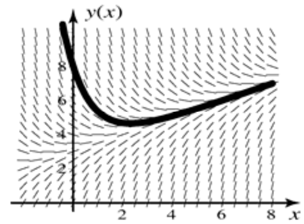
$\lim_{x\to\infty} y(x) = \infty$ and $y(2) \approx 13$

5.



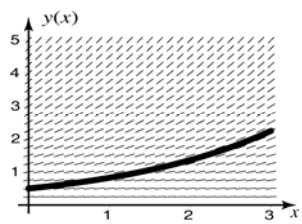
The oblique asymptote is $y = x$.

6.



The oblique asymptote is $y = 3 + x/2$.

7.



$$\frac{dy}{dx} = \frac{1}{2}y; \quad y(0) = \frac{1}{2}$$

$$\frac{dy}{y} = \frac{1}{2}dx$$

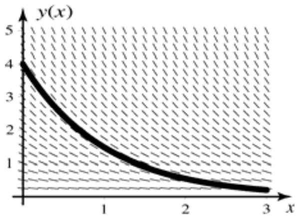
$$\ln y = \frac{x}{2} + C$$

$$y = C_1 e^{x/2}$$

To find C_1, apply the initial condition:

$$\frac{1}{2} = y(0) = C_1 e^0 = C_1$$

$$y = \frac{1}{2}e^{x/2}$$

8. 

$\frac{dy}{dx} = -y; \qquad y(0) = 4$

$\frac{dy}{y} = -dx$

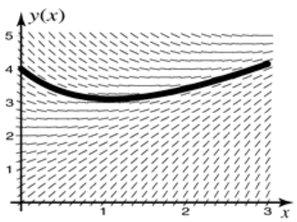
$\ln y = -x + C$

$y = C_1 e^{-x}$

To find C_1, apply the initial condition:

$4 = y(0) = C_1 e^{-0} = C_1$

$y = 4e^{-x}$

9. 

$y' + y = x + 2$

The integrating factor is $e^{\int 1 dx} = e^x$.

$e^x y' + ye^x = e^x(x+2)$

$\frac{d}{dx}\left(e^x y\right) = (x+2)e^x$

$e^x y = \int (x+2)e^x \, dx$

Integrate by parts: let $u = x+2, \quad dv = e^x \, dx$.

Then $du = dx$ and $v = e^x$. Thus

$e^x y = (x+2)e^x - \int e^x \, dx$

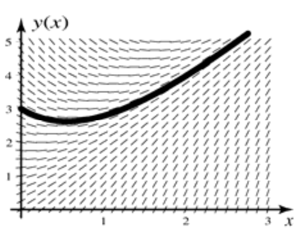
$e^x y = (x+2)e^x - e^x + C$

$y = x + 2 - 1 + Ce^{-x}$

To find C, apply the initial condition:

$4 = y(0) = 0 + 1 + Ce^{-0} = 1 + C \quad \to \quad C = 3$

Thus, $y = x + 1 + 3e^{-x}$.

10. 

$y' + y = 2x + \frac{3}{2}$

$e^x y' + ye^x = \left(2x + \frac{3}{2}\right)e^x$

$\frac{d}{dx}\left(e^x y\right) = \left(2x + \frac{3}{2}\right)e^x$

$e^x y = \int \left(2x + \frac{3}{2}\right)e^x \, dx$

Integrate by parts: let $u = 2x + \frac{3}{2}$,

$dv = e^x \, dx$. Then $du = 2dx$ and $v = e^x$.

Thus,

$e^x y = \left(2x + \frac{3}{2}\right)e^x - \int 2e^x \, dx$

$e^x y = \left(2x + \frac{3}{2}\right)e^x - 2e^x + C$

$y = 2x - \frac{1}{2} + Ce^{-x}$

To find C, apply the initial condition:

$3 = y(0) = 0 - \frac{1}{2} + Ce^{-0} = C - \frac{1}{2}$

Thus $C = \frac{7}{2}$, so the solution is

$y = 2x - \frac{1}{2} + \frac{7}{2}e^{-x}$

Note: Solutions to Problems 22-28 are given along with the corresponding solutions to 11-16.

11., 22.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	3.0	3.0
0.2	4.2	4.44
0.4	5.88	6.5712
0.6	8.232	9.72538
0.8	11.5248	14.39356
1.0	16.1347	21.30246

12., 23.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	2.0	2.0
0.2	1.6	1.64
0.4	1.28	1.3448
0.6	1.024	1.10274
0.8	0.8195	0.90424
1.0	0.65536	0.74148

13., 24.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	0.0	0.0
0.2	0.0	0.02
0.4	0.04	0.08
0.6	0.12	0.18
0.8	0.24	0.32
1.0	0.40	0.50

14., 25.

x_n	Euler's Method y_n	Improved Euler Method y_n
0.0	0.0	0.0
0.2	0.0	0.004
0.4	0.008	0.024
0.6	0.040	0.076
0.8	0.112	0.176
1.0	0.240	0.340

15., 26.

x_n	Euler's Method y_n	Improved Euler Method y_n
1.0	1.0	1.0
1.2	1.2	1.244
1.4	1.488	1.60924
1.6	1.90464	2.16410
1.8	2.51412	3.02455
2.0	3.41921	4.391765

16., 27.

x_n	Euler's Method y_n	Improved Euler Method y_n
1.0	2.0	2.0
1.2	1.2	1.312
1.4	0.624	0.80609
1.6	0.27456	0.46689
1.8	0.09884	0.25698
2.0	0.02768	0.13568

17. a.

$$\begin{aligned}
y_0 &= 1 \\
y_1 &= y_0 + hf(x_0, y_0) \\
&= y_0 + hy_0 = (1+h)y_0 \\
y_2 &= y_1 + hf(x_1, y_1) = y_1 + hy_1 \\
&= (1+h)y_1 = (1+h)^2 y_0 \\
y_3 &= y_2 + hf(x_2, y_2) = y_2 + hy_2 \\
&= (1+h)y_2 = (1+h)^3 y_0 \\
&\vdots \\
y_n &= y_{n-1} + hf(x_{n-1}, y_{n-1}) = y_{n-1} + hy_{n-1} \\
&= (1+h)y_{n-1} = (1+h)^n y_0 = (1+h)^n
\end{aligned}$$

b. Let $N = 1/h$. Then y_N is an approximation to the solution at $x = Nh = (1/h)h = 1$. The exact solution is $y(1) = e$. Thus, $(1+1/N)^N \approx e$ for large *N*. From Chapter 7, we know that $\lim_{N\to\infty}(1+1/N)^N = e$.

18.

$$\begin{aligned}
y_0 &= y(x_0) = 0 \\
y_1 &= y_0 + hf(x_0) = 0 + hf(x_0) = hf(x_0) \\
y_2 &= y_1 + hf(x_1) = hf(x_0) + hf(x_1) \\
&= h\left(f(x_0) + f(x_1)\right) \\
y_3 &= y_2 + hf(x_2) \\
&= h\left[f(x_0) + f(x_1)\right] + hf(x_2) \\
&= h\left[f(x_0) + f(x_1) + f(x_2)\right] = h\sum_{i=0}^{3-1} f(x_i)
\end{aligned}$$

At the *n*th step of Euler's method,

$$y_n = y_{n-1} + hf(x_{n-1}) = h\sum_{i=0}^{n-1} f(x_i)$$

19. a.

$$\begin{aligned}
&\int_{x_0}^{x_1} y'(x)dx = \int_{x_0}^{x_1} \sin x^2 dx \\
&y(x_1) - y(x_0) \approx (x_1 - x_0)\sin x_0^2 \\
&y(x_1) - y(0) = h\sin x_0^2 \\
&y(x_1) - 0 \approx 0.1\sin 0^2 \\
&y(x_1) \approx 0
\end{aligned}$$

b.

$$\begin{aligned}
&\int_{x_0}^{x_2} y'(x)dx = \int_{x_0}^{x_2} \sin x^2 dx \\
&y(x_2) - y(x_0) \approx (x_1 - x_0)\sin x_0^2 \\
&\qquad + (x_2 - x_1)\sin x_1^2 \\
&y(x_2) - y(0) = h\sin x_0^2 + h\sin x_1^2 \\
&y(x_2) - 0 \approx 0.1\sin 0^2 + 0.1\sin 0.1^2 \\
&y(x_2) \approx 0.00099998
\end{aligned}$$

c. $\int_{x_0}^{x_3} y'(x)dx = \int_{x_0}^{x_3} \sin x^2 dx$

$y(x_3) - y(x_0) \approx (x_1 - x_0)\sin x_0^2$
$\quad + (x_2 - x_1)\sin x_1^2 + (x_3 - x_2)\sin x_1^2$

$y(x_3) - y(0) = h\sin x_0^2 + h\sin x_1^2 + h\sin x_2^2$

$y(x_3) - 0 \approx 0.1\sin 0^2 + 0.1\sin 0.1^2$
$\quad + 0.1\sin 0.2^2$

$y(x_3) \approx 0.004999$

Continuing in this fashion, we have

$\int_{x_0}^{x_n} y'(x)dx = \int_{x_0}^{x_n} \sin x^2 dx$

$y(x_n) - y(x_0) \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i)\sin x_i^2$

$y(x_n) \approx h\sum_{i=0}^{n-1} f(x_{i-1})$

When $n = 10$, this becomes

$y(x_{10}) = y(1) \approx 0.269097$

d. The result $y(x_n) \approx h\sum_{i=0}^{n-1} f(x_{i-1})$ is the same as that given in Problem 18. Thus, when $f(x, y)$ depends only on x, then the two methods (1) Euler's method for approximating the solution to $y' = f(x)$ at x_n, and (2) the left-endpoint Riemann sum for approximating $\int_0^{x_n} f(x)\,dx$, are equivalent.

20. a. $\int_{x_0}^{x_1} y'(x)dx = \int_{x_0}^{x_1} \sqrt{x+1}\,dx$

$y(x_1) - y(x_0) \approx (x_1 - x_0)\sqrt{x_0 + 1}$

$y(x_1) - y(0) = h\sqrt{x_0 + 1}$

$y(x_1) - 0 \approx 0.1\sqrt{0+1}$

$y(x_1) \approx 0.1$

b. $\int_{x_0}^{x_2} y'(x)dx = \int_{x_0}^{x_2} \sqrt{x+1}\,dx$

$y(x_2) - y(x_0) \approx (x_1 - x_0)\sqrt{x_0 + 1}$
$\quad + (x_2 - x_1)\sqrt{x_1 + 1}$

$y(x_2) - y(0) = h\sqrt{x_0 + 1} + h\sqrt{x_1 + 1}$

$y(x_2) - 0 \approx 0.1\sqrt{0+1} + 0.1\sqrt{0.1+1}$

$y(x_2) \approx 0.204881$

c. $\int_{x_0}^{x_3} y'(x)dx = \int_{x_0}^{x_3} \sqrt{x+1}\,dx$

$y(x_3) - y(x_0) \approx (x_1 - x_0)\sqrt{x_0 + 1}$
$\quad + (x_2 - x_1)\sqrt{x_1 + 1} + (x_3 - x_2)\sqrt{x_2 + 1}$

$y(x_3) - y(0) = 0.1\sqrt{0+1} + 0.1\sqrt{0.1+1}$
$\quad + 0.1\sqrt{0.2+1}$

$y(x_3) \approx 0.314425$

Continuing in this fashion, we have

$\int_{x_0}^{x_n} y'(x)dx = \int_{x_0}^{x_n} \sqrt{x+1}\,dx$

$y(x_n) - y(x_0) \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i)\sqrt{x_{i-1} + 1}$

$y(x_n) \approx h\sum_{i=0}^{n-1} \sqrt{x_{i-1} + 1}$

When $n = 10$, this becomes

$y(x_{10}) = y(1) \approx 1.198119$

21. a. $\frac{\Delta y}{\Delta x} = \frac{1}{2}[f(x_0, y_0) + f(x_1 + \hat{y}_1)]$

b. $\frac{y_1 - y_0}{h} = \frac{\Delta y}{\Delta x} = \frac{1}{2}[f(x_0, y_0) + f(x_1 + \hat{y}_1)] \Rightarrow$

$2(y_1 - y_0) = h[f(x_0, y_0) + f(x_1 + \hat{y}_1)] \Rightarrow$

$y_1 - y_0 = \frac{h}{2}[f(x_0, y_0) + f(x_1 + \hat{y}_1)] \Rightarrow$

$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1 + \hat{y}_1)]$

c. 1. $x_{n-1} + h$

2. $y_{n-1} + hf(x_{n-1}, y_{n-1})$

3. $y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, \hat{y}_n)]$

22-27. See problems 11-16

28.

h	Error from Euler's Method	Error from Improved Euler Method
0.2	0.229962	0.015574
0.1	0.124539	0.004201
0.05	0.064984	0.001091
0.01	0.013468	0.000045
0.005	0.006765	0.000011

For Euler's method, the error is halved as the step size *h* is halved. Thus, the error is proportional to *h*. For the improved Euler method, when *h* is halved, the error decreases to approximately one-fourth of what is was. Hence, for the improved Euler method, the error is proportional to h^2

6.8 Concepts Review

1. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; arcsin

2. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; arctan

3. 1

4. π

Problem Set 6.8

1. $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ since $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

3. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ since $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

4. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

5. $\arctan(\sqrt{3}) = \frac{\pi}{3}$ since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

6. $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so $\sec\left(\frac{\pi}{3}\right) = 2$

7. $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ since $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

8. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$ since $\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

9. $\sin(\sin^{-1} 0.4567) = 0.4567$ by definition

10. $\cos(\sin^{-1} 0.56) = \sqrt{1-\sin^2(\sin^{-1} 0.56)}$
$= \sqrt{1-(0.56)^2} \approx 0.828$

11. $\sin^{-1}(0.1113) \approx 0.1115$

12. $\arccos(0.6341) \approx 0.8840$

13. $\cos(\text{arccot}\, 3.212) = \cos\left(\arctan\frac{1}{3.212}\right)$
$\approx \cos 0.3018 \approx 0.9548$

14. $\sec(\arccos 0.5111) = \frac{1}{\cos(\arccos 0.5111)}$
$= \frac{1}{0.5111} \approx 1.957$

15. $\sec^{-1}(-2.222) = \cos^{-1}\left(\frac{1}{-2.222}\right) \approx 2.038$

16. $\tan^{-1}(-60.11) \approx -1.554$

17. $\cos(\sin(\tan^{-1} 2.001)) \approx 0.6259$

18. $\sin^2(\ln(\cos 0.5555)) \approx 0.02632$

19. $\theta = \sin^{-1}\frac{x}{8}$

20. $\theta = \tan^{-1}\frac{x}{6}$

21. $\theta = \sin^{-1}\frac{5}{x}$

22. $\theta = \cos^{-1}\frac{9}{x}$ or $\theta = \sec^{-1}\frac{x}{9}$

23. Let θ_1 be the angle opposite the side of length 3, and $\theta_2 = \theta_1 - \theta$, so $\theta = \theta_1 - \theta_2$. Then $\tan\theta_1 = \frac{3}{x}$ and $\tan\theta_2 = \frac{1}{x}$. $\theta = \tan^{-1}\frac{3}{x} - \tan^{-1}\frac{1}{x}$.

24. Let θ_1 be the angle opposite the side of length 5, and $\theta_2 = \theta_1 - \theta$, and y the length of the unlabeled side. Then $\theta = \theta_1 - \theta_2$ and $y = \sqrt{x^2 - 25}$.
$\tan\theta_1 = \frac{5}{y} = \frac{5}{\sqrt{x^2-25}}$, $\tan\theta_2 = \frac{2}{y} = \frac{2}{\sqrt{x^2-25}}$,
$\theta = \tan^{-1}\frac{5}{\sqrt{x^2-25}} - \tan^{-1}\frac{2}{\sqrt{x^2-25}}$

25. $\cos\left[2\sin^{-1}\left(-\frac{2}{3}\right)\right] = 1 - 2\sin^2\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$
$= 1 - 2\left(-\frac{2}{3}\right)^2 = \frac{1}{9}$

26. $\tan\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]=\frac{2\tan\left[\tan^{-1}\left(\frac{1}{3}\right)\right]}{1-\tan^2\left[\tan^{-1}\left(\frac{1}{3}\right)\right]}$

$=\frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}=\frac{3}{4}$

27. $\sin\left[\cos^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{5}{13}\right)\right]=\sin\left[\cos^{-1}\left(\frac{3}{5}\right)\right]\cos\left[\cos^{-1}\left(\frac{5}{13}\right)\right]+\cos\left[\cos^{-1}\left(\frac{3}{5}\right)\right]\sin\left[\cos^{-1}\left(\frac{5}{13}\right)\right]$

$=\sqrt{1-\left(\frac{3}{5}\right)^2}\cdot\frac{5}{13}+\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}=\frac{56}{65}$

28. $\cos\left[\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{12}{13}\right)\right]=\cos\left[\cos^{-1}\left(\frac{4}{5}\right)\right]\cos\left[\sin^{-1}\left(\frac{12}{13}\right)\right]-\sin\left[\cos^{-1}\left(\frac{4}{5}\right)\right]\sin\left[\sin^{-1}\left(\frac{12}{13}\right)\right]$

$=\frac{4}{5}\cdot\sqrt{1-\left(\frac{12}{13}\right)^2}-\sqrt{1-\left(\frac{4}{5}\right)^2}\cdot\frac{12}{13}=-\frac{16}{65}$

29. $\tan(\sin^{-1}x)=\frac{\sin(\sin^{-1}x)}{\cos(\sin^{-1}x)}=\frac{x}{\sqrt{1-x^2}}$

30. $\sin(\tan^{-1}x)=\frac{1}{\csc(\tan^{-1}x)}=\frac{1}{\sqrt{1+\cot^2(\tan^{-1}x)}}$

$=\frac{1}{\sqrt{1+\frac{1}{\tan^2(\tan^{-1}x)}}}=\frac{1}{\sqrt{1+\frac{1}{x^2}}}=\frac{x}{\sqrt{x^2+1}}$

31. $\cos(2\sin^{-1}x)=1-2\sin^2(\sin^{-1}x)=1-2x^2$

32. $\tan(2\tan^{-1}x)=\frac{2\tan(\tan^{-1}x)}{1-\tan^2(\tan^{-1}x)}=\frac{2x}{1-x^2}$

33. **a.** $\lim\limits_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}$ since $\lim\limits_{\theta\to\pi/2^-}\tan\theta=\infty$

b. $\lim\limits_{x\to-\infty}\tan^{-1}x=-\frac{\pi}{2}$ since $\lim\limits_{\theta\to-\pi/2^+}\tan\theta=-\infty$

34. **a.** $\lim\limits_{x\to\infty}\sec^{-1}x=\lim\limits_{x\to\infty}\cos^{-1}\left(\frac{1}{x}\right)$

$=\lim\limits_{z\to0^+}\cos^{-1}z=\frac{\pi}{2}$

b. $\lim\limits_{x\to-\infty}\sec^{-1}x=\lim\limits_{x\to-\infty}\cos^{-1}\left(\frac{1}{x}\right)$

$=\lim\limits_{z\to0^-}\cos^{-1}z=\frac{\pi}{2}$

35. **a.** Let $L=\lim\limits_{x\to1^-}\sin^{-1}x$. Since

$\sin(\sin^{-1}x)=x$, $\lim\limits_{x\to1^-}\sin(\sin^{-1}x)=\lim\limits_{x\to1^-}x=1$.

Thus, since sin is continuous, the Composite Limit Theorem gives us

$\lim\limits_{x\to1^-}\sin(\sin^{-1}x)=\lim\limits_{x\to1^-}\sin(L)$; hence

$\sin L=1$ and since the range of $\sin^{-1}$ is

$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $L=\frac{\pi}{2}$.

b. Let $L = \lim_{x \to -1^+} \sin^{-1} x$. Since

$\sin(\sin^{-1} x) = x,$

$$\lim_{x \to -1^+} \sin(\sin^{-1} x) = \lim_{x \to -1^+} x = -1.$$

Thus, since sin is continuous, the Composite Limit Theorem gives us

$$\lim_{x \to -1^+} \sin(\sin^{-1} x) = \lim_{x \to -1^+} \sin(L);$$

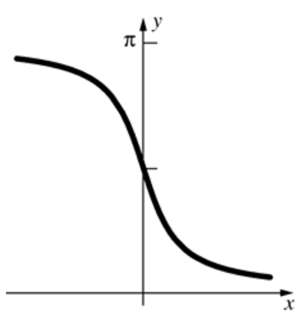
hence

$\sin L = -1$ and since the range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $L = -\frac{\pi}{2}$.

36. No. Since $\sin^{-1} x$ is not defined on $(1, \infty)$, $\lim_{x \to 1^+} \sin^{-1} x$ does not exist so neither can the two-sided limit $\lim_{x \to 1} \sin^{-1} x$.

37. Let $f(x) = y = \sin^{-1} x$; then the slope of the tangent line to the graph of y at c is $f'(c) = \frac{1}{\sqrt{1-c^2}}$. Hence, $\lim_{c \to 1^-} f'(c) = \infty$ so that the tangent lines approach the vertical.

38.



39. $y = \ln(2 + \sin x)$. Let $u = 2 + \sin x$; then $y = \ln u$ so by the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{u}\right)\frac{du}{dx} = \left(\frac{1}{2+\sin x}\right) \cdot \cos x$$

$$= \frac{\cos x}{2 + \sin x}$$

40. $$\frac{d}{dx} e^{\tan x} = e^{\tan x} \frac{d}{dx} \tan x = e^{\tan x} \sec^2 x$$

41. $$\frac{d}{dx} \ln(\sec x + \tan x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{(\sec x)(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

42. $$\frac{d}{dx}[-\ln(\csc x + \cot x)] = -\frac{-\csc x \cot x - \csc^2 x}{\csc x + \cot x}$$

$$= \frac{\csc x(\cot x + \csc x)}{\cot x + \csc x} = \csc x$$

43. $$\frac{d}{dx} \sin^{-1}(2x^2) = \frac{1}{\sqrt{1-(2x^2)^2}} \cdot 4x = \frac{4x}{\sqrt{1-4x^4}}$$

44. $$\frac{d}{dx} \arccos(e^x) = -\frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = -\frac{e^x}{\sqrt{1-e^{2x}}}$$

45. $$\frac{d}{dx}[x^3 \tan^{-1}(e^x)] = x^3 \cdot \frac{e^x}{1+(e^x)^2} + 3x^2 \tan^{-1}(e^x)$$

$$= x^2 \left[\frac{xe^x}{1+e^{2x}} + 3\tan^{-1}(e^x)\right]$$

46. $$\frac{d}{dx}(e^x \arcsin x^2) = e^x \cdot \frac{2x}{\sqrt{1-(x^2)^2}} + e^x \arcsin x^2$$

$$= e^x \left(\frac{2x}{\sqrt{1-x^4}} + \arcsin x^2\right)$$

47. $$\frac{d}{dx}(\tan^{-1} x)^3 = 3(\tan^{-1} x)^2 \cdot \frac{1}{1+x^2} = \frac{3(\tan^{-1} x)^2}{1+x^2}$$

48. $$\frac{d}{dx} \tan(\cos^{-1} x) = \frac{d}{dx} \frac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x)} = \frac{d}{dx} \frac{\sqrt{1-x^2}}{x}$$

$$= \frac{x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}}(-2x) - \sqrt{1-x^2} \cdot 1}{x^2}$$

$$= \frac{-x^2 - (1-x^2)}{x^2\sqrt{1-x^2}} = -\frac{1}{x^2\sqrt{1-x^2}}$$

49. $\frac{d}{dx}\sec^{-1}(x^3)=\frac{1}{|x^3|\sqrt{(x^3)^2-1}}\cdot 3x^2=\frac{3}{|x|\sqrt{x^6-1}}$

50. $\frac{d}{dx}(\sec^{-1}x)^3=3(\sec^{-1}x)^2\cdot\frac{1}{|x|\sqrt{x^2-1}}$

$=\frac{3(\sec^{-1}x)^2}{|x|\sqrt{x^2-1}}$

51. $\frac{d}{dx}(1+\sin^{-1}x)^3=3(1+\sin^{-1}x)^2\cdot\frac{1}{\sqrt{1-x^2}}$

$=\frac{3(1+\sin^{-1}x)^2}{\sqrt{1-x^2}}$

52. $y=\sin^{-1}\left(\frac{1}{x^2+4}\right)$

Let $u=\frac{1}{x^2+4}$; then $y=\sin^{-1}(u(x))$ so by the Chain Rule:

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}=$

$\frac{1}{\sqrt{1-\left(\frac{1}{x^2+4}\right)^2}}\cdot\left(\frac{-2x}{(x^2+4)^2}\right)=$

$\left(\frac{(x^2+4)}{\sqrt{x^4+8x^2+15}}\right)\cdot\left(\frac{-2x}{(x^2+4)^2}\right)=$

$\frac{-2x}{(x^2+4)\sqrt{x^4+8x^2+15}}$

53. $y=\tan^{-1}\left(\ln x^2\right)$

Let $u=x^2$, $v=\ln u$; then $y=\tan^{-1}(v(u(x)))$ so by the Chain Rule:

$\frac{dy}{dx}=\frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}=\frac{1}{1+v^2}\cdot\frac{1}{u}\cdot 2x=$

$\frac{1}{1+(\ln x^2)^2}\cdot\frac{1}{x^2}\cdot 2x=$

$\frac{2}{x[1+(\ln x^2)^2]}$

54. $y=x\,\mathrm{arc}\sec(x^2+1)$

$\frac{dy}{dx}=x\left[\frac{d}{dx}\mathrm{arcsec}(x^2+1)\right]+\left(\frac{d}{dx}x\right)\cdot\mathrm{arcsec}(x^2+1)$

$=x\left[\frac{2x}{\left(x^2+1\right)\sqrt{(x^2+1)^2-1}}\right]+1\cdot\mathrm{arcsec}(x^2+1)$

$=\left[\frac{2x^2}{\left(x^2+1\right)\sqrt{x^4+2x^2}}\right]+\mathrm{arcsec}(x^2+1)$

$=\left[\frac{2x^2}{\left(x^2+1\right)\cdot|x|\sqrt{x^2+2}}\right]+\mathrm{arcsec}(x^2+1)$

$=\left[\frac{2|x|}{\left(x^2+1\right)\sqrt{x^2+2}}\right]+\mathrm{arcsec}(x^2+1)$

55. $\int\cos 3x\,dx$

Let $u=3x,\ du=3dx$; then

$\int\cos 3x\,dx=\frac{1}{3}\int\cos 3x\,(3dx)=$

$\frac{1}{3}\int\cos u\,du=\frac{1}{3}\sin u+C=\frac{1}{3}\sin 3x+C$

56. Let $u=x^2$, so $du=2x\,dx$.

$\int x\sin(x^2)dx=\frac{1}{2}\int\sin(x^2)\cdot 2x\,dx$

$=\frac{1}{2}\int\sin u\,du=-\frac{1}{2}\cos u+C$

$=-\frac{1}{2}\cos(x^2)+C$

57. Let $u=\sin 2x$, so $du=2\cos 2x\,dx$.

$\int\sin 2x\cos 2x\,dx=\frac{1}{2}\int\sin 2x(2\cos 2x)dx$

$=\frac{1}{2}\int u\,du$

$=\frac{u^2}{4}+C=\frac{1}{4}\sin^2 2x+C$

58. Let $u=\cos x$, so $du=-\sin x\,dx$.

$\int\tan x\,dx=\int\frac{\sin x}{\cos x}dx=-\int\frac{1}{\cos x}(-\sin x)dx$

$=-\int\frac{1}{u}du=-\ln|u|+C=-\ln|\cos x|+C$

$=\ln|\sec x|+C$

59. Let $u = e^{2x}$, so $du = 2e^{2x}dx$.

$$\int e^{2x}\cos(e^{2x})dx = \frac{1}{2}\int \cos(e^{2x})(2e^{2x})dx$$

$$= \frac{1}{2}\int \cos u\, du$$

$$= \frac{1}{2}\sin u + C = \frac{1}{2}\sin(e^{2x}) + C$$

$$\int_0^1 e^{2x}\cos(e^{2x})\,dx = \left[\frac{1}{2}\sin(e^{2x})\right]_0^1$$

$$= \left[\frac{1}{2}\sin(e^2) - \frac{1}{2}\sin(e^0)\right]$$

$$= \frac{\sin e^2 - \sin 1}{2} \approx 0.0262$$

60. Let $u = \sin x$, so $du = \cos x\, dx$.

$$\int \sin^2 x\cos x\,dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

$$\int_0^{\pi/2}\sin^2 x\cos x\,dx = \left[\frac{\sin^3 x}{3}\right]_0^{\pi/2} = \frac{1}{3} - 0 = \frac{1}{3}$$

61. $\int_0^{\sqrt{2}/2}\frac{1}{\sqrt{1-x^2}}dx = [\arcsin x]_0^{\sqrt{2}/2}$

$$= \arcsin\frac{\sqrt{2}}{2} - \arcsin 0 = \frac{\pi}{4}$$

62. $\int_{\sqrt{2}}^{2}\frac{dx}{x\sqrt{x^2-1}} = \int_{\sqrt{2}}^{2}\frac{dx}{|x|\sqrt{x^2-1}} = \left[\sec^{-1} x\right]_{\sqrt{2}}^{2}$

$$= \sec^{-1} 2 - \sec^{-1}\sqrt{2}$$

$$= \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

63. $\int_{-1}^{1}\frac{1}{1+x^2}dx = \left[\tan^{-1} x\right]_{-1}^{1} = \tan^{-1}1 - \tan^{-1}(-1)$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

64. Let $u = \cos\theta$, so $du = -\sin\theta\, d\theta$.

$$\int \frac{\sin\theta}{1+\cos^2\theta}d\theta = -\int\frac{1}{1+\cos^2\theta}(-\sin\theta)d\theta$$

$$= -\int\frac{1}{1+u^2}du = -\tan^{-1}u + C$$

$$= -\tan^{-1}(\cos\theta) + C$$

$$\int_0^{\pi/2}\frac{\sin\theta}{1+\cos^2\theta}d\theta = \left[-\tan^{-1}(\cos\theta)\right]_0^{\pi/2}$$

$$= -\tan^{-1}0 + \tan^{-1}1 = -0 + \frac{\pi}{4} = \frac{\pi}{4}$$

65. Let $u = 2x$, so $du = 2\,dx$.

$$\int\frac{1}{1+4x^2}dx = \frac{1}{2}\int\frac{1}{1+(2x)^2}2dx$$

$$= \frac{1}{2}\int\frac{1}{1+u^2}du = \frac{1}{2}\arctan u + C$$

$$= \frac{1}{2}\arctan 2x + C$$

66. Let $u = e^x$, so $du = e^x dx$.

$$\int\frac{e^x}{1+e^{2x}}dx = \int\frac{e^x}{1+(e^x)^2}dx = \int\frac{1}{1+u^2}du$$

$= \arctan u + C = \arctan e^x + C$

67. $\int\frac{1}{\sqrt{12-9x^2}}dx = \int\frac{1}{\sqrt{12\left(1-\frac{3}{4}x^2\right)}}dx$

$$= \frac{1}{2\sqrt{3}}\int\frac{1}{\sqrt{1-\left(\frac{\sqrt{3}}{2}x\right)^2}}dx$$

Let $u = \frac{\sqrt{3}}{2}x,\ du = \frac{\sqrt{3}}{2}dx$; then

$$\frac{1}{2\sqrt{3}}\int\frac{1}{\sqrt{1-\left(\frac{\sqrt{3}}{2}x\right)^2}}dx = \frac{1}{2\sqrt{3}}\left(\frac{2}{\sqrt{3}}\right)\int\frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\sin^{-1}u + C = \frac{1}{3}\sin^{-1}\left(\frac{\sqrt{3}}{2}x\right) + C$$

68. $\int\frac{x}{\sqrt{12-9x^2}}dx$. Let $u = 12-9x^2,\ du = -18x\,dx$;

then

$$\int\frac{x}{\sqrt{12-9x^2}}dx = -\frac{1}{18}\int\frac{1}{\sqrt{12-9x^2}}(-18\,dx)$$

$$= -\frac{1}{18}\int\frac{1}{\sqrt{u}}du = \left(-\frac{1}{18}\right)(2\sqrt{u}) + C$$

$$= -\frac{\sqrt{12-9x^2}}{9} + C$$

69. $\int \frac{1}{x^2-6x+13}dx = \int \frac{1}{(x^2-6x+9)+4}dx$

$= \int \frac{1}{(x-3)^2+4}dx$

Let $u = x-3,\ du = dx,\ a = 2;$ then

$\int \frac{1}{(x-3)^2+4}dx = \int \frac{1}{u^2+a^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$

$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C$

70. $\int \frac{1}{2x^2+8x+25}dx = \int \frac{1}{2(x^2+4x+4+\frac{17}{2})}dx =$

$\frac{1}{2}\int \frac{1}{(x+2)^2+\left(\sqrt{\frac{17}{2}}\right)^2}dx$

Let $u = x+2,\ du = dx,\ a = \sqrt{\frac{17}{2}};$ then

$\frac{1}{2}\int \frac{1}{(x+2)^2+\frac{17}{2}}dx = \frac{1}{2}\int \frac{1}{u^2+a^2}du =$

$\frac{1}{2}\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C = \frac{1}{2}\cdot\frac{\sqrt{2}}{\sqrt{17}}\tan^{-1}\left(\frac{x+2}{\sqrt{17/2}}\right)+C$

$= \frac{\sqrt{34}}{34}\tan^{-1}\left[\frac{\sqrt{34}\cdot(x+2)}{17}\right]+C$

71. $\int \frac{1}{x\sqrt{4x^2-9}}dx$. Let $u = 2x,\ du = 2\,dx, a = 3$;

then $\int \frac{1}{x\sqrt{4x^2-9}}dx = \int \frac{1}{2x\sqrt{4x^2-9}}(2\,dx) =$

$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right)+C =$

$\frac{1}{3}\sec^{-1}\left(\frac{2|x|}{3}\right)+C$

72. $\int \frac{x+1}{\sqrt{4-9x^2}}dx = \int \frac{x}{\sqrt{4-9x^2}}dx + \int \frac{1}{\sqrt{4-9x^2}}dx$

These integrals are evaluated the same as those in problems 67 and 68 (with a constant of 4 rather than 12). Thus

$\int \frac{x+1}{\sqrt{4-9x^2}}dx = -\frac{1}{9}\sqrt{4-9x^2}+\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C$

73. The top of the picture is 7.6 ft above eye level, and the bottom of the picture is 2.6 ft above eye level. Let θ_1 be the angle between the viewer's line of sight to the top of the picture and the horizontal. Then call $\theta_2 = \theta_1 - \theta$, so $\theta = \theta_1 - \theta_2$.

$\tan\theta_1 = \frac{7.6}{b};\ \tan\theta_2 = \frac{2.6}{b};$

$\theta = \tan^{-1}\frac{7.6}{b} - \tan^{-1}\frac{2.6}{b}$

If $b = 12.9$, $\theta \approx 0.3335$ or $19.1°$.

74. a. Restrict $2x$ to $[0, \pi]$, i.e., restrict x to $\left[0, \frac{\pi}{2}\right]$.

Then $y = 3\cos 2x$

$\frac{y}{3} = \cos 2x$

$2x = \arccos\frac{y}{3}$

$x = f^{-1}(y) = \frac{1}{2}\arccos\frac{y}{3}$

$f^{-1}(x) = \frac{1}{2}\arccos\frac{x}{3}$

b. Restrict $3x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e., restrict x to $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

Then $y = 2\sin 3x$

$\frac{y}{2} = \sin 3x$

$3x = \arcsin\frac{y}{2}$

$x = f^{-1}(y) = \frac{1}{3}\arcsin\frac{y}{2}$

$f^{-1}(x) = \frac{1}{3}\arcsin\frac{x}{2}$

c. Restrict x to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = \frac{1}{2}\tan x$

$2y = \tan x$

$x = f^{-1}(y) = \arctan 2y$

$f^{-1}(x) = \arctan 2x$

d. Restrict x to $\left(-\infty, -\frac{2}{\pi}\right) \cup \left(\frac{2}{\pi}, \infty\right)$ so $\frac{1}{x}$ is restricted to $\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$

then $y = \sin\frac{1}{x}$

$\frac{1}{x} = \arcsin y$

$x = f^{-1}(y) = \frac{1}{\arcsin y}$

$f^{-1}(x) = \frac{1}{\arcsin x}$

75. $\tan\left[2\tan^{-1}\left(\frac{1}{4}\right)\right] = \dfrac{2\tan\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}{1-\tan^2\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}$

$= \dfrac{2\cdot\frac{1}{4}}{1-\left(\frac{1}{4}\right)^2} = \dfrac{8}{15}$

$\tan\left[3\tan^{-1}\left(\frac{1}{4}\right)\right] = \tan\left[2\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{4}\right)\right]$

$= \dfrac{\tan\left[2\tan^{-1}\left(\frac{1}{4}\right)\right]+\tan\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}{1-\tan\left[2\tan^{-1}\left(\frac{1}{4}\right)\right]\tan\left[\tan^{-1}\left(\frac{1}{4}\right)\right]}$

$= \dfrac{\frac{8}{15}+\frac{1}{4}}{1-\frac{8}{15}\cdot\frac{1}{4}} = \dfrac{47}{52}$

$\tan\left[3\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{5}{99}\right)\right]$

$= \dfrac{\tan\left[3\tan^{-1}\left(\frac{1}{4}\right)\right]+\tan\left[\tan^{-1}\left(\frac{5}{99}\right)\right]}{1-\tan\left[3\tan^{-1}\left(\frac{1}{4}\right)\right]\tan\left[\tan^{-1}\left(\frac{5}{99}\right)\right]}$

$= \dfrac{\frac{47}{52}+\frac{5}{99}}{1-\frac{47}{52}\cdot\frac{5}{99}} = \dfrac{4913}{4913} = 1 = \tan\frac{\pi}{4}$

Thus, $3\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{5}{99}\right) = \tan^{-1}(1) = \frac{\pi}{4}$.

76. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right] = \dfrac{2\tan\left[\tan^{-1}\left(\frac{1}{5}\right)\right]}{1-\tan^2\left[\tan^{-1}\left(\frac{1}{5}\right)\right]}$

$= \dfrac{2\cdot\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2} = \dfrac{5}{12}$

$\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)\right] = \tan\left[2\cdot 2\tan^{-1}\left(\frac{1}{5}\right)\right]$

$= \dfrac{2\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right]}{1-\tan^2\left[2\tan^{-1}\left(\frac{1}{5}\right)\right]} = \dfrac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} = \dfrac{120}{119}$

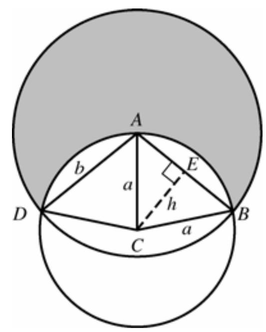
$\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right)\right]$

$= \dfrac{\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)\right]-\tan\left[\tan^{-1}\left(\frac{1}{239}\right)\right]}{1+\tan\left[4\tan^{-1}\left(\frac{1}{5}\right)\right]\tan\left[\tan^{-1}\left(\frac{1}{239}\right)\right]}$

$= \dfrac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}} = \dfrac{28,561}{28,561} = 1 = \tan\frac{\pi}{4}$

Thus, $4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

77.



Let θ represent $\angle DAB$, then $\angle CAB$ is $\frac{\theta}{2}$. Since ΔABC is isosceles, $|AE| = \frac{b}{2}$, $\cos\frac{\theta}{2} = \frac{\frac{b}{2}}{a} = \frac{b}{2a}$ and $\theta = 2\cos^{-1}\frac{b}{2a}$. Thus sector ADB has area $\frac{1}{2}\left(2\cos^{-1}\frac{b}{2a}\right)b^2 = b^2\cos^{-1}\frac{b}{2a}$. Let ϕ represent $\angle DCB$, then $\angle ACB$ is $\frac{\phi}{2}$ and $\angle ECA$ is $\frac{\phi}{4}$, so $\sin\frac{\phi}{4} = \frac{\frac{b}{2}}{a} = \frac{b}{2a}$ and $\phi = 4\sin^{-1}\frac{b}{2a}$. Thus sector DCB has area $\frac{1}{2}\left(4\sin^{-1}\frac{b}{2a}\right)a^2 = 2a^2\sin^{-1}\frac{b}{2a}$.

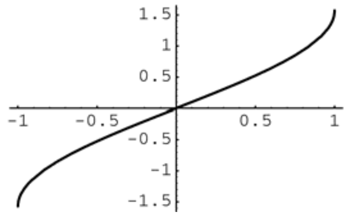
These sectors overlap on the triangles ΔDAC and ΔCAB, each of which has area

$\frac{1}{2}|AB|h = \frac{1}{2}b\sqrt{a^2-\left(\frac{b}{2}\right)^2} = \frac{1}{2}b\frac{\sqrt{4a^2-b^2}}{2}$.

The large circle has area πb^2, hence the shaded region has area

$\pi b^2 - b^2\cos^{-1}\frac{b}{2a} - 2a^2\sin^{-1}\frac{b}{2a} + \frac{1}{2}b\sqrt{4a^2-b^2}$

78.



They have the same graph.

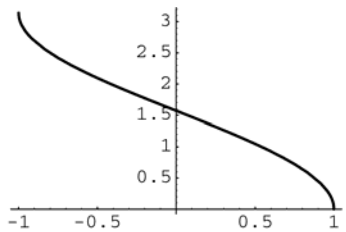
Conjecture: $\arcsin x = \arctan \dfrac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

Proof: Let $\theta = \arcsin x$, so $x = \sin\theta$.

Then $\dfrac{x}{\sqrt{1-x^2}} = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

so $\theta = \arctan \dfrac{x}{\sqrt{1-x^2}}$.

79.



It is the same graph as $y = \arccos x$.

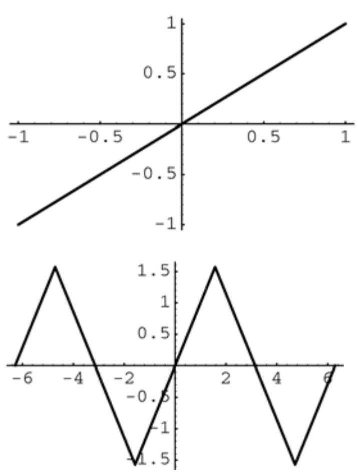
Conjecture: $\dfrac{\pi}{2} - \arcsin x = \arccos x$

Proof: Let $\theta = \dfrac{\pi}{2} - \arcsin x$

Then $x = \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

so $\theta = \arccos x$.

80.



$y = \sin(\arcsin x)$ is the line $y = x$, but only defined for $-1 \le x \le 1$.

$y = \arcsin(\sin x)$ is defined for all x, but only the portion for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ is the line $y = x$.

81. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dx}{\sqrt{a^2\left[1-\left(\frac{x}{a}\right)^2\right]}}$

$\displaystyle = \int \frac{1}{|a|} \cdot \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{1}{a} \cdot \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$ since $a > 0$

Let $u = \dfrac{x}{a}$, so $du = \dfrac{1}{a}dx$.

$\displaystyle\int \frac{1}{a} \cdot \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$

$= \sin^{-1} \dfrac{x}{a} + C$

82. $D_x \sin^{-1} \dfrac{x}{a} = \dfrac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \dfrac{1}{a}$

$= \dfrac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}} \cdot \dfrac{1}{a} = \dfrac{|a|}{\sqrt{a^2 - x^2}} \cdot \dfrac{1}{a}$

$= \dfrac{a}{\sqrt{a^2 - x^2}} \cdot \dfrac{1}{a}$, since $a > 0$

$= \dfrac{1}{\sqrt{a^2 - x^2}}$

83. Let $u = \dfrac{x}{a}$, so $du = \dfrac{1}{a}dx$

$\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \int \frac{1}{1+\left(\frac{x}{a}\right)^2} \frac{1}{a} dx$

$\displaystyle = \frac{1}{a} \int \frac{1}{1+u^2} du = \frac{1}{a} \tan^{-1} u + C$

$= \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right) + C$

84. Let $u=\frac{x}{a}$, so $du=(1/a)dx$. Since $a>0$,

$$\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\int \frac{1}{\left(\frac{x}{a}\right)\sqrt{\left(\frac{x}{a}\right)^2-1}}\frac{1}{a}dx$$

$$=\frac{1}{a}\int \frac{1}{u\sqrt{u^2-1}}du$$

$$=\frac{1}{a}\sec^{-1}|u|+C=\frac{1}{a}\sec^{-1}\frac{|x|}{a}+C$$

85. Note that $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$ (See Problem 67).

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C\right]$$

$$=\frac{1}{2}\sqrt{a^2-x^2}+\frac{x}{2}\frac{1}{2\sqrt{a^2-x^2}}(-2x)$$

$$+\frac{a^2}{2}\frac{1}{\sqrt{a^2-x^2}}+0$$

$$=\frac{1}{2}\sqrt{a^2-x^2}+\frac{1}{2}\frac{-x^2+a^2}{\sqrt{a^2-x^2}}=\sqrt{a^2-x^2}$$

86. $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=2\int_0^a\sqrt{a^2-x^2}\,dx$

$$=2\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=2\left[\frac{a}{2}(0)+\frac{a^2}{2}\sin^{-1}(1)-\frac{0}{2}\sqrt{a^2}-\frac{a^2}{2}\sin^{-1}(0)\right]$$

$$=a^2\sin^{-1}(1)=\frac{\pi a^2}{2}$$

This result is expected because the integral should be half the area of a circle with radius a.

87. Let θ be the angle subtended by viewer's eye.

$$\theta=\tan^{-1}\left(\frac{12}{b}\right)-\tan^{-1}\left(\frac{2}{b}\right)$$

$$\frac{d\theta}{db}=\frac{1}{1+\left(\frac{12}{b}\right)^2}\left(-\frac{12}{b^2}\right)-\frac{1}{1+\left(\frac{2}{b}\right)^2}\left(-\frac{2}{b^2}\right)$$

$$=\frac{2}{b^2+4}-\frac{12}{b^2+144}=\frac{10(24-b^2)}{(b^2+4)(b^2+144)}$$

Since $\frac{d\theta}{db}>0$ for b in $\left[0,2\sqrt{6}\right)$ and $\frac{d\theta}{db}<0$ for b in $\left(2\sqrt{6},\infty\right)$, the angle is maximized for $b=2\sqrt{6}\approx 4.899$.

The ideal distance is about 4.9 ft from the wall.

88. **a.** $\theta=\cos^{-1}\left(\frac{x}{b}\right)-\cos^{-1}\left(\frac{x}{a}\right)$

$$\frac{d\theta}{dt}=\left(\frac{-1}{\sqrt{1-\left(\frac{x}{b}\right)^2}}\right)\left(\frac{1}{b}\right)\left(\frac{dx}{dt}\right)-\left(\frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right)\left(\frac{1}{a}\right)\left(\frac{dx}{dt}\right)=\left(\frac{1}{\sqrt{a^2-x^2}}-\frac{1}{\sqrt{b^2-x^2}}\right)\frac{dx}{dt}$$

b. $\theta=\tan^{-1}\left(\frac{a+x}{\sqrt{b^2-x^2}}\right)-\sin^{-1}\left(\frac{x}{b}\right)$

$$\frac{d\theta}{dt}=\left(\frac{1}{1+\left(\frac{a+x}{\sqrt{b^2-x^2}}\right)^2}\right)\left(\frac{\sqrt{b^2-x^2}+\frac{(a+x)x}{\sqrt{b^2-x^2}}}{b^2-x^2}\right)\left(\frac{dx}{dt}\right)-\left(\frac{1}{\sqrt{1-\left(\frac{x}{b}\right)^2}}\right)\left(\frac{1}{b}\right)\left(\frac{dx}{dt}\right)$$

$$=\left[\left(\frac{b^2-x^2}{b^2-x^2+(a+x)^2}\right)\left(\frac{b^2+ax}{(b^2-x^2)^{3/2}}\right)-\frac{1}{\sqrt{b^2-x^2}}\right]\frac{dx}{dt}$$

$$=\left[\frac{b^2+ax}{(b^2+a^2+2ax)\sqrt{b^2-x^2}}-\frac{1}{\sqrt{b^2-x^2}}\right]\frac{dx}{dt}=\left[-\frac{a^2+ax}{(b^2+a^2+2ax)\sqrt{b^2-x^2}}\right]\frac{dx}{dt}$$

89. Let $h(t)$ represent the height of the elevator (the number of feet above the spectator's line of sight) t seconds after the line of sight passes horizontal, and let $\theta(t)$ denote the angle of elevation.

Then $h(t) = 15t$, so $\theta(t) = \tan^{-1}\left(\frac{15t}{60}\right) = \tan^{-1}\left(\frac{t}{4}\right)$.

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{t}{4}\right)^2}\left(\frac{1}{4}\right) = \frac{4}{16+t^2}$$

At $t = 6$, $\frac{d\theta}{dt} = \frac{4}{16+6^2} = \frac{1}{13}$ radians per second or about 4.41° per second.

90. Let $x(t)$ be the *horizontal* distance from the observer to the plane, in miles, at time t., in minutes.
Let $t = 0$ when the distance to the plane is 3 miles. Then
$x(0) = \sqrt{3^2 - 2^2} = \sqrt{5}$. The speed of the plane is 10 miles per minute, so $x(t) = \sqrt{5} - 10t$. The angle of elevation is $\theta(t) = \tan^{-1}\left(\frac{2}{x(t)}\right) = \tan^{-1}\left(\frac{2}{\sqrt{5}-10t}\right)$,

so $$\frac{d\theta}{dt} = \frac{1}{1+\left(2/\left(\sqrt{5}-10t\right)\right)^2}\left(\frac{-2}{(\sqrt{5}-10t)^2}\right)(-10)$$
$$= \frac{20}{(\sqrt{5}-10t)^2+4}.$$

When $t = 0$, $\frac{d\theta}{dt} = \frac{20}{9} \approx 2.22$ radians per minute.

91. Let x represent the position on the shoreline and let θ represent the angle of the beam ($x = 0$ and $\theta = 0$ when the light is pointed at P). Then
$\theta = \tan^{-1}\left(\frac{x}{2}\right)$, so $\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{1}{2}\frac{dx}{dt} = \frac{2}{4+x^2}\frac{dx}{dt}$

When $x = 1$,
$\frac{dx}{dt} = 5\pi$, so $\frac{d\theta}{dt} = \frac{2}{4+1^2}(5\pi) = 2\pi$ The beacon revolves at a rate of 2π radians per minute or 1 revolution per minute.

92 Let x represent the length of the rope and let θ represent the angle of depression of the rope.
Then $\theta = \sin^{-1}\left(\frac{8}{x}\right)$, so

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{1-\left(\frac{8}{x}\right)^2}} - \frac{8}{x^2}\frac{dx}{dt} = -\frac{8}{x\sqrt{x^2-64}}\frac{dx}{dt}.$$

When $x = 17$ and $\frac{dx}{dt} = -5$, we obtain

$$\frac{d\theta}{dt} = -\frac{8}{17\sqrt{17^2-64}}(-5) = \frac{8}{51}.$$

The angle of depression is increasing at a rate of $8/51 \approx 0.16$ radians per second.

93. Let x represent the distance to the *center* of the earth and let θ represent the angle subtended by the earth. Then $\theta = 2\sin^{-1}\left(\frac{6376}{x}\right)$, so

$$\frac{d\theta}{dt} = 2\frac{1}{\sqrt{1-\left(\frac{6376}{x}\right)^2}}\left(-\frac{6376}{x^2}\right)\frac{dx}{dt}$$
$$= -\frac{12,752}{x\sqrt{x^2-6376^2}}\frac{dx}{dt}$$

When she is 3000 km from the surface
$x = 3000 + 6376 = 9376$ and $\frac{dx}{dt} = -2$. Substituting these values, we obtain $\frac{d\theta}{dt} \approx 3.96\times 10^{-4}$ radians per second.

6.9 Concepts Review

1. $\frac{e^x - e^{-x}}{2}; \frac{e^x + e^{-x}}{2}$

2. $\cosh^2 x - \sinh^2 x = 1$

3. the graph of $x^2 - y^2 = 1$, a hyperbola

4. catenary; a hanging cable or chain

Problem Set 6.9

1. $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$

$= \frac{2e^x}{2} = e^x$

2. $\cosh 2x + \sinh 2x = \frac{e^{2x} + e^{-2x}}{2} + \frac{e^{2x} - e^{-2x}}{2}$

$= \frac{2e^{2x}}{2} = e^{2x}$

3. $\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}$

$= \frac{2e^{-x}}{2} = e^{-x}$

4. $\cosh 2x - \sinh 2x = \frac{e^{2x} + e^{-2x}}{2} - \frac{e^{2x} - e^{-2x}}{2} = \frac{2e^{-2x}}{2} = e^{-2x}$

5. $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} + \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4}$

$= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$

6. $\sinh x \cosh y - \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} - \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} - \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}}{4}$

$= \frac{2e^{x-y} - 2e^{-x+y}}{4} = \frac{e^{x-y} - e^{-(x-y)}}{2} = \sinh(x-y)$

7. $\cosh x\cosh y+\sinh x\sinh y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}}{4}+\frac{e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}}{4}$

$=\frac{2e^{x+y}+2e^{-x-y}}{4}=\frac{e^{x+y}+e^{-(x+y)}}{2}=\cosh(x+y)$

8. $\cosh x\cosh y-\sinh x\sinh y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}-\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}}{4}-\frac{e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}}{4}$

$=\frac{2e^{x-y}+2e^{-x+y}}{4}=\frac{e^{x-y}+e^{-(x-y)}}{2}=\cosh(x-y)$

9. $\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}=\frac{\frac{\sinh x}{\cosh x}+\frac{\sinh y}{\cosh y}}{1+\frac{\sinh x}{\cosh x}\cdot\frac{\sinh y}{\cosh y}}$

$=\frac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y+\sinh x\sinh y}=\frac{\sinh(x+y)}{\cosh(x+y)}$

$=\tanh\,(x+y)$

10. $\frac{\tanh x-\tanh y}{1-\tanh x\tanh y}=\frac{\frac{\sinh x}{\cosh x}-\frac{\sinh y}{\cosh y}}{1-\frac{\sinh x}{\cosh x}\cdot\frac{\sinh y}{\cosh y}}$

$=\frac{\sinh x\cosh y-\cosh x\sinh y}{\cosh x\cosh y-\sinh x\sinh y}=\frac{\sinh(x-y)}{\cosh(x-y)}$

$=\tanh(x-y)$

11. $2\sinh x\cosh x=\sinh x\cosh x+\cosh x\sinh x$
$=\sinh\,(x+x)=\sinh 2x$

12. $\cosh^2 x+\sinh^2 x=\cosh x\cosh x+\sinh x\sinh x$
$=\cosh(x+x)=\cosh 2x$

13. $D_x\sinh^2 x=2\sinh x\cosh x=\sinh 2x$

14. $D_x\cosh^2 x=2\cosh x\sinh x=\sinh 2x$

15. $D_x(5\sinh^2 x)=10\sinh x\cdot\cosh x=5\sinh 2x$

16. $D_x\cosh^3 x=3\cosh^2 x\sinh x$

17. $D_x\cosh(3x+1)=\sinh(3x+1)\cdot 3=3\sinh(3x+1)$

18. $D_x\sinh(x^2+x)=\cosh(x^2+x)\cdot(2x+1)$
$=(2x+1)\cosh(x^2+x)$

19. $D_x\ln(\sinh x)=\frac{1}{\sinh x}\cdot\cosh x=\frac{\cosh x}{\sinh x}$
$=\coth x$

20. $D_x\ln(\coth x)=\frac{1}{\coth x}(-\operatorname{csch}^2 x)$

$=-\frac{\sinh x}{\cosh x}\cdot\frac{1}{\sinh^2 x}=-\frac{1}{\sinh x\cosh x}$

$=-\operatorname{csch} x\operatorname{sech} x$

21. $D_x(x^2\cosh x)=x^2\cdot\sinh x+\cosh x\cdot 2x$
$=x^2\sinh x+2x\cosh x$

22. $D_x(x^{-2}\sinh x)=x^{-2}\cdot\cosh x+\sinh x\cdot(-2x^{-3})$
$=x^{-2}\cosh x-2x^{-3}\sinh x$

23. $D_x(\cosh 3x\sinh x)=\cosh 3x\cdot\cosh x+\sinh x\cdot\sinh 3x\cdot 3=\cosh 3x\cosh x+3\sinh 3x\sinh x$

24. $D_x(\sinh x\cosh 4x)=\sinh x\cdot\sinh 4x\cdot 4+\cosh 4x\cdot\cosh x=4\sinh x\sinh 4x+\cosh x\cosh 4x$

25. $D_x(\tanh x\sinh 2x)=\tanh x\cdot\cosh 2x\cdot 2+\sinh 2x\cdot\operatorname{sech}^2 x=2\tanh x\cosh 2x+\sinh 2x\operatorname{sech}^2 x$

26. $D_x(\coth 4x\sinh x)=\coth 4x\cdot\cosh x+\sinh x(-\operatorname{csch}^2 4x)\cdot 4=\cosh x\coth 4x-4\sinh x\operatorname{csch}^2 4x$

27. $D_x \sinh^{-1}(x^2) = \dfrac{1}{\sqrt{(x^2)^2+1}} \cdot 2x = \dfrac{2x}{\sqrt{x^4+1}}$

28. $D_x \cosh^{-1}(x^3) = \dfrac{1}{\sqrt{(x^3)^2-1}} \cdot 3x^2 = \dfrac{3x^2}{\sqrt{x^6-1}}$

29. $D_x \tanh^{-1}(2x-3) = \dfrac{1}{1-(2x-3)^2} \cdot 2 = \dfrac{2}{1-(4x^2-12x+9)} = \dfrac{2}{-4x^2+12x-8} = -\dfrac{1}{2(x^2-3x+2)}$

30. $D_x \coth^{-1}(x^5) = D_x \tanh^{-1}\left(\dfrac{1}{x^5}\right) = \dfrac{1}{1-\left(\frac{1}{x^5}\right)^2} \cdot \left(-\dfrac{5}{x^6}\right) = \dfrac{x^{10}}{x^{10}-1} \cdot \left(-\dfrac{5}{x^6}\right) = -\dfrac{5x^4}{x^{10}-1}$

31. $D_x[x\cosh^{-1}(3x)] = x \cdot \dfrac{1}{\sqrt{(3x)^2-1}} \cdot 3 + \cosh^{-1}(3x) \cdot 1 = \dfrac{3x}{\sqrt{9x^2-1}} + \cosh^{-1} 3x$

32. $D_x(x^2 \sinh^{-1} x^5) = x^2 \cdot \dfrac{1}{\sqrt{(x^5)^2+1}} \cdot 5x^4 + \sinh^{-1} x^5 \cdot 2x = \dfrac{5x^6}{\sqrt{x^{10}+1}} + 2x \sinh^{-1} x^5$

33. $D_x \ln(\cosh^{-1} x) = \dfrac{1}{\cosh^{-1} x} \cdot \dfrac{1}{\sqrt{x^2-1}}$

$= \dfrac{1}{\sqrt{x^2-1}\cosh^{-1} x}$

34. $\cosh^{-1}(\cos x)$ does not have a derivative, since $D_u \cosh^{-1} u$ is only defined for $u > 1$ while $\cos x \le 1$ for all x.

35. $D_x \tanh(\cot x) = \operatorname{sech}^2(\cot x) \cdot (-\csc^2 x)$

$= -\csc^2 x \operatorname{sech}^2(\cot x)$

36. $D_x \coth^{-1}(\tanh x) = D_x \tanh^{-1}\left(\dfrac{1}{\tanh x}\right)$

$= D_x \tanh^{-1}(\coth x)$

$= \dfrac{1}{1-(\coth x)^2}(-\operatorname{csch}^2 x) = \dfrac{-\operatorname{csch}^2 x}{-\operatorname{csch}^2 x} = 1$

37. Area $= \int_0^{\ln 3} \cosh 2x\,dx = \left[\dfrac{1}{2}\sinh 2x\right]_0^{\ln 3}$

$= \dfrac{1}{2}\left(\dfrac{e^{2\ln 3}-e^{-2\ln 3}}{2} - \dfrac{e^0-e^{-0}}{2}\right)$

$= \dfrac{1}{4}(e^{\ln 9} - e^{\ln\frac{1}{9}}) = \dfrac{1}{4}\left(9-\dfrac{1}{9}\right) = \dfrac{20}{9}$

38. Let $u = 3x+2$, so $du = 3\,dx$.

$\int \sinh(3x+2)dx = \dfrac{1}{3}\int \sinh u\,du = \dfrac{1}{3}\cosh u + C$

$= \dfrac{1}{3}\cosh(3x+2) + C$

39. Let $u = \pi x^2 + 5$, so $du = 2\pi x dx$.

$\int x\cosh(\pi x^2+5)dx = \dfrac{1}{2\pi}\int \cosh u\,du$

$= \dfrac{1}{2\pi}\sinh u + C = \dfrac{1}{2\pi}\sinh(\pi x^2+5) + C$

40. Let $u = \sqrt{z}$, so $du = \dfrac{1}{2\sqrt{z}}dz$.

$\int \dfrac{\cosh\sqrt{z}}{\sqrt{z}}dz = 2\int \cosh u\,du = 2\sinh u + C$

$= 2\sinh\sqrt{z} + C$

41. Let $u = 2z^{1/4}$, so $du = \dfrac{1}{4} \cdot 2z^{-3/4}dz = \dfrac{1}{2\sqrt[4]{z^3}}dz.$

$\int \dfrac{\sinh(2z^{1/4})}{\sqrt[4]{z^3}}dz = 2\int \sinh u\,du = 2\cosh u + C$

$= 2\cosh(2z^{1/4}) + C$

42. Let $u = e^x$, so $du = e^x dx$.

$$\int e^x \sinh e^x dx = \int \sinh u\, du = \cosh u + C$$

$$= \cosh e^x + C$$

43. Let $u = \sin x$, so $du = \cos x\, dx$

$$\int \cos x \sinh(\sin x) dx = \int \sinh u\, du = \cosh u + C$$

$$= \cosh(\sin x) + C$$

44. Let $u = \ln(\cosh x)$, so

$$du = \frac{1}{\cosh x} \cdot \sinh x = \tanh x\, dx.$$

$$\int \tanh x \ln(\cosh x) dx = \int u\, du = \frac{u^2}{2} + C$$

$$= \frac{1}{2}[\ln(\cosh x)]^2 + C$$

45. Let $u = \ln(\sinh x^2)$, so

$$du = \frac{1}{\sinh x^2} \cdot \cosh x^2 \cdot 2x dx = 2x \coth x^2 dx.$$

$$\int x \coth x^2 \ln(\sinh x^2) dx = \frac{1}{2}\int u\, du = \frac{1}{2} \cdot \frac{u^2}{2} + C$$

$$= \frac{1}{4}[\ln(\sinh x^2)]^2 + C$$

46. Area $= \int_{-\ln 5}^{\ln 5} \cosh 2x\, dx = 2\int_0^{\ln 5} \cosh 2x\, dx$

$$= 2\left[\frac{1}{2}\sinh 2x\right]_0^{\ln 5}$$

$$= \sinh(2\ln 5) = \frac{1}{2}(e^{2\ln 5} - e^{-2\ln 5})$$

$$= \frac{1}{2}(e^{\ln 25} - e^{\ln \frac{1}{25}}) = \frac{1}{2}\left(25 - \frac{1}{25}\right)$$

$$= \frac{312}{25} = 12.48$$

47. Note that the graphs of $y = \sinh x$ and $y = 0$ intersect at the origin.

Area $= \int_0^{\ln 2} \sinh x\, dx = [\cosh x]_0^{\ln 2}$

$$= \frac{e^{\ln 2} + e^{-\ln 2}}{2} - \frac{e^0 + e^0}{2} = \frac{1}{2}\left(2 + \frac{1}{2}\right) - 1 = \frac{1}{4}$$

48. $\tanh x = 0$ when $\sinh x = 0$, which is when $x = 0$.

Area $= \int_{-8}^{0}(-\tanh x)\, dx + \int_0^8 \tanh x\, dx$

$$= 2\int_0^8 \tanh x\, dx = 2\int_0^8 \frac{\sinh x}{\cosh x} dx$$

Let $u = \cosh x$, so $du = \sinh x dx$.

$$2\int \frac{\sinh x}{\cosh x} dx = 2\int \frac{1}{u} du = 2\ln|u| + C$$

$$2\int_0^8 \frac{\sinh x}{\cosh x} dx = \left[2\ln|\cosh x|\right]_0^8$$

$$= 2(\ln|\cosh 8| - \ln 1) = 2\ln(\cosh 8) \approx 14.61$$

49. Volume $= \int_0^1 \pi \cosh^2 x\, dx = \frac{\pi}{2}\int_0^1 (1 + \cosh 2x) dx$

$$= \frac{\pi}{2}\left[x + \frac{\sinh 2x}{2}\right]_0^1$$

$$= \frac{\pi}{2}\left(1 + \frac{\sinh 2}{2} - 0\right)$$

$$= \frac{\pi}{2} + \frac{\pi \sinh 2}{4} \approx 4.42$$

50. Volume $= \int_0^{\ln 10} \pi \sinh^2 x dx$

$$= \pi\int_0^{\ln 10}\left(\frac{e^x - e^{-x}}{2}\right)^2 dx$$

$$= \pi\int_0^{\ln 10} \frac{e^{2x} - 2 + e^{-2x}}{4} dx = \frac{\pi}{4}\int_0^{\ln 10}(e^{2x} - 2 + e^{-2x}) dx$$

$$= \frac{\pi}{4}\left[\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right]_0^{\ln 10}$$

$$= \frac{\pi}{8}[e^{2x} - 4x - e^{-2x}]_0^{\ln 10}$$

$$= \frac{\pi}{8}\left(100 - 4\ln 10 - \frac{1}{100}\right) \approx 35.65$$

51. Note that $1 + \sinh^2 x = \cosh^2 x$ and

$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

Surface area $= \int_0^1 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$$= \int_0^1 2\pi \cosh x \sqrt{1 + \sinh^2 x}\, dx$$

$$= \int_0^1 2\pi \cosh x \cosh x\, dx$$

$$= \int_0^1 \pi(1 + \cosh 2x) dx$$

$$= \left[\pi x + \frac{\pi}{2}\sinh 2x\right]_0^1 = \pi + \frac{\pi}{2}\sinh 2 \approx 8.84$$

52. Surface area = $\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi \sinh x\sqrt{1+\cosh^2 x}\,dx$

Let $u = \cosh x$, so $du = \sinh x\, dx$

$\int 2\pi \sinh x\sqrt{1+\cosh^2 x}\,dx = 2\pi\int \sqrt{1+u^2}\,du = 2\pi\left[\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\ln\left|u+\sqrt{1+u^2}\right|+C\right]$

$= \pi\cosh x\sqrt{1+\cosh^2 x}+\pi\ln\left|\cosh x+\sqrt{1+\cosh^2 x}\right|+C$ (The integration of $\int\sqrt{1+u^2}\,du$ is shown in Formula 44 of the Tables in the back of the text, which is covered in Chapter 8.)

$\int_0^1 2\pi\sinh x\sqrt{1+\cosh^2 x}\,dx = \pi\left[\cosh x\sqrt{1+\cosh^2 x}+\ln\left|\cosh x+\sqrt{1+\cosh^2 x}\right|\right]_0^1$

$= \pi\left[\cosh 1\sqrt{1+\cosh^2 1}+\ln\left|\cosh 1+\sqrt{1+\cos^2 1}\right|-\left(\sqrt{2}+\ln\left|1+\sqrt{2}\right|\right)\right]\approx 5.53$

53. $y = a\cosh\left(\frac{x}{a}\right)+C$

$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$

$\frac{d^2y}{dx^2} = \frac{1}{a}\cosh\left(\frac{x}{a}\right)$

We need to show that $\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Note that $1+\sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$ and $\cosh\left(\frac{x}{a}\right) > 0$. Therefore,

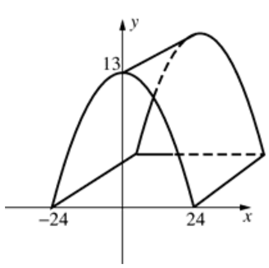
$\frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{a}\sqrt{1+\sinh^2\left(\frac{x}{a}\right)} = \frac{1}{a}\sqrt{\cosh^2\left(\frac{x}{a}\right)} = \frac{1}{a}\cosh\left(\frac{x}{a}\right) = \frac{d^2y}{dx^2}$

54. **a.** The graph of $y = b - a\cosh\left(\frac{x}{a}\right)$ is symmetric about the y-axis, so if its width along the x-axis is $2a$, its x-intercepts are $(\pm a, 0)$. Therefore, $y(a) = b - a\cosh\left(\frac{a}{a}\right) = 0$, so $b = a\cosh 1 \approx 1.54308a$.

b. The height is $y(0) \approx 1.54308a - a\cosh 0 = 0.54308a$.

c. If $2a = 48$, the height is about $0.54308a = (0.54308)(24) \approx 13$.

55. **a.**



b. Area under the curve is

$\int_{-24}^{24}\left[37-24\cosh\left(\frac{x}{24}\right)\right]dx = \left[37x-576\sinh\left(\frac{x}{24}\right)\right]_{-24}^{24} \approx 422$

Volume is about $(422)(100) = 42{,}200$ ft^3.

c. Length of the curve is

$$\int_{-24}^{24}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_{-24}^{24}\sqrt{1+\sinh^2\left(\frac{x}{24}\right)}dx=\int_{-24}^{24}\cosh\left(\frac{x}{24}\right)dx=\left[24\sinh\left(\frac{x}{24}\right)\right]_{-24}^{24}=48\sinh 1\approx 56.4$$

Surface area $\approx (56.4)(100)=5640\text{ ft}^2$

56. Area $=\frac{1}{2}\cosh t\sinh t-\int_1^{\cosh t}\sqrt{x^2-1}\,dx=\frac{1}{2}\cosh t\sinh t-\left[\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|\right]_1^{\cosh t}$

$=\frac{1}{2}\cosh t\sinh t-\left[\frac{1}{2}\cosh t\sqrt{\cosh^2 t-1}-\frac{1}{2}\ln\left|\cosh t+\sqrt{\cosh^2 t-1}\right|-0\right]$

$=\frac{1}{2}\cosh t\sinh t-\frac{1}{2}\cosh t\sinh t+\frac{1}{2}\ln\left|\cosh t+\sinh t\right|=\frac{1}{2}\ln e^t=\frac{t}{2}$

57. a. $(\sinh x+\cosh x)^r=\left(\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}\right)^r=\left(\frac{2e^x}{2}\right)^r=e^{rx}$

$\sinh rx+\cosh rx=\frac{e^{rx}-e^{-rx}}{2}+\frac{e^{rx}+e^{-rx}}{2}=\frac{2e^{rx}}{2}=e^{rx}$

b. $(\cosh x-\sinh x)^r=\left(\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}\right)^r=\left(\frac{2e^{-x}}{2}\right)^r=e^{-rx}$

$\cosh rx-\sinh rx=\frac{e^{rx}+e^{-rx}}{2}-\frac{e^{rx}-e^{-rx}}{2}=\frac{2e^{-rx}}{2}=e^{-rx}$

c. $(\cos x+i\sin x)^r=\left(\frac{e^{ix}+e^{-ix}}{2}+i\frac{e^{ix}-e^{-ix}}{2i}\right)^r=\left(\frac{2e^{ix}}{2}\right)^r=e^{irx}$

$\cos rx+i\sin rx=\frac{e^{irx}+e^{-irx}}{2}+i\frac{e^{irx}-e^{-irx}}{2i}=\frac{2e^{irx}}{2}=e^{irx}$

d. $(\cos x-i\sin x)^r=\left(\frac{e^{ix}+e^{-ix}}{2}-i\frac{e^{ix}-e^{-ix}}{2i}\right)^r=\left(\frac{2e^{-ix}}{2}\right)^r=e^{-irx}$

$\cos rx-i\sin rx=\frac{e^{irx}+e^{-irx}}{2}-i\frac{e^{irx}-e^{-irx}}{2i}=\frac{2e^{-irx}}{2}=e^{-irx}$

58. a. $gd(-t)=\tan^{-1}[\sinh(-t)]$

$=\tan^{-1}(-\sinh t)=-\tan^{-1}(\sinh t)=-gd(t)$

so gd is odd.

$D_t[gd(t)]=\frac{1}{1+\sinh^2 t}\cdot\cosh t=\frac{\cosh t}{\cosh^2 t}$

$=\operatorname{sech} t>0$ for all t, so gd is increasing.

$D_t^2[gd(t)]=D_t(\operatorname{sech} t)=-\operatorname{sech} t\tanh t$

$D_t^2[gd(t)]=0$ when $\tanh t=0$, since $\operatorname{sech} t>0$ for all t. $\tanh t=0$ at $t=0$ and $\tanh t<0$ for $t<0$, thus $D_t^2[gd(t)]>0$ for $t<0$ and $D_t^2[gd(t)]<0$ for $t>0$. Hence $gd(t)$ has an inflection point at $(0, gd(0))=(0,\tan^{-1}0)=(0,0)$.

b. If $y=\tan^{-1}(\sinh t)$ then $\tan y=\sinh t$ so

$\sin y=\frac{\tan y}{\sqrt{\tan^2 y+1}}=\frac{\sinh t}{\sqrt{\sinh^2 t+1}}$

$=\frac{\sinh t}{\cosh t}=\tanh t$ so $y=\sin^{-1}(\tanh t)$

Also, $D_t y=\frac{1}{1+\sinh^2 t}\cdot\cosh t$

$=\frac{\cosh t}{\cosh^2 t}=\frac{1}{\cosh t}=\operatorname{sech} t$,

so $y=\int_0^t \operatorname{sech} u\,du$ by the Fundamental Theorem of Calculus.

59. Area = $\int_0^x \cosh t\,dt = [\sinh t]_0^x = \sinh x$

Arc length =

$\int_0^x \sqrt{1+[D_t \cosh t]^2}\,dt = \int_0^x \sqrt{1+\sinh^2 t}\,dt$

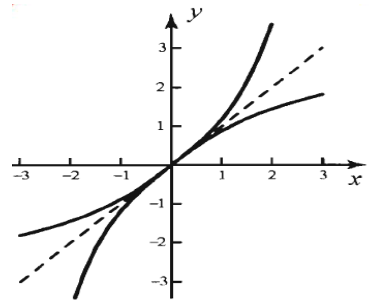
$= \int_0^x \cosh t\,dt = [\sinh t]_0^x = \sinh x$

60. From Problem 54, the equation of an inverted catenary is $y = b - a\cosh\frac{x}{a}$. Given the information about the Gateway Arch, the curve passes through the points (±315, 0) and (0, 630). Thus, $b = a\cosh\frac{315}{a}$ and 630 = *b* – *a*, so *b* = *a* + 630.

$a + 630 = a\cosh\frac{315}{a} \Rightarrow a \approx 128$, so $b \approx 758$.

The equation is $y = 758 - 128\cosh\frac{x}{128}$.

61.



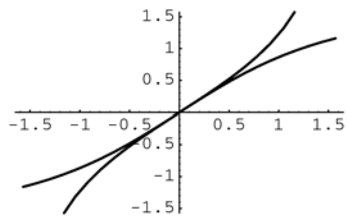
The functions $y = \sinh x$ and $y = \ln(x + \sqrt{x^2+1})$ are inverse functions.

62. $y = gd(x) = \tan^{-1}(\sinh x)$

$\tan y = \sinh x$

$x = gd^{-1}(y) = \sinh^{-1}(\tan y)$

Thus, $y = gd^{-1}(x) = \sinh^{-1}(\tan x)$



6.10 Chapter Review

Concepts Test

1. False: ln 0 is undefined.

2. True: $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ for all $x > 0$.

3. True: $\int_1^{e^3} \frac{1}{t}\,dt = \left[\ln|t|\right]_1^{e^3} = \ln e^3 - \ln 1 = 3$

4. False: The graph is intersected *at most* once by every horizontal line.

5. True: The range of *y* = ln *x* is the set of all real numbers.

6. False: $\ln x - \ln y = \ln\left(\frac{x}{y}\right)$

7. False: $4\ln x = \ln(x^4)$

8. True: $\ln(2e^{x+1}) - \ln(2e^x) = \ln\frac{2e^{x+1}}{2e^x}$

$= \ln e = 1$

9. True: $f(g(x)) = 4 + e^{\ln(x-4)}$

$= 4 + (x-4) = x$

and

$g(f(x)) = \ln(4 + e^x - 4) = \ln e^x = x$

10. False: $\exp(x+y) = \exp x \exp y$

11. True: ln *x* is an increasing function.

12. False: Only true for *x* > 1, or ln *x* > 0.

13. True: $e^z > 0$ for all *z*.

14. True: e^x is an increasing function.

15. True: $\lim_{x\to 0^+}(\ln\sin x - \ln x)$

$= \lim_{x\to 0^+} \ln\left(\frac{\sin x}{x}\right) = \ln 1 = 0$

16. True: $\pi^{\sqrt{2}} = e^{\sqrt{2}\ln\pi}$

17. False: $\ln\pi$ is a constant so $\frac{d}{dx}\ln\pi = 0$.

18. True: $\frac{d}{dx}(\ln 3|x| + C)$

$= \frac{d}{dx}(\ln|x| + \ln 3 + C) = \frac{1}{x}$

19. True: e is a number.

20. True: $\exp[g(x)] \neq 0$ because 0 is not in the range of the function $y = e^x$.

21. False: $D_x(x^x) = x^x(1 + \ln x)$

22. True: $2(\tan x + \sec x)' - (\tan x + \sec x)^2$
$= 2(\sec^2 x + \sec x \tan x)$
$- \tan^2 x - 2\tan x \sec x - \sec^2 x$
$= \sec^2 x - \tan^2 x = 1$

23. True: The integrating factor is
$e^{\int 4/x\,dx} = e^{4\ln x} = \left(e^{\ln x}\right)^4 = x^4$

24. True: The solution is $y(x) = e^{-4} \cdot e^{2x}$. Thus, slope $= 2e^{-4} \cdot e^{2x}$ and at $x = 2$ the slope is 2.

25. False: The solution is $y(x) = e^{2x}$, so $y'(x) = 2e^{2x}$. In general, Euler's method will underestimate the solution if the slope of the solution is increasing as it is in this case.

26. False: $\sin(\arcsin(2))$ is undefined

27. False: $\arcsin(\sin 2\pi) = \arcsin 0 = 0$

28. True: $\sinh x$ is increasing.

29. False: $\cosh x$ is not increasing.

30. True: $\cosh(0) = 1 = e^0$
If $x > 0$, $e^x > 1$ while $e^{-x} < 1 < e^x$ so
$\cosh x = \frac{1}{2}(e^x + e^{-x}) < \frac{1}{2}(2e^x)$
$= e^x = e^{|x|}$. If $x < 0$, $-x > 0$ and
$e^{-x} > 1$ while $e^x < 1 < e^{-x}$ so
$\cosh x = \frac{1}{2}(e^x + e^{-x}) < \frac{1}{2}(2e^{-x})$
$= e^{-x} = e^{|x|}$.

31. True: $|\sinh x| \le \frac{1}{2}e^{|x|}$ is equivalent to
$\left|e^x - e^{-x}\right| \le e^{|x|}$. When $x = 0$,
$\sinh x = 0 < \frac{1}{2}e^0 = \frac{1}{2}$. If $x > 0$,
$e^x > 1$ and $e^{-x} < 1 < e^x$, thus
$\left|e^x - e^{-x}\right| = e^x - e^{-x} < e^x = e^{|x|}$.
If $x < 0$, $e^{-x} > 1$ and $e^x < 1 < e^{-x}$, thus
$\left|e^x - e^{-x}\right| = -(e^x - e^{-x})$
$= e^{-x} - e^x < e^{-x} = e^{|x|}$.

32. False: $\tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636$
but $\dfrac{\sin^{-1}\left(\frac{1}{2}\right)}{\cos^{-1}\left(\frac{1}{2}\right)} = \dfrac{1}{2}$

33. False: $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2}$
$= \frac{1}{2}\left(3 + \frac{1}{3}\right) = \frac{5}{3}$

34. False: $\lim_{x\to 0} \ln\left(\dfrac{\sin x}{x}\right) = \ln 1 = 0$

35. True: $\lim_{x\to -\infty} \tan^{-1} x = -\dfrac{\pi}{2}$, since
$\lim_{x\to -\frac{\pi}{2}^+} \tan x = -\infty$.

36. False: $\cosh x > 1$ for $x \neq 0$, while $\sin^{-1} u$ is only defined for $-1 \le u \le 1$.

37. True: $\tanh x = \dfrac{\sinh x}{\cosh x}$; $\sinh x$ is an odd function and $\cosh x$ is an even function.

38. False: Both functions satisfy $y'' - y = 0$.

39. True: $\ln 3^{100} = 100 \ln 3 > 100 \cdot 1$ since $\ln 3 > 1$.

40. False: $\ln(x - 3)$ is not defined for $x < 3$.

41. True: y triples every time t increases by t_1.

42. False: $x(0) = C$; $\frac{1}{2}C = Ce^{-kt}$ when
$\frac{1}{2} = e^{-kt}$, so $\ln\frac{1}{2} = -kt$ or
$t = \dfrac{\ln\frac{1}{2}}{-k} = \dfrac{-\ln 2}{-k} = \dfrac{\ln 2}{k}$

43. True: $(y(t)+z(t))' = y'(t)+z'(t)$
$= ky(t)+kz(t) = k(y(t)+z(t))$

44. False: Only true if $C = 0$;
$(y_1(t)+y_2(t))' = y_1'(t)+y_2'(t)$
$= ky_1(t)+C+ky_2(t)+C$
$= k(y_1(t)+y_2(t))+2C$.

45. False: Use the substitution $u = -h$.
$\lim_{h\to 0}(1-h)^{-1/h} = \lim_{u\to 0}(1+u)^{1/u} = e$
by Theorem 6.5.A.

46. False: $e^{0.05} \approx 1.051 < \left(1+\frac{0.06}{12}\right)^{12} \approx 1.062$

47. True: If $D_x(a^x) = a^x \ln a = a^x$, then $\ln a = 1$, so $a = e$.

Sample Test Problems

1. $\ln\frac{x^4}{2} = 4\ln x - \ln 2$

$$\frac{d}{dx}\ln\frac{x^4}{2} = \frac{d}{dx}(4\ln x - \ln 2) = \frac{4}{x}$$

2. $\frac{d}{dx}\sin^2(x^3) = 2\sin(x^3)\frac{d}{dx}\sin(x^3)$

$$= 2\sin(x^3)\cos(x^3)\frac{d}{dx}x^3 = 6x^2\sin(x^3)\cos(x^3)$$

3. $\frac{d}{dx}e^{x^2-4x} = e^{x^2-4x}\frac{d}{dx}(x^2-4x)$

$$= (2x-4)e^{x^2-4x}$$

4. $\frac{d}{dx}\log_{10}(x^5-1) = \frac{1}{(x^5-1)\ln 10}\frac{d}{dx}(x^5-1)$

$$= \frac{5x^4}{(x^5-1)\ln 10}$$

5. $\frac{d}{dx}\tan(\ln e^x) = \frac{d}{dx}\tan x = \sec^2 x$

6. $\frac{d}{dx}e^{\ln\cot x} = \frac{d}{dx}\cot x = -\csc^2 x$

7. $\frac{d}{dx}2\tanh\sqrt{x} = 2\operatorname{sech}^2\sqrt{x}\,\frac{d}{dx}\sqrt{x} = \frac{\operatorname{sech}^2\sqrt{x}}{\sqrt{x}}$

8. $\frac{d}{dx}\tanh^{-1}(\sin x) = \frac{1}{1-\sin^2 x}\frac{d}{dx}\sin x = \frac{\cos x}{1-\sin^2 x}$

$$= \frac{\cos x}{\cos^2 x} = \sec x$$

9. $\frac{d}{dx}\sinh^{-1}(\tan x) = \frac{1}{\sqrt{\tan^2 x+1}}\frac{d}{dx}\tan x$

$$= \frac{\sec^2 x}{\sqrt{\tan^2 x+1}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = |\sec x|$$

10. $\frac{d}{dx}2\sin^{-1}\sqrt{3x} = \frac{2}{\sqrt{1-\left(\sqrt{3x}\right)^2}}\frac{d}{dx}\sqrt{3x}$

$$= \frac{2}{\sqrt{1-3x}}\frac{3}{2\sqrt{3x}} = \frac{3}{\sqrt{3x-9x^2}}$$

11. $\frac{d}{dx}\sec^{-1}e^x = \frac{1}{|e^x|\sqrt{(e^x)^2-1}}\frac{d}{dx}e^x$

$$= \frac{e^x}{e^x\sqrt{e^{2x}-1}} = \frac{1}{\sqrt{e^{2x}-1}}$$

12. $\frac{d}{dx}\ln\sin^2\left(\frac{x}{2}\right) = \frac{1}{\sin^2\left(\frac{x}{2}\right)}\frac{d}{dx}\sin^2\left(\frac{x}{2}\right)$

$$= \frac{1}{\sin^2\left(\frac{x}{2}\right)}2\sin\left(\frac{x}{2}\right)\frac{d}{dx}\sin\left(\frac{x}{2}\right)$$

$$= \frac{1}{\sin^2\left(\frac{x}{2}\right)}\left[2\sin\left(\frac{x}{2}\right)\right]\frac{1}{2}\cos\left(\frac{x}{2}\right) = \cot\left(\frac{x}{2}\right)$$

13. $\frac{d}{dx}3\ln(e^{5x}+1) = \frac{3}{e^{5x}+1}(5e^{5x}) = \frac{15e^{5x}}{e^{5x}+1}$

14. $\frac{d}{dx}\ln(2x^3-4x+5)$

$$= \frac{1}{2x^3-4x+5}\frac{d}{dx}(2x^3-4x+5) = \frac{6x^2-4}{2x^3-4x+5}$$

15. $\frac{d}{dx}\cos e^{\sqrt{x}} = -\sin e^{\sqrt{x}}\frac{d}{dx}e^{\sqrt{x}}$

$$= (-\sin e^{\sqrt{x}})e^{\sqrt{x}}\frac{d}{dx}\sqrt{x}$$

$$= -\frac{e^{\sqrt{x}}\sin e^{\sqrt{x}}}{2\sqrt{x}}$$

16. $\frac{d}{dx}\ln(\tanh x) = \frac{1}{\tanh x}\frac{d}{dx}\tanh x$

$= \frac{1}{\tanh x}\operatorname{sech}^2 x = \operatorname{csch} x \operatorname{sech} x$

17. $\frac{d}{dx}2\cos^{-1}\sqrt{x} = \frac{-2}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}\sqrt{x}$

$= \frac{-2}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} = -\frac{1}{\sqrt{x-x^2}}$

18. $\frac{d}{dx}\left[4^{3x}+(3x)^4\right] = \frac{d}{dx}(64^x + 81x^4)$

$= 64^x \ln 64 + 324x^3$

19. $\frac{d}{dx}2\csc e^{\ln\sqrt{x}} = \frac{d}{dx}2\csc\sqrt{x}$

$= -2\csc\sqrt{x}\cot\sqrt{x}\frac{d}{dx}\sqrt{x}$

$= -\frac{\csc\sqrt{x}\cot\sqrt{x}}{\sqrt{x}}$

20. $\frac{d}{dx}(\log_{10} 2x)^{2/3}$

$= \frac{2}{3}(\log_{10} 2x)^{-1/3}\frac{d}{dx}(\log_{10} 2 + \log_{10} x)$

$= \frac{2}{3}(\log_{10} 2x)^{-1/3}\frac{1}{x\ln 10}$

$= \frac{2}{3x\ln 10\sqrt[3]{\log_{10} 2x}}$

21. $\frac{d}{dx}4\tan 5x \sec 5x$

$= 20\sec^2 5x \sec 5x + 20\tan 5x \sec 5x \tan 5x$

$= 20\sec 5x(\sec^2 5x + \tan^2 5x)$

$= 20\sec 5x(2\sec^2 5x - 1)$

22 $\frac{d}{dx}\tan^{-1}\left(\frac{x^2}{2}\right) = \frac{1}{\left(\frac{x^2}{2}\right)^2+1}\frac{d}{dx}\left(\frac{x^2}{2}\right)$

$= \frac{x}{\left(\frac{x^4}{4}\right)+1} = \frac{4x}{x^4+4}$

$\frac{d}{dx}\left[x\tan^{-1}\left(\frac{x^2}{2}\right)\right] = (1)\tan^{-1}\left(\frac{x^2}{2}\right) + (x)\left(\frac{4x}{x^4+4}\right)$

$= \tan^{-1}\left(\frac{x^2}{2}\right) + \frac{4x^2}{x^4+4}$

23. $\frac{d}{dx}x^{1+x} = \frac{d}{dx}e^{(1+x)\ln x}$

$= e^{(1+x)\ln x}\frac{d}{dx}[(1+x)\ln x]$

$= x^{1+x}\left[(1)(\ln x) + (1+x)\left(\frac{1}{x}\right)\right]$

$= x^{1+x}\left(\ln x + 1 + \frac{1}{x}\right)$

24. $\frac{d}{dx}(1+x^2)^e = e(1+x^2)^{e-1}\frac{d}{dx}(1+x^2)$

$= 2xe(1+x^2)^{e-1}$

25. Let $u = 3x - 1$, so $du = 3\,dx$.

$\int e^{3x-1}dx = \frac{1}{3}\int e^{3x-1}3dx = \frac{1}{3}\int e^u du$

$= \frac{1}{3}e^u + C = \frac{1}{3}e^{3x-1} + C$

Check:

$\frac{d}{dx}\left(\frac{1}{3}e^{3x-1}+C\right) = \frac{1}{3}e^{3x-1}\frac{d}{dx}(3x-1) = e^{3x-1}$

26. Let $u = \sin 3x$, so $du = 3\cos 3x\,dx$.

$\int 6\cot 3x\,dx = 2\int \frac{1}{\sin 3x}3\cos 3x\,dx = 2\int \frac{1}{u}du$

$= 2\ln|u| + C = 2\ln|\sin 3x| + C$

Check:

$\frac{d}{dx}(2\ln|\sin 3x| + C) = \frac{2}{\sin 3x}\frac{d}{dx}\sin 3x$

$= \frac{2(3\cos 3x)}{\sin 3x} = 6\cot 3x$

27. Let $u = e^x$, so $du = e^x dx$.

$\int e^x \sin e^x dx = \int \sin u\,du = -\cos u + C$

$= -\cos e^x + C$

Check:

$\frac{d}{dx}(-\cos e^x + C) = (\sin e^x)\frac{d}{dx}e^x = e^x \sin e^x$

28. Let $u = x^2 + x - 5$, so $du = (2x+1)dx$.

$\int \frac{6x+3}{x^2+x-5}dx = 3\int \frac{1}{x^2+x-5}(2x+1)dx$

$= 3\int \frac{1}{u}du = 3\ln|u| + C = 3\ln\left|x^2+x-5\right| + C$

Check:

$\frac{d}{dx}\left(3\ln\left|x^2+x-5\right| + C\right) = \frac{3}{x^2+x-5}\frac{d}{dx}(x^2+x-5)$

$= \frac{6x+3}{x^2+x-5}$

29. Let $u = e^{x+3} + 1$, so $du = e^{x+3}dx$.

$$\int \frac{e^{x+2}}{e^{x+3}+1}dx = \frac{1}{e}\int \frac{1}{e^{x+3}+1}e^{x+3}dx = \frac{1}{e}\int \frac{1}{u}du$$

$$= \frac{1}{e}\ln|u| + C = \frac{\ln(e^{x+3}+1)}{e} + C$$

Check:

$$\frac{d}{dx}\left(\frac{\ln(e^{x+3}+1)}{e} + C\right) = \frac{1}{e}\frac{1}{e^{x+3}+1}\frac{d}{dx}(e^{x+3}+1)$$

$$= \frac{e^{x+3}e^{-1}}{e^{x+3}+1} = \frac{e^{x+2}}{e^{x+3}+1}$$

30. Let $u = x^2$, so $du = 2x\, dx$.

$$\int 4x\cos x^2 dx = 2\int (\cos x^2)2x\, dx = 2\int \cos u\, du$$

$$= 2\sin u + C = 2\sin x^2 + C$$

Check:

$$\frac{d}{dx}(2\sin x^2 + C) = 2\cos x^2 \frac{d}{dx}x^2 = 4x\cos x^2$$

31. Let $u = 2x$, so $du = 2\, dx$.

$$\int \frac{4}{\sqrt{1-4x^2}}dx = 2\int \frac{1}{\sqrt{1-(2x)^2}}2dx$$

$$= 2\int \frac{1}{\sqrt{1-u^2}}du$$

$$= 2\sin^{-1} u + C = 2\sin^{-1} 2x + C$$

Check:

$$\frac{d}{dx}(2\sin^{-1} 2x + C) = 2\left(\frac{1}{\sqrt{1-(2x)^2}}\right)\frac{d}{dx}2x$$

$$= \frac{4}{\sqrt{1-4x^2}}$$

32. Let $u = \sin x$, so $du = \cos x\, dx$.

$$\int \frac{\cos x}{1+\sin^2 x}dx = \int \frac{1}{1+u^2}du = \tan^{-1} u + C$$

$$= \tan^{-1}(\sin x) + C$$

Check:

$$\frac{d}{dx}\left[\tan^{-1}(\sin x) + C\right] = \frac{1}{1+\sin^2 x}\frac{d}{dx}\sin x$$

$$= \frac{\cos x}{1+\sin^2 x}$$

33. Let $u = \ln x$, so $du = \frac{1}{x}dx$.

$$\int \frac{-1}{x+x(\ln x)^2}dx = -\int \frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}dx$$

$$= -\int \frac{1}{1+u^2}du = -\tan^{-1} u + C = -\tan^{-1}(\ln x) + C$$

Check:

$$\frac{d}{dx}[-\tan^{-1}(\ln x) + C] = -\frac{1}{1+(\ln x)^2}\frac{d}{dx}\ln x$$

$$= \frac{-1}{x+x(\ln x)^2}$$

34. Let $u = x - 3$, so $du = dx$.

$$\int \operatorname{sech}^2(x-3)dx = \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$= \tanh(x-3) + C$$

Check:

$$\frac{d}{dx}[\tanh(x-3)] = \operatorname{sech}^2(x-3)\frac{d}{dx}(x-3)$$

$$= \operatorname{sech}^2(x-3)$$

35. $f'(x) = \cos x - \sin x$; $f'(x) = 0$ when $\tan x = 1$, $x = \frac{\pi}{4}$

$f'(x) > 0$ when $\cos x > \sin x$ which occurs when $-\frac{\pi}{2} \le x < \frac{\pi}{4}$.

$f''(x) = -\sin x - \cos x$; $f''(x) = 0$ when $\tan x = -1$, $x = -\frac{\pi}{4}$

$f''(x) > 0$ when $\cos x < -\sin x$ which occurs when $-\frac{\pi}{2} \le x < -\frac{\pi}{4}$.

Increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right]$

Decreasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

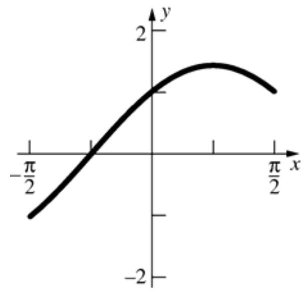
Concave up on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$

Concave down on $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$

Inflection point at $\left(-\frac{\pi}{4}, 0\right)$

Global maximum at $\left(\frac{\pi}{4}, \sqrt{2}\right)$

Global minimum at $\left(-\frac{\pi}{2}, -1\right)$



36. $f(x) = \frac{x^2}{e^x}$

$f'(x) = \frac{e^x(2x) - x^2(e^x)}{(e^x)^2} = \frac{2x - x^2}{e^x}$

f is increasing on [0, 2] because $f'(x) > 0$ on (0, 2).

f is decreasing on $(-\infty, 0] \cup [2, \infty)$ because $f'(x) < 0$ on $(-\infty, 0) \cup (2, \infty)$.

$f''(x) = \frac{e^x(2-2x) - (2x - x^2)e^x}{(e^x)^2} = \frac{x^2 - 4x + 2}{e^x}$

Inflection points are at

$x = \frac{4 \pm \sqrt{16 - 4 \cdot 2}}{2} = 2 \pm \sqrt{2}$.

The graph of f is concave up on $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$ because $f''(x) > 0$ on these intervals.

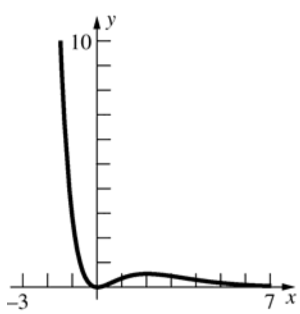
The graph of f is concave down on $(2-\sqrt{2}, 2+\sqrt{2})$ because $f''(x) < 0$ on this interval.

The absolute minimum value is $f(0) = 0$.

The relative maximum value is $f(2) = \frac{4}{e^2}$.

The inflection points are

$\left(2-\sqrt{2}, \frac{6-4\sqrt{2}}{e^{2-\sqrt{2}}}\right)$ and $\left(2+\sqrt{2}, \frac{6+4\sqrt{2}}{e^{2+\sqrt{2}}}\right)$.



37. a. $f'(x) = 5x^4 + 6x^2 + 4 \geq 4 > 0$ for all x, so $f(x)$ is increasing.

b. $f(1) = 7$, so $g(7) = f^{-1}(7) = 1$.

c. $g'(7) = \frac{1}{f'(1)} = \frac{1}{15}$

38. $\frac{1}{2} = e^{10k}$

$k = \frac{\ln\left(\frac{1}{2}\right)}{10} \approx -0.06931$

$y = 100e^{-0.06931t}$

$1 = 100e^{-0.06931t}$

$t = \frac{\ln\left(\frac{1}{100}\right)}{-0.06931} \approx 66.44$

It will take about 66.44 years.

39.

x_n	y_n
1.0	2.0
1.2	2.4
1.4	2.976
1.6	3.80928
1.8	5.02825
2.0	6.83842

40. Let x be the horizontal distance from the airplane to the searchlight, $\frac{dx}{dt} = 300$.

$\tan\theta = \frac{500}{x}$, so $\theta = \tan^{-1}\frac{500}{x}$.

$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{500}{x}\right)^2}\left(-\frac{500}{x^2}\right)\frac{dx}{dt}$

$= -\frac{500}{x^2 + 250{,}000}\frac{dx}{dt}$

When $\theta = 30°$, $x = \frac{500}{\tan 30°} = 500\sqrt{3}$ and

$\frac{d\theta}{dt} = -\frac{500}{(500\sqrt{3})^2 + (500)^2}(300)$

$= -\frac{300}{2000} = -\frac{3}{20}$. The angle is decreasing at the rate of 0.15 rad/s $\approx$ 8.59°/s.

41. $y = (\cos x)^{\sin x} = e^{\sin x \ln(\cos x)}$

$$\frac{dy}{dx} = e^{\sin x \ln(\cos x)} \frac{d}{dx}[\sin x \ln(\cos x)]$$

$$= e^{\sin x \ln(\cos x)}\left[\cos x \ln(\cos x) + (\sin x)\left(\frac{1}{\cos x}\right)(-\sin x)\right]$$

$$= (\cos x)^{\sin x}\left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$$

At $x = 0$, $\frac{dy}{dx} = 1^0(1\ln 1 - 0) = 0$.

The tangent line has slope 0, so it is horizontal: $y = 1$.

42. Let t represent the number of years since 1990.

$$14,000 = 10,000e^{10k}$$

$$k = \frac{\ln(1.4)}{10} \approx 0.03365$$

$$y = 10,000e^{0.03365t}$$

$$y(20) = 10,000e^{(0.03365)(20)} \approx 19,601$$

The population will be about 19,600.

43. Integrating factor is x. $D[yx] = 0; y = Cx^{-1}$

44. Integrating factor is x^2.

$$D[yx^2] = x^3; y = \left(\frac{1}{4}\right)x^2 + Cx^{-2}$$

45. (Linear first-order) $y' + 2xy = 2x$

Integrating factor: $e^{\int 2x dx} = e^{x^2}$

$$D[ye^{x^2}] = 2xe^{x^2}; ye^{x^2} = e^{x^2} + C;$$

$$y = 1 + Ce^{-x^2}$$

If $x = 0$, $y = 3$, then $3 = 1 + C$, so $C = 2$.

Therefore, $y = 1 + 2e^{-x^2}$.

46. Integrating factor is e^{-ax}.

$$D[ye^{-ax}] = 1; y = e^{ax}(x + C)$$

47. Integrating factor is e^{-2x}.

$$D[ye^{-2x}] = e^{-x}; y = -e^{x} + Ce^{2x}$$

48. **a.** $Q'(t) = 3 - 0.02Q$

b. $Q'(t) + 0.02Q = 3$

Integrating factor is $e^{0.02t}$

$$D[Qe^{0.02t}] = 3e^{0.02t}$$

$$Q(t) = 150 + Ce^{-0.02t}$$

$Q(t) = 150 - 30e^{-0.02t}$ goes through (0, 120).

c. $Q \to 150$ g, as $t \to \infty$.

Review and Preview Problems

1. $\underset{\substack{u=2x \\ du=2\,dx}}{\int \sin 2x\, dx} = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C =$

$$-\frac{1}{2}\cos 2x + C$$

2. $\underset{\substack{u=3t \\ du=3\,dt}}{\int e^{3t}\, dt} = \frac{1}{3}\int e^u\, du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3t} + C$

3. $\underset{\substack{u=x^2 \\ du=2x\,dx}}{\int x \sin x^2\, dx} = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C =$

$$-\frac{1}{2}\cos x^2 + C$$

4. $\underset{\substack{u=3x^2 \\ du=6x\,dx}}{\int xe^{3x^2}\, dx} = \frac{1}{6}\int e^u\, du = \frac{1}{6}e^u + C = \frac{1}{6}e^{3x^2} + C$

5. $\underset{\substack{u=\cos t \\ du=-\sin t\,dt}}{\int \frac{\sin t}{\cos t}\, dt} = -\int \frac{1}{u}\, du = -\ln|u| + C =$

$$\ln\left|\frac{1}{u}\right| + C = \ln\left|\frac{1}{\cos t}\right| + C = \ln|\sec t| + C$$

6. $\underset{\substack{u=\sin x \\ du=\cos x\,dx}}{\int \sin^2 x \cos x\, dx} = \int u^2\, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$

7. $\underset{\substack{u=x^2+2 \\ du=2x\,dx}}{\int x\sqrt{x^2+2}\, dx} = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{3}u^{3/2} + C$

$$= \frac{1}{3}\left(x^2 + 2\right)^{3/2} + C$$

8. $\underset{\substack{u=x^2+1 \\ du=2x\,dx}}{\int \frac{x}{x^2+1}\, dx} = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C$

$$= \ln\sqrt{|u|} + C = \ln\sqrt{x^2+1} + C$$

9. $f'(x) = \left[x\left(\frac{1}{x}\right) + (\ln x)(1)\right] - 1 = \ln x$

10. $f'(x) = \left[\frac{x}{\sqrt{1-x^2}} + (1)\arcsin x\right] + \frac{-2x}{2\sqrt{1-x^2}}$
$= \arcsin x$

11. $f'(x) = \left[(-2x)(\cos x) + (-x^2)(-\sin x)\right] +$
$\left[(2)(\sin x) + (2x)(\cos x)\right] + \left[2(-\sin x)\right]$
$= x^2 \sin x$

12. $f'(x) = e^x\left(\cos x + \sin x\right) + e^x(\sin x - \cos x)$
$= 2e^x \sin x$

13. $\cos 2x = 1 - 2\sin^2 x$**;** thus $\sin^2 x = \frac{1-\cos 2x}{2}$

14. $\cos 2x = 2\cos^2 x - 1$**;** thus $\cos^2 x = \frac{1+\cos 2x}{2}$

15. $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

16. $\sin u \cos v = \frac{\sin(u+v)+\sin(u-v)}{2} \Rightarrow$
$\sin 3x \cos 4x = \frac{\sin(7x)+\sin(-x)}{2} = \frac{\sin 7x - \sin x}{2}$

17. $\cos u \cos v = \frac{\cos(u+v)+\cos(u-v)}{2} \Rightarrow$
$\cos 3x \cos 5x = \frac{\cos(8x)+\cos(-2x)}{2}$
$= \frac{\cos 8x + \cos 2x}{2}$

18. $\sin u \sin v = \frac{\cos(u-v)-\cos(u+v)}{2} \Rightarrow$
$\sin 2x \sin 3x = \frac{\cos(-x)-\cos(5x)}{2}$
$= \frac{\cos x - \cos 5x}{2}$

19. $\sqrt{a^2 - (a\sin t)^2} = \sqrt{a^2(1-\sin^2 t)} =$
$|a|\sqrt{\cos^2 t} = |a|\cos t$

20. $\sqrt{a^2 + (a\tan t)^2} = \sqrt{a^2(1+\tan^2 t)} =$
$|a|\sqrt{\sec^2 t} = |a|\sec t$

21. $\sqrt{(a\sec t)^2 - a^2 -} = \sqrt{a^2(\sec^2 t - 1)} =$
$|a|\sqrt{\tan^2 t} = |a| \cdot |\tan t|$

22. $\int_0^a e^{-x}\,dx = \frac{1}{2} \Rightarrow \left[-e^{-x}\right]_0^a = \frac{1}{2} \Rightarrow$
$\left[-e^{-a} + 1\right] = \frac{1}{2} \Rightarrow \frac{1}{e^a} = \frac{1}{2} \Rightarrow$
$e^a = 2 \Rightarrow a = \ln 2$

23. $\frac{1}{1-x} - \frac{1}{x} = \frac{x-(1-x)}{(1-x)x} = \frac{2x-1}{x(1-x)}$

24. $\frac{7}{5(x+2)} + \frac{8}{5(x-3)} = \frac{7(x-3)+8(x+2)}{5(x+2)(x-3)} =$
$\frac{15x-5}{5(x+2)(x-3)} = \frac{5(3x-1)}{5(x+2)(x-3)} =$
$\frac{(3x-1)}{(x+2)(x-3)}$

25. $-\frac{1}{x} - \frac{1}{2(x+1)} + \frac{3}{2(x-3)}$
$= \frac{-2(x+1)(x-3) - x(x-3) + 3x(x+1)}{2x(x+1)(x-3)}$
$= \frac{-2(x^2-2x-3) - (x^2-3x) + (3x^2+3x)}{2x(x+1)(x-3)}$
$= \frac{10x+6}{2x(x+1)(x-3)} = \frac{2(5x+3)}{2x(x+1)(x-3)} =$
$\frac{(5x+3)}{x(x+1)(x-3)}$

26. $\frac{1}{y} + \frac{1}{2000-y} = \frac{(2000-y)+y}{y(2000-y)} = \frac{2000}{y(2000-y)}$

CHAPTER 7	Techniques of Integration

7.1 Concepts Review

1. elementary function

2. $\int u^5 du$

3. e^x

4. $\int_1^2 u^3 du$

Problem Set 7.1

1. $\int (x-2)^5 dx = \frac{1}{6}(x-2)^6 + C$

2. $\int \sqrt{3x}\, dx = \frac{1}{3}\int \sqrt{3x} \cdot 3dx = \frac{2}{9}(3x)^{3/2} + C$

3. $u = x^2 + 1, du = 2x\, dx$
When $x = 0$, $u = 1$ and when $x = 2$, $u = 5$.
$\int_0^2 x(x^2+1)^5 dx = \frac{1}{2}\int_0^2 (x^2+1)^5 (2x\, dx)$
$= \frac{1}{2}\int_1^5 u^5\, du$
$= \left[\frac{u^6}{12}\right]_1^5 = \frac{5^6 - 1^6}{12}$
$= \frac{15624}{12} = 1302$

4. $u = 1 - x^2, du = -2x\, dx$
When $x = 0$, $u = 1$ and when $x = 1$, $u = 0$.
$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_0^1 \sqrt{1-x^2}\,(-2x\, dx)$
$= -\frac{1}{2}\int_1^0 u^{1/2}\, du$
$= \frac{1}{2}\int_0^1 u^{1/2}\, du$
$= \left[\frac{1}{3}u^{3/2}\right]_0^1 = \frac{1}{3}$

5. $\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

6. $u = 2 + e^x, du = e^x dx$
$\int \frac{e^x}{2+e^x} dx = \int \frac{du}{u}$
$= \ln|u| + C$
$= \ln\left|2 + e^x\right| + C$
$= \ln(2 + e^x) + C$

7. $u = x^2 + 4,\ du = 2x\, dx$
$\int \frac{x}{x^2+4} dx = \frac{1}{2}\int \frac{du}{u}$
$= \frac{1}{2}\ln|u| + C$
$= \frac{1}{2}\ln\left|x^2+4\right| + C$
$= \frac{1}{2}\ln(x^2+4) + C$

8. $\int \frac{2t^2}{2t^2+1} dt = \int \frac{2t^2+1-1}{2t^2+1} dt$
$= \int dt - \int \frac{1}{2t^2+1} dt$
$u = \sqrt{2}t, du = \sqrt{2}dt$
$t - \int \frac{1}{2t^2+1} dt = t - \frac{1}{\sqrt{2}}\int \frac{du}{1+u^2}$
$= t - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t) + C$

9. $u = 4 + z^2,\ du = 2z\, dz$
$\int 6z\sqrt{4+z^2} dz = 3\int \sqrt{u}\, du$
$= 2u^{3/2} + C$
$= 2(4+z^2)^{3/2} + C$

10. $u = 2t+1, du = 2dt$

$$\int \frac{5}{\sqrt{2t+1}}dt = \frac{5}{2}\int \frac{du}{\sqrt{u}}$$
$$= 5\sqrt{u} + C$$
$$= 5\sqrt{2t+1} + C$$

11. $\int \frac{\tan z}{\cos^2 z}dz = \int \tan z \sec^2 z\,dz$

$u = \tan z,\ du = \sec^2 z\,dz$

$$\int \tan z \sec^2 z\,dz = \int u\,du$$
$$= \frac{1}{2}u^2 + C$$
$$= \frac{1}{2}\tan^2 z + C$$

12. $u = \cos z, du = -\sin z\,dz$

$$\int e^{\cos z}\sin z\,dz = -\int e^{\cos z}(-\sin z\,dz)$$
$$= -\int e^u du = -e^u + C$$
$$= -e^{\cos z} + C$$

13. $u = \sqrt{t}, du = \frac{1}{2\sqrt{t}}dt$

$$\int \frac{\sin\sqrt{t}}{\sqrt{t}}dt = 2\int \sin u\,du$$
$$= -2\cos u + C$$
$$= -2\cos\sqrt{t} + C$$

14. $u = x^2,\ du = 2x\,dx$

$$\int \frac{2x\,dx}{\sqrt{1-x^4}} = \int \frac{du}{\sqrt{1-u^2}}$$
$$= \sin^{-1}u + C$$
$$= \sin^{-1}(x^2) + C$$

15. $u = \sin x, du = \cos x\,dx$

$$\int_0^{\pi/4}\frac{\cos x}{1+\sin^2 x}dx = \int_0^{\sqrt{2}/2}\frac{du}{1+u^2}$$
$$= [\tan^{-1}u]_0^{\sqrt{2}/2}$$
$$= \tan^{-1}\frac{\sqrt{2}}{2}$$
$$\approx 0.6155$$

16. $u = \sqrt{1-x}, du = -\frac{1}{2\sqrt{1-x}}dx$

$$\int_0^{3/4}\frac{\sin\sqrt{1-x}}{\sqrt{1-x}}dx = -2\int_1^{1/2}\sin u\,du$$
$$= 2\int_{1/2}^{1}\sin u\,du$$
$$= [-2\cos u]_{1/2}^{1}$$
$$= -2\left(\cos 1 - \cos\frac{1}{2}\right)$$
$$\approx 0.6746$$

17. $\int \frac{3x^2+2x}{x+1}dx = \int (3x-1)dx + \int \frac{1}{x+1}dx$

$$= \frac{3}{2}x^2 - x + \ln|x+1| + C$$

18. $\int \frac{x^3+7x}{x-1}dx = \int (x^2+x+8)dx + 8\int \frac{1}{x-1}dx$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 8x + 8\ln|x-1| + C$$

19. $u = \ln 4x^2, du = \frac{2}{x}dx$

$$\int \frac{\sin(\ln 4x^2)}{x}dx = \frac{1}{2}\int \sin u\,du$$
$$= -\frac{1}{2}\cos u + C$$
$$= -\frac{1}{2}\cos(\ln 4x^2) + C$$

20. $u = \ln x,\ du = \frac{1}{x}dx$

$$\int \frac{\sec^2(\ln x)}{2x}dx = \frac{1}{2}\int \sec^2 u\,du$$
$$= \frac{1}{2}\tan u + C$$
$$= \frac{1}{2}\tan(\ln x) + C$$

21. $u = e^x,\quad du = e^x dx$

$$\int \frac{6e^x}{\sqrt{1-e^{2x}}}dx = 6\int \frac{du}{\sqrt{1-u^2}}du$$
$$= 6\sin^{-1}u + C$$
$$= 6\sin^{-1}(e^x) + C$$

22. $u = x^2,\ du = 2x\,dx$

$$\int \frac{x}{x^4+4}dx = \frac{1}{2}\int \frac{du}{4+u^2}$$
$$= \frac{1}{4}\tan^{-1}\frac{u}{2} + C$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{x^2}{2}\right) + C$$

23. $u = 1 - e^{2x},\ du = -2e^{2x}dx$

$$\int \frac{3e^{2x}}{\sqrt{1-e^{2x}}}dx = -\frac{3}{2}\int \frac{du}{\sqrt{u}}$$
$$= -3\sqrt{u} + C$$
$$= -3\sqrt{1-e^{2x}} + C$$

24. $\int \frac{x^3}{x^4+4}dx = \frac{1}{4}\int \frac{4x^3}{x^4+4}dx$

$$= \frac{1}{4}\ln\left|x^4+4\right| + C$$
$$= \frac{1}{4}\ln(x^4+4) + C$$

25. $\int_0^1 t\,3^{t^2}\,dt = \frac{1}{2}\int_0^1 2t\,3^{t^2}\,dt$

$$= \left[\frac{3^{t^2}}{2\ln 3}\right]_0^1 = \frac{3}{2\ln 3} - \frac{1}{2\ln 3}$$
$$= \frac{1}{\ln 3} \approx 0.9102$$

26. $\int_0^{\pi/6} 2^{\cos x}\sin x\,dx = -\int_0^{\pi/6} 2^{\cos x}(-\sin x\,dx)$

$$= \left[-\frac{2^{\cos x}}{\ln 2}\right]_0^{\pi/6}$$
$$= -\frac{1}{\ln 2}(2^{\sqrt{3}/2} - 2)$$
$$= \frac{2 - 2^{\sqrt{3}/2}}{\ln 2}$$
$$\approx 0.2559$$

27. $\int \frac{\sin x - \cos x}{\sin x}dx = \int\left(1 - \frac{\cos x}{\sin x}\right)dx$

$u = \sin x,\ du = \cos x\,dx$

$$\int \frac{\sin x - \cos x}{\sin x}dx = x - \int \frac{du}{u}$$
$$= x - \ln|u| + C$$
$$= x - \ln|\sin x| + C$$

28. $u = \cos(4t-1),\ du = -4\sin(4t-1)dt$

$$\int \frac{\sin(4t-1)}{1-\sin^2(4t-1)}dt = \int \frac{\sin(4t-1)}{\cos^2(4t-1)}dt$$
$$= -\frac{1}{4}\int \frac{1}{u^2}du$$
$$= \frac{1}{4}u^{-1} + C = \frac{1}{4}\sec(4t-1) + C$$

29. $u = e^x,\ du = e^x dx$

$$\int e^x \sec e^x dx = \int \sec u\,du$$
$$= \ln|\sec u + \tan u| + C$$
$$= \ln\left|\sec e^x + \tan e^x\right| + C$$

30. $u = e^x,\ du = e^x dx$

$$\int e^x \sec^2(e^x)dx = \int \sec^2 u\,du = \tan u + C$$
$$= \tan(e^x) + C$$

31. $\int \frac{\sec^3 x + e^{\sin x}}{\sec x}dx = \int(\sec^2 x + e^{\sin x}\cos x)\,dx$

$$= \tan x + \int e^{\sin x}\cos x\,dx$$

$u = \sin x,\ du = \cos x\,dx$

$$\tan x + \int e^{\sin x}\cos x\,dx = \tan x + \int e^u\,du$$
$$= \tan x + e^u + C = \tan x + e^{\sin x} + C$$

32. $u = \sqrt{3t^2 - t - 1}$,

$$du = \frac{1}{2}(3t^2 - t - 1)^{-1/2}(6t-1)dt$$
$$\int \frac{(6t-1)\sin\sqrt{3t^2-t-1}}{\sqrt{3t^2-t-1}}dt = 2\int \sin u\,du$$
$$= -2\cos u + C$$
$$= -2\cos\sqrt{3t^2-t-1} + C$$

33. $u = t^3 - 2$, $du = 3t^2 dt$

$$\int \frac{t^2 \cos(t^3-2)}{\sin^2(t^3-2)} dt = \frac{1}{3}\int \frac{\cos u}{\sin^2 u} du$$

$v = \sin u$, $dv = \cos u\, du$

$$\frac{1}{3}\int \frac{\cos u}{\sin^2 u} du = \frac{1}{3}\int v^{-2} dv = -\frac{1}{3}v^{-1} + C$$

$$= -\frac{1}{3\sin u} + C$$

$$= -\frac{1}{3\sin(t^3-2)} + C.$$

34. $\int \frac{1+\cos 2x}{\sin^2 2x} dx = \int \frac{1}{\sin^2 2x} dx + \int \frac{\cos 2x}{\sin^2 2x} dx$

$$= \int \csc^2 2x\, dx + \int \cot 2x \csc 2x\, dx$$

$$= -\frac{1}{2}\cot 2x - \frac{1}{2}\csc 2x + C$$

35. $u = t^3 - 2$, $du = 3t^2 dt$

$$\int \frac{t^2 \cos^2(t^3-2)}{\sin^2(t^3-2)} dt = \frac{1}{3}\int \frac{\cos^2 u}{\sin^2 u} du$$

$$= \frac{1}{3}\int \cot^2 u\, du = \frac{1}{3}\int (\csc^2 u - 1) du$$

$$= \frac{1}{3}[-\cot u - u] + C_1$$

$$= \frac{1}{3}[-\cot(t^3-2) - (t^3-2)] + C_1$$

$$= -\frac{1}{3}[\cot(t^3-2) + t^3] + C$$

36. $u = 1 + \cot 2t$, $du = -2\csc^2 2t$

$$\int \frac{\csc^2 2t}{\sqrt{1+\cot 2t}} dt = -\frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{1+\cot 2t} + C$$

37. $u = \tan^{-1} 2t$, $du = \frac{2}{1+4t^2} dt$

$$\int \frac{e^{\tan^{-1} 2t}}{1+4t^2} dt = \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u + C = \frac{1}{2}e^{\tan^{-1} 2t} + C$$

38. $u = -t^2 - 2t - 5$,

$du = (-2t - 2)dt = -2(t + 1)dt$

$$\int (t+1)e^{-t^2-2t-5} = -\frac{1}{2}\int e^u du$$

$$= -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-t^2-2t-5} + C$$

39. $u = 3y^2$, $du = 6y\, dy$

$$\int \frac{y}{\sqrt{16-9y^4}} dy = \frac{1}{6}\int \frac{1}{\sqrt{4^2-u^2}} du$$

$$= \frac{1}{6}\sin^{-1}\left(\frac{u}{4}\right) + C$$

$$= \frac{1}{6}\sin^{-1}\left(\frac{3y^2}{4}\right) + C$$

40. $u = 3x$, $du = 3\,dx$

$$\int \cosh 3x\, dx$$

$$= \frac{1}{3}\int (\cosh u) du = \frac{1}{3}\sinh u + C$$

$$= \frac{1}{3}\sinh 3x + C$$

41. $u = x^3$, $du = 3x^2 dx$

$$\int x^2 \sinh x^3 dx = \frac{1}{3}\int \sinh u\, du$$

$$= \frac{1}{3}\cosh u + C$$

$$= \frac{1}{3}\cosh x^3 + C$$

42. $u = 2x$, $du = 2\, dx$

$$\int \frac{5}{\sqrt{9-4x^2}} dx = \frac{5}{2}\int \frac{1}{\sqrt{3^2-u^2}} du$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

43. $u = e^{3t}$, $du = 3e^{3t} dt$

$$\int \frac{e^{3t}}{\sqrt{4-e^{6t}}} dt = \frac{1}{3}\int \frac{1}{\sqrt{2^2-u^2}} du$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{e^{3t}}{2}\right) + C$$

44. $u = 2t$, $du = 2dt$

$$\int \frac{dt}{2t\sqrt{4t^2-1}} = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}} du$$

$$= \frac{1}{2}\left[\sec^{-1}|u|\right] + C$$

$$= \frac{1}{2}\sec^{-1}|2t| + C$$

45. $u = \cos x$, $du = -\sin x\,dx$

$$\int_0^{\pi/2} \frac{\sin x}{16+\cos^2 x}dx = -\int_1^0 \frac{1}{16+u^2}du$$
$$= \int_0^1 \frac{1}{16+u^2}du$$
$$= \left[\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right)\right]_0^1 = \left[\frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{4}\tan^{-1} 0\right]$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) \approx 0.0612$$

46. $u = e^{2x}+e^{-2x}$, $du = (2e^{2x}-2e^{-2x})dx$

$= 2(e^{2x}-e^{-2x})dx$

$$\int_0^1 \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx = \frac{1}{2}\int_2^{e^2+e^{-2}} \frac{1}{u}du$$
$$= \frac{1}{2}\left[\ln|u|\right]_2^{e^2+e^{-2}} = \frac{1}{2}\ln\left|e^2+e^{-2}\right| - \frac{1}{2}\ln 2$$
$$= \frac{1}{2}\ln\left|\frac{e^4+1}{e^2}\right| - \frac{1}{2}\ln 2$$
$$= \frac{1}{2}\ln(e^4+1) - \frac{1}{2}\ln(e^2) - \frac{1}{2}\ln 2$$
$$= \frac{1}{2}\left(\ln\left(\frac{e^4+1}{2}\right) - 2\right) \approx 0.6625$$

47. $$\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{x^2+2x+1+4}dx$$
$$= \int \frac{1}{(x+1)^2+2^2}d(x+1)$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

48. $$\int \frac{1}{x^2-4x+9}dx = \int \frac{1}{x^2-4x+4+5}dx$$
$$= \int \frac{1}{(x-2)^2+(\sqrt{5})^2}d(x-2)$$
$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x-2}{\sqrt{5}}\right) + C$$

49. $$\int \frac{dx}{9x^2+18x+10} = \int \frac{dx}{9x^2+18x+9+1}$$
$$= \int \frac{dx}{(3x+3)^2+1^2}$$

$u = 3x + 3$, $du = 3\,dx$

$$\int \frac{dx}{(3x+3)^2+1^2} = \frac{1}{3}\int \frac{du}{u^2+1^2}$$
$$= \frac{1}{3}\tan^{-1}(3x+3) + C$$

50. $$\int \frac{dx}{\sqrt{16+6x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-6x+9-25)}}$$
$$= \int \frac{dx}{\sqrt{-(x-3)^2+5^2}} = \int \frac{dx}{\sqrt{5^2-(x-3)^2}}$$
$$= \sin^{-1}\left(\frac{x-3}{5}\right) + C$$

51. $$\int \frac{x+1}{9x^2+18x+10}dx = \frac{1}{18}\int \frac{18x+18}{9x^2+18x+10}dx$$
$$= \frac{1}{18}\ln\left|9x^2+18x+10\right| + C$$
$$= \frac{1}{18}\ln\left(9x^2+18x+10\right) + C$$

52. $$\int \frac{3-x}{\sqrt{16+6x-x^2}}dx = \frac{1}{2}\int \frac{6-2x}{\sqrt{16+6x-x^2}}dx$$
$$= \sqrt{16+6x-x^2} + C$$

53. $u = \sqrt{2}t$, $du = \sqrt{2}dt$

$$\int \frac{dt}{t\sqrt{2t^2-9}} = \int \frac{du}{u\sqrt{u^2-3^2}}$$
$$= \frac{1}{3}\sec^{-1}\left(\frac{\left|\sqrt{2}t\right|}{3}\right) + C$$

54. $$\int \frac{\tan x}{\sqrt{\sec^2 x - 4}}dx = \int \frac{\cos x}{\cos x}\frac{\tan x}{\sqrt{\sec^2 x-4}}dx$$
$$= \int \frac{\sin x}{\sqrt{1-4\cos^2 x}}dx$$

$u = 2\cos x$, $du = -2\sin x\,dx$

$$\int \frac{\sin x}{\sqrt{1-4\cos^2 x}}dx = -\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$$
$$= -\frac{1}{2}\sin^{-1} u + C = -\frac{1}{2}\sin^{-1}(2\cos x) + C$$

55. The length is given by

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \int_0^{\pi/4} \sqrt{1+\left[\frac{1}{\cos x}(-\sin x)\right]^2}\,dx$$
$$= \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$$
$$= \int_0^{\pi/4} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$$
$$= \ln\left|\sqrt{2}+1\right| - \ln|1| = \ln\left|\sqrt{2}+1\right| \approx 0.881$$

56. $\sec x = \frac{1}{\cos x} = \frac{1+\sin x}{\cos x(1+\sin x)}$

$= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1+\sin x)} = \frac{\sin x(1+\sin x) + \cos^2 x}{\cos x(1+\sin x)}$

$= \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$

$\int \sec x = \int \left(\frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x} \right) dx$

$= \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1+\sin x} dx$

For the first integral use $u = \cos x$, $du = -\sin x\, dx$, and for the second integral use $v = 1 + \sin x$, $dv = \cos x\, dx$.

$\int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1+\sin x} dx = -\int \frac{du}{u} + \int \frac{dv}{v}$

$= -\ln|u| + \ln|v| + C$

$= -\ln|\cos x| + \ln|1+\sin x| + C$

$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C$

$= \ln|\sec x + \tan x| + C$

57. $u = x - \pi$, $du = dx$

$\int_0^{2\pi} \frac{x|\sin x|}{1+\cos^2 x} dx = \int_{-\pi}^{\pi} \frac{(u+\pi)|\sin(u+\pi)|}{1+\cos^2(u+\pi)} du$

$= \int_{-\pi}^{\pi} \frac{(u+\pi)|\sin u|}{1+\cos^2 u} du$

$= \int_{-\pi}^{\pi} \frac{u|\sin u|}{1+\cos^2 u} du + \int_{-\pi}^{\pi} \frac{\pi|\sin u|}{1+\cos^2 u} du$

$\int_{-\pi}^{\pi} \frac{u|\sin u|}{1+\cos^2 u} du = 0$ by symmetry.

$\int_{-\pi}^{\pi} \frac{\pi|\sin u|}{1+\cos^2 u} du = 2\int_0^{\pi} \frac{\pi \sin u}{1+\cos^2 u} du$

$v = \cos u$, $dv = -\sin u\, du$

$-2\int_1^{-1} \frac{\pi}{1+v^2} dv = 2\pi\int_{-1}^{1} \frac{1}{1+v^2} dv$

$= 2\pi[\tan^{-1} v]_{-1}^{1} = 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$

$= 2\pi\left(\frac{\pi}{2}\right) = \pi^2$

58. $V = 2\pi\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(x + \frac{\pi}{4}\right)|\sin x - \cos x|\, dx$

$u = x - \frac{\pi}{4}, du = dx$

$V = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2}\right)\left|\sin\left(u + \frac{\pi}{4}\right) - \cos\left(u + \frac{\pi}{4}\right)\right| du$

$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2}\right)\left|\frac{\sqrt{2}}{2}\sin u + \frac{\sqrt{2}}{2}\cos u - \frac{\sqrt{2}}{2}\cos u + \frac{\sqrt{2}}{2}\sin u\right| du$

$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2}\right)\left|\sqrt{2}\sin u\right| du = 2\sqrt{2}\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u|\sin u|\, du + \sqrt{2}\pi^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin u|\, du$

$2\sqrt{2}\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u|\sin u|\, du = 0$ by symmetry. Therefore,

$V = \sqrt{2}\pi^2 2\int_0^{\frac{\pi}{2}} \sin u\, du = 2\sqrt{2}\pi^2[-\cos u]_0^{\frac{\pi}{2}} = 2\sqrt{2}\pi^2$

7.2 Concepts Review

1. $uv - \int v\,du$

2. x; $\sin x\,dx$

3. 1

4. reduction

Problem Set 7.2

1. $u = x \qquad dv = e^x dx$
$du = dx \qquad v = e^x$
$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

2. $u = x \qquad dv = e^{3x} dx$
$du = dx \qquad v = \frac{1}{3}e^{3x}$
$\int xe^{3x} dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x} dx$
$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

3. $u = t \qquad dv = e^{5t+\pi} dt$
$du = dt \qquad v = \frac{1}{5}e^{5t+\pi}$
$\int te^{5t+\pi} dt = \frac{1}{5}te^{5t+\pi} - \int \frac{1}{5}e^{5t+\pi} dt$
$= \frac{1}{5}te^{5t+\pi} - \frac{1}{25}e^{5t+\pi} + C$

4. $u = t + 7 \quad dv = e^{2t+3} dt$
$du = dt \qquad v = \frac{1}{2}e^{2t+3}$
$\int (t+7)e^{2t+3} dt = \frac{1}{2}(t+7)e^{2t+3} - \int \frac{1}{2}e^{2t+3} dt$
$= \frac{1}{2}(t+7)e^{2t+3} - \frac{1}{4}e^{2t+3} + C$
$= \frac{t}{2}e^{2t+3} + \frac{13}{4}e^{2t+3} + C$

5. $u = x \qquad dv = \cos x\,dx$
$du = dx \qquad v = \sin x$
$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$
$= x \sin x + \cos x + C$

6. $u = x \qquad dv = \sin 2x\,dx$
$du = dx \qquad v = -\frac{1}{2}\cos 2x$
$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\,dx$
$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

7. $u = t - 3 \qquad dv = \cos (t - 3)dt$
$du = dt \qquad v = \sin (t - 3)$
$\int (t-3)\cos(t-3)dt = (t-3)\sin(t-3) - \int \sin(t-3)dt$
$= (t - 3) \sin (t - 3) + \cos (t - 3) + C$

8. $u = x - \pi \qquad dv = \sin(x)dx$
$du = dx \qquad v = -\cos x$
$\int (x-\pi)\sin(x)dx = -(x-\pi)\cos x + \int \cos x\,dx$
$= (\pi - x) \cos x + \sin x + C$

9. $u = t \qquad dv = \sqrt{t+1}\,dt$
$du = dt \qquad v = \frac{2}{3}(t+1)^{3/2}$
$\int t\sqrt{t+1}\,dt = \frac{2}{3}t(t+1)^{3/2} - \int \frac{2}{3}(t+1)^{3/2} dt$
$= \frac{2}{3}t(t+1)^{3/2} - \frac{4}{15}(t+1)^{5/2} + C$

10. $u = t \qquad dv = \sqrt[3]{2t+7}dt$
$du = dt \qquad v = \frac{3}{8}(2t+7)^{4/3}$
$\int t\sqrt[3]{2t+7}dt = \frac{3}{8}t(2t+7)^{4/3} - \int \frac{3}{8}(2t+7)^{4/3} dt$
$= \frac{3}{8}t(2t+7)^{4/3} - \frac{9}{112}(2t+7)^{7/3} + C$

11. $u = \ln 3x \qquad dv = dx$
$du = \frac{1}{x}dx \qquad v = x$
$\int \ln 3x\,dx = x\ln 3x - \int x\frac{1}{x}dx = x\ln 3x - x + C$

12. $u = \ln(7x^5) \qquad dv = dx$
$du = \frac{5}{x}dx \qquad v = x$
$\int \ln(7x^5)dx = x\ln(7x^5) - \int x\frac{5}{x}dx$
$= x\ln(7x^5) - 5x + C$

13. $u = \arctan x \qquad dv = dx$

$du = \dfrac{1}{1+x^2}dx \qquad v = x$

$$\int \arctan x = x\arctan x - \int \frac{x}{1+x^2}dx$$

$$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

14. $u = \arctan 5x \qquad dv = dx$

$du = \dfrac{5}{1+25x^2}dx \qquad v = x$

$$\int \arctan 5x\,dx = x\arctan 5x - \int \frac{5x}{1+25x^2}dx$$

$$= x\arctan 5x - \frac{1}{10}\int \frac{50x\,dx}{1+25x^2}$$

$$= x\arctan 5x - \frac{1}{10}\ln(1+25x^2) + C$$

15. $u = \ln x \qquad dv = \dfrac{dx}{x^2}$

$du = \dfrac{1}{x}dx \qquad v = -\dfrac{1}{x}$

$$\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \int -\frac{1}{x}\left(\frac{1}{x}\right)dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

16. $u = \ln 2x^5 \qquad dv = \dfrac{1}{x^2}dx$

$du = \dfrac{5}{x}dx \qquad v = -\dfrac{1}{x}$

$$\int_2^3 \frac{\ln 2x^5}{x^2}dx = \left[-\frac{1}{x}\ln 2x^5\right]_2^3 + 5\int_2^3 \frac{1}{x^2}dx$$

$$= \left[-\frac{1}{x}\ln 2x^5 - \frac{5}{x}\right]_2^3$$

$$= \left(-\frac{1}{3}\ln(2\cdot 3^5) - \frac{5}{3}\right) - \left(-\frac{1}{2}\ln(2\cdot 2^5) - \frac{5}{2}\right)$$

$$= -\frac{1}{3}\ln 2 - \frac{5}{3}\ln 3 - \frac{5}{3} + 3\ln 2 + \frac{5}{2}$$

$$= \frac{8}{3}\ln 2 - \frac{5}{3}\ln 3 + \frac{5}{6} \approx 0.8507$$

17. $u = \ln t \qquad dv = \sqrt{t}\,dt$

$du = \dfrac{1}{t}dt \qquad v = \dfrac{2}{3}t^{3/2}$

$$\int_1^e \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^e - \int_1^e \frac{2}{3}t^{1/2}dt$$

$$= \frac{2}{3}e^{3/2}\ln e - \frac{2}{3}\cdot 1\ln 1 - \left[\frac{4}{9}t^{3/2}\right]_1^e$$

$$= \frac{2}{3}e^{3/2} - 0 - \frac{4}{9}e^{3/2} + \frac{4}{9} = \frac{2}{9}e^{3/2} + \frac{4}{9} \approx 1.4404$$

18. $u = \ln x^3 \qquad dv = \sqrt{2x}dx$

$du = \dfrac{3}{x}dx \qquad v = \dfrac{1}{3}(2x)^{3/2}$

$$\int_1^5 \sqrt{2x}\ln x^3\,dx = \left[\frac{1}{3}(2x)^{3/2}\ln x^3\right]_1^5 - \int_1^5 2^{3/2}\sqrt{x}\,dx$$

$$= \left[\frac{1}{3}(2x)^{3/2}\ln x^3 - \frac{2^{5/2}}{3}x^{3/2}\right]_1^5$$

$$= \frac{1}{3}(10)^{3/2}\ln 5^3 - \frac{2^{5/2}}{3}5^{3/2} - \left(\frac{1}{3}(2)^{3/2}\ln 1^3 - \frac{2^{5/2}}{3}\right)$$

$$= -\frac{4\sqrt{2}}{3}5^{3/2} + \frac{4\sqrt{2}}{3} + 10^{3/2}\ln 5 \approx 31.699$$

19. $u = \ln z \qquad dv = z^3dz$

$du = \dfrac{1}{z}dz \qquad v = \dfrac{1}{4}z^4$

$$\int z^3\ln z\,dz = \frac{1}{4}z^4\ln z - \int \frac{1}{4}z^4\cdot\frac{1}{z}dz$$

$$= \frac{1}{4}z^4\ln z - \frac{1}{4}\int z^3dz$$

$$= \frac{1}{4}z^4\ln z - \frac{1}{16}z^4 + C$$

20. $u = \arctan t \qquad dv = t\,dt$

$du = \dfrac{1}{1+t^2}dt \qquad v = \dfrac{1}{2}t^2$

$$\int t\arctan t\,dt = \frac{1}{2}t^2\arctan t - \frac{1}{2}\int \frac{t^2}{1+t^2}dt$$

$$= \frac{1}{2}t^2\arctan t - \frac{1}{2}\int \frac{1+t^2-1}{1+t^2}dt$$

$$= \frac{1}{2}t^2\arctan t - \frac{1}{2}\int dt + \frac{1}{2}\int \frac{1}{1+t^2}dt$$

$$= \frac{1}{2}t^2\arctan t - \frac{1}{2}t + \frac{1}{2}\arctan t + C$$

21. $u = \arctan\left(\frac{1}{t}\right) \qquad dv = dt$

$du = -\frac{1}{1+t^2}dt \qquad v = t$

$$\int \arctan\left(\frac{1}{t}\right)dt = t\arctan\left(\frac{1}{t}\right) + \int \frac{t}{1+t^2}dt$$

$$= t\arctan\left(\frac{1}{t}\right) + \frac{1}{2}\ln(1+t^2) + C$$

22. $u = \ln(t^7) \qquad dv = t^5 dt$

$du = \frac{7}{t}dt \qquad v = \frac{1}{6}t^6$

$$\int t^5 \ln(t^7)dt = \frac{1}{6}t^6 \ln(t^7) - \frac{7}{6}\int t^5 dt$$

$$= \frac{1}{6}t^6 \ln(t^7) - \frac{7}{36}t^6 + C$$

23. $u = x \qquad dv = \csc^2 x\, dx$

$du = dx \qquad v = -\cot x$

$$\int_{\pi/6}^{\pi/2} x\csc^2 x\,dx = \left[-x\cot x\right]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2} \cot x\,dx = \left[-x\cot x + \ln\left|\sin x\right|\right]_{\pi/6}^{\pi/2}$$

$$= -\frac{\pi}{2}\cdot 0 + \ln 1 + \frac{\pi}{6}\sqrt{3} - \ln\frac{1}{2} = \frac{\pi}{2\sqrt{3}} + \ln 2 \approx 1.60$$

24. $u = x \qquad dv = \sec^2 x\, dx$

$du = dx \qquad v = \tan x$

$$\int_{\pi/6}^{\pi/4} x\sec^2 x\,dx = \left[x\tan x\right]_{\pi/6}^{\pi/4} - \int_{\pi/6}^{\pi/4} \tan x\,dx = \left[x\tan x + \ln\left|\cos x\right|\right]_{\pi/6}^{\pi/4} = \frac{\pi}{4} + \ln\frac{\sqrt{2}}{2} - \left(\frac{\pi}{6\sqrt{3}} + \ln\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6\sqrt{3}} + \frac{1}{2}\ln\frac{2}{3} \approx 0.28$$

25. $u = x^3 \qquad dv = x^2\sqrt{x^3+4}\,dx$

$du = 3x^2 dx \qquad v = \frac{2}{9}(x^3+4)^{3/2}$

$$\int x^5\sqrt{x^3+4}\,dx = \frac{2}{9}x^3(x^3+4)^{3/2} - \int \frac{2}{3}x^2(x^3+4)^{3/2}dx = \frac{2}{9}x^3(x^3+4)^{3/2} - \frac{4}{45}(x^3+4)^{5/2} + C$$

26. $u = x^7 \qquad dv = x^6\sqrt{x^7+1}\,dx$

$du = 7x^6 dx \qquad v = \frac{2}{21}(x^7+1)^{3/2}$

$$\int x^{13}\sqrt{x^7+1}\,dx = \frac{2}{21}x^7(x^7+1)^{3/2} - \int \frac{2}{3}x^6(x^7+1)^{3/2}dx = \frac{2}{21}x^7(x^7+1)^{3/2} - \frac{4}{105}(x^7+1)^{5/2} + C$$

27. $u = t^4 \qquad dv = \frac{t^3}{(7-3t^4)^{3/2}}dt$

$du = 4t^3\,dt \qquad v = \frac{1}{6(7-3t^4)^{1/2}}$

$$\int \frac{t^7}{(7-3t^4)^{3/2}}dt = \frac{t^4}{6(7-3t^4)^{1/2}} - \frac{2}{3}\int \frac{t^3}{(7-3t^4)^{1/2}}dt = \frac{t^4}{6(7-3t^4)^{1/2}} + \frac{1}{9}(7-3t^4)^{1/2} + C$$

28. $u = x^2$ $\quad dv = x\sqrt{4-x^2}\,dx$

$du = 2x\,dx$ $\quad v = -\frac{1}{3}(4-x^2)^{3/2}$

$$\int x^3\sqrt{4-x^2}\,dx = -\frac{1}{3}x^2(4-x^2)^{3/2} + \frac{2}{3}\int x(4-x^2)^{3/2}\,dx = -\frac{1}{3}x^2(4-x^2)^{3/2} - \frac{2}{15}(4-x^2)^{5/2} + C$$

29. $u = z^4$ $\quad dv = \frac{z^3}{(4-z^4)^2}\,dz$

$du = 4z^3 dz$ $\quad v = \frac{1}{4(4-z^4)}$

$$\int \frac{z^7}{(4-z^4)^2}\,dz = \frac{z^4}{4(4-z^4)} - \int \frac{z^3}{4-z^4}dz = \frac{z^4}{4(4-z^4)} + \frac{1}{4}\ln\left|4-z^4\right| + C$$

30. $u = x$ $\quad dv = \cosh x\,dx$

$du = dx$ $\quad v = \sinh x$

$$\int x\cosh x\,dx = x\sinh x - \int \sinh x\,dx = x\sinh x - \cosh x + C$$

31. $u = x$ $\quad dv = \sinh x\,dx$

$du = dx$ $\quad v = \cosh x$

$$\int x\sinh x\,dx = x\cosh x - \int \cosh x\,dx = x\cosh x - \sinh x + C$$

32. $u = \ln x$ $\quad dv = x^{-1/2}dx$

$du = \frac{1}{x}dx$ $\quad v = 2x^{1/2}$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 2\int \frac{1}{x^{1/2}}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

33. $u = x$ $\quad dv = (3x+10)^{49}\,dx$

$du = dx$ $\quad v = \frac{1}{150}(3x+10)^{50}$

$$\int x(3x+10)^{49}\,dx = \frac{x}{150}(3x+10)^{50} - \frac{1}{150}\int (3x+10)^{50}\,dx = \frac{x}{150}(3x+10)^{50} - \frac{1}{22{,}950}(3x+10)^{51} + C$$

34. $u = t$ $\quad dv = \left(t-1\right)^{12}\,dt$

$du = dt$ $\quad v = \frac{1}{13}\left(t-1\right)^{13}$

$$\int_0^1 t(t-1)^{12}\,dt = \left[\frac{t}{13}\left(t-1\right)^{13}\right]_0^1 - \frac{1}{13}\int_0^1 \left(t-1\right)^{13}\,dt$$

$$= \left[\frac{t}{13}\left(t-1\right)^{13} - \frac{1}{182}\left(t-1\right)^{14}\right]_0^1 = \frac{1}{182}$$

35. $u = x$ $\quad dv = 2^x\,dx$

$du = dx$ $\quad v = \frac{1}{\ln 2}2^x$

$$\int x2^x\,dx = \frac{x}{\ln 2}2^x - \frac{1}{\ln 2}\int 2^x\,dx$$

$$= \frac{x}{\ln 2}2^x - \frac{1}{(\ln 2)^2}2^x + C$$

36. $u = z$ $\quad dv = a^z dz$

$du = dz$ $\quad v = \frac{1}{\ln a}a^z$

$$\int za^z dz = \frac{z}{\ln a}a^z - \frac{1}{\ln a}\int a^z dz$$

$$= \frac{z}{\ln a}a^z - \frac{1}{(\ln a)^2}a^z + C$$

37. $u = x^2$ $\quad dv = e^x dx$

$du = 2x\,dx$ $\quad v = e^x$

$$\int x^2e^x dx = x^2e^x - \int 2xe^x dx$$

$u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad v = e^x$

$$\int x^2e^x\,dx = x^2e^x - 2\left(xe^x - \int e^x\,dx\right)$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

38. $u = x^4 \qquad dv = xe^{x^2}dx$

$du = 4x^3dx \qquad v = \frac{1}{2}e^{x^2}$

$\int x^5e^{x^2}dx = \frac{1}{2}x^4e^{x^2} - \int 2x^3e^{x^2}dx$

$u = x^2 \qquad dv = 2xe^{x^2}dx$

$du = 2x\,dx \qquad v = e^{x^2}$

$\int x^5e^{x^2}dx = \frac{1}{2}x^4e^{x^2} - \left(x^2e^{x^2} - \int 2xe^{x^2}dx\right)$

$= \frac{1}{2}x^4e^{x^2} - x^2e^{x^2} + e^{x^2} + C$

39. $u = \ln^2 z \qquad dv = dz$

$du = \frac{2\ln z}{z}dz \qquad v = z$

$\int \ln^2 z\,dz = z\ln^2 z - 2\int \ln z\,dz$

$u = \ln z \qquad dv = dz$

$du = \frac{1}{z}dz \qquad v = z$

$\int \ln^2 z\,dz = z\ln^2 z - 2\left(z\ln z - \int dz\right)$

$= z\ln^2 z - 2z\ln z + 2z + C$

40. $u = \ln^2 x^{20} \qquad dv = dx$

$du = \frac{40\ln x^{20}}{x}dx \qquad v = x$

$\int \ln^2 x^{20}dx = x\ln^2 x^{20} - 40\int \ln x^{20}dx$

$u = \ln x^{20} \qquad dv = dx$

$du = \frac{20}{x}dx \qquad v = x$

$\int \ln^2 x^{20}dx = x\ln^2 x^{20} - 40\left(x\ln x^{20} - 20\int dx\right)$

$= x\ln^2 x^{20} - 40x\ln x^{20} + 800x + C$

41. $u = e^t \qquad dv = \cos t\,dt$

$du = e^t dt \qquad v = \sin t$

$\int e^t \cos t\,dt = e^t \sin t - \int e^t \sin t\,dt$

$u = e^t \qquad dv = \sin t\,dt$

$du = e^t dt \qquad v = -\cos t$

$\int e^t \cos t\,dt = e^t \sin t - \left[-e^t \cos t + \int e^t \cos t\,dt\right]$

$\int e^t \cos t\,dt = e^t \sin t + e^t \cos t - \int e^t \cos t\,dt$

$2\int e^t \cos t\,dt = e^t \sin t + e^t \cos t + C$

$\int e^t \cos t\,dt = \frac{1}{2}e^t(\sin t + \cos t) + C$

42. $u = e^{at} \qquad dv = \sin t\,dt$

$du = ae^{at}dt \qquad v = -\cos t$

$\int e^{at} \sin t\,dt = -e^{at}\cos t + a\int e^{at}\cos t\,dt$

$u = e^{at} \qquad dv = \cos t\,dt$

$du = ae^{at}dt \qquad v = \sin t$

$\int e^{at} \sin t\,dt = -e^{at}\cos t + a\left(e^{at}\sin t - a\int e^{at}\sin t\,dt\right)$

$\int e^{at} \sin t\,dt = -e^{at}\cos t + ae^{at}\sin t - a^2\int e^{at}\sin t\,dt$

$(1+a^2)\int e^{at}\sin t\,dt = -e^{at}\cos t + ae^{at}\sin t + C$

$\int e^{at}\sin t\,dt = \frac{-e^{at}\cos t}{a^2+1} + \frac{ae^{at}\sin t}{a^2+1} + C$

43. $u = x^2 \qquad dv = \cos x\,dx$

$du = 2x\,dx \qquad v = \sin x$

$\int x^2 \cos x\,dx = x^2 \sin x - \int 2x\sin x\,dx$

$u = 2x \qquad dv = \sin x\,dx$

$du = 2dx \qquad v = -\cos x$

$\int x^2 \cos x\,dx = x^2 \sin x - \left(-2x\cos x + \int 2\cos x\,dx\right)$

$= x^2 \sin x + 2x\cos x - 2\sin x + C$

44. $u = r^2 \qquad dv = \sin r\,dr$

$du = 2r\,dr \qquad v = -\cos r$

$\int r^2 \sin r\,dr = -r^2\cos r + 2\int r\cos r\,dr$

$u = r \qquad dv = \cos r\,dr$

$du = dr \qquad v = \sin r$

$\int r^2 \sin r\,dr = -r^2\cos r + 2\left(r\sin r - \int \sin r\,dr\right) = -r^2\cos r + 2r\sin r + 2\cos r + C$

45. $u = \sin(\ln x)$ $\qquad dv = dx$

$du = \cos(\ln x)\cdot\frac{1}{x}dx$ $\qquad v = x$

$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$

$u = \cos\,(\ln x)$ $\qquad dv = dx$

$du = -\sin(\ln x)\cdot\frac{1}{x}dx$ $\qquad v = x$

$\int \sin(\ln x)dx = x\sin(\ln x) - \left[x\cos(\ln x) - \int -\sin(\ln x)dx\right]$

$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$

$2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) + C$

$\int \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C$

46. $u = \cos(\ln x)$ $\qquad dv = dx$

$du = -\sin(\ln x)\frac{1}{x}dx$ $\qquad v = x$

$\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$

$u = \sin(\ln x)$ $\qquad dv = dx$

$du = \cos(\ln x)\frac{1}{x}dx$ $\qquad v = x$

$\int \cos(\ln x)dx = x\cos(\ln x) + \left[x\sin(\ln x) - \int \cos(\ln x)dx\right]$

$2\int \cos(\ln x)dx = x[\cos(\ln x) + \sin(\ln x)] + C$

$\int \cos(\ln x)dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C$

47. $u = (\ln x)^3$ $\qquad dv = dx$

$du = \frac{3\ln^2 x}{x}dx$ $\qquad v = x$

$\int (\ln x)^3 dx = x(\ln x)^3 - 3\int \ln^2 x\,dx$

$= x\ln^3 x - 3(x\ln^2 x - 2x\ln x + 2x + C)$

$= x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + C$

48. $u = (\ln x)^4$ $\qquad dv = dx$

$du = \frac{4\ln^3 x}{x}dx$ $\qquad v = x$

$\int (\ln x)^4 dx = x(\ln x)^4 - 4\int \ln^3 x\,dx = x\ln^4 x - 4(x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + C)$

$= x\ln^4 x - 4x\ln^3 x + 12x\ln^2 x - 24x\ln x + 24x + C$

49. $u = \sin x$ $\qquad dv = \sin(3x)dx$

$du = \cos x\,dx$ $\qquad v = -\frac{1}{3}\cos(3x)$

$\int \sin x\sin(3x)dx = -\frac{1}{3}\sin x\cos(3x) + \frac{1}{3}\int \cos x\cos(3x)dx$

$u = \cos x$ $\qquad dv = \cos(3x)dx$

$du = -\sin x\,dx$ $\qquad v = \frac{1}{3}\sin(3x)$

$$\int \sin x \sin(3x)dx = -\frac{1}{3}\sin x \cos(3x) + \frac{1}{3}\left[\frac{1}{3}\cos x \sin(3x) + \frac{1}{3}\int \sin x \sin(3x)dx\right]$$
$$= -\frac{1}{3}\sin x \cos(3x) + \frac{1}{9}\cos x \sin(3x) + \frac{1}{9}\int \sin x \sin(3x)dx$$
$$\frac{8}{9}\int \sin x \sin(3x)dx = -\frac{1}{3}\sin x \cos(3x) + \frac{1}{9}\cos x \sin(3x) + C$$
$$\int \sin x \sin(3x)dx = -\frac{3}{8}\sin x \cos(3x) + \frac{1}{8}\cos x \sin(3x) + C$$

50. $u = \cos(5x)$ $\quad dv = \sin(7x)dx$

$du = -5\sin(5x)dx \quad v = -\frac{1}{7}\cos(7x)$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7}\int \sin(5x)\cos(7x)dx$$

$u = \sin(5x)$ $\quad dv = \cos(7x)dx$

$du = 5\cos(5x)dx \quad v = \frac{1}{7}\sin(7x)$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7}\left[\frac{1}{7}\sin(5x)\sin(7x) - \frac{5}{7}\int \cos(5x)\sin(7x)dx\right]$$
$$= -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + \frac{25}{49}\int \cos(5x)\sin(7x)dx$$
$$\frac{24}{49}\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + C$$
$$\int \cos(5x)\sin(7x)dx = -\frac{7}{24}\cos(5x)\cos(7x) - \frac{5}{24}\sin(5x)\sin(7x) + C$$

51. $u = e^{\alpha z}$ $\quad dv = \sin \beta z\, dz$

$du = \alpha e^{\alpha z}dz \quad v = -\frac{1}{\beta}\cos \beta z$

$$\int e^{\alpha z}\sin \beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta}\int e^{\alpha z}\cos \beta z\, dz$$

$u = e^{\alpha z}$ $\quad dv = \cos \beta z\, dz$

$du = \alpha e^{\alpha z}dz \quad v = \frac{1}{\beta}\sin \beta z$

$$\int e^{\alpha z}\sin \beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta}\left[\frac{1}{\beta}e^{\alpha z}\sin \beta z - \frac{\alpha}{\beta}\int e^{\alpha z}\sin \beta z\, dz\right]$$
$$= -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\sin \beta z - \frac{\alpha^2}{\beta^2}\int e^{\alpha z}\sin \beta z\, dz$$
$$\frac{\beta^2+\alpha^2}{\beta^2}\int e^{\alpha z}\sin \beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\sin \beta z + C$$
$$\int e^{\alpha z}\sin \beta z\, dz = \frac{-\beta}{\alpha^2+\beta^2}e^{\alpha z}\cos \beta z + \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha z}\sin \beta z + C = \frac{e^{\alpha z}(\alpha \sin \beta z - \beta \cos \beta z)}{\alpha^2+\beta^2} + C$$

52. $u = e^{\alpha z}$ $\quad dv = \cos \beta z \; dz$

$du = \alpha e^{\alpha z} dz \quad v = \frac{1}{\beta} \sin \beta z$

$$\int e^{\alpha z} \cos \beta z \, dz = \frac{1}{\beta} e^{\alpha z} \sin \beta z - \frac{\alpha}{\beta} \int e^{\alpha z} \sin \beta z \, dz$$

$u = e^{\alpha z}$ $\quad dv = \sin \beta z \, dz$

$du = \alpha e^{\alpha z} dz \quad v = -\frac{1}{\beta} \cos \beta z$

$$\int e^{\alpha z} \cos \beta z \, dz = \frac{1}{\beta} e^{\alpha z} \sin \beta z - \frac{\alpha}{\beta} \left[-\frac{1}{\beta} e^{\alpha z} \cos \beta z + \frac{\alpha}{\beta} \int e^{\alpha z} \cos \beta z \, dz \right]$$

$$= \frac{1}{\beta} e^{\alpha z} \sin \beta z + \frac{\alpha}{\beta^2} e^{\alpha z} \cos \beta z - \frac{\alpha^2}{\beta^2} \int e^{\alpha z} \cos \beta z \, dz$$

$$\frac{\alpha^2 + \beta^2}{\beta^2} \int e^{\alpha z} \cos \beta z \, dz = \frac{\alpha}{\beta^2} e^{\alpha z} \cos \beta z + \frac{1}{\beta} e^{\alpha z} \sin \beta z + C$$

$$\int e^{\alpha z} \cos \beta z \, dz = \frac{e^{\alpha z} (\alpha \cos \beta z + \beta \sin \beta z)}{\alpha^2 + \beta^2} + C$$

53. $u = \ln x$ $\quad dv = x^{\alpha} dx$

$du = \frac{1}{x} dx \quad v = \frac{x^{\alpha+1}}{\alpha+1}, \; \alpha \neq -1$

$$\int x^{\alpha} \ln x \, dx = \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{1}{\alpha+1} \int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} + C, \alpha \neq -1$$

54. $u = (\ln x)^2$ $\quad dv = x^{\alpha} dx$

$du = \frac{2 \ln x}{x} dx \quad v = \frac{x^{\alpha+1}}{\alpha+1}, \; \alpha \neq -1$

$$\int x^{\alpha} (\ln x)^2 dx = \frac{x^{\alpha+1}}{\alpha+1} (\ln x)^2 - \frac{2}{\alpha+1} \int x^{\alpha} \ln x \, dx = \frac{x^{\alpha+1}}{\alpha+1} (\ln x)^2 - \frac{2}{\alpha+1} \left[\frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} \right] + C$$

$$= \frac{x^{\alpha+1}}{\alpha+1} (\ln x)^2 - 2 \frac{x^{\alpha+1}}{(\alpha+1)^2} \ln x + 2 \frac{x^{\alpha+1}}{(\alpha+1)^3} + C, \alpha \neq -1$$

Problem 53 was used for $\int x^{\alpha} \ln x \, dx$.

55. $u = x^{\alpha}$ $\quad dv = e^{\beta x} dx$

$du = \alpha x^{\alpha-1} dx \quad v = \frac{1}{\beta} e^{\beta x}$

$$\int x^{\alpha} e^{\beta x} dx = \frac{x^{\alpha} e^{\beta x}}{\beta} - \frac{\alpha}{\beta} \int x^{\alpha-1} e^{\beta x} dx$$

56. $u = x^{\alpha}$ $\quad dv = \sin \beta x \, dx$

$du = \alpha x^{\alpha-1} dx \quad v = -\frac{1}{\beta} \cos \beta x$

$$\int x^{\alpha} \sin \beta x \, dx = -\frac{x^{\alpha} \cos \beta x}{\beta} + \frac{\alpha}{\beta} \int x^{\alpha-1} \cos \beta x \, dx$$

57. $u = x^{\alpha}$ $\quad dv = \cos \beta x\, dx$

$du = \alpha x^{\alpha-1} dx \quad v = \frac{1}{\beta}\sin \beta x$

$$\int x^{\alpha} \cos \beta x\, dx = \frac{x^{\alpha} \sin \beta x}{\beta} - \frac{\alpha}{\beta}\int x^{\alpha-1} \sin \beta x\, dx$$

58. $u = (\ln x)^{\alpha}$ $\quad dv = dx$

$du = \frac{\alpha(\ln x)^{\alpha-1}}{x} dx \quad v = x$

$$\int (\ln x)^{\alpha}\, dx = x(\ln x)^{\alpha} - \alpha\int (\ln x)^{\alpha-1} dx$$

59. $u = (a^2 - x^2)^{\alpha}$ $\quad dv = dx$

$du = -2\alpha x(a^2 - x^2)^{\alpha-1} dx \quad v = x$

$$\int (a^2 - x^2)^{\alpha}\, dx = x(a^2 - x^2)^{\alpha} + 2\alpha\int x^2 (a^2 - x^2)^{\alpha-1} dx$$

60. $u = \cos^{\alpha-1} x$ $\quad dv = \cos x\, dx$

$du = -(\alpha - 1)\cos^{\alpha-2} x \sin x\, dx \quad v = \sin x$

$$\int \cos^{\alpha} x\, dx = \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x \sin^2 x\, dx$$

$$= \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x(1 - \cos^2 x)\, dx \quad = \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x\, dx - (\alpha - 1)\int \cos^{\alpha} x\, dx$$

$$\alpha\int \cos^{\alpha} x\, dx = \cos^{\alpha-1} x \sin x + (\alpha - 1)\int \cos^{\alpha-2} x\, dx$$

$$\int \cos^{\alpha} x\, dx = \frac{\cos^{\alpha-1} x \sin x}{\alpha} + \frac{\alpha - 1}{\alpha}\int \cos^{\alpha-2} x\, dx$$

61. $u = \cos^{\alpha-1} \beta x$ $\quad dv = \cos \beta x\, dx$

$du = -\beta(\alpha - 1)\cos^{\alpha-2} \beta x \sin \beta x\, dx \quad v = \frac{1}{\beta}\sin \beta x$

$$\int \cos^{\alpha} \beta x\, dx = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x \sin^2 \beta x\, dx$$

$$= \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x(1 - \cos^2 \beta x)\, dx$$

$$= \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x\, dx - (\alpha - 1)\int \cos^{\alpha} \beta x\, dx$$

$$\alpha\int \cos^{\alpha} \beta x = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha - 1)\int \cos^{\alpha-2} \beta x\, dx$$

$$\int \cos^{\alpha} \beta x\, dx = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\alpha\beta} + \frac{\alpha - 1}{\alpha}\int \cos^{\alpha-2} \beta x\, dx$$

62. $\int x^4 e^{3x} dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\int x^3 e^{3x} dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\left[\frac{1}{3}x^3 e^{3x} - \int x^2 e^{3x} dx\right]$

$$= \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{3}\left[\frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} dx\right] \quad = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{9}\left[\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x} dx\right]$$

$$= \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{27}x e^{3x} + \frac{8}{81}e^{3x} + C$$

63. $\int x^4 \cos 3x\,dx = \frac{1}{3}x^4 \sin 3x - \frac{4}{3}\int x^3 \sin 3x\,dx \;= \frac{1}{3}x^4 \sin 3x - \frac{4}{3}\left[-\frac{1}{3}x^3 \cos 3x + \int x^2 \cos 3x\,dx\right]$

$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{3}\left[\frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x\,dx\right]$

$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{9}x^2 \sin 3x + \frac{8}{9}\left[-\frac{1}{3}x \cos 3x + \frac{1}{3}\int \cos 3x\,dx\right]$

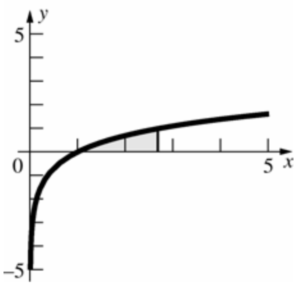
$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{9}x^2 \sin 3x - \frac{8}{27}x \cos 3x + \frac{8}{81}\sin 3x + C$

64. $\int \cos^6 3x\,dx = \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{6}\int \cos^4 3x\,dx \;= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{6}\left[\frac{1}{12}\cos^3 3x \sin 3x + \frac{3}{4}\int \cos^2 3x\,dx\right]$

$= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{72}\cos^3 3x \sin 3x + \frac{5}{8}\left[\frac{1}{6}\cos 3x \sin 3x + \frac{1}{2}\int dx\right]$

$= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{72}\cos^3 3x \sin 3x + \frac{5}{48}\cos 3x \sin 3x + \frac{5x}{16} + C$

65. First make a sketch.



From the sketch, the area is given by

$\int_1^e \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$\int_1^e \ln x\,dx = [x \ln x]_1^e - \int_1^e dx \;= [x \ln x - x]_1^e = (e - e) - (1 \cdot 0 - 1) = 1$

66. $V = \int_1^e \pi(\ln x)^2\,dx$

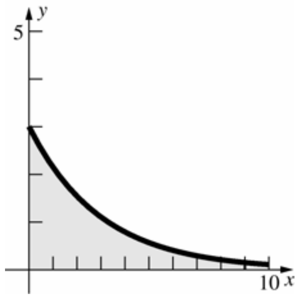
$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2 \ln x}{x}dx \qquad v = x$

$\pi\int_1^e (\ln x)^2\,dx = \pi\left(\left[x(\ln x)^2\right]_1^e - 2\int_1^e \ln x\,dx\right) = \pi\left[x(\ln x)^2 - 2(x \ln x - x)\right]_1^e = \pi[x(\ln x)^2 - 2x \ln x + 2x]_1^e$

$= \pi[(e - 2e + 2e) - (0 - 0 + 2)] = \pi(e - 2) \approx 2.26$

67.



$$\int_0^9 3e^{-x/3}dx = -9\int_0^9 e^{-x/3}\left(-\frac{1}{3}dx\right) = -9[e^{-x/3}]_0^9 = -\frac{9}{e^3}+9 \approx 8.55$$

68. $V = \int_0^9 \pi(3e^{-x/3})^2\,dx = 9\pi\int_0^9 e^{-2x/3}dx$

$$= 9\pi\left(-\frac{3}{2}\right)\int_0^9 e^{-2x/3}\left(-\frac{2}{3}dx\right) = -\frac{27\pi}{2}[e^{-2x/3}]_0^9 = -\frac{27\pi}{2e^6}+\frac{27\pi}{2} \approx 42.31$$

69. $\int_0^{\pi/4}(x\cos x - x\sin x)dx = \int_0^{\pi/4} x\cos x\,dx - \int_0^{\pi/4} x\sin x\,dx$

$$= \left(\left[x\sin x\right]_0^{\pi/4} - \int_0^{\pi/4}\sin x\,dx\right) - \left([-x\cos x]_0^{\pi/4} + \int_0^{\pi/4}\cos x\,dx\right)$$

$$= [x\sin x + \cos x + x\cos x - \sin x]_0^{\pi/4} = \frac{\sqrt{2}\pi}{4} - 1 \approx 0.11$$

Use Problems 60 and 61 for $\int x\sin x\,dx$ and $\int x\cos x\,dx$.

70. $V = 2\pi\int_0^{2\pi} x\sin\left(\frac{x}{2}\right)dx$

$u = x \qquad dv = \sin\frac{x}{2}dx$

$du = dx \quad v = -2\cos\frac{x}{2}$

$$V = 2\pi\left(\left[-2x\cos\frac{x}{2}\right]_0^{2\pi} + \int_0^{2\pi} 2\cos\frac{x}{2}dx\right) = 2\pi\left(4\pi + \left[4\sin\frac{x}{2}\right]_0^{2\pi}\right) = 8\pi^2$$

71. $\int_1^e \ln x^2 dx = 2\int_1^e \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$2\int_1^e \ln x\,dx = 2\left([x\ln x]_1^e - \int_1^e dx\right) = 2\left(e - [x]_1^e\right) = 2$$

$\int_1^e x\ln x^2 dx = 2\int_1^e x\ln x\,dx$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$2\int_1^e x\ln x\,dx = 2\left(\left[\frac{1}{2}x^2\ln x\right]_1^e - \int_1^e \frac{1}{2}x\,dx\right) = 2\left(\frac{1}{2}e^2 - \left[\frac{1}{4}x^2\right]_1^e\right) = \frac{1}{2}(e^2+1)$$

$$\frac{1}{2}\int_1^e (\ln x)^2\,dx$$

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x}dx \qquad v = x$

$$\frac{1}{2}\int_1^e (\ln x)^2\,dx = \frac{1}{2}\left([x(\ln x)^2]_1^e - 2\int_1^e \ln x\,dx\right) = \frac{1}{2}(e-2)$$

$$\overline{x} = \frac{\frac{1}{2}(e^2+1)}{2} = \frac{e^2+1}{4}$$

$$\overline{y} = \frac{\frac{1}{2}(e-2)}{2} = \frac{e-2}{4}$$

72. a. $u = \cot x \qquad dv = \csc^2 x\,dx$

$du = -\csc^2 x\,dx \qquad v = -\cot x$

$$\int \cot x \csc^2 x\,dx = -\cot^2 x - \int \cot x \csc^2 x\,dx$$

$$2\int \cot x \csc^2 x\,dx = -\cot^2 x + C$$

$$\int \cot x \csc^2 x\,dx = -\frac{1}{2}\cot^2 x + C$$

b. $u = \csc x \qquad dv = \cot x \csc x\,dx$
$du = -\cot x \csc x\,dx \qquad v = -\csc x$

$$\int \cot x \csc^2 x\,dx = -\csc^2 x - \int \cot x \csc^2 x\,dx$$

$$2\int \cot x \csc^2 x\,dx = -\csc^2 x + C$$

$$\int \cot x \csc^2 x\,dx = -\frac{1}{2}\csc^2 x + C$$

c. $-\frac{1}{2}\cot^2 x = -\frac{1}{2}(\csc^2 x - 1) = -\frac{1}{2}\csc^2 x + \frac{1}{2}$

73. a. $p(x) = x^3 - 2x$

$g(x) = e^x$

All antiderivatives of $g(x) = e^x$

$$\int (x^3 - 2x)e^x dx = (x^3 - 2x)e^x - (3x^2 - 2)e^x + 6xe^x - 6e^x + C$$

b. $p(x) = x^2 - 3x + 1$

$g(x) = \sin x$

$G_1(x) = -\cos x$

$G_2(x) = -\sin x$

$G_3(x) = \cos x$

$$\int (x^2 - 3x + 1)\sin x\,dx = (x^2 - 3x + 1)(-\cos x) - (2x - 3)(-\sin x) + 2\cos x + C$$

74. a. We note that the nth arch extends from $x = 2\pi(n-1)$ to $x = \pi(2n-1)$, so the area of the nth arch is $A(n) = \int_{2\pi(n-1)}^{\pi(2n-1)} x\sin x\,dx$. Using integration by parts:

$u = x \qquad dv = \sin x\,dx$

$du = dx \qquad v = -\cos x$

$$A(n) = \int_{2\pi(n-1)}^{\pi(2n-1)} x\sin x\,dx = \left[-x\cos x\right]_{2\pi(n-1)}^{\pi(2n-1)} - \int_{2\pi(n-1)}^{\pi(2n-1)} -\cos x\,dx = \left[-x\cos x\right]_{2\pi(n-1)}^{\pi(2n-1)} + \left[\sin x\right]_{2\pi(n-1)}^{\pi(2n-1)}$$

$$= \left[-\pi(2n-1)\cos(\pi(2n-1)) + 2\pi(n-1)\cos(2\pi(n-1))\right] + \left[\sin(\pi(2n-1)) - \sin(2\pi(n-1))\right]$$

$$= -\pi(2n-1)(-1) + 2\pi(n-1)(1) + 0 - 0 = \pi\left[(2n-1)+(2n-2)\right]. \quad \text{So} \quad A(n) = (4n-3)\pi$$

b. $V = 2\pi\int_{2\pi}^{3\pi} x^2 \sin x\,dx$

$u = x^2 \qquad dv = \sin x\,dx$

$du = 2x\,dx \qquad v = -\cos x$

$$V = 2\pi\left(\left[-x^2\cos x\right]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} 2x\cos x\,dx\right) = 2\pi\left(9\pi^2 + 4\pi^2 + \int_{2\pi}^{3\pi} 2x\cos x\,dx\right)$$

$u = 2x \qquad dv = \cos x\,dx$

$du = 2\,dx \qquad v = \sin x$

$$V = 2\pi\left(13\pi^2 + [2x\sin x]_{2\pi}^{3\pi} - \int_{2\pi}^{3\pi} 2\sin x\right)$$

$$= 2\pi\left(13\pi^2 + [2\cos x]_{2\pi}^{3\pi}\right) = 2\pi(13\pi^2 - 4) \approx 781$$

75. $u = f(x) \qquad dv = \sin nx\,dx$

$du = f'(x)dx \qquad v = -\frac{1}{n}\cos nx$

$$a_n = \frac{1}{\pi}\left[\underbrace{\left[-\frac{1}{n}\cos(nx)\,f(x)\right]_{-\pi}^{\pi}}_{\text{Term 1}} + \underbrace{\frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)\,f'(x)dx}_{\text{Term 2}}\right]$$

Term 1 = $\frac{1}{n}\cos(n\pi)(f(-\pi) - f(\pi)) = \pm\frac{1}{n}(f(-\pi) - f(\pi))$

Since $f'(x)$ is continuous on $[-\infty, \infty]$, it is bounded. Thus, $\int_{-\pi}^{\pi}\cos(nx)\,f'(x)dx$ is bounded so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\pi n}\left[\pm(f(-\pi) - f(\pi)) + \int_{-\pi}^{\pi}\cos(nx)\,f'(x)\,dx\right] = 0.$$

76. $\frac{G_n}{n} = \frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{[n^n]^{1/n}} = \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$

$$\ln\left(\frac{G_n}{n}\right) = \frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]$$

$$= \frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + \cdots + \ln\left(1+\frac{n}{n}\right)\right]$$

$$\lim_{n\to\infty}\ln\left(\frac{G_n}{n}\right) = \int_1^2 \ln x\,dx = 2\ln 2 - 1$$

$$\lim_{n\to\infty}\left(\frac{G_n}{n}\right) = e^{2\ln 2 - 1} = 4e^{-1} = \frac{4}{e}$$

77. The proof fails to consider the constants when integrating $\frac{1}{t}$.

The symbol $\int (1/t)\,dt$ is a *family* of functions, all of who whom have derivative $\frac{1}{t}$. We know that any two of these functions will differ by a constant, so it is perfectly correct (notationally) to write $\int (1/t)\,dt = \int (1/t)\,dt + 1$

78. $\frac{d}{dx}[e^{5x}(C_1 \cos 7x + C_2 \sin 7x) + C_3] = 5e^{5x}(C_1 \cos 7x + C_2 \sin 7x) + e^{5x}(-7C_1 \sin 7x + 7C_2 \cos 7x)$

$= e^{5x}[(5C_1 + 7C_2)\cos 7x + (5C_2 - 7C_1)\sin 7x]$

Thus, $5C_1 + 7C_2 = 4$ and $5C_2 - 7C_1 = 6$.

Solving, $C_1 = -\frac{11}{37}; C_2 = \frac{29}{37}$

79. $u = f(x)$ $\quad dv = dx$

$du = f'(x)dx$ $\quad v = x$

$\int_a^b f(x)dx = [xf(x)]_a^b - \int_a^b xf'(x)dx$

Starting with the same integral,

$u = f(x)$ $\quad dv = dx$

$du = f'(x)dx$ $\quad v = x - a$

$\int_a^b f(x)\,dx = [(x-a)\,f(x)]_a^b - \int_a^b (x-a)\,f'(x)dx$

80. $u = f'(x)$ $\quad dv = dx$

$du = f''(x)dx$ $\quad v = x - a$

$f(b) - f(a) = \int_a^b f'(x)dx = [(x-a)\,f'(x)]_a^b - \int_a^b (x-a)\,f''(x)dx = f'(b)(b-a) - \int_a^b (x-a)\,f''(x)dx$

Starting with the same integral,

$u = f'(x)$ $\quad dv = dx$

$du = f''(x)dx$ $\quad v = x - b$

$f(b) - f(a) = \int_a^b f'(x)dx = [(x-b)\,f'(x)]_a^b - \int_a^b (x-b)\,f''(x)dx = f'(a)(b-a) - \int_a^b (x-b)\,f''(x)dx$

81. Use proof by induction.

$n = 1$: $f(a) + f'(a)(t-a) + \int_a^t (t-x)\,f''(x)dx = f(a) + f'(a)(t-a) + [\,f'(x)(t-x)]_a^t + \int_a^t f'(x)dx$

$= f(a) + f'(a)(t-a) - f'(a)(t-a) + [\,f(x)]_a^t = f(t)$

Thus, the statement is true for $n = 1$. Note that integration by parts was used with $u = (t-x)$, $dv = f''(x)dx$.

Suppose the statement is true for n.

$$f(t) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i + \int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$$

Integrate $\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$ by parts.

$u = f^{(n+1)}(x)$ $\quad dv = \frac{(t-x)^n}{n!}dx$

$du = f^{(n+2)}(x)$ $\quad v = -\frac{(t-x)^{n+1}}{(n+1)!}$

$$\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx = \left[-\frac{(t-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\right]_a^t + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

$$= \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

Thus $f(t) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i + \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$

$$= f(a) + \sum_{i=1}^{n+1} \frac{f^{(i)}(a)}{i!}(t-a)^i + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

Thus, the statement is true for $n + 1$.

82. a. $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ where $\alpha \ge 1, \beta \ge 1$

$x = 1 - u$, $dx = -du$

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \int_1^0 (1-u)^{\alpha-1}(u)^{\beta-1}(-du) = \int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du = B(\beta,\alpha)$$

Thus, $B(\alpha, \beta) = B(\beta, \alpha)$.

b. $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$

$u = x^{\alpha-1}$ $\quad$ $dv = (1-x)^{\beta-1}dx$

$du = (\alpha-1)x^{\alpha-2}dx$ $\quad$ $v = -\frac{1}{\beta}(1-x)^{\beta}$

$$B(\alpha,\beta) = \left[-\frac{1}{\beta}x^{\alpha-1}(1-x)^{\beta}\right]_0^1 + \frac{\alpha-1}{\beta}\int_0^1 x^{\alpha-2}(1-x)^{\beta}dx = \frac{\alpha-1}{\beta}\int_0^1 x^{\alpha-2}(1-x)^{\beta}dx \qquad (*)$$

$$= \frac{\alpha-1}{\beta}B(\alpha-1, \beta+1)$$

Similarly,

$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$

$u = (1-x)^{\beta-1}$ $\quad$ $dv = x^{\alpha-1}dx$

$du = -(\beta-1)(1-x)^{\beta-2}dx$ $\quad$ $v = \frac{1}{\alpha}x^{\alpha}$

$$B(\alpha,\beta) = \left[\frac{1}{\alpha}x^{\alpha}(1-x)^{\beta-1}\right]_0^1 + \frac{\beta-1}{\alpha}\int_0^1 x^{\alpha}(1-x)^{\beta-2}dx = \frac{\beta-1}{\alpha}\int_0^1 x^{\alpha}(1-x)^{\beta-2}dx = \frac{\beta-1}{\alpha}B(\alpha+1, \beta-1)$$

c. Assume that $n \le m$. Using part (b) n times,

$$B(n, m) = \frac{n-1}{m}B(n-1, m+1) = \frac{(n-1)(n-2)}{m(m+1)}B(n-2, m+2)$$

$$= \ldots = \frac{(n-1)(n-2)(n-3)\ldots\cdot 2\cdot 1}{m(m+1)(m+2)\ldots(m+n-2)}B(1, m+n-1).$$

$$B(1, m+n-1) = \int_0^1 (1-x)^{m+n-2}dx = -\frac{1}{m+n-1}[(1-x)^{m+n-1}]_0^1 = \frac{1}{m+n-1}$$

Thus, $B(n, m) = \frac{(n-1)(n-2)(n-3)\ldots\cdot 2\cdot 1}{m(m+1)(m+2)\ldots(m+n-2)(m+n-1)} = \frac{(n-1)!(m-1)!}{(m+n-1)!} = \frac{(n-1)!(m-1)!}{(n+m-1)!}$

If $n > m$, then $B(n, m) = B(m, n) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$ by the above reasoning.

83. $u = f(t)$ $\qquad dv = f''(t)dt$

$du = f'(t)dt$ $\qquad v = f'(t)$

$$\int_a^b f''(t)f(t)dt = \left[f(t)f'(t)\right]_a^b - \int_a^b [f'(t)]^2\,dt$$

$$= f(b)f'(b) - f(a)f'(a) - \int_a^b [f'(t)]^2\,dt = -\int_a^b [f'(t)]^2\,dt$$

$[f'(t)]^2 \ge 0$, so $-\int_a^b [f'(t)]^2 \le 0$.

84. $\int_0^x \left(\int_0^t f(z)dz\right)dt$

$u = \int_0^t f(z)dz \quad dv = dt$

$du = f(t)dt$ $\qquad v = t$

$$\int_0^x \left(\int_0^t f(z)dz\right)dt = \left[t\int_0^t f(z)dz\right]_0^x - \int_0^x t\,f(t)dt = \int_0^x x\,f(z)dz - \int_0^x t\,f(t)dt$$

By letting $z = t$, $\int_0^x x\,f(z)dz = \int_0^x x\,f(t)dt$, so

$$\int_0^x \left(\int_0^t f(z)dz\right)dt = \int_0^x x\,f(t)dt - \int_0^x t\,f(t)dt = \int_0^x (x-t)f(t)dt$$

85. Let $I = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_2 dt_1$ be the iterated integral. Note that for $i \ge 2$, the limits of integration of the integral with respect to t_i are 0 to t_{i-1} so that any change of variables in an outer integral affects the limits, and hence the variables in all interior integrals. We use induction on n, noting that the case $n = 2$ is solved in the previous problem.

Assume we know the formula for $n-1$, and we want to show it for n.

$$I = \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\,dt_n \ldots dt_3 dt_2\,dt_1 = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} F(t_{n-1})\,dt_{n-1} \ldots dt_3 dt_2 dt_1$$

where $F(t_{n-1}) = \int_0^{t_{n-1}} f(t_n)\,dn$.

By induction,

$$I = \frac{1}{(n-2)!}\int_0^x F(t_1)(x-t_1)^{n-2}\,dt_1$$

$u = F(t_1) = \int_0^{t_1} f(t_n)\,dt_n$, $\qquad dv = (x-t_1)^{n-2}$

$du = f(t_1)\,dt_1$, $\qquad v = -\frac{1}{n-1}(x-t_1)^{n-1}$

$$I = \frac{1}{(n-2)!}\left\{\left[-\frac{1}{n-1}(x-t_1)^{n-1}\int_0^{t_1} f(t_n)\,dt_n\right]_{t_1=0}^{t_1=x} + \frac{1}{n-1}\int_0^x f(t_1)(x-t_1)^{n-1}\,dt_1\right\}.$$

$$= \frac{1}{(n-1)!}\int_0^x f(t_1)(x-t_1)^{n-1}dt_1$$

(note: that the quantity in square brackets equals 0 when evaluated at the given limits)

86. Proof by induction.
$n = 1$:

$u = P_1(x) \qquad dv = e^x dx$

$du = \frac{dP_1(x)}{dx}dx \qquad v = e^x$

$$\int e^x P_1(x)dx = e^x P_1(x) - \int e^x \frac{dP_1(x)}{dx}dx = e^x P_1(x) - \frac{dP_1(x)}{dx}\int e^x dx = e^x P_1(x) - e^x \frac{dP_1(x)}{dx}$$

Note that $\frac{dP_1(x)}{dx}$ is a constant.

Suppose the formula is true for n. By using integration by parts with $u = P_{n+1}(x)$ and $dv = e^x dx$,

$$\int e^x P_{n+1}(x)dx = e^x P_{n+1}(x) - \int e^x \frac{dP_{n+1}(x)}{dx}dx$$

Note that $\frac{dP_{n+1}(x)}{dx}$ is a polynomial of degree n, so

$$\int e^x P_{n+1}(x)dx = e^x P_{n+1}(x) - \left[e^x \sum_{j=0}^{n}(-1)^j \frac{d^j}{dx^j}\left(\frac{dP_{n+1}(x)}{dx}\right)\right] = e^x P_{n+1}(x) - e^x \sum_{j=0}^{n}(-1)^j \frac{d^{j+1}P_{n+1}(x)}{dx^{j+1}}$$

$$= e^x P_{n+1}(x) + e^x \sum_{j=1}^{n+1}(-1)^j \frac{d^j P_{n+1}(x)}{dx^j} = e^x \sum_{j=0}^{n+1}(-1)^j \frac{d^j P_{n+1}(x)}{dx^j}$$

87. $\int (3x^4 + 2x^2)e^x dx = e^x \sum_{j=0}^{4}(-1)^j \frac{d^j(3x^4 + 2x^2)}{dx^j}$

$= e^x[3x^4 + 2x^2 - 12x^3 - 4x + 36x^2 + 4 - 72x + 72]$

$= e^x(3x^4 - 12x^3 + 38x^2 - 76x + 76)$

7.3 Concepts Review

1. $\int \frac{1+\cos 2x}{2}dx$

2. $\int (1-\sin^2 x)\cos x\, dx$

3. $\int \sin^2 x(1-\sin^2 x)\cos x\, dx$

4. $\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

Problem Set 7.3

1. $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

2. $u = 6x$, $du = 6\, dx$

$\int \sin^4 6x\, dx = \frac{1}{6}\int \sin^4 u\, du$

$= \frac{1}{6}\int \left(\frac{1-\cos 2u}{2}\right)^2 du$

$= \frac{1}{24}\int (1 - 2\cos 2u + \cos^2 2u)du$

$= \frac{1}{24}\int du - \frac{1}{24}\int 2\cos 2u\, du + \frac{1}{48}\int (1+\cos 4u)du$

$= \frac{3}{48}\int du - \frac{1}{24}\int 2\cos 2u\, du + \frac{1}{192}\int 4\cos 4u\, du$

$= \frac{3}{48}(6x) - \frac{1}{24}\sin 12x + \frac{1}{192}\sin 24x + C$

$= \frac{3}{8}x - \frac{1}{24}\sin 12x + \frac{1}{192}\sin 24x + C$

3. $\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)dx$

$= \int \sin x\,dx - \int \sin x \cos^2 x\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

4. $\int \cos^3 x\,dx =$

$= \int \cos x(1-\sin^2 x)dx$

$= \int \cos x\,dx - \int \cos x \sin^2 x\,dx$

$= \sin x - \frac{1}{3}\sin^3 x + C$

5. $\int_0^{\pi/2} \cos^5 \theta\,d\theta = \int_0^{\pi/2} (1-\sin^2\theta)^2 \cos\theta\,d\theta$

$= \int_0^{\pi/2} (1-2\sin^2\theta + \sin^4\theta)\cos\theta\,d\theta$

$= \left[\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\right]_0^{\pi/2}$

$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$

6. $\int_0^{\pi/2} \sin^6\theta\,d\theta = \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right)^3 d\theta$

$= \frac{1}{8}\int_0^{\pi/2} (1 - 3\cos 2\theta + 3\cos^2 2\theta - \cos^3 2\theta)d\theta$

$= \frac{1}{8}\int_0^{\pi/2} d\theta - \frac{3}{16}\int_0^{\pi/2} 2\cos 2\theta\,d\theta + \frac{3}{8}\int_0^{\pi/2} \cos^2 2\theta - \frac{1}{8}\int_0^{\pi/2} \cos^3 2\theta\,d\theta$

$= \frac{1}{8}[\theta]_0^{\pi/2} - \frac{3}{16}[\sin 2\theta]_0^{\pi/2} + \frac{3}{8}\int_0^{\pi/2} \left(\frac{1+\cos 4\theta}{2}\right) d\theta - \frac{1}{8}\int_0^{\pi/2} (1-\sin^2 2\theta)\cos 2\theta\,d\theta$

$= \frac{1}{8}\cdot\frac{\pi}{2} + \frac{3}{16}\int_0^{\pi/2} d\theta + \frac{3}{64}\int_0^{\pi/2} 4\cos 4\theta\,d\theta - \frac{1}{16}\int_0^{\pi/2} 2\cos 2\theta\,d\theta + \frac{1}{16}\int_0^{\pi/2} \sin^2 2\theta \cdot 2\cos 2\theta\,d\theta$

$= \frac{\pi}{16} + \frac{3\pi}{32} + \frac{3}{64}[\sin 4\theta]_0^{\pi/2} - \frac{1}{16}[\sin 2\theta]_0^{\pi/2} + \frac{1}{48}[\sin^3 2\theta]_0^{\pi/2} = \frac{5\pi}{32}$

7. $\int \sin^5 4x \cos^2 4x\,dx = \int (1-\cos^2 4x)^2 \cos^2 4x \sin 4x\,dx = \int (1 - 2\cos^2 4x + \cos^4 4x)\cos^2 4x \sin 4x\,dx$

$= -\frac{1}{4}\int (\cos^2 4x - 2\cos^4 4x + \cos^6 4x)(-4\sin 4x)dx = -\frac{1}{12}\cos^3 4x + \frac{1}{10}\cos^5 4x - \frac{1}{28}\cos^7 4x + C$

8. $\int (\sin^3 2t)\sqrt{\cos 2t}\,dt = \int (1-\cos^2 2t)(\cos 2t)^{1/2} \sin 2t\,dt = -\frac{1}{2}\int [(\cos 2t)^{1/2} - (\cos 2t)^{5/2}](-2\sin 2t)dt$

$= -\frac{1}{3}(\cos 2t)^{3/2} + \frac{1}{7}(\cos 2t)^{7/2} + C$

9. $\int \cos^3 3\theta \sin^{-2} 3\theta\,d\theta = \int (1-\sin^2 3\theta)\sin^{-2} 3\theta \cos 3\theta\,d\theta = \frac{1}{3}\int (\sin^{-2} 3\theta - 1)3\cos 3\theta\,d\theta$

$= -\frac{1}{3}\csc 3\theta - \frac{1}{3}\sin 3\theta + C$

10. $\int \sin^{1/2} 2z \cos^3 2z\,dz = \int (1-\sin^2 2z)\sin^{1/2} 2z \cos 2z\,dz$

$= \frac{1}{2}\int (\sin^{1/2} 2z - \sin^{5/2} 2z)2\cos 2z\,dz = \frac{1}{3}\sin^{3/2} 2z - \frac{1}{7}\sin^{7/2} 2z + C$

11. $\int \sin^4 3t \cos^4 3t\,dt = \int \left(\frac{1-\cos 6t}{2}\right)^2 \left(\frac{1+\cos 6t}{2}\right)^2 dt \ = \frac{1}{16}\int (1-2\cos^2 6t+\cos^4 6t)dt$

$= \frac{1}{16}\int \left[1-(1+\cos 12t)+\frac{1}{4}(1+\cos 12t)^2\right]dt \ = -\frac{1}{16}\int \cos 12t\,dt + \frac{1}{64}\int (1+2\cos 12t+\cos^2 12t)dt$

$= -\frac{1}{192}\int 12\cos 12t\,dt + \frac{1}{64}\int dt + \frac{1}{384}\int 12\cos 12t\,dt + \frac{1}{128}\int (1+\cos 24t)dt$

$= -\frac{1}{192}\sin 12t + \frac{1}{64}t + \frac{1}{384}\sin 12t + \frac{1}{128}t + \frac{1}{3072}\sin 24t + C \ = \frac{3}{128}t - \frac{1}{384}\sin 12t + \frac{1}{3072}\sin 24t + C$

12. $\int \cos^6\theta \sin^2\theta\,d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right)^3 \left(\frac{1-\cos 2\theta}{2}\right) d\theta \ = \frac{1}{16}\int (1+2\cos 2\theta - 2\cos^3 2\theta - \cos^4 2\theta)d\theta$

$= \frac{1}{16}\int d\theta + \frac{1}{16}\int 2\cos 2\theta\,d\theta - \frac{1}{8}\int (1-\sin^2 2\theta)\cos 2\theta\,d\theta - \frac{1}{64}\int (1+\cos 4\theta)^2\,d\theta$

$= \frac{1}{16}\int d\theta + \frac{1}{16}\int 2\cos 2\theta\,d\theta - \frac{1}{16}\int 2\cos 2\theta\,d\theta + \frac{1}{16}\int 2\sin^2 2\theta\cos 2\theta\,d\theta - \frac{1}{64}\int (1+2\cos 4\theta+\cos^2 4\theta)d\theta$

$= \frac{1}{16}\int d\theta + \frac{1}{16}\int \sin^2 2\theta\cdot 2\cos 2\theta\,d\theta - \frac{1}{64}\int d\theta - \frac{1}{128}\int 4\cos 4\theta\,d\theta - \frac{1}{128}\int (1+\cos 8\theta)d\theta$

$= \frac{1}{16}\theta + \frac{1}{48}\sin^3 2\theta - \frac{1}{64}\theta - \frac{1}{128}\sin 4\theta - \frac{1}{128}\theta - \frac{1}{1024}\sin 8\theta + C$

$= \frac{5}{128}\theta + \frac{1}{48}\sin^3 2\theta - \frac{1}{128}\sin 4\theta - \frac{1}{1024}\sin 8\theta \ + C$

13. $\int \sin 4y \cos 5y\,dy = \frac{1}{2}\int \left[\sin 9y + \sin(-y)\right]dy = \frac{1}{2}\int (\sin 9y - \sin y)dy$

$= \frac{1}{2}\left(-\frac{1}{9}\cos 9y + \cos y\right) + C = \frac{1}{2}\cos y - \frac{1}{18}\cos 9y + C$

14. $\int \cos y \cos 4y\,dy = \frac{1}{2}\int [\cos 5y + \cos(-3y)]dy \ = \frac{1}{10}\sin 5y - \frac{1}{6}\sin(-3y) + C \ = \frac{1}{10}\sin 5y + \frac{1}{6}\sin 3y + C$

15. $\int \sin^4\left(\frac{w}{2}\right)\cos^2\left(\frac{w}{2}\right)dw = \int \left(\frac{1-\cos w}{2}\right)^2 \left(\frac{1+\cos w}{2}\right)dw \ = \frac{1}{8}\int (1-\cos w - \cos^2 w + \cos^3 w)dw$

$= \frac{1}{8}\int \left[1-\cos w - \frac{1}{2}(1+\cos 2w) + (1-\sin^2 w)\cos w\right]dw \ = \frac{1}{8}\int \left[\frac{1}{2} - \frac{1}{2}\cos 2w - \sin^2 w\cos w\right]dw$

$= \frac{1}{16}w - \frac{1}{32}\sin 2w - \frac{1}{24}\sin^3 w + C$

16. $\int \sin 3t \sin t\ dt = \int -\frac{1}{2}\left[\cos 4t - \cos 2t\right]dt$

$= -\frac{1}{2}\left(\int \cos 4t dt - \int \cos 2t dt\right)$

$= -\frac{1}{2}\left(\frac{1}{4}\sin 4t - \frac{1}{2}\sin 2t\right) + C$

$= -\frac{1}{8}\sin 4t + \frac{1}{4}\sin 2t + C$

17. $\int x\cos^2 x\sin x\,dx$

$u = x \qquad du = 1\,dx$

$dv = \cos^2 x\sin x\ dx$

$v = -\int\underset{t=\cos x}{(\cos x)^2(-\sin x)\,dx} = -\frac{1}{3}\cos^3 x$

Thus

$\int x\cos^2 x\sin x\,dx =$

$x(-\frac{1}{3}\cos^3 x) - \int(1)(-\frac{1}{3}\cos^3 x)\,dx =$

$\frac{1}{3}\left[-x\cos^3 x + \int\cos^3 x\,dx\right] =$

$\frac{1}{3}\left[-x\cos^3 x + \int\cos x(1-\sin^2 x)\,dx\right] =$

$\frac{1}{3}\left[-x\cos^3 x + \int\underset{t=\sin x}{(\cos x - \cos x\sin^2 x)}\ dx\right] =$

$\frac{1}{3}\left[-x\cos^3 x + \sin x - \frac{1}{3}\sin^3 x\right] + C$

18. $\int x\sin^3 x\cos x\,dx$

$u = x \qquad du = 1\,dx$

$dv = \sin^3 x\cos x\ dx$

$v = \int\underset{t=\sin x}{(\sin x)^3(\cos x)\,dx} = \frac{1}{4}\sin^4 x$

Thus

$\int x\sin^3 x\cos x\,dx =$

$x(\frac{1}{4}\sin^4 x) - \int(1)(\frac{1}{4}\sin^4 x)\,dx =$

$\frac{1}{4}\left[x\sin^4 x - \int(\sin^2 x)^2\,dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{1}{4}\int(1-\cos 2x)^2\,dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{1}{4}\int(1 - 2\cos 2x + \cos^2 2x)\,dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{1}{4}x + \frac{1}{4}\sin 2x - \frac{1}{8}\int(1+\cos 4x)\,dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{3}{8}x + \frac{1}{4}\sin 2x - \frac{1}{32}\sin 4x\right] + C$

19. $\int\tan^4 x\ dx = \int\left(\tan^2 x\right)\left(\tan^2 x\right)\ dx$

$= \int\left(\tan^2 x\right)(\sec^2 x - 1)\ dx$

$= \int\left(\tan^2 x\sec^2 x - \tan^2 x\right)dx$

$= \int\tan^2 x\sec^2 x\ dx - \int(\sec^2 x - 1)dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

20. $\int\cot^4 x\ dx = \int\left(\cot^2 x\right)\left(\cot^2 x\right)\ dx$

$= \int\left(\cot^2 x\right)(\csc^2 x - 1)\ dx$

$= \int\left(\cot^2 x\csc^2 x - \cot^2 x\right)dx$

$= \int\cot^2 x\csc^2 x\ dx - \int(\csc^2 x - 1)dx$

$= -\frac{1}{3}\cot^3 x + \cot x + x + C$

21. $\tan^3 \mathrm{x} = \int\left(\tan x\right)\left(\tan^2 x\right)dx$

$= \int\left(\tan x\right)\left(\sec^2 x - 1\right)dx$

$= \frac{1}{2}\tan^2 x + \ln\left|\cos x\right| + C$

22. $\int\cot^3 2t\,dt = \int\left(\cot 2t\right)\left(\cot^2 2t\right)dt$

$= \int\left(\cot 2t\right)\left(\csc^2 2t - 1\right)dt$

$= \int\cot 2t\csc^2 2t\,dt - \int\cot 2t\,dt$

$= -\frac{1}{4}\cot^2 2t - \frac{1}{2}\ln\left|\sin 2t\right| + C$

23. $\int \tan^5\left(\frac{\theta}{2}\right)d\theta$

$u=\left(\frac{\theta}{2}\right);\ du=\frac{d\theta}{2}$

$$\begin{aligned}\int \tan^5\left(\frac{\theta}{2}\right)d\theta &= 2\int \tan^5 u\ du\\ &= 2\int\left(\tan^3 u\right)\left(\sec^2 u-1\right)du\\ &= 2\int \tan^3 u\sec^2 u\ du-2\int \tan^3 u\ du\\ &= 2\int \tan^3 u\sec^2 u\ du-2\int \tan u\left(\sec^2 u-1\right)du\\ &= 2\int \tan^3 u\sec^2 u\ du-2\int \tan u\sec^2 u\ du+2\int \tan u\ \mathrm{du}\\ &= \frac{1}{2}\tan^4\left(\frac{\theta}{2}\right)-\tan^2\left(\frac{\theta}{2}\right)-2\ln\left|\cos\frac{\theta}{2}\right|+C\end{aligned}$$

24. $\int \cot^5 2t\ dt$

$u=2t; du=2dt$

$$\begin{aligned}\int \cot^5 2t\ dt &= \frac{1}{2}\int \cot^5 u\ du\\ &= \frac{1}{2}\int\left(\cot^3 u\right)\left(\cot^2 u\right)du=\frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2-1\right)du\\ &= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du-\frac{1}{2}\int \cot^3 u\ du\\ &= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du-\frac{1}{2}\int\left(\cot u\right)\left(\csc^2 u-1\right)\ du\\ &= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du-\frac{1}{2}\int\left(\cot u\right)\left(\csc^2 u\right)\ du+\frac{1}{2}\int \cot u\\ &= -\frac{1}{8}\cot^4 u+\frac{1}{4}\cot^2 u+\frac{1}{2}\ln\left|\sin u\right|+C\\ &= -\frac{1}{8}\cot^4 2t+\frac{1}{4}\cot^2 2t+\frac{1}{2}\ln\left|\sin 2t\right|+C\end{aligned}$$

25. $$\begin{aligned}\int \tan^{-3} x\sec^4 x dx &= \int\left(\tan^{-3} x\right)\left(\sec^2 x\right)\left(\sec^2 x\right)dx\\ &= \int\left(\tan^{-3} x\right)\left(1+\tan^2 x\right)\left(\sec^2 x\right)dx\\ &= \int \tan^{-3} x\sec^2 x\ \mathrm{dx}+\int\left(\tan x\right)^{-1}\sec^2 x\ dx\\ &= -\frac{1}{2}\tan^{-2} x+\ln\left|\tan x\right|+C\end{aligned}$$

26. $$\begin{aligned}\int \tan^{-3/2} x\sec^4 x\ dx &= \int\left(\tan^{-3/2} x\right)\left(\sec^2 x\right)\left(\sec^2 x\right)\\ &= \int\left(\tan^{-3/2} x\right)\left(1+\tan^2 x\right)\left(\sec^2 x\right)dx\\ &= \int \tan^{-3/2} x\sec^2 x\ dx+\int \tan^{1/2} x\sec^2 x\ dx\\ &= -2\tan^{-1/2} x+\frac{2}{3}\tan^{3/2} x+C\end{aligned}$$

27. $\int \tan^3 x \sec^2 x\, dx$

Let $u = \tan x$. Then $du = \sec^2 x\, dx$.

$$\int \tan^3 x \sec^2 x\, dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4 x + C$$

28. $\int \tan^3 x \sec^{-1/2} x\, dx = \int \tan^2 x \sec^{-3/2} x(\sec x \tan x)dx$

$$= \int \left(\sec^2 x - 1\right)\left(\sec^{-3/2} x\right)\left(\sec x \tan x\right) dx$$

$$= \int \sec^{1/2} x\left(\sec x \tan x\right) dx - \int \sec^{-3/2} x\left(\sec x \tan x\right) dx$$

$$= \frac{2}{3}\sec^{3/2} x + 2\sec^{-1/2} x + C$$

29. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos[(m+n)x] + \cos[(m-n)x])dx = \frac{1}{2}\left[\frac{1}{m+n}\sin[(m+n)x] + \frac{1}{m-n}\sin[(m-n)x]\right]_{-\pi}^{\pi}$

= 0 for $m \neq n$, since $\sin k\pi = 0$ for all integers k.

30. If we let $u = \frac{\pi x}{L}$ then $du = \frac{\pi}{L}dx$. Making the substitution and changing the limits as necessary, we get

$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\, dx = \frac{L}{\pi}\int_{-\pi}^{\pi} \cos mu \cos nu\, du = 0$ (See Problem 29

31. $\int_0^{\pi} \pi(x + \sin x)^2 dx = \pi\int_0^{\pi} (x^2 + 2x\sin x + \sin^2 x)\, dx = \pi\int_0^{\pi} x^2\, dx + 2\pi\int_0^{\pi} x\sin x\, dx + \frac{\pi}{2}\int_0^{\pi} (1 - \cos 2x)dx$

$$= \pi\left[\frac{1}{3}x^3\right]_0^{\pi} + 2\pi\left[\sin x - x\cos x\right]_0^{\pi} + \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{3}\pi^4 + 2\pi(0 + \pi - 0) + \frac{\pi}{2}(\pi - 0 - 0) = \frac{1}{3}\pi^4 + \frac{5}{2}\pi^2 \approx 57.1437$$

Use Formula 40 with $u = x$ for $\int x \sin x\, dx$

32. $V = 2\pi\int_0^{\sqrt{\pi/2}} x\sin^2(x^2)dx$

$u = x^2,\ du = 2x\, dx$

$$V = \pi\int_0^{\pi/2} \sin^2 u\, du = \pi\int_0^{\pi/2} \frac{1 - \cos 2u}{2} du = \pi\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{\pi/2} = \frac{\pi^2}{4} \approx 2.4674$$

33. **a.** $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N} a_n \sin(nx)\right)\sin(mx)dx = \frac{1}{\pi}\sum_{n=1}^{N} a_n \int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx$

From Example 6,

$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } n = m \end{cases}$ so every term in the sum is 0 except for when $n = m$.

If $m > N$, there is no term where $n = m$, while if $m \le N$, then $n = m$ occurs. When $n = m$

$a_n\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = a_m\pi$ so when $m \le N$,

$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\, dx = \frac{1}{\pi}\cdot a_m \cdot \pi = a_m$.

b. $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N} a_n \sin(nx)\right)\left(\sum_{m=1}^{N} a_m \sin(mx)\right)dx = \frac{1}{\pi}\sum_{n=1}^{N} a_n \sum_{m=1}^{N} a_m \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx$

From Example 6, the integral is 0 except when $m = n$. When $m = n$, we obtain

$\frac{1}{\pi}\sum_{n=1}^{N} a_n(a_n\pi) = \sum_{n=1}^{N} a_n^2$.

34. a. Proof by induction

$n = 1$: $\cos\frac{x}{2} = \cos\frac{x}{2}$

Assume true for $k \le n$.

$$\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\cdot\cos\frac{x}{2^{n+1}} = \left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{1}{2^{n-1}}\cos\frac{x}{2^{n+1}}$$

Note that

$$\left(\cos\frac{k}{2^n}x\right)\left(\cos\frac{1}{2^{n+1}}x\right) = \frac{1}{2}\left[\cos\frac{2k+1}{2^{n+1}}x + \cos\frac{2k-1}{2^{n+1}}x\right], \text{ so}$$

$$\left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\left(\cos\frac{1}{2^{n+1}}x\right)\frac{1}{2^{n-1}} = \left[\cos\frac{1}{2^{n+1}}x + \cos\frac{3}{2^{n+1}}x + \cdots + \cos\frac{2^{n+1}-1}{2^{n+1}}x\right]\frac{1}{2^n}$$

b. $$\lim_{n\to\infty}\left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{1}{2^{n-1}} = \frac{1}{x}\lim_{n\to\infty}\left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{x}{2^{n-1}}$$

$$= \frac{1}{x}\int_0^x \cos t\,dt$$

c. $\frac{1}{x}\int_0^x \cos t\,dt = \frac{1}{x}[\sin t]_0^x = \frac{\sin x}{x}$

35. Using the half-angle identity $\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$, we see that since

$$\cos\frac{\pi}{4} = \cos\frac{\frac{\pi}{2}}{2} = \frac{\sqrt{2}}{2}$$

$$\cos\frac{\pi}{8} = \cos\frac{\frac{\pi}{2}}{4} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\cos\frac{\pi}{16} = \cos\frac{\frac{\pi}{2}}{8} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}, \text{ etc.}$$

Thus, $\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots = \cos\left(\frac{\frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{2}}{4}\right)\cos\left(\frac{\frac{\pi}{2}}{8}\right)\cdots$

$$= \lim_{n\to\infty}\cos\left(\frac{\frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{2}}{4}\right)\cdots\cos\left(\frac{\frac{\pi}{2}}{2^n}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$$

36. Since $(k-\sin x)^2=(\sin x-k)^2$, the volume of S is $\int_0^\pi \pi(k-\sin x)^2=\pi\int_0^\pi (k^2-2k\sin x+\sin^2 x)\,dx$

$$=\pi k^2\int_0^\pi dx-2k\pi\int_0^\pi \sin x\,dx+\frac{\pi}{2}\int_0^\pi (1-\cos 2x)\,dx=\pi k^2\left[x\right]_0^\pi+2k\pi\left[\cos x\right]_0^\pi+\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^\pi$$

$$=\pi^2k^2+2k\pi(-1-1)+\frac{\pi}{2}(\pi-0)=\pi^2k^2-4k\pi+\frac{\pi^2}{2}$$

Let $f(k)=\pi^2k^2-4k\pi+\frac{\pi^2}{2}$, then $f'(k)=2\pi^2k-4\pi$ and $f'(k)=0$ when $k=\frac{2}{\pi}$.

The critical points of $f(k)$ on $0\le k\le 1$ are 0, $\frac{2}{\pi}$, 1.

$$f(0)=\frac{\pi^2}{2}\approx 4.93,\ f\left(\frac{2}{\pi}\right)=4-8+\frac{\pi^2}{2}\approx 0.93,\ f(1)=\pi^2-4\pi+\frac{\pi^2}{2}\approx 2.24$$

a. S has minimum volume when $k=\frac{2}{\pi}$.

b. S has maximum volume when $k=0$.

7.4 Concepts Review

1. $\sqrt{x-3}$

2. 2 sin t

3. 2 tan t

4. 2 sec t

Problem Set 7.4

1. $u=\sqrt{x+1}, u^2=x+1, 2u\,du=dx$

$$\int x\sqrt{x+1}dx=\int (u^2-1)u(2u\,du)$$

$$=\int (2u^4-2u^2)du=\frac{2}{5}u^5-\frac{2}{3}u^3+C$$

$$=\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+C$$

2. $u=\sqrt[3]{x+\pi}, u^3=x+\pi, 3u^2du=dx$

$$\int x\sqrt[3]{x+\pi}dx=\int (u^3-\pi)u(3u^2du)$$

$$=\int (3u^6-3\pi u^3)du=\frac{3}{7}u^7-\frac{3\pi}{4}u^4+C$$

$$=\frac{3}{7}(x+\pi)^{7/3}-\frac{3\pi}{4}(x+\pi)^{4/3}+C$$

3. $u=\sqrt{3t+4}, u^2=3t+4,\ 2u\,du=3\,dt$

$$\int \frac{t\,dt}{\sqrt{3t+4}}=\int \frac{\frac{1}{3}(u^2-4)\frac{2}{3}u\,du}{u}=\frac{2}{9}\int (u^2-4)du$$

$$=\frac{2}{27}u^3-\frac{8}{9}u+C$$

$$=\frac{2}{27}(3t+4)^{3/2}-\frac{8}{9}(3t+4)^{1/2}+C$$

4. $u=\sqrt{x+4}, u^2=x+4,\ 2u\,du=dx$

$$\int \frac{x^2+3x}{\sqrt{x+4}}dx=\int \frac{(u^2-4)^2+3(u^2-4)}{u}2u\,du$$

$$=2\int (u^4-5u^2+4)du=\frac{2}{5}u^5-\frac{10}{3}u^3+8u+C$$

$$=\frac{2}{5}(x+4)^{5/2}-\frac{10}{3}(x+4)^{3/2}+8(x+4)^{1/2}+C$$

5. $u=\sqrt{t}, u^2=t,\ 2u\,du=dt$

$$\int_1^2 \frac{dt}{\sqrt{t}+e}=\int_1^{\sqrt{2}}\frac{2u\,du}{u+e}=2\int_1^{\sqrt{2}}\frac{u+e-e}{u+e}du$$

$$=2\int_1^{\sqrt{2}}du-2\int_1^{\sqrt{2}}\frac{e}{u+e}du$$

$$=2[u]_1^{\sqrt{2}}-2e\left[\ln|u+e|\right]_1^{\sqrt{2}}$$

$$=2(\sqrt{2}-1)-2e[\ln(\sqrt{2}+e)-\ln(1+e)]$$

$$=2\sqrt{2}-2-2e\ln\left(\frac{\sqrt{2}+e}{1+e}\right)$$

6. $u=\sqrt{t}, u^2=t,\ 2u\,du=dt$

$$\int_0^1 \frac{\sqrt{t}}{t+1}dt=\int_0^1 \frac{u}{u^2+1}(2u\,du)$$

$$=2\int_0^1 \frac{u^2}{u^2+1}du=2\int_0^1 \frac{u^2+1-1}{u^2+1}du$$

$$=2\int_0^1 du-2\int_0^1 \frac{1}{u^2+1}du=2[u]_0^1-2[\tan^{-1}u]_0^1$$

$$=2-2\tan^{-1}1=2-\frac{\pi}{2}\approx 0.4292$$

7. $u = (3t+2)^{1/2}, u^2 = 3t+2, 2u\,du = 3dt$

$$\int t(3t+2)^{3/2}\,dt = \int \frac{1}{3}(u^2-2)u^3\left(\frac{2}{3}u\,du\right)$$

$$= \frac{2}{9}\int (u^6 - 2u^4)du = \frac{2}{63}u^7 - \frac{4}{45}u^5 + C$$

$$= \frac{2}{63}(3t+2)^{7/2} - \frac{4}{45}(3t+2)^{5/2} + C$$

8. $u = (1-x)^{1/3}, u^3 = 1-x, 3u^2 du = -dx$

$$\int x(1-x)^{2/3}dx = \int (1-u^3)u^2(-3u^2)du$$

$$= 3\int (u^7 - u^4)du = \frac{3}{8}u^8 - \frac{3}{5}u^5 + C$$

$$= \frac{3}{8}(1-x)^{8/3} - \frac{3}{5}(1-x)^{5/3} + C$$

9. $x = 2\sin t$, $dx = 2\cos t\,dt$

$$\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{2\cos t}{2\sin t}(2\cos t\,dt)$$

$$= 2\int \frac{1-\sin^2 t}{\sin t}dt = 2\int \csc t\,dt - 2\int \sin t\,dt$$

$$= 2\ln|\csc t - \cot t| + 2\cos t + C$$

$$= 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$$

10. $x = 4\sin t, dx = 4\cos t\,dt$

$$\int \frac{x^2 dx}{\sqrt{16-x^2}} = 16\int \frac{\sin^2 t\cos t}{\cos t}dt$$

$$= 16\int \sin^2 t\,dt = 8\int (1-\cos 2t)dt$$

$$= 8t - 4\sin 2t + C = 8t - 8\sin t\cos t + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$$

11. $x = 2\tan t, dx = 2\sec^2 t\,dt$

$$\int \frac{dx}{(x^2+4)^{3/2}} = \int \frac{2\sec^2 t\,dt}{(4\sec^2 t)^{3/2}} = \frac{1}{4}\int \cos t\,dt$$

$$= \frac{1}{4}\sin t + C = \frac{x}{4\sqrt{x^2+4}} + C$$

12. $t = \sec x$, $dt = \sec x\tan x\,dx$

Note that $0 \le x < \frac{\pi}{2}$.

$$\sqrt{t^2-1} = |\tan x| = \tan x$$

$$\int_2^3 \frac{dt}{t^2\sqrt{t^2-1}} = \int_{\pi/3}^{\sec^{-1}(3)} \frac{\sec x\tan x}{\sec^2 x\tan x}dx$$

$$= \int_{\pi/3}^{\sec^{-1}(3)} \cos x\,dx$$

$$= [\sin x]_{\pi/3}^{\sec^{-1}(3)} = \sin[\sec^{-1}(3)] - \sin\frac{\pi}{3}$$

$$= \sin\left[\cos^{-1}\left(\frac{1}{3}\right)\right] - \frac{\sqrt{3}}{2} = \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \approx 0.0768$$

13. $t = \sec x$, $dt = \sec x\tan x\,dx$

Note that $\frac{\pi}{2} < x \le \pi$.

$$\sqrt{t^2-1} = |\tan x| = -\tan x$$

$$\int_{-2}^{-3} \frac{\sqrt{t^2-1}}{t^3}dt = \int_{2\pi/3}^{\sec^{-1}(-3)} \frac{-\tan x}{\sec^3 x}\sec x\tan x\,dx$$

$$= \int_{2\pi/3}^{\sec^{-1}(-3)} -\sin^2 x\,dx = \int_{2\pi/3}^{\sec^{-1}(-3)}\left(\frac{1}{2}\cos 2x - \frac{1}{2}\right)dx$$

$$= \left[\frac{1}{4}\sin 2x - \frac{1}{2}x\right]_{2\pi/3}^{\sec^{-1}(-3)}$$

$$= \left[\frac{1}{2}\sin x\cos x - \frac{1}{2}x\right]_{2\pi/3}^{\sec^{-1}(-3)}$$

$$= -\frac{\sqrt{2}}{9} - \frac{1}{2}\sec^{-1}(-3) + \frac{\sqrt{3}}{8} + \frac{\pi}{3} \approx 0.151252$$

14. $t = \sin x$, $dt = \cos x\,dx$

$$\int \frac{t}{\sqrt{1-t^2}}dt = \int \sin x\,dx = -\cos x + C$$

$$= -\sqrt{1-t^2} + C$$

15. $z = \sin t$, $dz = \cos t\,dt$

$$\int \frac{2z-3}{\sqrt{1-z^2}}dz = \int (2\sin t - 3)dt$$

$$= -2\cos t - 3t + C$$

$$= -2\sqrt{1-z^2} - 3\sin^{-1} z + C$$

16. $x = \pi \tan t,\ dx = \pi \sec^2 t\,dt$

$$\int \frac{\pi x - 1}{\sqrt{x^2+\pi^2}}dx = \int (\pi^2 \tan t - 1)\sec t\,dt$$
$$= \pi^2 \int \tan t \sec t\,dt - \int \sec t\,dt$$
$$= \pi^2 \sec t - \ln|\sec t + \tan t| + C$$
$$= \pi\sqrt{x^2+\pi^2} - \ln\left|\frac{1}{\pi}\sqrt{x^2+\pi^2} + \frac{x}{\pi}\right| + C$$
$$\int_0^{\pi} \frac{\pi x - 1}{\sqrt{x^2+\pi^2}}dx$$
$$= \left[\pi\sqrt{x^2+\pi^2} - \ln\left|\frac{\sqrt{x^2+\pi^2}}{\pi} + \frac{x}{\pi}\right|\right]_0^{\pi}$$
$$= [\sqrt{2}\pi^2 - \ln(\sqrt{2}+1)] - [\pi^2 - \ln 1]$$
$$= (\sqrt{2}-1)\pi^2 - \ln(\sqrt{2}+1) \approx 3.207$$

17. $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$

$u = x + 1,\ du = dx$
$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{du}{\sqrt{u^2+4}}$$
$u = 2\tan t,\ du = 2\sec^2 t\,dt$
$$\int \frac{du}{\sqrt{u^2+4}} = \int \sec t\,dt = \ln|\sec t + \tan t| + C_1$$
$$= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C_1$$
$$= \ln\left|\frac{\sqrt{x^2+2x+5} + x + 1}{2}\right| + C_1$$
$$= \ln\left|\sqrt{x^2+2x+5} + x + 1\right| + C$$

18. $x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x+2)^2 + 1$

$u = x + 2,\ du = dx$
$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{du}{\sqrt{u^2+1}}$$
$u = \tan t,\ du = \sec^2 t\,dt$
$$\int \frac{du}{\sqrt{u^2+1}} = \int \sec t\,dt = \ln|\sec t + \tan t| + C$$
$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \ln\left|\sqrt{u^2+1} + u\right| + C$$
$$= \ln\left|\sqrt{x^2+4x+5} + x + 2\right| + C$$

19. $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$

$u = x + 1,\ du = dx$
$$\int \frac{3x}{\sqrt{x^2+2x+5}}dx = \int \frac{3u-3}{\sqrt{u^2+4}}du$$
$$= 3\int \frac{u}{\sqrt{u^2+4}}du - 3\int \frac{du}{\sqrt{u^2+4}}$$
(Use the result of Problem 17.)
$$= 3\sqrt{u^2+4} - 3\ln\left|\sqrt{u^2+4} + u\right| + C$$
$$= 3\sqrt{x^2+2x+5} - 3\ln\left|\sqrt{x^2+2x+5} + x + 1\right| + C$$

20. $x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x+2)^2 + 1$

$u = x + 2,\ du = dx$
$$\int \frac{2x-1}{\sqrt{x^2+4x+5}}dx = \int \frac{2u-5}{\sqrt{u^2+1}}du$$
$$= \int \frac{2u\,du}{\sqrt{u^2+1}} - 5\int \frac{du}{\sqrt{u^2+1}}$$
(Use the result of Problem 18.)
$$= 2\sqrt{u^2+1} - 5\ln\left|\sqrt{u^2+1} + u\right| + C$$
$$= 2\sqrt{x^2+4x+5} - 5\ln\left|\sqrt{x^2+4x+5} + x + 2\right| + C$$

21. $5 - 4x - x^2 = 9 - (4 + 4x + x^2) = 9 - (x+2)^2$

$u = x + 2,\ du = dx$
$$\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-u^2}\,du$$
$u = 3\sin t,\ du = 3\cos t\,dt$
$$\int \sqrt{9-u^2}\,du = 9\int \cos^2 t\,dt = \frac{9}{2}\int (1+\cos 2t)dt$$
$$= \frac{9}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{9}{2}(t + \sin t \cos t) + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{u}{3}\right) + \frac{1}{2}u\sqrt{9-u^2} + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + \frac{x+2}{2}\sqrt{5-4x-x^2} + C$$

22. $16 + 6x - x^2 = 25 - (9 - 6x + x^2) = 25 - (x-3)^2$

$u = x - 3,\ du = dx$
$$\int \frac{dx}{\sqrt{16+6x-x^2}} = \int \frac{du}{\sqrt{25-u^2}}$$
$u = 5\sin t,\ du = 5\cos t$
$$\int \frac{du}{\sqrt{25-u^2}} = \int dt = t + C = \sin^{-1}\left(\frac{u}{5}\right) + C$$
$$= \sin^{-1}\left(\frac{x-3}{5}\right) + C$$

23. $4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (x-2)^2$

$u = x - 2$, $du = dx$

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{du}{\sqrt{4 - u^2}}$$

$u = 2 \sin t$, $du = 2 \cos t\, dt$

$$\int \frac{du}{\sqrt{4-u^2}} = \int dt = t + C = \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

24. $4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (x-2)^2$

$u = x - 2$, $du = dx$

$$\int \frac{x}{\sqrt{4x - x^2}} dx = \int \frac{u+2}{\sqrt{4-u^2}} du$$

$$= -\int \frac{-u\, du}{\sqrt{4-u^2}} + 2\int \frac{du}{\sqrt{4-u^2}}$$

(Use the result of Problem 23.)

$$= -\sqrt{4-u^2} + 2\sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4x - x^2} + 2\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

25. $x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x+1)^2 + 1$

$u = x + 1$, $du = dx$

$$\int \frac{2x+1}{x^2 + 2x + 2} dx = \int \frac{2u - 1}{u^2 + 1} du$$

$$= \int \frac{2u}{u^2+1} du - \int \frac{du}{u^2+1}$$

$$= \ln\left|u^2 + 1\right| - \tan^{-1} u + C$$

$$= \ln\left(x^2 + 2x + 2\right) - \tan^{-1}(x+1) + C$$

26. $x^2 - 6x + 18 = x^2 - 6x + 9 + 9 = (x-3)^2 + 9$

$u = x - 3$, $du = dx$

$$\int \frac{2x-1}{x^2 - 6x + 18} dx = \int \frac{2u+5}{u^2+9} du$$

$$= \int \frac{2u\, du}{u^2+9} + 5\int \frac{du}{u^2+9}$$

$$= \ln\left(u^2+9\right) + \frac{5}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \ln\left(x^2 - 6x + 18\right) + \frac{5}{3}\tan^{-1}\left(\frac{x-3}{3}\right) + C$$

27. $V = \pi\int_0^1 \left(\frac{1}{x^2 + 2x + 5}\right)^2 dx$

$$= \pi\int_0^1 \left[\frac{1}{(x+1)^2 + 4}\right]^2 dx$$

$x + 1 = 2 \tan t$, $dx = 2\sec^2 t\, dt$

$$V = \pi\int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{4\sec^2 t}\right)^2 2\sec^2 t\, dt$$

$$= \frac{\pi}{8}\int_{\tan^{-1}(1/2)}^{\pi/4} \frac{1}{\sec^2 t} dt = \frac{\pi}{8}\int_{\tan^{-1}(1/2)}^{\pi/4} \cos^2 t\, dt$$

$$= \frac{\pi}{8}\int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt$$

$$= \frac{\pi}{8}\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_{\tan^{-1}(1/2)}^{\pi/4}$$

$$= \frac{\pi}{8}\left[\frac{1}{2}t + \frac{1}{2}\sin t \cos t\right]_{\tan^{-1}(1/2)}^{\pi/4}$$

$$= \frac{\pi}{8}\left[\left(\frac{\pi}{8} + \frac{1}{4}\right) - \left(\frac{1}{2}\tan^{-1}\frac{1}{2} + \frac{1}{5}\right)\right]$$

$$= \frac{\pi}{16}\left(\frac{1}{10} + \frac{\pi}{4} - \tan^{-1}\frac{1}{2}\right) \approx 0.082811$$

28. $V = 2\pi\int_0^1 \frac{1}{x^2 + 2x + 5} x\, dx$

$$= 2\pi\int_0^1 \frac{x}{(x+1)^2 + 4} dx$$

$$= 2\pi\int_0^1 \frac{x+1}{(x+1)^2+4} dx - 2\pi\int_0^1 \frac{1}{(x+1)^2+4} dx$$

$$= 2\pi\left[\frac{1}{2}\ln[(x+1)^2 + 4]\right]_0^1 - 2\pi\left[\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\right]_0^1$$

$$= \pi[\ln 8 - \ln 5] - \pi\left[\tan^{-1} 1 - \tan^{-1}\frac{1}{2}\right]$$

$$= \pi\left(\ln\frac{8}{5} - \frac{\pi}{4} + \tan^{-1}\frac{1}{2}\right) \approx 0.465751$$

29. **a.** $u = x^2 + 9$, $du = 2x\, dx$

$$\int \frac{x\, dx}{x^2+9} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln\left|x^2+9\right| + C = \frac{1}{2}\ln\left(x^2+9\right) + C$$

b. $x = 3 \tan t$, $dx = 3\sec^2 t\, dt$

$$\int \frac{x\, dx}{x^2+9} = \int \tan t\, dt = -\ln|\cos t| + C$$

$$= -\ln\left|\frac{3}{\sqrt{x^2+9}}\right| + C_1 = -\ln\left(\frac{3}{\sqrt{x^2+9}}\right) + C_1$$

$$= \ln\left(\sqrt{x^2+9}\right) - \ln 3 + C_1$$

$$= \ln\left((x^2+9)^{1/2}\right) + C = \frac{1}{2}\ln\left(x^2+9\right) + C$$

30. $u = \sqrt{9+x^2}, u^2 = 9+x^2,\ 2u\,du = 2x\,dx$

$$\int_0^3 \frac{x^3 dx}{\sqrt{9+x^2}} = \int_0^3 \frac{x^2}{\sqrt{9+x^2}} x\,dx = \int_3^{3\sqrt{2}} \frac{u^2-9}{u} u\,du$$

$$= \int_3^{3\sqrt{2}} (u^2-9)\,du = \left[\frac{u^3}{3} - 9u\right]_3^{3\sqrt{2}} = 18 - 9\sqrt{2}$$

≈ 5.272

31. a. $u = \sqrt{4-x^2}, u^2 = 4-x^2,\ 2u\,du = -2x\,dx$

$$\int \frac{\sqrt{4-x^2}}{x} dx = \int \frac{\sqrt{4-x^2}}{x^2} x\,dx = -\int \frac{u^2 du}{4-u^2}$$

$$= \int \frac{-4+4-u^2}{4-u^2} du = -4\int \frac{1}{4-u^2} du + \int du$$

$$= -4 \cdot \frac{1}{4} \ln\left|\frac{u+2}{u-2}\right| + u + C$$

$$= -\ln\left|\frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2}\right| + \sqrt{4-x^2} + C$$

b. $x = 2 \sin t$, $dx = 2 \cos t\,dt$

$$\int \frac{\sqrt{4-x^2}}{x} dx = 2\int \frac{\cos^2 t}{\sin t} dt$$

$$= 2\int \frac{(1-\sin^2 t)}{\sin t} dt$$

$$= 2\int \csc t\,dt - 2\int \sin t\,dt$$

$$= 2\ln|\csc t - \cot t| + 2\cos t + C$$

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$$

$$= 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$$

To reconcile the answers, note that

$$-\ln\left|\frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2}\right| = \ln\left|\frac{\sqrt{4-x^2}-2}{\sqrt{4-x^2}+2}\right|$$

$$= \ln\left|\frac{(\sqrt{4-x^2}-2)^2}{(\sqrt{4-x^2}+2)(\sqrt{4-x^2}-2)}\right|$$

$$= \ln\left|\frac{(2-\sqrt{4-x^2})^2}{4-x^2-4}\right| = \ln\left|\frac{(2-\sqrt{4-x^2})^2}{-x^2}\right|$$

$$= \ln\left|\left(\frac{2-\sqrt{4-x^2}}{x}\right)^2\right| = 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right|$$

32. The equation of the circle with center $(-a, 0)$ is $(x+a)^2 + y^2 = b^2$, so $y = \pm\sqrt{b^2-(x+a)^2}$. By symmetry, the area of the overlap is four times the area of the region bounded by $x = 0$, $y = 0$, and $y = \sqrt{b^2-(x+a)^2}dx$.

$$A = 4\int_0^{b-a} \sqrt{b^2-(x+a)^2}dx$$

$x + a = b \sin t$, $dx = b \cos t\,dt$

$$A = 4\int_{\sin^{-1}(a/b)}^{\pi/2} b^2 \cos^2 t\,dt$$

$$= 2b^2 \int_{\sin^{-1}(a/b)}^{\pi/2} (1+\cos 2t)dt$$

$$= 2b^2 \left[t + \frac{1}{2}\sin 2t\right]_{\sin^{-1}(a/b)}^{\pi/2}$$

$$= 2b^2[t + \sin t \cos t]_{\sin^{-1}(a/b)}^{\pi/2}$$

$$= 2b^2\left[\frac{\pi}{2} - \left(\sin^{-1}\left(\frac{a}{b}\right) + \frac{a}{b}\frac{\sqrt{b^2-a^2}}{b}\right)\right]$$

$$= \pi b^2 - 2b^2 \sin^{-1}\left(\frac{a}{b}\right) - 2a\sqrt{b^2-a^2}$$

33. a. The coordinate of C is $(0, -a)$. The lower arc of the lune lies on the circle given by the equation $x^2 + (y+a)^2 = 2a^2$ or $y = \pm\sqrt{2a^2-x^2} - a.$ The upper arc of the lune lies on the circle given by the equation $x^2 + y^2 = a^2$ or $y = \pm\sqrt{a^2-x^2}.$

$$A = \int_{-a}^{a} \sqrt{a^2-x^2}dx - \int_{-a}^{a}\left(\sqrt{2a^2-x^2} - a\right)dx$$

$$= \int_{-a}^{a} \sqrt{a^2-x^2}dx - \int_{-a}^{a} \sqrt{2a^2-x^2}dx + 2a^2$$

Note that $\int_{-a}^{a}\sqrt{a^2-x^2}dx$ is the area of a semicircle with radius a, so

$$\int_{-a}^{a}\sqrt{a^2-x^2}dx = \frac{\pi a^2}{2}.$$

For $\int_{-a}^{a}\sqrt{2a^2-x^2}dx,$ let

$x = \sqrt{2}a \sin t, dx = \sqrt{2}a \cos t\,dt$

$$\int_{-a}^{a}\sqrt{2a^2-x^2}dx = \int_{-\pi/4}^{\pi/4} 2a^2 \cos^2 t\,dt$$

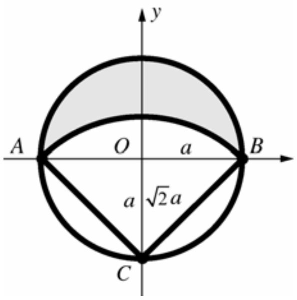
$$= a^2 \int_{-\pi/4}^{\pi/4}(1+\cos 2t)dt = a^2\left[t + \frac{1}{2}\sin 2t\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi a^2}{2} + a^2$$

$$A = \frac{\pi a^2}{2} - \left(\frac{\pi a^2}{2} + a^2\right) + 2a^2 = a^2$$

Thus, the area of the lune is equal to the area of the square.

b. Without using calculus, consider the following labels on the figure.



Area of the lune = Area of the semicircle of radius a at O + Area (ΔABC) – Area of the sector ABC.

$$A=\frac{1}{2}\pi a^2+a^2-\frac{1}{2}\left(\frac{\pi}{2}\right)(\sqrt{2}a)^2$$

$$=\frac{1}{2}\pi a^2+a^2-\frac{1}{2}\pi a^2=a^2$$

Note that since BC has length $\sqrt{2}a,$ the measure of angle OCB is $\frac{\pi}{4},$ so the measure of angle ACB is $\frac{\pi}{2}.$

34. Using reasoning similar to Problem 33 b, the area is

$$\frac{1}{2}\pi a^2+\frac{1}{2}(2a)\sqrt{b^2-a^2}-\frac{1}{2}\left(2\sin^{-1}\frac{a}{b}\right)b^2$$

$$=\frac{1}{2}\pi a^2+a\sqrt{b^2-a^2}-b^2\sin^{-1}\frac{a}{b}.$$

35. $\frac{dy}{dx}=-\frac{\sqrt{a^2-x^2}}{x}; \quad y=\int -\frac{\sqrt{a^2-x^2}}{x}dx$

$x = a \sin t,\ dx = a \cos t\, dt$

$$y=\int -\frac{a\cos t}{a\sin t}a\cos t\,dt=-a\int\frac{\cos^2 t}{\sin t}dt$$

$$=-a\int\frac{1-\sin^2 t}{\sin t}dt=a\int(\sin t-\csc t)dt$$

$$=a\left(-\cos t-\ln\left|\csc t-\cot t\right|\right)+C$$

$$\cos t=\frac{\sqrt{a^2-x^2}}{a},\ \csc t=\frac{a}{x},\ \cot t=\frac{\sqrt{a^2-x^2}}{x}$$

$$y=a\left(-\frac{\sqrt{a^2-x^2}}{a}-\ln\left|\frac{a}{x}-\frac{\sqrt{a^2-x^2}}{x}\right|\right)+C$$

$$=-\sqrt{a^2-x^2}-a\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+C$$

Since $y = 0$ when $x = a$,
$0 = 0 - a \ln 1 + C$, so $C = 0$.

$$y=-\sqrt{a^2-x^2}-a\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|$$

7.5 Concepts Review

1. proper

2. $x-1+\frac{5}{x+1}$

3. $a = 2;\ b = 3;\ c = -1$

4. $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}$

Problem Set 7.5

1. $\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$

$1 = A(x + 1) + Bx$
$A = 1, B = -1$

$$\int\frac{1}{x(x+1)}dx=\int\frac{1}{x}dx-\int\frac{1}{x+1}dx$$

$$=\ln|x|-\ln|x+1|+C$$

2. $\frac{2}{x^2+3x}=\frac{2}{x(x+3)}=\frac{A}{x}+\frac{B}{x+3}$

$2 = A(x + 3) + Bx$

$A=\frac{2}{3},\ B=-\frac{2}{3}$

$$\int\frac{2}{x^2+3x}dx=\frac{2}{3}\int\frac{1}{x}dx-\frac{2}{3}\int\frac{B}{x+3}dx$$

$$=\frac{2}{3}\ln|x|-\frac{2}{3}\ln|x+3|+C$$

3. $\frac{3}{x^2-1}=\frac{3}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$

$3 = A(x - 1) + B(x + 1)$

$A=-\frac{3}{2},\ B=\frac{3}{2}$

$$\int\frac{3}{x^2-1}dx=-\frac{3}{2}\int\frac{1}{x+1}dx+\frac{3}{2}\int\frac{1}{x-1}dx$$

$$=-\frac{3}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|+C$$

4. $\frac{5x}{2x^3+6x^2}=\frac{5x}{2x^2(x+3)}=\frac{5}{2x(x+3)}$

$=\frac{A}{x}+\frac{B}{x+3}$

$\frac{5}{2}=A(x+3)+Bx$

$A=\frac{5}{6}, B=-\frac{5}{6}$

$\int\frac{5x}{2x^3+6x^2}=\frac{5}{6}\int\frac{1}{x}dx-\frac{5}{6}\int\frac{1}{x+3}dx$

$=\frac{5}{6}\ln|x|-\frac{5}{6}\ln|x+3|+C$

5. $\frac{x-11}{x^2+3x-4}=\frac{x-11}{(x+4)(x-1)}=\frac{A}{x+4}+\frac{B}{x-1}$

$x-11=A(x-1)+B(x+4)$

$A=3, B=-2$

$\int\frac{x-11}{x^2+3x-4}dx=3\int\frac{1}{x+4}dx-2\int\frac{1}{x-1}dx$

$=3\ln|x+4|-2\ln|x-1|+C$

6. $\frac{x-7}{x^2-x-12}=\frac{x-7}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$

$x-7=A(x+3)+B(x-4)$

$A=-\frac{3}{7}, B=\frac{10}{7}$

$\int\frac{x-7}{x^2-x-12}dx=-\frac{3}{7}\int\frac{1}{x-4}dx+\frac{10}{7}\int\frac{1}{x+3}dx$

$=-\frac{3}{7}\ln|x-4|+\frac{10}{7}\ln|x+3|+C$

7. $\frac{3x-13}{x^2+3x-10}=\frac{3x-13}{(x+5)(x-2)}=\frac{A}{x+5}+\frac{B}{x-2}$

$3x-13=A(x-2)+B(x+5)$

$A=4, B=-1$

$\int\frac{3x-13}{x^2+3x-10}dx=4\int\frac{1}{x+5}dx-\int\frac{1}{x-2}dx$

$=4\ln|x+5|-\ln|x-2|+C$

8. $\frac{x+\pi}{x^2-3\pi x+2\pi^2}=\frac{x+\pi}{(x-2\pi)(x-\pi)}=\frac{A}{x-2\pi}+\frac{B}{x-\pi}$

$x+\pi=A(x-\pi)+B(x-2\pi)$

$A=3, B=-2$

$\int\frac{x+\pi}{x^2-3\pi x+2\pi^2}dx=\int\frac{3}{x-2\pi}dx-\int\frac{2}{x-\pi}dx$

$=3\ln|x-2\pi|-2\ln|x-\pi|+C$

9. $\frac{2x+21}{2x^2+9x-5}=\frac{2x+21}{(2x-1)(x+5)}=\frac{A}{2x-1}+\frac{B}{x+5}$

$2x+21=A(x+5)+B(2x-1)$

$A=4, B=-1$

$\int\frac{2x+21}{2x^2+9x-5}dx=\int\frac{4}{2x-1}dx-\int\frac{1}{x+5}dx$

$=2\ln|2x-1|-\ln|x+5|+C$

10. $\frac{2x^2-x-20}{x^2+x-6}=\frac{2(x^2+x-6)-3x-8}{x^2+x-6}$

$=2-\frac{3x+8}{x^2+x-6}$

$\frac{3x+8}{x^2+x-6}=\frac{3x+8}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$

$3x+8=A(x-2)+B(x+3)$

$A=\frac{1}{5}, B=\frac{14}{5}$

$\int\frac{2x^2-x-20}{x^2+x-6}dx$

$=\int 2\,dx-\frac{1}{5}\int\frac{1}{x+3}dx-\frac{14}{5}\int\frac{1}{x-2}dx$

$=2x-\frac{1}{5}\ln|x+3|-\frac{14}{5}\ln|x-2|+C$

11. $\frac{17x-3}{3x^2+x-2}=\frac{17x-3}{(3x-2)(x+1)}=\frac{A}{3x-2}+\frac{B}{x+1}$

$17x-3=A(x+1)+B(3x-2)$

$A=5, B=4$

$\int\frac{17x-3}{3x^2+x-2}dx=\int\frac{5}{3x-2}dx+\int\frac{4}{x+1}dx=\frac{5}{3}\ln|3x-2|+4\ln|x+1|+C$

12. $\dfrac{5-x}{x^2-x(\pi+4)+4\pi}=\dfrac{5-x}{(x-\pi)(x-4)}=\dfrac{A}{x-\pi}+\dfrac{B}{x-4}$

$5-x=A(x-4)+B(x-\pi)$

$A=\dfrac{5-\pi}{\pi-4},\ B=\dfrac{1}{4-\pi}$

$\displaystyle\int\frac{5-x}{x^2-x(\pi+4)+4\pi}dx=\frac{5-\pi}{\pi-4}\int\frac{1}{x-\pi}dx+\frac{1}{4-\pi}\int\frac{1}{x-4}dx=\frac{5-\pi}{\pi-4}\ln|x-\pi|+\frac{1}{4-\pi}\ln|x-4|+C$

13. $\dfrac{2x^2+x-4}{x^3-x^2-2x}=\dfrac{2x^2+x-4}{x(x+1)(x-2)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-2}$

$2x^2+x-4=A(x+1)(x-2)+Bx(x-2)+Cx(x+1)$

$A=2,\ B=-1,\ C=1$

$\displaystyle\int\frac{2x^2+x-4}{x^3-x^2-2x}dx=\int\frac{2}{x}dx-\int\frac{1}{x+1}dx+\int\frac{1}{x-2}dx=2\ln|x|-\ln|x+1|+\ln|x-2|+C$

14. $\dfrac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}=\dfrac{A}{2x-1}+\dfrac{B}{3x+2}+\dfrac{C}{x-3}$

$7x^2+2x-3=A(3x+2)(x-3)+B(2x-1)(x-3)+C(2x-1)(3x+2)$

$A=\dfrac{1}{35},\ B=-\dfrac{1}{7},\ C=\dfrac{6}{5}$

$\displaystyle\int\frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}dx=\frac{1}{35}\int\frac{1}{2x-1}dx-\frac{1}{7}\int\frac{1}{3x+2}dx+\frac{6}{5}\int\frac{1}{x-3}dx$

$=\dfrac{1}{70}\ln|2x-1|-\dfrac{1}{21}\ln|3x+2|+\dfrac{6}{5}\ln|x-3|+C$

15. $\dfrac{6x^2+22x-23}{(2x-1)(x^2+x-6)}=\dfrac{6x^2+22x-23}{(2x-1)(x+3)(x-2)}=\dfrac{A}{2x-1}+\dfrac{B}{x+3}+\dfrac{C}{x-2}$

$6x^2+22x-23=A(x+3)(x-2)+B(2x-1)(x-2)+C(2x-1)(x+3)$

$A=2,\ B=-1,\ C=3$

$\displaystyle\int\frac{6x^2+22x-23}{(2x-1)(x^2+x-6)}dx=\int\frac{2}{2x-1}dx-\int\frac{1}{x+3}dx+\int\frac{3}{x-2}dx=\ln|2x-1|-\ln|x+3|+3\ln|x-2|+C$

16. $\dfrac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32}=\dfrac{1}{4}\left(\dfrac{x^3-6x^2+11x-6}{x^3-7x^2+14x-8}\right)=\dfrac{1}{4}\left(1+\dfrac{x^2-3x+2}{x^3-7x^2+14x-8}\right)$

$=\dfrac{1}{4}\left(1+\dfrac{(x-1)(x-2)}{(x-1)(x-2)(x-4)}\right)=\dfrac{1}{4}\left(1+\dfrac{1}{x-4}\right)$

$\displaystyle\int\frac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32}dx=\int\frac{1}{4}dx+\frac{1}{4}\int\frac{1}{x-4}dx=\frac{1}{4}x+\frac{1}{4}\ln|x-4|+C$

17. $\frac{x^3}{x^2+x-2} = x-1+\frac{3x-2}{x^2+x-2}$

$\frac{3x-2}{x^2+x-2} = \frac{3x-2}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1}$

$3x-2 = A(x-1) + B(x+2)$

$A = \frac{8}{3}, B = \frac{1}{3}$

$$\int \frac{x^3}{x^2+x-2}dx = \int (x-1)\,dx + \frac{8}{3}\int \frac{1}{x+2}dx + \frac{1}{3}\int \frac{1}{x-1}dx = \frac{1}{2}x^2 - x + \frac{8}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

18. $\frac{x^3+x^2}{x^2+5x+6} = x-4+\frac{14x+24}{(x+3)(x+2)}$

$\frac{14x+24}{(x+3)(x+2)} = \frac{A}{x+3}+\frac{B}{x+2}$

$14x + 24 = A(x+2) + B(x+3)$

$A = 18, B = -4$

$$\int \frac{x^3+x^2}{x^2+5x+6}dx = \int (x-4)\,dx + \int \frac{18}{x+3}dx - \int \frac{4}{x+2}dx = \frac{1}{2}x^2 - 4x + 18\ln|x+3| - 4\ln|x+2| + C$$

19. $\frac{x^4+8x^2+8}{x^3-4x} = x+\frac{12x^2+8}{x(x+2)(x-2)}$

$\frac{12x^2+8}{x(x+2)(x-2)} = \frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}$

$12x^2+8 = A(x+2)(x-2)+Bx(x-2)+Cx(x+2)$

$A = -2, B = 7, C = 7$

$$\int \frac{x^4+8x^2+8}{x^3-4x}dx = \int x\,dx - 2\int \frac{1}{x}dx + 7\int \frac{1}{x+2}dx + 7\int \frac{1}{x-2}dx = \frac{1}{2}x^2 - 2\ln|x| + 7\ln|x+2| + 7\ln|x-2| + C$$

20. $\frac{x^6+4x^3+4}{x^3-4x^2} = x^3+4x^2+16x+68+\frac{272x^2+4}{x^3-4x^2}$

$\frac{272x^2+4}{x^2(x-4)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-4}$

$272x^2+4 = Ax(x-4)+B(x-4)+Cx^2$

$A = -\frac{1}{4}, B = -1, C = \frac{1089}{4}$

$$\int \frac{x^6+4x^3+4}{x^3-4x^2}dx = \int (x^3+4x^2+16x+68)\,dx - \frac{1}{4}\int \frac{1}{x}dx - \int \frac{1}{x^2}dx + \frac{1089}{4}\int \frac{1}{x-4}dx$$

$$= \frac{1}{4}x^4 + \frac{4}{3}x^3 + 8x^2 + 68x - \frac{1}{4}\ln|x| + \frac{1}{x} + \frac{1089}{4}\ln|x-4| + C$$

21. $\frac{x+1}{(x-3)^2} = \frac{A}{x-3}+\frac{B}{(x-3)^2}$

$x + 1 = A(x-3) + B$

$A = 1, B = 4$

$$\int \frac{x+1}{(x-3)^2}dx = \int \frac{1}{x-3}dx + \int \frac{4}{(x-3)^2}dx = \ln|x-3| - \frac{4}{x-3} + C$$

22. $\frac{5x+7}{x^2+4x+4}=\frac{5x+7}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$

$5x + 7 = A(x + 2) + B$

$A = 5$, $B = –3$

$\int\frac{5x+7}{x^2+4x+4}dx=\int\frac{5}{x+2}dx-\int\frac{3}{(x+2)^2}dx=5\ln|x+2|+\frac{3}{x+2}+C$

23. $\frac{3x+2}{x^3+3x^2+3x+1}=\frac{3x+2}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$

$3x+2=A(x+1)^2+B(x+1)+C$

$A=0, B=3, C=-1$

$\int\frac{3x+2}{x^3+3x^2+3x+1}dx=\int\frac{3}{(x+1)^2}dx-\int\frac{1}{(x+1)^3}dx=-\frac{3}{x+1}+\frac{1}{2(x+1)^2}+C$

24. $\frac{x^6}{(x-2)^2(1-x)^5}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{1-x}+\frac{D}{(1-x)^2}+\frac{E}{(1-x)^3}+\frac{F}{(1-x)^4}+\frac{G}{(1-x)^5}$

$A = 128$, $B = –64$, $C = 129$, $D = –72$, $E = 30$, $F = –8$, $G = 1$

$\int\frac{x^6}{(x-2)^2(1-x)^5}dx=\int\left[\frac{128}{x-2}-\frac{64}{(x-2)^2}+\frac{129}{1-x}-\frac{72}{(1-x)^2}+\frac{30}{(1-x)^3}-\frac{8}{(1-x)^4}+\frac{1}{(1-x)^5}\right]dx$

$=128\ln|x-2|+\frac{64}{x-2}-129\ln|1-x|+\frac{72}{1-x}-\frac{15}{(1-x)^2}+\frac{8}{3(1-x)^3}-\frac{1}{4(1-x)^4}+C$

25. $\frac{3x^2-21x+32}{x^3-8x^2+16x}=\frac{3x^2-21x+32}{x(x-4)^2}=\frac{A}{x}+\frac{B}{x-4}+\frac{C}{(x-4)^2}$

$3x^2-21x+32=A(x-4)^2+Bx(x-4)+Cx$

$A = 2$, $B = 1$, $C = –1$

$\int\frac{3x^2-21x+32}{x^3-8x^2+16}dx=\int\frac{2}{x}dx+\int\frac{1}{x-4}dx-\int\frac{1}{(x-4)^2}dx=2\ln|x|+\ln|x-4|+\frac{1}{x-4}+C$

26. $\frac{x^2+19x+10}{2x^4+5x^3}=\frac{x^2+19x+10}{x^3(2x+5)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{2x+5}$

$A = –1$, $B = 3$, $C = 2$, $D = 2$

$\int\frac{x^2+19x+10}{2x^4+5x^3}dx=\int\left(-\frac{1}{x}+\frac{3}{x^2}+\frac{2}{x^3}+\frac{2}{2x+5}\right)dx=-\ln|x|-\frac{3}{x}-\frac{1}{x^2}+\ln|2x+5|+C$

27. $\frac{2x^2+x-8}{x^3+4x}=\frac{2x^2+x-8}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$

$A = –2$, $B = 4$, $C = 1$

$\int\frac{2x^2+x-8}{x^3+4x}dx=-2\int\frac{1}{x}dx+\int\frac{4x+1}{x^2+4}dx=-2\int\frac{1}{x}dx+2\int\frac{2x}{x^2+4}dx+\int\frac{1}{x^2+4}dx$

$=-2\ln|x|+2\ln|x^2+4|+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

28. $\dfrac{3x+2}{x(x+2)^2+16x}=\dfrac{3x+2}{x(x^2+4x+20)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4x+20}$

$A=\dfrac{1}{10}, B=-\dfrac{1}{10}, C=\dfrac{13}{5}$

$\int\dfrac{3x+2}{x(x+2)^2+16x}dx=\dfrac{1}{10}\int\dfrac{1}{x}dx+\int\dfrac{-\frac{1}{10}x+\frac{13}{5}}{x^2+4x+20}dx=\dfrac{1}{10}\int\dfrac{1}{x}dx+\dfrac{14}{5}\int\dfrac{1}{(x+2)^2+16}dx-\dfrac{1}{20}\int\dfrac{2x+4}{x^2+4x+20}dx$

$=\dfrac{1}{10}\ln|x|+\dfrac{7}{10}\tan^{-1}\left(\dfrac{x+2}{4}\right)-\dfrac{1}{20}\ln\left|x^2+4x+20\right|+C$

29. $\dfrac{2x^2-3x-36}{(2x-1)(x^2+9)}=\dfrac{A}{2x-1}+\dfrac{Bx+C}{x^2+9}$

$A = –4, B = 3, C = 0$

$\int\dfrac{2x^2-3x-36}{(2x-1)(x^2+9)}dx=-4\int\dfrac{1}{2x-1}dx+\int\dfrac{3x}{x^2+9}dx=-2\ln|2x-1|+\dfrac{3}{2}\ln\left|x^2+9\right|+C$

30. $\dfrac{1}{x^4-16}=\dfrac{1}{(x-2)(x+2)(x^2+4)}$

$=\dfrac{A}{x-2}+\dfrac{B}{x+2}+\dfrac{Cx+D}{x^2+4}$

$A=\dfrac{1}{32}, B=-\dfrac{1}{32}, C=0, D=-\dfrac{1}{8}$

$\int\dfrac{1}{x^4-16}dx=\dfrac{1}{32}\int\dfrac{1}{x-2}dx-\dfrac{1}{32}\int\dfrac{1}{x+2}dx-\dfrac{1}{8}\int\dfrac{1}{x^2+4}dx=\dfrac{1}{32}\ln|x-2|-\dfrac{1}{32}\ln|x+2|-\dfrac{1}{16}\tan^{-1}\left(\dfrac{x}{2}\right)+C$

31. $\dfrac{1}{(x-1)^2(x+4)^2}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+4}+\dfrac{D}{(x+4)^2}$

$A=-\dfrac{2}{125}, B=\dfrac{1}{25}, C=\dfrac{2}{125}, D=\dfrac{1}{25}$

$\int\dfrac{1}{(x-1)^2(x+4)^2}dx=-\dfrac{2}{125}\int\dfrac{1}{x-1}dx+\dfrac{1}{25}\int\dfrac{1}{(x-1)^2}dx+\dfrac{2}{125}\int\dfrac{1}{x+4}dx+\dfrac{1}{25}\int\dfrac{1}{(x+4)^2}dx$

$=-\dfrac{2}{125}\ln|x-1|-\dfrac{1}{25(x-1)}+\dfrac{2}{125}\ln|x+4|-\dfrac{1}{25(x+4)}+C$

32. $\dfrac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}=1+\dfrac{-7x^2+7x-16}{(x+3)(x^2-4x+5)}$

$\dfrac{-7x^2+7x-16}{(x+3)(x^2-4x+5)}=\dfrac{A}{x+3}+\dfrac{Bx+C}{x^2-4x+5}$

$A=-\dfrac{50}{13}, B=-\dfrac{41}{13}, C=\dfrac{14}{13}$

$\int\dfrac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}dx=\int\left[1-\dfrac{50}{13}\left(\dfrac{1}{x+3}\right)+\dfrac{-\frac{41}{13}x+\frac{14}{13}}{x^2-4x+5}\right]dx$

$=\int dx-\dfrac{50}{13}\int\dfrac{1}{x+3}dx-\dfrac{68}{13}\int\dfrac{1}{(x-2)^2+1}dx-\dfrac{41}{26}\int\dfrac{2x-4}{x^2-4x+5}dx$

$=x-\dfrac{50}{13}\ln|x+3|-\dfrac{68}{13}\tan^{-1}(x-2)-\dfrac{41}{26}\ln\left|x^2-4x+5\right|+C$

33. $x = \sin t$, $dx = \cos t\,dt$

$$\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)}dt = \int \frac{x^3 - 8x^2 - 1}{(x+3)(x^2 - 4x + 5)}dx$$

$$= x - \frac{50}{13}\ln|x+3| - \frac{68}{13}\tan^{-1}(x-2) - \frac{41}{26}\ln\left|x^2 - 4x + 5\right| + C$$

which is the result of Problem 32.

$$\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)}dt = \sin t - \frac{50}{13}\ln|\sin t + 3| - \frac{68}{13}\tan^{-1}(\sin t - 2) - \frac{41}{26}\ln\left|\sin^2 t - 4\sin t + 5\right| + C$$

34. $x = \sin t$, $dx = \cos t\,dt$

$$\int \frac{\cos t}{\sin^4 t - 16}dt = \int \frac{1}{x^4 - 16}dx = \frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C$$

which is the result of Problem 30.

$$\int \frac{\cos t}{\sin^4 t - 16}dt = \frac{1}{32}\ln|\sin t - 2| - \frac{1}{32}\ln|\sin t + 2| - \frac{1}{16}\tan^{-1}\left(\frac{\sin t}{2}\right) + C$$

35. $\dfrac{x^3 - 4x}{(x^2+1)^2} = \dfrac{Ax + B}{x^2 + 1} + \dfrac{Cx + D}{(x^2+1)^2}$

$A = 1, B = 0, C = –5, D = 0$

$$\int \frac{x^3 - 4x}{(x^2+1)^2}dx = \int \frac{x}{x^2+1}dx - 5\int \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\ln\left|x^2+1\right| + \frac{5}{2(x^2+1)} + C$$

36. $x = \cos t$, $dx = –\sin t\,dt$

$$\int \frac{(\sin t)(4\cos^2 t - 1)}{(\cos t)(1 + 2\cos^2 t + \cos^4 t)}dt = -\int \frac{4x^2 - 1}{x(1 + 2x^2 + x^4)}dx$$

$$\frac{4x^2 - 1}{x(1 + 2x^2 + x^4)} = \frac{4x^2 - 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2+1)^2}$$

$A = –1$, $B = 1$, $C = 0$, $D = 5$, $E = 0$

$$-\int\left[-\frac{1}{x} + \frac{x}{x^2+1} + \frac{5x}{(x^2+1)^2}\right]dx = \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + \frac{5}{2(x^2+1)} + C = \ln|\cos t| - \frac{1}{2}\ln\left|\cos^2 t + 1\right| + \frac{5}{2(\cos^2 t + 1)} + C$$

37. $\dfrac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} = \dfrac{x(2x^2 + 5x + 16)}{x(x^4 + 8x^2 + 16)} = \dfrac{2x^2 + 5x + 16}{(x^2+4)^2} = \dfrac{Ax + B}{x^2 + 4} + \dfrac{Cx + D}{(x^2+4)^2}$

$A = 0$, $B = 2$, $C = 5$, $D = 8$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x}dx = \int \frac{2}{x^2+4}dx + \int \frac{5x + 8}{(x^2+4)^2}dx = \int \frac{2}{x^2+4}dx + \int \frac{5x}{(x^2+4)^2}dx + \int \frac{8}{(x^2+4)^2}dx$$

To integrate $\displaystyle\int \frac{8}{(x^2+4)^2}dx$, let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$.

$$\int \frac{8}{(x^2+4)^2}dx = \int \frac{16\sec^2\theta}{16\sec^4\theta}d\theta = \int \cos^2\theta\,d\theta = \int\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C = \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{x}{x^2+4} + C$$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x}dx = \tan^{-1}\frac{x}{2} - \frac{5}{2(x^2+4)} + \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{x}{x^2+4} + C = \frac{3}{2}\tan^{-1}\frac{x}{2} + \frac{2x - 5}{2(x^2+4)} + C$$

38. $\dfrac{x-17}{x^2+x-12}=\dfrac{x-17}{(x+4)(x-3)}=\dfrac{A}{x+4}+\dfrac{B}{x-3}$

$A = 3$, $B = -2$

$\int_4^6 \dfrac{x-17}{x^2+x-12}dx=\int_4^6\left(\dfrac{3}{x+4}-\dfrac{2}{x-3}\right)dx = \left[3\ln|x+4|-2\ln|x-3|\right]_4^6=(3\ln 10-2\ln 3)-(3\ln 8-2\ln 1)$

$=3\ln 10-2\ln 3-3\ln 8\ \approx -1.53$

39. $u = \sin\theta$, $du = \cos\theta\, d\theta$

$\int_0^{\pi/4}\dfrac{\cos\theta}{(1-\sin^2\theta)(\sin^2\theta+1)^2}d\theta=\int_0^{1/\sqrt{2}}\dfrac{1}{(1-u^2)(u^2+1)^2}du\ =\int_0^{1/\sqrt{2}}\dfrac{1}{(1-u)(1+u)(u^2+1)^2}du$

$\dfrac{1}{(1-u^2)(u^2+1)^2}=\dfrac{A}{1-u}+\dfrac{B}{1+u}+\dfrac{Cu+D}{u^2+1}+\dfrac{Eu+F}{(u^2+1)^2}$

$A=\dfrac{1}{8}, B=\dfrac{1}{8}, C=0, D=\dfrac{1}{4}, E=0, F=\dfrac{1}{2}$

$\int_0^{1/\sqrt{2}}\dfrac{1}{(1-u^2)(u^2+1)^2}du=\dfrac{1}{8}\int_0^{1/\sqrt{2}}\dfrac{1}{1-u}du+\dfrac{1}{8}\int_0^{1/\sqrt{2}}\dfrac{1}{1+u}du+\dfrac{1}{4}\int_0^{1/\sqrt{2}}\dfrac{1}{u^2+1}du+\dfrac{1}{2}\int_0^{1/\sqrt{2}}\dfrac{1}{(u^2+1)^2}du$

$=\left[-\dfrac{1}{8}\ln|1-u|+\dfrac{1}{8}\ln|1+u|+\dfrac{1}{4}\tan^{-1}u+\dfrac{1}{4}\left(\tan^{-1}u+\dfrac{u}{u^2+1}\right)\right]_0^{1/\sqrt{2}}=\left[\dfrac{1}{8}\ln\left|\dfrac{1+u}{1-u}\right|+\dfrac{1}{2}\tan^{-1}u+\dfrac{u}{4(u^2+1)}\right]_0^{1/\sqrt{2}}$

$=\dfrac{1}{8}\ln\left|\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right|+\dfrac{1}{2}\tan^{-1}\dfrac{1}{\sqrt{2}}+\dfrac{1}{6\sqrt{2}}\approx 0.65$

(To integrate $\int\dfrac{1}{(u^2+1)^2}du$, let $u = \tan t$.)

40. $\dfrac{3x+13}{x^2+4x+3}=\dfrac{3x+13}{(x+3)(x+1)}=\dfrac{A}{x+3}+\dfrac{B}{x+1}$

$A = -2$, $B = 5$

$\int_1^5\dfrac{3x+13}{x^2+4x+3}dx=\left[-2\ln|x+3|+5\ln|x+1|\right]_1^5 = -2\ln 8+5\ln 6+2\ln 4-5\ln 2 = 5\ln 3-2\ln 2\ \approx\ 4.11$

41. $\dfrac{dy}{dt}=y(1-y)$ so that

$\int\dfrac{1}{y(1-y)}dy=\int 1\,dt=t+C_1$

a. Using partial fractions:

$\dfrac{1}{y(1-y)}=\dfrac{A}{y}+\dfrac{B}{1-y}=\dfrac{A(1-y)+By}{y(1-y)}\Rightarrow$

$A+(B-A)y=1+0y\Rightarrow A=1,\ B-A=0\Rightarrow A=1,\ B=1\Rightarrow\dfrac{1}{y(1-y)}=\dfrac{1}{y}+\dfrac{1}{1-y}$

Thus: $t+C_1=\int\left(\dfrac{1}{y}+\dfrac{1}{1-y}\right)dy=\ \ln y-\ln(1-y)=\ln\left(\dfrac{y}{1-y}\right)$ so that

$\dfrac{y}{1-y}=e^{t+C_1}=\underset{(C=e^{C_1})}{Ce^t}$ or $y(t)=\dfrac{e^t}{\frac{1}{C}+e^t}$

Since $y(0)=0.5,\ 0.5=\dfrac{1}{\frac{1}{C}+1}$ or $C=1$; thus $y(t)=\dfrac{e^t}{1+e^t}$

b. $y(3)=\dfrac{e^3}{1+e^3}\approx 0.953$

42. $\frac{dy}{dt} = \frac{1}{10} y(12-y)$ so that

$\int \frac{1}{y(12-y)} dy = \int \frac{1}{10} dt = \frac{1}{10} t + C_1$

a. Using partial fractions:

$\frac{1}{y(12-y)} = \frac{A}{y} + \frac{B}{12-y} = \frac{A(12-y)+By}{y(12-y)} \Rightarrow$

$12A + (B-A)y = 1 + 0y \Rightarrow 12A = 1,\ B - A = 0$

$\Rightarrow A = \frac{1}{12},\ B = \frac{1}{12}$

$\Rightarrow \frac{1}{y(12-y)} = \frac{1}{12y} + \frac{1}{12(12-y)}$

Thus: $\frac{1}{10} t + C_1 = \int \left(\frac{1}{12y} + \frac{1}{12(12-y)} \right) dy =$

$\frac{1}{12}\left[\ln y - \ln(12-y)\right] = \frac{1}{12} \ln\left(\frac{y}{12-y} \right)$ so that

$\frac{y}{12-y} = e^{1.2t + 12C_1} \underset{(C = e^{12C_1})}{=} Ce^{1.2t}$ or

$y(t) = \frac{12e^{1.2t}}{\frac{1}{C} + e^{1.2t}}$

Since $y(0) = 2.0,\ 2.0 = \frac{12}{\frac{1}{C}+1}$ or $C = 0.2$;

thus $y(t) = \frac{12e^{1.2t}}{5 + e^{1.2t}}$

b. $y(3) = \frac{12e^{3.6}}{5 + e^{3.6}} \approx 10.56$

43. $\frac{dy}{dt} = 0.0003 y(8000-y)$ so that

$\int \frac{1}{y(8000-y)} dy = \int 0.0003\, dt = 0.0003t + C_1$

a. Using partial fractions:

$\frac{1}{y(8000-y)} = \frac{A}{y} + \frac{B}{8000-y} = \frac{A(8000-y)+By}{y(8000-y)}$

$\Rightarrow 8000A + (B-A)y = 1 + 0y$

$\Rightarrow 8000A = 1,\ B - A = 0$

$\Rightarrow A = \frac{1}{8000},\ B = \frac{1}{8000}$

$\Rightarrow \frac{1}{y(8000-y)} = \frac{1}{8000}\left[\frac{1}{y} + \frac{1}{(8000-y)} \right]$

Thus:

$0.0003t + C_1 = \frac{1}{8000} \int \left(\frac{1}{y} + \frac{1}{(8000-y)} \right) dy =$

$\frac{1}{8000}\left[\ln y - \ln(8000-y)\right] = \frac{1}{8000} \ln\left(\frac{y}{8000-y} \right)$

so that

$\frac{y}{8000-y} = e^{2.4t + 8000C_1} \underset{(C = e^{8000C_1})}{=} Ce^{2.4t}$ or

$y(t) = \frac{8000e^{2.4t}}{\frac{1}{C} + e^{2.4t}}$

Since $y(0) = 1000,\ 1000 = \frac{8000}{\frac{1}{C}+1}$ or $C = \frac{1}{7}$;

thus $y(t) = \frac{8000e^{2.4t}}{7 + e^{2.4t}}$

b. $y(3) = \frac{8000e^{7.2}}{7 + e^{7.2}} \approx 7958.4$

44. $\frac{dy}{dt} = 0.001 y(4000-y)$ so that

$\int \frac{1}{y(4000-y)} dy = \int 0.001\, dt = 0.001t + C_1$

a. Using partial fractions:

$\frac{1}{y(4000-y)} = \frac{A}{y} + \frac{B}{4000-y}$

$= \frac{A(4000-y)+By}{y(4000-y)}$

$\Rightarrow 4000A + (B-A)y = 1 + 0y$

$\Rightarrow 4000A = 1,\ B - A = 0$

$\Rightarrow A = \frac{1}{4000},\ B = \frac{1}{4000}$

$\Rightarrow \frac{1}{y(4000-y)} = \frac{1}{4000}\left[\frac{1}{y} + \frac{1}{(4000-y)} \right]$

Thus:

$0.001t + C_1 = \frac{1}{4000} \int \left(\frac{1}{y} + \frac{1}{(4000-y)} \right) dy =$

$\frac{1}{4000}\left[\ln y - \ln(4000-y)\right] = \frac{1}{4000} \ln\left(\frac{y}{4000-y} \right)$

so that

$\frac{y}{4000-y} = e^{4t + 4000C_1} \underset{(C = e^{4000C_1})}{=} Ce^{4t}$ or

$y(t) = \frac{4000e^{4t}}{\frac{1}{C} + e^{4t}}$

Since $y(0) = 100,\ 100 = \frac{4000}{\frac{1}{C}+1}$ or $C = \frac{1}{39}$;

thus $y(t) = \frac{4000e^{4t}}{39 + e^{4t}}$

b. $y(3) = \frac{4000e^{12}}{39 + e^{12}} \approx 3999.04$

45. $\frac{dy}{dt} = ky(L-y)$ so that

$$\int \frac{1}{y(L-y)}dy = \int k\,dt = kt + C_1$$

Using partial fractions:

$$\frac{1}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y} = \frac{A(L-y)+By}{y(L-y)} \Rightarrow$$

$$LA + (B-A)y = 1 + 0y \Rightarrow LA = 1,\ B - A = 0 \Rightarrow$$

$$A = \frac{1}{L},\ B = \frac{1}{L} \Rightarrow \frac{1}{y(L-y)} = \frac{1}{L}\left[\frac{1}{y} + \frac{1}{L-y}\right]$$

Thus: $kt + C_1 = \frac{1}{L}\int\left(\frac{1}{y} + \frac{1}{L-y}\right)dy =$

$\frac{1}{L}\left[\ln y - \ln(L-y)\right] = \frac{1}{L}\ln\left(\frac{y}{L-y}\right)$ so that

$\frac{y}{L-y} = e^{kLt + LC_1} = \underset{(C = e^{LC_1})}{Ce^{kLt}}$ or $y(t) = \frac{Le^{kLt}}{\frac{1}{C} + e^{kLt}}$

If $y_0 = y(0) = \frac{L}{\frac{1}{C}+1}$ then $\frac{1}{\text{C}} = \frac{L-y_0}{y_0}$; so our

final formula is $y(t) = \frac{Le^{kLt}}{\left(\frac{L-y_0}{y_0}\right) + e^{kLt}}$.

(Note: if $y_0 < L$, then $u = \frac{L-y_0}{y_0} > 0$ and

$\frac{e^{kLt}}{u + e^{kLt}} < 1$; thus $y(t) < L$ for all t)

46. Since $y'(0) = ky_0(L-y_0)$ is negative if $y_0 > L$, the population would be decreasing at time $t = 0$. Further, since

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\frac{L}{\left(\frac{L-y_0}{y_0e^{kLt}}\right)+1} = \frac{L}{0+1} = L$$

(no matter how y_0 and L compare), and since $\frac{L-y_0}{y_0e^{kLt}}$ is monotonic as $t \to \infty$,we conclude that the population would *decrease* toward a limiting value of L.

47. If $y_0 < L$, then $y'(0) = ky_0(L-y_0) > 0$ and the population is increasing initially.

48. The graph will be concave up for values of t that make $y''(t) > 0$. Now

$$y''(t) = \frac{dy'}{dt} = \frac{d}{dt}\left[ky(L-y)\right] =$$

$$k\left[-yy' + (L-y)y'\right] = k\left[ky(L-y)\right]\left[L-2y\right]$$

Thus if $y_0 < L$, then $y(t) < L$ for all positive t (see note at the end of problem 45 solution) and so the graph will be concave up as long as $L - 2y > 0$; that is, as long as the population is less than half the capacity.

49. a. $\frac{dy}{dt} = ky(16-y)$

$$\frac{dy}{y(16-y)} = kdt$$

$$\int\frac{dy}{y(16-y)} = \int kdt$$

$$\frac{1}{16}\int\left(\frac{1}{y} + \frac{1}{16-y}\right)dy = kt + C$$

$$\frac{1}{16}\left(\ln|y| - \ln|16-y|\right) = kt + C$$

$$\ln\left|\frac{y}{16-y}\right| = 16kt + C$$

$$\frac{y}{16-y} = Ce^{16kt}$$

$y(0) = 2$: $\frac{1}{7} = C;\ \frac{y}{16-y} = \frac{1}{7}e^{16kt}$

$y(50) = 4$: $\frac{1}{3} = \frac{1}{7}e^{800k}$, so $k = \frac{1}{800}\ln\frac{7}{3}$

$$\frac{y}{16-y} = \frac{1}{7}e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}$$

$$7y = 16e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t} - ye^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}$$

$$y = \frac{16e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}{7 + e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}} = \frac{16}{1 + 7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}$$

b. $y(90) = \frac{16}{1 + 7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)90}} \approx 6.34$ billion

c. $9 = \frac{16}{1 + 7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}$

$$7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t} = \frac{16}{9} - 1$$

$$e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t} = \frac{1}{9}$$

$$-\left(\frac{1}{50}\ln\frac{7}{3}\right)t = \ln\frac{1}{9}$$

$$t = -50\left(\frac{\ln\frac{1}{9}}{\ln\frac{7}{3}}\right) \approx 129.66$$

The population will be 9 billion in 2055.

50. a. $\frac{dy}{dt} = ky(10-y)$

$\frac{dy}{y(10-y)} = kdt$

$\frac{1}{10}\int\left(\frac{1}{y}+\frac{1}{10-y}\right)dy = \int kdt$

$\ln\left|\frac{y}{10-y}\right| = 10kt + C$

$\frac{y}{10-y} = Ce^{10kt}$

$y(0) = 2$: $\frac{1}{4} = C$; $\frac{y}{10-y} = \frac{1}{4}e^{10kt}$

$y(50) = 4$: $\frac{2}{3} = \frac{1}{4}e^{500k}$, $k = \frac{1}{500}\ln\frac{8}{3}$

$\frac{y}{10-y} = \frac{1}{4}e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}$

$4y = 10e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t} - ye^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}$

$y = \frac{10e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}{4+e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}} = \frac{10}{1+4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}$

b. $y(90) = \frac{10}{1+4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)90}} \approx 5.94$ billion

c. $9 = \frac{10}{1+4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}$

$4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t} = \frac{10}{9} - 1$

$e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t} = \frac{1}{36}$

$-\left(\frac{1}{50}\ln\frac{8}{3}\right)t = \ln\frac{1}{36}$

$t = -50\left(\frac{\ln\frac{1}{36}}{\ln\frac{8}{3}}\right) \approx 182.68$

The population will be 9 billion in 2108.

51. a. Separating variables, we obtain

$\frac{dx}{(a-x)(b-x)} = k\,dt$

$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$

$A = -\frac{1}{a-b}$, $B = \frac{1}{a-b}$

$\int\frac{dx}{(a-x)(b-x)}$

$= \frac{1}{a-b}\int\left(-\frac{1}{a-x}+\frac{1}{b-x}\right)dx = \int k\,dt$

$\frac{\ln|a-x| - \ln|b-x|}{a-b} = kt + C$

$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| = kt + C$

$\frac{a-x}{b-x} = Ce^{(a-b)kt}$

Since $x = 0$ when $t = 0$, $C = \frac{a}{b}$, so

$a - x = (b-x)\frac{a}{b}e^{(a-b)kt}$

$a\left(1-e^{(a-b)kt}\right) = x\left(1-\frac{a}{b}e^{(a-b)kt}\right)$

$x(t) = \frac{a(1-e^{(a-b)kt})}{1-\frac{a}{b}e^{(a-b)kt}} = \frac{ab(1-e^{(a-b)kt})}{b-ae^{(a-b)kt}}$

b. Since $b > a$ and $k > 0$, $e^{(a-b)kt} \to 0$ as $t \to \infty$. Thus,

$x \to \frac{ab(1)}{b-0} = a$.

c. $x(t) = \frac{8(1-e^{-2kt})}{4-2e^{-2kt}}$

$x(20) = 1$, so $4 - 2e^{-40k} = 8 - 8e^{-40k}$

$6e^{-40k} = 4$

$k = -\frac{1}{40}\ln\frac{2}{3}$

$e^{-2kt} = e^{t/20\ln 2/3} = e^{\ln(2/3)^{t/20}} = \left(\frac{2}{3}\right)^{t/20}$

$x(t) = \frac{4\left(1-\left(\frac{2}{3}\right)^{t/20}\right)}{2-\left(\frac{2}{3}\right)^{t/20}}$

$x(60) = \frac{4\left(1-\left(\frac{2}{3}\right)^3\right)}{2-\left(\frac{2}{3}\right)^3} = \frac{38}{23} \approx 1.65$ grams

d. If $a = b$, the differential equation is, after separating variables

$$\frac{dx}{(a-x)^2} = k\,dt$$

$$\int \frac{dx}{(a-x)^2} = \int k\,dt$$

$$\frac{1}{a-x} = kt + C$$

$$\frac{1}{kt+C} = a - x$$

$$x(t) = a - \frac{1}{kt+C}$$

Since $x = 0$ when $t = 0$, $C = \frac{1}{a}$, so

$$x(t) = a - \frac{1}{kt+\frac{1}{a}} = a - \frac{a}{akt+1}$$

$$= a\left(1 - \frac{1}{akt+1}\right) = a\left(\frac{akt}{akt+1}\right).$$

52. Separating variables, we obtain

$$\frac{dy}{(y-m)(M-y)} = k\,dt\,.$$

$$\frac{1}{(y-m)(M-y)} = \frac{A}{y-m} + \frac{B}{M-y}$$

$$A = \frac{1}{M-m},\ B = \frac{1}{M-m}$$

$$\int \frac{dy}{(y-m)(M-y)} = \frac{1}{M-m}\int\left(\frac{1}{y-m} + \frac{1}{M-y}\right)dy$$

$$= \int k\,dt$$

$$\frac{\ln|y-m| - \ln|M-y|}{M-m} = kt + C$$

$$\frac{1}{M-m}\ln\left|\frac{y-m}{M-y}\right| = kt + C$$

$$\frac{y-m}{M-y} = Ce^{(M-m)kt}$$

$$y - m = (M-y)Ce^{(M-m)kt}$$

$$y(1+Ce^{(M-m)kt}) = m + MCe^{(M-m)kt}$$

$$y = \frac{m + MCe^{(M-m)kt}}{1+Ce^{(M-m)kt}} = \frac{me^{-(M-m)kt} + MC}{e^{-(M-m)kt} + C}$$

As $t \to \infty, e^{-(M-m)kt} \to 0$ since $M > m$.

Thus $y \to \frac{MC}{C} = M$ as $t \to \infty$.

53. Separating variables, we obtain

$$\frac{dy}{(A-y)(B+y)} = k\,dt$$

$$\frac{1}{(A-y)(B+y)} = \frac{C}{A-y} + \frac{D}{B+y}$$

$$C = \frac{1}{A+B},\ D = \frac{1}{A+B}$$

$$\int \frac{dy}{(A-y)(B+y)} = \frac{1}{A+B}\int\left(\frac{1}{A-y} + \frac{1}{B+y}\right)dy$$

$$= \int k\,dt$$

$$\frac{-\ln(A-y) + \ln(B+y)}{A+B} = kt + C$$

$$\frac{1}{A+B}\ln\left|\frac{B+y}{A-y}\right| = kt + C$$

$$\frac{B+y}{A-y} = Ce^{(A+B)kt}$$

$$B + y = (A-y)Ce^{(A+B)kt}$$

$$y(1+Ce^{(A+B)kt}) = ACe^{(A+B)kt} - B$$

$$y(t) = \frac{ACe^{(A+B)kt} - B}{1+Ce^{(A+B)kt}}$$

54. $u = \sin x$, $du = \cos x\,dx$

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(\sin^2 x + 1)^2}\,dx = \int_{\frac{1}{2}}^{1} \frac{1}{u(u^2+1)^2}\,du$$

$$\frac{1}{u(u^2+1)^2} = \frac{A}{u} + \frac{Bu+C}{u^2+1} + \frac{Du+E}{(u^2+1)^2}$$

$A = 1$, $B = -1$, $C = 0$, $D = -1$, $E = 0$

$$\int_{\frac{1}{2}}^{1} \frac{1}{u(u^2+1)^2}\,du$$

$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}\,du - \int_{\frac{1}{2}}^{1} \frac{u}{u^2+1}\,du - \int_{\frac{1}{2}}^{1} \frac{u}{(u^2+1)^2}\,du$$

$$= \left[\ln u - \frac{1}{2}\ln(u^2+1) + \frac{1}{2(u^2+1)}\right]_{\frac{1}{2}}^{1}$$

$$= 0 - \frac{1}{2}\ln 2 + \frac{1}{4} - \left(\ln\frac{1}{2} - \frac{1}{2}\ln\frac{5}{4} + \frac{2}{5}\right) \approx 0.308$$

7.6 Concepts Review

1. substitution

2. 53

3. approximation

4. 0

Problem Set 7.6

Note: Throughout this section, the notation Fxxx refers to integration formula number xxx in the back of the book.

1. Integration by parts.

$u = x \qquad dv = e^{-5x}$

$du = 1dx \qquad v = -\frac{1}{5}e^{-5x}$

$$\int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \int -\frac{1}{5}e^{-5x}\,dx$$

$$= -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$$

$$= -\frac{1}{5}e^{-5x}\left(x + \frac{1}{5}\right) + C$$

2. Substitution

$$\int \underbrace{\frac{x}{x^2+9}}_{\substack{u=x^2+9\\ du=2x\,dx}} dx = \frac{1}{2}\int \frac{1}{u}\,du = \ln|u| + C = \ln\left(x^2+9\right) + C$$

3. Substitution

$$\int_1^2 \underbrace{\frac{\ln x}{x}}_{\substack{u=\ln x\\ du=\frac{1}{x}dx}} dx = \int_0^{\ln 2} u\,du = \left[\frac{u^2}{2}\right]_0^{\ln 2} = \frac{(\ln 2)^2}{2} \approx 0.2402$$

4. Partial fractions

$$\int \frac{x}{x^2-5x+6}\,dx = \int \frac{x}{(x-3)(x-2)}\,dx$$

$$\frac{x}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)} =$$

$$\frac{A(x-2)+B(x-3)}{(x-3)(x-2)} = \frac{(A+B)x+(-2A-3B)}{(x-3)(x-2)} \Rightarrow$$

$$A+B=1, -2A-3B=0 \Rightarrow A=3, B=-2$$

$$\int \frac{x}{x^2-5x+6}\,dx = \int \frac{3}{(x-3)} - \frac{2}{(x-2)}\,dx =$$

$$3\ln|x-3| - 2\ln|x-2| = \ln\left|\frac{(x-3)^3}{(x-2)^2}\right| + C$$

5. Trig identity $\cos^2 u = \frac{1+\cos 2u}{2}$ and substitution.

$$\int \cos^4 2x\,dx = \int \left(\frac{1+\cos 4x}{2}\right)^2 dx =$$

$$\frac{1}{4}\int \left[1 + 2\underbrace{\cos 4x}_{\substack{u=4x\\ du=4dx}} + \cos^2 4x\right] dx =$$

$$\frac{1}{4}\left[x + \frac{1}{2}\sin 4x + \int \underbrace{\left(\frac{1+\cos 8x}{2}\right)}_{\substack{v=8x\\ dv=8dx}} dx\right] =$$

$$\frac{1}{4}\left[x + \frac{1}{2}\sin 4x + \frac{1}{2}x + \frac{1}{16}\sin 8x\right] + C =$$

$$\frac{1}{64}\left[24x + 8\sin 4x + \sin 8x\right] + C$$

6. Substitution

$$\int \underbrace{\sin^3 x\cos x\,dx}_{\substack{u=\sin x\\ du=\cos x\,dx}} = \int u^3\,du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

7. Partial fractions

$$\int \frac{1}{x^2+6x+8}\,dx = \int \frac{1}{(x+4)(x+2)}\,dx$$

$$\frac{1}{(x+4)(x+2)} = \frac{A}{(x+4)} + \frac{B}{(x+2)} =$$

$$\frac{A(x+2)+B(x+4)}{(x+4)(x+2)} = \frac{(A+B)x+(2A+4B)}{(x+4)(x+2)} \Rightarrow$$

$$A+B=0,\ 2A+4B=1 \Rightarrow A=-\frac{1}{2}, B=\frac{1}{2}$$

$$\int_1^2 \frac{1}{x^2+6x+8} = \frac{1}{2}\int_1^2 \left(\frac{1}{x+2} - \frac{1}{x+4}\right) dx$$

$$= \frac{1}{2}\left[\ln|x+2| - \ln|x+4|\right]_1^2 = \frac{1}{2}\left[\ln\left|\frac{(x+2)}{(x+4)}\right|\right]_1^2$$

$$= \frac{1}{2}\left(\ln\frac{4}{6} - \ln\frac{3}{5}\right) = \frac{1}{2}\ln\frac{10}{9} \approx 0.0527$$

8. Partial fractions

$$\int \frac{1}{1-t^2}\,dt = \int \frac{1}{(1-t)(1+t)}\,dt$$

$$\frac{1}{(1-t)(1+t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} =$$

$$\frac{A(1+t)+B(1-t)}{(1-t)(1+t)} = \frac{(A-B)t+(A+B)}{(1-t)(1+t)} \Rightarrow$$

$$A+B=1,\ \ A-B=0 \Rightarrow A=\frac{1}{2}, B=\frac{1}{2}$$

$$\int_0^{1/2} \frac{1}{1-t^2}\,dt = \frac{1}{2}\int_0^{1/2}\left(\frac{1}{1-t}+\frac{1}{1+t}\right)dx$$

$$\frac{1}{2}\Big[-\ln|1-t|+\ln|1+t|\Big]_0^{1/2} =$$

$$\frac{1}{2}\left[\ln\left|\frac{(1+t)}{(1-t)}\right|\right]_0^{1/2} \approx 0.5493$$

9. Substitution

$$\int_0^5 x\sqrt{x+2}\,dx \underset{\substack{u=\sqrt{x+2}\\ u^2=x+2\\ 2u\,du=dx}}{=} \int_{\sqrt{2}}^{\sqrt{7}} (u^2-2)(u)2u\,du =$$

$$\int_{\sqrt{2}}^{\sqrt{7}} 2u^4-4u^2\,du = 2\left[\frac{u^5}{5}-\frac{2u^3}{3}\right]_{\sqrt{2}}^{\sqrt{7}} =$$

$$\frac{2}{15}\Big[3u^5-10u^3\Big]_{\sqrt{2}}^{\sqrt{7}} = \frac{2}{15}\Big[77\sqrt{7}+8\sqrt{2}\Big] \approx 28.67$$

10. Substitution

$$\int_3^4 \frac{1}{t-\sqrt{2t}}\,dt \underset{\substack{u=\sqrt{2t},\ u^2=2t\\ u\,du\ =\ dt}}{=} \int_{\sqrt{6}}^{\sqrt{8}} \frac{u}{\frac{u^2}{2}-u}\,du = 2\int_{\sqrt{6}}^{\sqrt{8}} \frac{1}{u-2}\,du =$$

$$2\Big[\ln|u-2|\Big]_{\sqrt{6}}^{\sqrt{8}} = 2\ln\left|\frac{\sqrt{8}-2}{\sqrt{6}-2}\right| \approx 1.223$$

11. Use of symmetry; this is an odd function, so

$$\int_{-\pi/2}^{\pi/2} \cos^2 x \sin x\,dx = 0$$

12. Use of symmetry; substitution

$$\int_0^{2\pi} |\sin 2x|\,dx = 8\int_0^{\pi/4} \underset{\substack{u=2x\\ du=2\,dx}}{\sin 2x}\ dx =$$

$$4\int_0^{\pi/2} \sin u\,du = 4\Big[-\cos u\Big]_0^{\pi/2} = 4$$

13. a. Formula 96

$$\int x\sqrt{3x+1}\,dx \underset{\substack{F96\\ a=3,b=1}}{=} \frac{2}{135}(9x-2)(3x+1)^{3/2}+C$$

b. Substitution; Formula 96

$$\int\limits_{u=e^x,\ du=e^x\,dx} e^x\sqrt{3e^x+1}\ dx = \int u\sqrt{3u+1}\,du \underset{\substack{F96\\ a=3,b=1}}{=}$$

$$\frac{2}{135}\left(9e^x-2\right)\left(3e^x+1\right)^{3/2}+C$$

14. a. Formula 96

$$\int 2t(3-4t)\,dt = 2\int t(3-4t)\,dt \underset{\substack{F96\\ a=-4,\\ b=3}}{=}$$

$$2\left[\frac{2}{240}(-12t-6)(3-4t)^{3/2}\right]+C =$$

$$-\frac{1}{10}(2t+1)(3-4t)^{3/2}+C$$

b. Substitution; Formula 96

$$\int\limits_{u=\cos t,\ du=-\sin t\,dt} \cos t\sqrt{3-4\cos t}\,\sin t\,dt = -\int u\sqrt{3-4u}\,du \underset{\text{part a.}}{=}$$

$$\frac{1}{20}(2\cos t+1)(3-4\cos t)^{3/2}+C$$

15. a. Substitution, Formula 18

$$\int\limits_{u=4x,\,du=4dx} \frac{dx}{9-16x^2} = \frac{1}{4}\int \frac{du}{9-u^2} \underset{\substack{F18\\ a=3}}{=}$$

$$\frac{1}{4}\left[\frac{1}{6}\ln\left|\frac{u+3}{u-3}\right|\right]+C = \frac{1}{24}\ln\left|\frac{4x+3}{4x-3}\right|+C$$

b. Substitution, Formula 18

$$\int\limits_{u=4e^x,\,du=4e^x dx} \frac{e^x}{9-16e^{2x}}\ dx = \frac{1}{4}\int \frac{du}{9-u^2} \underset{\text{part a.}}{=}$$

$$\frac{1}{24}\ln\left|\frac{4e^x+3}{4e^x-3}\right|+C$$

16. a. Substitution, Formula 18

$$\int \frac{dx}{5x^2-11} = -\int\limits_{\substack{u=\sqrt{5}x,\\ du=\sqrt{5}\,dx}} \frac{dx}{11-5x^2} = -\frac{\sqrt{5}}{5}\int \frac{du}{11-u^2} \underset{\substack{F18\\ a=\sqrt{11}}}{=}$$

$$\frac{-\sqrt{5}}{5}\,\frac{\sqrt{11}}{22}\ln\left|\frac{\sqrt{5}x+\sqrt{11}}{\sqrt{5}x-\sqrt{11}}\right|+C$$

$$= \frac{\sqrt{55}}{110}\ln\left|\frac{\sqrt{5}x-\sqrt{11}}{\sqrt{5}x+\sqrt{11}}\right|+C$$

b. Substitution, Formula 18

$$\int \underset{\substack{u=\sqrt{5}x^2,\\ du=2\sqrt{5}\,x\,dx}}{\frac{x\,dx}{5x^4-11}} = -\frac{\sqrt{5}}{10}\int \frac{du}{11-u^2} \underset{\substack{F18\\ a=\sqrt{11}}}{=}$$

$$\frac{\sqrt{55}}{220}\ln\left|\frac{\sqrt{5}x^2-\sqrt{11}}{\sqrt{5}x^2+\sqrt{11}}\right|+C$$

17. a. Substitution, Formula 57

$$\int \underset{\substack{u=\sqrt{2}\,x\\ du=\sqrt{2}\,dx}}{x^2\sqrt{9-2x^2}}\,dx = \frac{\sqrt{2}}{4}\int u^2\sqrt{9-u^2}\,du \underset{\substack{F57\\ a=3}}{=}$$

$$\frac{1}{16}\left(x(4x^2-9)\sqrt{9-2x^2}\right)+\frac{81\sqrt{2}}{32}\sin^{-1}\left(\frac{\sqrt{2}x}{3}\right)+C$$

b. Substitution, Formula 57

$$\int \sin^2 \underset{\substack{u=\sqrt{2}\sin x\\ du=\sqrt{2}\cos x\,dx}}{x\cos x}\sqrt{9-2\sin^2 x}\,dx = \frac{\sqrt{2}}{4}\int u^2\sqrt{9-u^2}\,du$$

$$\underset{\substack{F57\\ a=3}}{=} \frac{1}{16}\left(\sin x(4\sin^2 x-9)\sqrt{9-2\sin^2 x}\right)$$

$$+\frac{81\sqrt{2}}{32}\sin^{-1}\left(\frac{\sqrt{2}\sin x}{3}\right)+C$$

18. a. Substitution, Formula 55

$$\int \underset{\substack{u=\sqrt{3}t\\ du=\sqrt{3}\,dt}}{\frac{\sqrt{16-3t^2}}{t}}\,dt = \int \frac{\sqrt{16-u^2}}{u}\,du \underset{\substack{F55\\ a=4}}{=}$$

$$\sqrt{16-3t^2}-4\ln\left|\frac{4+\sqrt{16-3t^2}}{\sqrt{3}\,t}\right|+C$$

b. Substitution, Formula 55

$$\int \frac{\sqrt{16-3t^6}}{t}\,dt = \int \underset{\substack{u=\sqrt{3}t^3\\ du=3\sqrt{3}\,t^2dt}}{\frac{t^2\sqrt{16-3t^6}}{t^3}}\,dt =$$

$$\frac{1}{3}\int \frac{\sqrt{16-u^2}}{u}\,du \underset{\substack{F55\\ a=4}}{=}$$

$$\frac{1}{3}\left\{\sqrt{16-3t^6}-4\ln\left|\frac{4+\sqrt{16-3t^6}}{\sqrt{3}\,t^3}\right|\right\}+C$$

19. a. Substitution, Formula 45

$$\int \underset{\substack{u=\sqrt{3}x\\ du=\sqrt{3}\,dx}}{\frac{dx}{\sqrt{5+3x^2}}} = \frac{\sqrt{3}}{3}\int \frac{du}{\sqrt{5+u^2}} \underset{\substack{F45\\ a=\sqrt{5}}}{=}$$

$$\frac{\sqrt{3}}{3}\ln\left|\sqrt{3}x+\sqrt{5+3x^2}\right|+C$$

b. Substitution, Formula 45

$$\int \underset{\substack{u=\sqrt{3}x^2\\ du=2\sqrt{3}\,x\,dx}}{\frac{x}{\sqrt{5+3x^4}}}\,dx = \frac{\sqrt{3}}{6}\int \frac{du}{\sqrt{5+u^2}} \underset{\substack{F45\\ a=\sqrt{5}}}{=}$$

$$\frac{\sqrt{3}}{6}\ln\left|\sqrt{3}x^2+\sqrt{5+3x^4}\right|+C$$

20. a. Substitution; Formula 48

$$\int \underset{\substack{u=\sqrt{5}\,t\\ du=\sqrt{5}\,dt}}{t^2\sqrt{3+5t^2}}\,dt = \frac{\sqrt{5}}{25}\int u^2\sqrt{3+u^2}\,du \underset{\substack{F48\\ a=\sqrt{3}}}{=}$$

$$\frac{\sqrt{5}}{25}\left\{\begin{matrix}\left(\frac{\sqrt{5}}{8}t\right)(10t^2+3)\left(\sqrt{3+5t^2}\right)-\\ \frac{9}{8}\ln\left|\sqrt{5}\,t+\sqrt{3+5t^2}\right|\end{matrix}\right\}+C=$$

$$\frac{1}{200}\left\{5t(10t^2+3)\sqrt{3+5t^2}-9\sqrt{5}\ln\left|\sqrt{5}t+\sqrt{3+5t^2}\right|\right\}+C$$

b. Substitution; Formula 48

$$\int t^8\sqrt{3+5t^6}\,dt = \int \underset{\substack{u=\sqrt{5}\,t^3\\ du=3\sqrt{5}\,t^2dt}}{t^6\sqrt{3+5t^6}}\,t^2dt =$$

$$\frac{\sqrt{5}}{75}\int u^2\sqrt{3+u^2}\,du \underset{\substack{F48\\ a=\sqrt{3}}}{=}$$

$$\frac{\sqrt{5}}{75}\left\{\begin{matrix}\left(\frac{\sqrt{5}}{8}t^3\right)(10t^6+3)\left(\sqrt{3+5t^6}\right)-\\ \frac{9}{8}\ln\left|\sqrt{5}\,t^3+\sqrt{3+5t^6}\right|\end{matrix}\right\}+C=$$

$$\frac{1}{600}\left\{\begin{matrix}5t^3(10t^6+3)\sqrt{3+5t^6}-\\ 9\sqrt{5}\ln\left|\sqrt{5}t^3+\sqrt{3+5t^6}\right|\end{matrix}\right\}+C$$

21. **a.** Complete the square; substitution; Formula 45.

$$\int \frac{dt}{\sqrt{t^2+2t-3}} = \int \underset{\substack{u=t+1\\du=dt}}{\frac{dt}{\sqrt{(t+1)^2-4}}} = \int \frac{du}{\sqrt{u^2-4}} \underset{\substack{F45\\a=2}}{=}$$

$$\ln\left|(t+1)+\sqrt{t^2+2t-3}\right|+C$$

b. Complete the square; substitution; Formula 45.

$$\int \frac{dt}{\sqrt{t^2+3t-5}} = \int \underset{\substack{u=t+3/2\\du=dt}}{\frac{dt}{\sqrt{(t+\frac{3}{2})^2-\frac{29}{4}}}} =$$

$$\int \frac{du}{\sqrt{u^2-\frac{29}{4}}} \underset{\substack{F45\\a=\sqrt{29}/2}}{=} \ln\left|(t+\tfrac{3}{2})+\sqrt{t^2+3t-5}\right|+C$$

22. **a.** Complete the square; substitution; Formula 47.

$$\int \frac{\sqrt{x^2+2x-3}}{x+1}\,dx = \int \underset{\substack{u=x+1\\du=dx}}{\frac{\sqrt{(x+1)^2-4}}{x+1}}\,dx =$$

$$\int \frac{\sqrt{u^2-4}}{u}\,du \underset{\substack{F47\\a=2}}{=}$$

$$\sqrt{x^2+2x-3}-2\sec^{-1}\left(\frac{x+1}{2}\right)+C$$

b. Complete the square; substitution; Formula 47.

$$\int \frac{\sqrt{x^2-4x}}{x-2}\,dx = \int \underset{\substack{u=x-2\\du=dx}}{\frac{\sqrt{(x-2)^2-4}}{x-2}}\,dx =$$

$$\int \frac{\sqrt{u^2-4}}{u}\,du \underset{\substack{F47\\a=2}}{=}$$

$$\sqrt{x^2-4x}-2\sec^{-1}\left(\frac{x-2}{2}\right)+C$$

23. **a.** Formula 98

$$\int \frac{y}{\sqrt{3y+5}}\,dy \underset{\substack{F98\\a=3,b=5}}{=} \frac{2}{27}(3y-10)\sqrt{3y+5}+C$$

b. Substitution, Formula 98

$$\int \underset{\substack{u=\sin t\\du=\cos t\,dt}}{\frac{\sin t\cos t}{\sqrt{3\sin t+5}}} = \int \frac{u}{\sqrt{3u+5}}\,du \underset{\substack{F98\\a=3,b=5}}{=}$$

$$\frac{2}{27}(3\sin t-10)\sqrt{3\sin t+5}+C$$

24. **a.** Formula 100a

$$\int \frac{dz}{z\sqrt{5-4z}} \underset{\substack{F100a\\a=-4\\b=5}}{=} \frac{\sqrt{5}}{5}\ln\left|\frac{\sqrt{5-4z}-\sqrt{5}}{\sqrt{5-4z}+\sqrt{5}}\right|+C$$

b. Substitution, Formula 100a

$$\int \underset{\substack{u=\cos x\\du=-\sin x\,dx}}{\frac{\sin x}{\cos x\sqrt{5-4\cos x}}}\,dx = -\int \frac{du}{u\sqrt{5-4u}} \underset{\substack{F100a\\a=-4\\b=5}}{=}$$

$$-\frac{\sqrt{5}}{5}\ln\left|\frac{\sqrt{5-4\cos x}-\sqrt{5}}{\sqrt{5-4\cos x}+\sqrt{5}}\right|+C =$$

$$\frac{\sqrt{5}}{5}\ln\left|\frac{\sqrt{5-4\cos x}+\sqrt{5}}{\sqrt{5-4\cos x}-\sqrt{5}}\right|+C$$

25. Substitution; Formula 84

$$\int \underset{\substack{u=3t\\du=3\,dt}}{\sinh^2 3t\,dt} = \frac{1}{3}\int \sinh^2 u\,du \underset{F84}{=}$$

$$\frac{1}{3}\left(\frac{1}{4}\sinh 6t-\frac{3}{2}t\right)+C = \frac{1}{12}\left(\sinh 6t-6t\right)+C$$

26. Substitution; Formula 82

$$\int \underset{\substack{u=\sqrt{x}\\du=\frac{1}{2\sqrt{x}}dx}}{\frac{\operatorname{sech}\sqrt{x}}{\sqrt{x}}}\,dx = 2\int \operatorname{sech} u\,du \underset{F82}{=}$$

$$2\tan^{-1}\left|\sinh\sqrt{x}\right|+C$$

27. Substitution; Formula 98

$$\int \underset{\substack{u=\cos t\\du=-\sin t\,dt}}{\frac{\cos t\sin t}{\sqrt{2\cos t+1}}}\,dt = -\int \frac{u}{\sqrt{2u+1}}\,du \underset{\substack{F98\\a=2\\b=1}}{=}$$

$$-\frac{1}{6}(2\cos t-2)\sqrt{2\cos t+1}+C =$$

$$\frac{1}{3}(1-\cos t)\sqrt{2\cos t+1}+C$$

28. Substitution; Formula 96

$$\int \underset{\substack{u=\cos t\\du=-\sin t\,dt}}{\cos t\sin t\sqrt{4\cos t-1}}\,dt = -\int u\sqrt{4u-1}\,du \underset{\substack{F96\\a=4\\b=-1}}{=}$$

$$-\frac{1}{60}(6\cos t+1)(4\cos t-1)^{3/2}+C$$

29. Substitution; Formula 99, Formula 98

$$\int \frac{\cos^2 t \sin t}{\sqrt{\cos t+1}}\,dt \underset{\substack{u=\cos t\\ du=-\sin t\,dt}}{=} -\int \frac{u^2}{\sqrt{u+1}}\,du \underset{\substack{F99\\ n=2\\ a=1\\ b=1}}{=}$$

$$-\frac{2}{5}\left[u^2\sqrt{u+1}-2\int \frac{u}{\sqrt{u+1}}\,du\right] \underset{F98}{=}$$

$$-\frac{2}{5}\left[u^2\sqrt{u+1}-2\left(\frac{2}{3}(u-2)\sqrt{u+1}\right)\right]+C=$$

$$-\frac{2}{5}\sqrt{\cos t+1}\left[\cos^2 t-\frac{4}{3}(\cos t-2)\right]+C$$

30. Formula 95, Formula 17

$$\int \frac{1}{(9+x^2)^3}\,dx \underset{\substack{F95\\ n=3\\ a=3}}{=}$$

$$\frac{1}{36}\left[\frac{x}{(9+x^2)^2}+3\int \frac{dx}{(9+x^2)^2}\right] \underset{\substack{F95\\ n=2\\ a=3}}{=}$$

$$\frac{1}{36}\left[\frac{x}{(9+x^2)^2}+3\left[\frac{1}{18}\left(\frac{x}{(9+x^2)}+\int \frac{dx}{9+x^2}\right)\right]\right]$$

$$\underset{\substack{F17\\ a=3}}{=} \frac{1}{36}\left\{\frac{x}{(9+x^2)^2}+\frac{x}{6\cdot(9+x^2)}+\tan^{-1}\left(\frac{x}{3}\right)\right\}+C$$

31. Using a CAS, we obtain:

$$\int_0^{\pi} \frac{\cos^2 x}{1+\sin x}\,dx = \pi - 2 \approx 1.14159$$

32. Using a CAS, we obtain:

$$\int_0^1 \operatorname{sech} \sqrt[3]{x}\,dx \approx 0.76803$$

33. Using a CAS, we obtain:

$$\int_0^{\pi/2} \sin^{12} x\,dx = \frac{231\pi}{2048} \approx 0.35435$$

34. Using a CAS, we obtain:

$$\int_0^{\pi} \cos^4 \frac{x}{2}\,dx = \frac{3\pi}{8} \approx 1.17810$$

35. Using a CAS, we obtain:

$$\int_1^4 \frac{\sqrt{t}}{1+t^8}\,dt \approx 0.11083$$

36. Using a CAS, we obtain:

$$\int_0^3 x^4 e^{-x/2}\,dx = 768 - 3378e^{-3/2} \approx 14.26632$$

37. Using a CAS, we obtain:

$$\int_0^{\pi/2} \frac{1}{1+2\cos^5 x}\,dx \approx 1.10577$$

38. Using a CAS, we obtain:

$$\int_{-\pi/4}^{\pi/4} \frac{x^3}{4+\tan x}\,dx \approx -0.00921$$

39. Using a CAS, we obtain:

$$\int_2^3 \frac{x^2+2x-1}{x^2-2x+1}\,dx = 4\ln(2)+2 \approx 4.77259$$

40. Using a CAS, we obtain:

$$\int_1^3 \frac{du}{u\sqrt{2u-1}} = 2\tan^{-1}\left(\sqrt{5}\right)-\frac{\pi}{2} \approx 0.72973$$

41. $\displaystyle\int_0^c \frac{1}{x+1}\,dx \underset{F3}{=} \left[\ln|x+1|\right]_0^c = \ln(c+1)$

$\ln(c+1)=1 \Rightarrow c+1=e \Rightarrow$

$c=e-1 \approx 1.71828$

42. Formula 17

$$\int_0^c \frac{2}{x^2+1}\,dx \underset{F17}{=} \left[2\tan^{-1} x\right]_0^c = 2\tan^{-1} c$$

$$2\tan^{-1} c = 1 \Rightarrow \tan^{-1} c = \frac{1}{2} \Rightarrow$$

$$c = \tan\frac{1}{2} \approx 0.5463$$

43. Substitution; Formula 65

$$\int \ln(x+1)\,dx \underset{\substack{u=x+1\\ du=dx}}{=} \int \ln u\,du \underset{F65}{=}$$

$(x+1)\left[\ln(x+1)-1\right]$. Thus

$$\int_0^c \ln(x+1)\,dx = (x+1)\left[\ln(x+1)-1\right]_0^c =$$

$(c+1)\ln(c+1)-c$ and

$(c+1)\ln(c+1)-c=1 \Rightarrow \ln(c+1)=1 \Rightarrow$

$c+1=e \Rightarrow c=e-1 \approx 1.71828$

44. Substitution ; Formula 3

$$\int_0^c \frac{x}{x^2+1}\,dx \underset{\substack{u=x^2+1\\ du=2x\,dx}}{=} \frac{1}{2}\int_1^{c^2+1} \frac{1}{u}\,du =$$

$$\frac{1}{2}\left[\ln u\right]_1^{c^2+1} = \frac{1}{2}\ln(c^2+1)$$

$$\frac{1}{2}\ln(c^2+1)=1 \Rightarrow c^2+1=e^2 \Rightarrow$$

$$c=\sqrt{e^2-1} \approx 2.528$$

45. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 0.59601$

46. Integration by parts; partial fractions; Formula 17

a. $$\int \underset{\substack{u=\ln(x^3+1)\\ du=\frac{3x^2}{x^3+1}\\ dv=dx,\,v=x}}{\ln(x^3+1)\,dx} = x\ln(x^3+1) - 3\int \frac{x^3}{x^3+1}dx =$$

$$x\ln(x^3+1) - 3\int\left(1-\frac{1}{x^3+1}\right)dx =$$

$$x\ln(x^3+1) - 3x + 3\int\left(\frac{1}{(x+1)(x^2-x+1)}\right)dx$$

b. $$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} =$$

$$\frac{(A+B)x^2 + (B+C-A)x + (A+C)}{(x+1)(x^2-x+1)} \Rightarrow$$

$$A+C=1 \quad B+C=A \quad A=-B \quad \Rightarrow$$

$$A=\frac{1}{3} \quad B=-\frac{1}{3} \quad C=\frac{2}{3}.$$

Therefore

$$3\int\frac{1}{(x+1)(x^2-x+1)}dx =$$

$$\int\frac{1}{x+1}dx - \int\frac{x-2}{x^2-x+1}dx =$$

$$\ln|x+1| - \int\underset{\substack{u=x-\frac{1}{2}\\ du=dx}}{\frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}dx} =$$

$$\ln|x+1| - \int\frac{u-\frac{3}{2}}{u^2+\frac{3}{4}}du \underset{F17}{=}$$

$$\ln|x+1| - \frac{1}{2}\ln\left|x^2-x+1\right| + \sqrt{3}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}(x-\tfrac{1}{2})\right)$$

c. Summarizing

$$\int_0^c \ln(x^3+1)\,dx =$$

$$\left[\begin{array}{l} x\ln(x^3+1) - 3x + \ln(x+1) - \\ \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)\end{array}\right]_0^c =$$

$$\left\{\begin{array}{l} c(\ln(c^3+1)-3) + \ln\left(\frac{c+1}{\sqrt{c^2-c+1}}\right) + \\ \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(c-\frac{1}{2})\right) + \frac{\sqrt{3}\pi}{6}\end{array}\right\}$$

Using Newton's Method , with

$$G(c) = \left\{\begin{array}{l} c(\ln(c^3+1)-3) + \ln\left(\frac{c+1}{\sqrt{c^2-c+1}}\right) + \\ \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(c-\frac{1}{2})\right) + \frac{\sqrt{3}\pi}{6} - 1\end{array}\right\}$$

and $G'(c) = \ln(c^3+1)$ we get

n	1	2	3	4	5
a_n	2.0000	1.6976	1.6621	1.6615	1.6615

Therefore

$$\int_0^c \ln(x^3+1)\,dx = 1 \Rightarrow c \approx 1.6615$$

47. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 0.16668$

48. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 0.2509$

49. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 9.2365$

50. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 1.96$

51. $f(x)=8-x \quad g(x)=cx \quad a=0 \quad b=\frac{8}{c+1}$

a. $\int_a^b x(f(x)-g(x))\,dx=\int_0^{8/(c+1)} 8x-(c+1)x^2\,dx=$

$\left[4x^2-\left(\frac{c+1}{3}\right)x^3\right]_0^{8/(c+1)}=\frac{256}{(c+1)^2}-\frac{512}{3(c+1)^2}=$

$\frac{256}{3(c+1)^2}$

b. $\int_0^{c+1}(f(x)-g(x))\,dx=\int_0^{8/(c+1)} 8-(c+1)x\,dx=$

$\left[8x-\left(\frac{c+1}{2}\right)x^2\right]_0^{8/(c+1)}=\frac{64}{(c+1)}-\frac{32}{(c+1)}=$

$\frac{32}{(c+1)}$

c. $\bar{x}=\left(\frac{256}{3(c+1)^2}\right)\left(\frac{c+1}{32}\right)=\frac{8}{3(c+1)}$

$\bar{x}=2\Rightarrow\frac{8}{3(c+1)}=2\Rightarrow c=\frac{1}{3}$

52. $f(x)=c \quad g(x)=x \quad a=0 \quad b=c$

a. $\int_a^b x(f(x)-g(x))\,dx=\int_0^c cx-x^2\,dx=$

$\left[\frac{cx^2}{2}-\frac{x^3}{3}\right]_0^c=\frac{c^3}{6}$

b. $\int_a^b(f(x)-g(x))\,dx=\int_0^c(c-x)\,dx=$

$\left[cx-\frac{x^2}{2}\right]_0^c=\frac{c^2}{2}$

c. $\bar{x}=\left(\frac{c^3}{6}\right)\left(\frac{2}{c^2}\right)=\frac{c}{3}$

$\bar{x}=2\Rightarrow c=6$

53. $f(x)=6e^{-x/3} \quad g(x)=0 \quad a=0 \quad b=c$

a. $\int_a^b x(f(x)-g(x))\,dx=6\int_0^c \underset{u=x}{x}e^{-x/3}\,dx=$

$dv=e^{-x/3}$
$du=dx$
$v=-3e^{-x/3}$

$6\left[-3xe^{-x/3}\right]_0^c+18\int_0^c e^{-x/3}\,dx=$

$\left[-18e^{-x/3}(x+3)\right]_0^c=-18e^{-c/3}(c+3)+54$

b. $\int_a^b(f(x)-g(x))\,dx=\int_0^c 6e^{-x/3}\,dx=$

$-18\left(e^{-c/3}-1\right)$

c. For notational convenience, let $u=-18e^{-c/3}$; then

$\bar{x}=\frac{u(c+3)+54}{u+18}=\frac{cu}{u+18}+\frac{3(u+18)}{u+18}=$

$\frac{cu}{u+18}+3$

$\bar{x}=2\Rightarrow\frac{cu}{u+18}=-1\Rightarrow\frac{c}{1+\frac{18}{u}}=-1\Rightarrow$

$c=\frac{1}{e^{-c/3}}-1\Rightarrow\frac{1}{c+1}=e^{-c/3}$

Let

$h(c)=\frac{1}{c+1}-e^{-c/3}, \quad h'(c)=\frac{1}{3}e^{-c/3}-\frac{1}{(c+1)^2}$

and apply Newton's Method

n	1	2	3	4	5	6
a_n	2.0000	5.0000	5.6313	5.7103	5.7114	5.7114

$c\approx 5.7114$

54. $f(x)=c\sin\left(\frac{\pi x}{2c}\right) \quad g(x)=x \quad a=0 \quad b=c$

(Note: the value for b is obtained by setting $c\sin\left(\frac{\pi x}{2c}\right)=x$ This requires that $\frac{x}{c}$ be a zero for the function $h(u)=u-\sin\left(\frac{\pi}{2}u\right)$. Applying Newton's Method to h we discover that the zeros of h are -1, 0, and 1. Since we are dealing with positive values, we conclude that $\frac{x}{c}=1$ or $x=c$.)

a. $\int_a^b x(f(x)-g(x))\,dx = \int_0^c \left[cx\sin\left(\frac{\pi x}{2c}\right)-x^2\right]dx$

$= \underbrace{\int_0^c \;cx\sin\left(\frac{\pi x}{2c}\right)\;dx}_{u=\frac{\pi}{2c}x,\ du=\frac{\pi}{2c}dx} - \left[\frac{x^3}{3}\right]_0^c$

$= \int_0^{\pi/2} c\left(\frac{2c}{\pi}u\right)\sin u\left(\frac{2c}{\pi}\right)du - \left[\frac{c^3}{3}\right]$

$\underset{F40}{=} \frac{4c^3}{\pi^2}\left[\sin u - u\cos u\right]_0^{\pi/2} - \left[\frac{c^3}{3}\right] = \frac{4c^3}{\pi^2}-\frac{c^3}{3}$

$= c^3\left(\frac{4}{\pi^2}-\frac{1}{3}\right)$

b. $\int_a^b (f(x)-g(x))\,dx = \int_0^c\left[c\sin\left(\frac{\pi x}{2c}\right)-x\right]dx =$

$\left[-\frac{2c^2}{\pi}\cos\left(\frac{\pi x}{2c}\right)-\frac{x^2}{2}\right]_0^c = \frac{2c^2}{\pi}-\frac{c^2}{2} =$

$c^2\left(\frac{2}{\pi}-\frac{1}{2}\right)$

c. $\overline{x} = \dfrac{c^3\left(\dfrac{12-\pi^2}{3\pi^2}\right)}{c^2\left(\dfrac{4-\pi}{2\pi}\right)} = c\left[\dfrac{2(12-\pi^2)}{3\pi(4-\pi)}\right]$

$\overline{x} = 2 \Rightarrow c = \frac{3\pi(4-\pi)}{12-\pi^2} \approx 3.798$

55. a. $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

$\therefore \frac{d}{dx}erf(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$

b. $Si(x) = \int_0^x \frac{\sin t}{t}dt$

$\therefore \frac{d}{dx}Si(x) = \frac{\sin x}{x}$

56. a. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$

$\therefore \frac{d}{dx}S(x) = \sin\left(\frac{\pi x^2}{2}\right)$

b. $C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right)dt$

$\therefore \frac{d}{dx}C(x) = \cos\left(\frac{\pi x^2}{2}\right)$

57. a. (See problem 55 a.) . Since $erf'(x) > 0$ for all x, $erf(x)$ is increasing on $(0,\infty)$.

b. $erf''(x) = \frac{-4x}{\sqrt{\pi}}e^{-x^2}$ which is negative on $(0,\infty)$, so $erf(x)$ is not concave up anywhere on the interval.

58. a. (See problem 56 a.) Since $S'(x) = \sin\left(\frac{\pi}{2}x^2\right)$, $S'(x) > 0$ when $0 < \frac{\pi}{2}x^2 < \pi$ or $0 < x^2 < 2$; thus $S(x)$ is increasing on $\left(0,\sqrt{2}\right)$.

b. Since $S''(x) = \pi x\cos\left(\frac{\pi}{2}x^2\right)$, $S''(x) > 0$ when $0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \frac{\pi}{2}x^2 < 2\pi$, or $0 < x^2 < 1$ and $3 < x^2 < 4$. Thus $S(x)$ is concave up on $(0,1)\cup(\sqrt{3},2)$.

59. a. (See problem 56 b.) Since $C'(x) = \cos\left(\frac{\pi}{2}x^2\right)$, $C'(x) > 0$ when $0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \frac{\pi}{2}x^2 < 2\pi$; thus $C(x)$ is increasing on $(0,1)\cup(\sqrt{3},2)$.

b. Since $C''(x) = -\pi x\sin\left(\frac{\pi}{2}x^2\right)$, $C''(x) > 0$ when $\pi < \frac{\pi}{2}x^2 < 2\pi$. Thus $C(x)$ is concave up on $(\sqrt{2},2)$.

60. From problem 58 we know that $S(x)$ is concave up on $(0,1)$ and concave down on $(1,\sqrt{3})$ so the first point of inflection occurs at $x = 1$. Now $S(1) = \int_0^1 \sin\left(\frac{\pi}{2}t^2\right)dt$. Since the integral cannot be integrated directly, we must use some approximation method. Methods may vary but the result will be $S(1) \approx 0.43826$. Thus the first point of inflection is $(1, 0.43826)$

7.7 Chapter Review

Concepts Test

1. True: The resulting integrand will be of the form sin u.

2. True: The resulting integrand will be of the form $\frac{1}{a^2+u^2}$.

3. False: Try the substitution $u = x^4, du = 4x^3\,dx$

4. False: Use the substitution $u = x^2 - 3x + 5$, $du = (2x - 3)dx$.

5. True: The resulting integrand will be of the form $\frac{1}{a^2+u^2}$.

6. True: The resulting integrand will be of the form $\frac{1}{\sqrt{a^2-x^2}}$.

7. True: This integral is most easily solved with a partial fraction decomposition.

8. False: This improper fraction should be reduced first, then a partial fraction decomposition can be used.

9. True: Because both exponents are even positive integers, half-angle formulas are used.

10. False: Use the substitution $u = 1 + e^x,\ du = e^x dx$

11. False: Use the substitution $u = -x^2 - 4x, du = (-2x - 4)dx$

12. True: This substitution eliminates the radical.

13. True: Then expand and use the substitution $u = \sin x$, $du = \cos x\,dx$

14. True: The trigonometric substitution $x = 3\sin t$ will eliminate the radical.

15. True: Let $u = \ln x$ $\quad dv = x^2 dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{3}x^3$

16. False: Use a product identity.

17. False: $\frac{x^2}{x^2-1} = 1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

18. True: $\frac{x^2+2}{x(x^2-1)} = -\frac{2}{x} + \frac{3}{2(x+1)} + \frac{3}{2(x-1)}$

19. True: $\frac{x^2+2}{x(x^2+1)} = \frac{2}{x} + \frac{-x}{x^2+1}$

20. False: $\frac{x+2}{x^2(x^2-1)}$

$= -\frac{1}{x} - \frac{2}{x^2} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$

21. False: To complete the square, add $\frac{b^2}{4a}$.

22. False: Polynomials can be factored into products of linear and quadratic polynomials with real coefficients.

23. True: Polynomials with the same values for all x will have identical coefficients for like degree terms.

24. True: Let $u = 2x$; then $du = 2dx$ and

$$\int x^2\sqrt{25-4x^2}\,dx = \frac{1}{8}\int u^2\sqrt{25-u^2}\,du$$

which can be evaluated using Formula 57.

25. False: It can, however, be solved by the substitution $u = 25 - 4x^2$; then $du = -8x\,dx$ and

$$\int x\sqrt{25-4x^2}\,dx = -\frac{1}{8}\int\sqrt{u}\,du =$$

$$-\frac{1}{12}(25-4x^2)^{3/2} + C$$

26. True: Since (see Section 7.6, prob 55 a.)

$erf\,'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2} > 0$ for all x,

$erf\,(x)$ is an increasing function.

27. True: by the First Fundamental Theorem of Calculus.

28. False: Since (see Section 7.6, prob 55 b.)

$Si'(x) = \frac{\sin x}{x}$, which is negative on, say, $(\pi, 2\pi)$, $Si(x)$ will be decreasing on that same interval.

Sample Test Problems

1. $\int_0^4 \frac{t}{\sqrt{9+t^2}}dt = \left[\sqrt{9+t^2}\right]_0^4 = 5-3=2$

2. $\int \cot^2(2\theta)d\theta = \int \frac{\cos^2 2\theta}{\sin^2 2\theta}d\theta$

$= \int \frac{1-\sin^2 2\theta}{\sin^2 2\theta}d\theta = \int (\csc^2 2\theta - 1)d\theta$

$= -\frac{1}{2}\cot 2\theta - \theta + C$

3. $\int_0^{\pi/2} e^{\cos x}\sin x\,dx = \left[-e^{\cos x}\right]_0^{\pi/2} = e-1 \approx 1.718$

4. $\int_0^{\pi/4} x\sin 2x\,dx = \left[\frac{\sin 2x}{4} - \frac{x}{2}\cos 2x\right]_0^{\pi/4} = \frac{1}{4}$

(Use integration by parts with $u = x$, $dv = \sin 2x\,dx$.)

5. $\int \frac{y^3+y}{y+1}dy = \int\left(y^2 - y + 2 - \frac{2}{1+y}\right)dy$

$= \frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y - 2\ln|1+y| + C$

6. $\int \sin^3(2t)dt = \int [1-\cos^2(2t)]\sin(2t)dt$

$= -\frac{1}{2}\cos(2t) + \frac{1}{6}\cos^3(2t) + C$

7. $\int \frac{y-2}{y^2-4y+2}dy = \frac{1}{2}\int \frac{2y-4}{y^2-4y+2}dy$

$= \frac{1}{2}\ln\left|y^2-4y+2\right| + C$

8. $\int_0^{3/2} \frac{dy}{\sqrt{2y+1}} = \left[\sqrt{2y+1}\right]_0^{3/2} = 2-1=1$

9. $\int \frac{e^{2t}}{e^t-2}dt = e^t + 2\ln\left|e^t-2\right| + C$

(Use the substitution $u = e^t - 2$,

$du = e^t dt$

which gives the integral $\int \frac{u+2}{u}du$.)

10. $\int \frac{\sin x + \cos x}{\tan x}dx = \int\left(\cos x + \frac{\cos^2 x}{\sin x}\right)dx$

$= \int\left(\cos x + \frac{1-\sin^2 x}{\sin x}\right)dx$

$= \int(\cos x + \csc x - \sin x)dx$

$= \sin x + \ln\left|\csc x - \cot x\right| + \cos x + C$

(Use Formula 15 for $\int \csc x\,dx$.)

11. $\int \frac{dx}{\sqrt{16+4x-2x^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x-1}{3}\right) + C$

(Complete the square.)

12. $\int x^2 e^x dx = e^x(2-2x+x^2) + C$

Use integration by parts twice.

13. $y = \sqrt{\frac{2}{3}}\tan t,\ dy = \sqrt{\frac{2}{3}}\sec^2 t\,dt$

$\int \frac{dy}{\sqrt{2+3y^2}} = \int \frac{\sqrt{\frac{2}{3}}\sec^2 t}{\sqrt{2}\sec t}dt$

$= \frac{1}{\sqrt{3}}\int \sec t\,dt = \frac{1}{\sqrt{3}}\ln|\sec t + \tan t| + C_1$

$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{y^2+\frac{2}{3}}}{\sqrt{\frac{2}{3}}} + \frac{y}{\sqrt{\frac{2}{3}}}\right| + C_1$

$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{y^2+\frac{2}{3}}+y}{\sqrt{\frac{2}{3}}}\right| + C_1$

$= \frac{1}{\sqrt{3}}\ln\left|\sqrt{y^2+\frac{2}{3}}+y\right| + C$

Note that $\tan t = \frac{y}{\sqrt{\frac{2}{3}}}$, so $\sec t = \frac{\sqrt{y^2+\frac{2}{3}}}{\sqrt{\frac{2}{3}}}$.

14. $\int \frac{w^3}{1-w^2}dw = -\frac{1}{2}w^2 - \frac{1}{2}\ln\left|1-w^2\right| + C$

Divide the numerator by the denominator.

15. $\int \frac{\tan x}{\ln|\cos x|}dx = -\ln\left|\ln|\cos x|\right| + C$

Use the substitution $u = \ln|\cos x|$.

16. $\int \frac{3dt}{t^3-1} = \int \frac{1}{t-1}dt - \int \frac{t+2}{t^2+t+1}dt$

$= \int \frac{1}{t-1}dt - \frac{1}{2}\int \frac{2t+4}{t^2+t+1}dt$

$= \int \frac{1}{t-1}dt - \frac{1}{2}\int \frac{2t+1+3}{t^2+t+1}dt$

$= \int \frac{1}{t-1}dt - \frac{1}{2}\int \frac{2t+1}{t^2+t+1}dt - \frac{3}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}dt$

$= \ln|t-1| - \frac{1}{2}\ln\left|t^2+t+1\right| - \sqrt{3}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$

17. $\int \sinh x\,dx = \cosh x + C$

18. $u = \ln y,\ du = \frac{1}{y}dy$

$\int \frac{(\ln y)^5}{y}dy = \int u^5 du = \frac{1}{6}(\ln y)^6 + C$

19. $u = x \qquad dv = \cot^2 x\,dx$

$du = dx \qquad v = -\cot x - x$

$\int x\cot^2 x\,dx = -x\cot x - x^2 - \int(-\cot x - x)dx$

$= -x\cot x - \frac{1}{2}x^2 + \ln|\sin x| + C$

Use $\cot^2 x = \csc^2 x - 1$ for $\int \cot^2 x\,dx$.

20. $u = \sqrt{x},\ du = \frac{1}{2}x^{-1/2}dx$

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin u\,du$

$= -2\cos\sqrt{x} + C$

21. $u = \ln t^2, du = \frac{2}{t}dt$

$\int \frac{\ln t^2}{t}dt = \frac{[\ln(t^2)]^2}{4} + C$

22. $u = \ln(y^2+9) \qquad dv = dy$

$du = \frac{2y}{y^2+9}dy \qquad v = y$

$\int \ln(y^2+9)dy = y\ln(y^2+9) - \int \frac{2y^2}{y^2+9}dy$

$= y\ln(y^2+9) - \int\left(2 - \frac{18}{y^2+9}\right)dy$

$= y\ln(y^2+9) - 2y + 6\tan^{-1}\left(\frac{y}{3}\right) + C$

23. $\int e^{t/3}\sin 3t\,dt = \frac{-3e^{t/3}(9\cos 3t - \sin 3t)}{82} + C$

Use integration by parts twice.

24. $\int \frac{t+9}{t^3+9t}dt = \int \frac{1}{t}dt + \int \frac{-t+1}{t^2+9}dt$

$= \int \frac{1}{t}dt - \int \frac{t}{t^2+9}dt + \int \frac{1}{t^2+9}dt$

$= \ln|t| - \frac{1}{2}\ln\left|t^2+9\right| + \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

25. $\int \sin\frac{3x}{2}\cos\frac{x}{2}dx = -\frac{\cos x}{2} - \frac{\cos 2x}{4} + C$

Use a product identity.

26. $\int \cos^4\left(\frac{x}{2}\right)dx = \int\left(\frac{1+\cos x}{2}\right)^2 dx$

$= \frac{1}{4}\int dx + \frac{1}{4}\int 2\cos x\,dx + \frac{1}{4}\int \cos^2 x\,dx$

$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos x\,dx + \frac{1}{8}\int(1+\cos 2x)dx$

$= \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$

27. $\int \tan^3 2x\sec 2x\,dx = \frac{1}{2}\int(\sec^2 2x - 1)d(\sec 2x)$

$= \frac{1}{6}\sec^3(2x) - \frac{1}{2}\sec(2x) + C$

28. $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx$

$\int \frac{\sqrt{x}}{1+\sqrt{x}}dx = \int \frac{2x}{1+\sqrt{x}}\left(\frac{1}{2\sqrt{x}}dx\right) = 2\int \frac{u^2}{1+u}du$

$= 2\int \frac{(u+1)(u-1)+1}{u+1}du = 2\int\left(u - 1 + \frac{1}{u+1}\right)du$

$= 2\left(\frac{u^2}{2} - u + \ln|u+1|\right) + C$

$= x - 2\sqrt{x} + 2\ln\left(1+\sqrt{x}\right) + C$

29. $\int \tan^{3/2} x\sec^4 x\,dx = \int \tan^{3/2} x(1+\tan^2 x)\sec^2 x\,dx = \int \tan^{3/2} x\sec^2 x\,dx + \int \tan^{7/2} x\sec^2 x\,dx$

$= \frac{2}{5}\tan^{5/2} x + \frac{2}{9}\tan^{9/2} x + C$

30. $u = t^{1/6} + 1, (u-1)^6 = t, 6(u-1)^5 du = dt$

$\int \frac{dt}{t(t^{1/6}+1)} = \int \frac{6(u-1)^5 du}{(u-1)^6 u} = \int \frac{6du}{u(u-1)} = -6\int \frac{1}{u}du + 6\int \frac{1}{u-1}du = -6\ln\left|t^{1/6}+1\right| + 6\ln\left|t^{1/6}\right| + C$

31. $u = 9 - e^{2y}$, $du = -2e^{2y}dy$

$\int \frac{e^{2y}}{\sqrt{9-e^{2y}}}dy = -\frac{1}{2}\int u^{-1/2}du = -\sqrt{u} + C = -\sqrt{9-e^{2y}} + C$

32. $\int \cos^5 x\sqrt{\sin x}dx = \int (1-\sin^2 x)^2(\sin^{1/2} x)\cos x\,dx = \int \sin^{1/2} x\cos x\,dx - 2\int \sin^{5/2} x\cos x\,dx + \int \sin^{9/2} x\cos x\,dx$

$= \frac{2}{3}\sin^{3/2} x - \frac{4}{7}\sin^{7/2} x + \frac{2}{11}\sin^{11/2} x + C$

33. $\int e^{\ln(3\cos x)}dx = \int 3\cos x\,dx = 3\sin x + C$

34. $y = 3\sin t$, $dy = 3\cos t\,dt$

$\int \frac{\sqrt{9-y^2}}{y}dy = \int \frac{3\cos t}{3\sin t}\cdot 3\cos t\,dt$

$= 3\int \frac{1-\sin^2 t}{\sin t} = 3\int (\csc t - \sin t)\,dt$

$= 3\left[\ln\left|\csc t - \cot t\right| + \cos t\right] + C$

$= 3\ln\left|\frac{3}{y} - \frac{\sqrt{9-y^2}}{y}\right| + \sqrt{9-y^2} + C$

Note that $\sin t = \frac{y}{3}$, so $\csc t = \frac{3}{y}$ and

$\cot t = \frac{\sqrt{9-y^2}}{y}$.

35. $u = e^{4x}$, $du = 4e^{4x}dx$

$\int \frac{e^{4x}}{1+e^{8x}}dx = \frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1}(e^{4x}) + C$

36. $x = a\tan t$, $dx = a\sec^2 t\,dt$

$\int \frac{\sqrt{x^2+a^2}}{x^4}dx = \int \frac{a\sec t}{a^4\tan^4 t}a\sec^2 t\,dt$

$= \frac{1}{a^2}\int \frac{\sec^3 t}{\tan^4 t}dt = \frac{1}{a^2}\int \frac{\cos t}{\sin^4 t}dt$

$= \frac{1}{a^2}\left(-\frac{1}{3}\frac{1}{\sin^3 t}\right) + C = -\frac{1}{3a^2}\csc^3 t + C$

$= -\frac{1}{3a^2}\frac{(x^2+a^2)^{3/2}}{x^3} + C$

Note that $\tan t = \frac{x}{a}$, so $\csc t = \frac{\sqrt{x^2+a^2}}{x}$.

37. $u = \sqrt{w+5}, u^2 = w+5$, $2u\,du = dw$

$\int \frac{w}{\sqrt{w+5}}dw = 2\int (u^2-5)du = \frac{2}{3}u^3 - 10u + C$

$= \frac{2}{3}(w+5)^{3/2} - 10(w+5)^{1/2} + C$

38. $u = 1 + \cos t$, $du = -\sin t\,dt$

$\int \frac{\sin t\,dt}{\sqrt{1+\cos t}} = -\int \frac{du}{\sqrt{u}} = -2\sqrt{1+\cos t} + C$

39. $u = \cos^2 y, du = -2\cos y\sin y\,dy$

$\int \frac{\sin y\cos y}{9+\cos^4 y}dy = -\frac{1}{2}\int \frac{du}{9+u^2}$

$= -\frac{1}{6}\tan^{-1}\left(\frac{\cos^2 y}{3}\right) + C$

40. $\int \frac{dx}{\sqrt{1-6x-x^2}} = \int \frac{dx}{\sqrt{10-(x+3)^2}}$

$= \sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right) + C$

41. $\frac{4x^2+3x+6}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}$

$A = 1$, $B = 2$, $C = -1$, $D = 2$

$\int \frac{4x^2+3x+6}{x^2(x^2+3)}dx = \int \frac{1}{x}dx + 2\int \frac{1}{x^2}dx + \int \frac{-x+2}{x^2+3}dx$

$= \int \frac{1}{x}dx + 2\int \frac{1}{x^2}dx - \frac{1}{2}\int \frac{2x}{x^2+3}dx + 2\int \frac{1}{x^2+3}dx$

$= \ln|x| - \frac{2}{x} - \frac{1}{2}\ln\left|x^2+3\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

42. $x = 4\tan t,\ dx = 4\sec^2 t\, dt$

$\int \frac{dx}{(16+x^2)^{3/2}} = \frac{1}{16}\int \cos t\, dt = \frac{1}{16}\sin t + C = \frac{1}{16}\left(\frac{x}{\sqrt{x^2+16}}\right) + C = \frac{x}{16\sqrt{x^2+16}} + C$

43. **a.** $\frac{3-4x^2}{(2x+1)^3} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3}$

b. $\frac{7x-41}{(x-1)^2(2-x)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2-x} + \frac{D}{(2-x)^2} + \frac{E}{(2-x)^3}$

c. $\frac{3x+1}{(x^2+x+10)^2} = \frac{Ax+B}{x^2+x+10} + \frac{Cx+D}{(x^2+x+10)^2}$

d. $\frac{(x+1)^2}{(x^2-x+10)^2(1-x^2)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} + \frac{Ex+F}{x^2-x+10} + \frac{Gx+H}{(x^2-x+10)^2}$

e. $\frac{x^5}{(x+3)^4(x^2+2x+10)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} + \frac{Ex+F}{x^2+2x+10} + \frac{Gx+H}{(x^2+2x+10)^2}$

f. $\frac{(3x^2+2x-1)^2}{(2x^2+x+10)^3} = \frac{Ax+B}{2x^2+x+10} + \frac{Cx+D}{(2x^2+x+10)^2} + \frac{Ex+F}{(2x^2+x+10)^3}$

44. **a.** $V = \pi\int_1^2 \left[\frac{1}{\sqrt{3x-x^2}}\right]^2 dx = \pi\int_1^2 \frac{1}{3x-x^2}dx$

$\frac{1}{3x-x^2} = \frac{A}{x} + \frac{B}{3-x}$

$A = \frac{1}{3}, B = \frac{1}{3}$

$V = \pi\int_1^2 \frac{1}{3}\left(\frac{1}{x} + \frac{1}{3-x}\right)dx = \frac{\pi}{3}\Big[\ln|x| - \ln|3-x|\Big]_1^2 = \frac{\pi}{3}(\ln 2 + \ln 2) = \frac{2\pi}{3}\ln 2 \approx 1.4517$

b. $V = 2\pi\int_1^2 \frac{x}{\sqrt{3x-x^2}}\,dx = -\pi\int_1^2 \frac{-2x+3-3}{\sqrt{3x-x^2}}\,dx = -\pi\int_1^2 \frac{3-2x}{\sqrt{3x-x^2}}\,dx + 3\pi\int_1^2 \frac{1}{\sqrt{3x-x^2}}\,dx$

$= -\pi\left[2\sqrt{3x-x^2}\right]_1^2 + 3\pi\int_1^2 \frac{1}{\sqrt{\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx = \left[-2\pi\sqrt{3x-x^2} + 3\pi\sin^{-1}\left(\frac{2x-3}{3}\right)\right]_1^2$

$= -2\pi\sqrt{2} + 3\pi\sin^{-1}\frac{1}{3} + 2\pi\sqrt{2} - 3\pi\sin^{-1}\left(-\frac{1}{3}\right) = 6\pi\sin^{-1}\frac{1}{3} \approx 6.4058$

45. $y = \frac{x^2}{16},\ y' = \frac{x}{8}$

$L = \int_0^4 \sqrt{1+\left(\frac{x}{8}\right)^2}\,dx = \int_0^4 \sqrt{1+\frac{x^2}{64}}\,dx$

$x = 8 \tan t,\ dx = 8\sec^2 t$

$L = \int_0^{\tan^{-1}\frac{1}{2}} \sec t \cdot 8\sec^2 t\,dt = 8\int_0^{\tan^{-1}\frac{1}{2}} \sec^3 t\,dt = 4\left[\sec t\tan t + \ln\left|\sec t + \tan t\right|\right]_0^{\tan^{-1}\frac{1}{2}}$

$= 4\left[\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right) + \ln\left|\frac{1}{2}+\frac{\sqrt{5}}{2}\right|\right] = \sqrt{5} + 4\ln\left(\frac{1+\sqrt{5}}{2}\right) \approx 4.1609$ Note: Use Formula 28 for $\int \sec^3 t\,dt$.

46. $V = \pi\int_0^3 \frac{1}{(x^2+5x+6)^2}\,dx = \pi\int_0^3 \frac{1}{(x+3)^2(x+2)^2}\,dx$

$\frac{1}{(x+3)^2(x+2)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$

$A = 2, B = 1, C = –2, D = 1$

$V = \pi\int_0^3 \left[\frac{2}{x+3} + \frac{1}{(x+3)^2} - \frac{2}{x+2} + \frac{1}{(x+2)^2}\right]dx = \pi\left[2\ln|x+3| - \frac{1}{x+3} - 2\ln|x+2| - \frac{1}{x+2}\right]_0^3$

$= \pi\left[\left(2\ln 6 - \frac{1}{6} - 2\ln 5 - \frac{1}{5}\right) - \left(2\ln 3 - \frac{1}{3} - 2\ln 2 - \frac{1}{2}\right)\right] = \pi\left(\frac{7}{15} + 2\ln\frac{4}{5}\right) \approx 0.06402$

47. $V = 2\pi\int_0^3 \frac{x}{x^2+5x+6}\,dx$

$\frac{x}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$

$A = –2, B = 3$

$V = 2\pi\int_0^3 \left[-\frac{2}{x+2} + \frac{3}{x+3}\right]dx = 2\pi\left[-2\ln(x+2) + 3\ln(x+3)\right]_0^3$

$= 2\pi[(-2\ln 5 + 3\ln 6) - (-2\ln 2 + 3\ln 3)] = 2\pi\left(3\ln 2 + 2\ln\frac{2}{5}\right) = 2\pi\ln\frac{32}{25} \approx 1.5511$

48. $V = 2\pi\int_0^2 4x^2\sqrt{2-x}\,dx$

$u = 2 - x \qquad du = –dx$

$x = 2 - u \qquad dx = –du$

$V = 2\pi\int_2^0 4(2-u)^2\sqrt{u}(-du) = 8\pi\int_0^2 (4u^{1/2} - 4u^{3/2} + u^{5/2})\,du = 8\pi\left[\frac{8}{3}u^{3/2} - \frac{8}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_0^2$

$= 8\pi\left(\frac{16\sqrt{2}}{3} - \frac{32\sqrt{2}}{5} + \frac{16\sqrt{2}}{7}\right) = 8\pi\left(\frac{128\sqrt{2}}{105}\right) = \frac{1024\sqrt{2}\pi}{105} \approx 43.3287$

49. $V = 2\pi\int_0^{\ln 3} 2(e^x - 1)(\ln 3 - x)dx = 4\pi\int_0^{\ln 3}[(\ln 3)e^x - xe^x - \ln 3 + x]dx$

Note that $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$ by using integration by parts.

$$V = 4\pi\left[(\ln 3)e^x - xe^x + e^x - (\ln 3)x + \frac{1}{2}x^2\right]_0^{\ln 3} = 4\pi\left[\left(3\ln 3 - 3\ln 3 + 3 - (\ln 3)^2 + \frac{1}{2}(\ln 3)^2\right) - (\ln 3 + 1)\right]$$

$$= 4\pi\left[2 - \ln 3 - \frac{1}{2}(\ln 3)^2\right] \approx 3.7437$$

50. $A = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{18}{x^2\sqrt{x^2+9}}dx$

$x = 3\tan t$, $dx = 3\sec^2 t\,dt$

$$A = \int_{\pi/6}^{\pi/3} \frac{18}{27\tan^2 t\sec t}3\sec^2 t\,dt = 2\int_{\pi/6}^{\pi/3}\frac{\cos t}{\sin^2 t}dt = 2\left[-\frac{1}{\sin t}\right]_{\pi/6}^{\pi/3} = 2\left(-\frac{2}{\sqrt{3}} + 2\right) = 4\left(1 - \frac{1}{\sqrt{3}}\right) \approx 1.6906$$

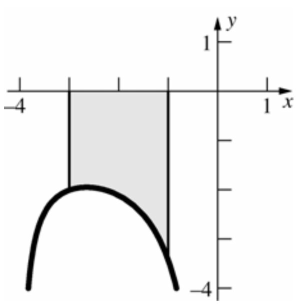
51. $A = -\int_{-6}^{0} \frac{t}{(t-1)^2}dt$

$$\frac{t}{(t-1)^2} = \frac{A}{(t-1)} + \frac{B}{(t-1)^2}$$

$A = 1$, $B = 1$

$$A = -\int_{-6}^{0}\left[\frac{1}{t-1} + \frac{1}{(t-1)^2}\right]dt = -\left[\ln|t-1| - \frac{1}{t-1}\right]_{-6}^{0} = -\left[(0+1) - \left(\ln 7 + \frac{1}{7}\right)\right] = \ln 7 - \frac{6}{7} \approx 1.0888$$

52.



$$V = \pi\int_{-3}^{-1}\left(\frac{6}{x\sqrt{x+4}}\right)^2 dx = \pi\int_{-3}^{-1}\frac{36}{x^2(x+4)}dx$$

$$\frac{36}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$A = -\frac{9}{4}, B = 9, C = \frac{9}{4}$$

$$V = \pi\int_{-3}^{-1}\left[-\frac{9}{4x} + \frac{9}{x^2} + \frac{9}{4(x+4)}\right]dx = \frac{9\pi}{4}\int_{-3}^{-1}\left(-\frac{1}{x} + \frac{4}{x^2} + \frac{1}{x+4}\right)dx = \frac{9\pi}{4}\left[-\ln|x| - \frac{4}{x} + \ln|x+4|\right]_{-3}^{-1}$$

$$= \frac{9\pi}{4}\left[(4 + \ln 3) - \left(-\ln 3 + \frac{4}{3}\right)\right] = \frac{9\pi}{4}\left(\frac{8}{3} + 2\ln 3\right) = \frac{3\pi}{2}(4 + 3\ln 3) \approx 34.3808$$

53. The length is given by

$$\int_{\pi/6}^{\pi/3}\sqrt{1 + [f'(x)]^2}\,dx = \int_{\pi/6}^{\pi/3}\sqrt{1 + \frac{\cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/6}^{\pi/3}\sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/6}^{\pi/3}\frac{1}{\sin x}dx = \int_{\pi/6}^{\pi/3}\csc x\,dx$$

$$= \left[\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3} = \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ln\left|2 - \sqrt{3}\right| = \ln\left(\frac{1}{\sqrt{3}}\right) - \ln(2 - \sqrt{3}) = \ln\left(\frac{2\sqrt{3}+3}{3}\right) \approx 0.768$$

54. a. First substitute $u = 2x,\ du = 2\,dx$ to obtain $\int \frac{\sqrt{81-4x^2}}{x}\,dx = \int \frac{\sqrt{81-u^2}}{u}\,du$, then use Formula 55:

$$\int \frac{\sqrt{81-4x^2}}{x}\,dx = \sqrt{81-4x^2} - 9\ln\left|\frac{9+\sqrt{81-4x^2}}{2x}\right| + C$$

b. First substitute $u = e^x,\ du = e^x\,dx$ to obtain $\int e^x\left(9-e^{2x}\right)^{3/2}dx = \int \left(9-u^2\right)^{3/2}du$, then use Formula 62:

$$\int e^x\left(9-e^{2x}\right)^{3/2}dx = \frac{e^x}{8}\left(45-2e^{2x}\right)\sqrt{9-e^{2x}} + \frac{243}{8}\sin^{-1}\left(\frac{e^x}{3}\right) + C$$

55. a. First substitute $u = \sin x,\ du = \cos x\,dx$ to obtain $\int \cos x\sqrt{\sin^2 x+4}\,dx = \int \sqrt{u^2+4}\,du$, then use Formula 44:

$$\int \cos x\sqrt{\sin^2 x+4}\,dx = \frac{\sin x}{2}\sqrt{\sin^2 x+4} + 2\ln\left|\sin x + \sqrt{\sin^2 x+4}\right| + C$$

b. First substitute $u = 2x$, $du = 2dx$ to obtain $\int \frac{1}{1-4x^2}dx = \frac{1}{2}\int \frac{du}{1-u^2}$.

Then use Formula 18: $\int \frac{1}{1-4x^2}dx = \frac{1}{4}\ln\left|\frac{2x+1}{2x-1}\right| + C$.

56. By the First Fundamental Theorem of Calculus,

$$Si'(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad Si''(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

57. Using partial fractions (see Section 7.6, prob 46 b.):

$$\frac{1}{1+x^3} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{(A+B)x^2+(B+C-A)x+(A+C)}{(x+1)(x^2-x+1)} \Rightarrow$$

$$A+C=1 \quad B+C=A \quad A=-B \quad \Rightarrow A=\frac{1}{3} \quad B=-\frac{1}{3} \quad C=\frac{2}{3}.$$

Therefore:

$$\int \frac{1}{1+x^3}dx = \frac{1}{3}\left[\int \frac{1}{x+1}dx - \int \frac{x-2}{x^2-x+1}dx\right] = \frac{1}{3}\left[\ln|x+1| - \underbrace{\int \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}dx}_{u=x-\frac{1}{2},\ du=dx}\right]$$

$$= \left[\ln|x+1| - \int \frac{u-\frac{3}{2}}{u^2+\frac{3}{4}}du\right] \underset{F17}{=} \frac{1}{3}\left[\ln\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + \sqrt{3}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}(x-\tfrac{1}{2})\right)\right]$$

so $\int_0^c \frac{1}{1+x^3}dx = \frac{1}{3}\left[\ln\left|\frac{c+1}{\sqrt{c^2-c+1}}\right| + \sqrt{3}\left[\tan^{-1}\left(\tfrac{2}{\sqrt{3}}(c-\tfrac{1}{2})\right) + \frac{\pi}{6}\right]\right]$.

Letting $G(c) = \frac{1}{3}\left[\ln\left|\frac{c+1}{\sqrt{c^2-c+1}}\right| + \sqrt{3}\left[\tan^{-1}\left(\tfrac{2}{\sqrt{3}}(c-\tfrac{1}{2})\right) + \frac{\pi}{6}\right]\right] - 0.5$ and $G'(c) = \frac{1}{1+c^3}$ we apply Newton's

Method to find the value of c such that $\int_0^c \frac{1}{1+x^3}dx = 0.5$:

n	1	2	3	4	5	6
a_n	1.0000	0.3287	0.5090	0.5165	0.5165	0.5165

Thus $c \approx 0.5165$.

Review and Preview Problems

1. $\lim\limits_{x\to 2}\dfrac{x^2+1}{x^2-1}=\dfrac{2^2+1}{2^2-1}=\dfrac{5}{3}$

2. $\lim\limits_{x\to 3}\dfrac{2x+1}{x+5}=\dfrac{2(3)+1}{3+5}=\dfrac{7}{8}$

3. $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=\lim\limits_{x\to 3}\dfrac{(x+3)(x-3)}{x-3}=$
$\lim\limits_{x\to 3}(x+3)=3+3=6$

4. $\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x-2}=\lim\limits_{x\to 2}\dfrac{(x-2)(x-3)}{x-2}=$
$\lim\limits_{x\to 2}(x-3)=2-3=-1$

5. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}=\lim\limits_{x\to 0}\dfrac{2\sin x\cos x}{x}=$
$\lim\limits_{x\to 0}2\left(\dfrac{\sin x}{x}\right)\cos x=2(1)(1)=2$

6. $\lim\limits_{x\to 0}\dfrac{\tan 3x}{x}=\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{\cos 3x}\right)\left(\dfrac{3}{3x}\right)=$
$\lim\limits_{x\to 0}3\left(\dfrac{\sin 3x}{3x}\right)\left(\dfrac{1}{\cos 3x}\right)=3(1)(1)=3$

7. $\lim\limits_{x\to\infty}\dfrac{x^2+1}{x^2-1}=\lim\limits_{x\to\infty}\dfrac{1+\dfrac{1}{x^2}}{1-\dfrac{1}{x^2}}=\dfrac{1+0}{1-0}=1$ or:

$\lim\limits_{x\to\infty}\dfrac{x^2+1}{x^2-1}=\lim\limits_{x\to\infty}1+\dfrac{2}{x^2-1}=1+0=1$

8. $\lim\limits_{x\to\infty}\dfrac{2x+1}{x+5}=\lim\limits_{x\to\infty}\dfrac{2+\dfrac{1}{x}}{1+\dfrac{5}{x}}=\dfrac{2+0}{1+0}=2$

9. $\lim\limits_{x\to\infty}e^{-x}=\lim\limits_{x\to\infty}\dfrac{1}{e^x}=0$

10. $\lim\limits_{x\to\infty}e^{-x^2}=\lim\limits_{x\to\infty}\dfrac{1}{e^{x^2}}=0$

11. $\lim\limits_{x\to\infty}e^{2x}=\infty$ (has no finite value)

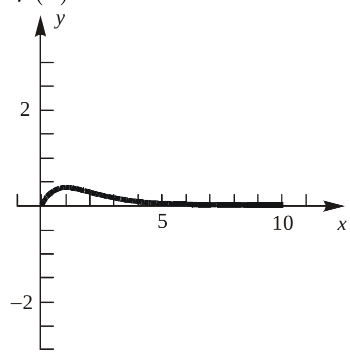
12. $\lim\limits_{x\to-\infty}e^{-2x}\underset{(u=-x)}{=}\lim\limits_{u\to\infty}e^{2u}=\infty$
(has no finite value)

13. $\lim\limits_{x\to\infty}\tan^{-1}x=\dfrac{\pi}{2}$

14. Note that, if $\theta=\sec^{-1}x$, then
$\sec\theta=x\Rightarrow\cos\theta=\dfrac{1}{x}\Rightarrow\theta=\cos^{-1}\dfrac{1}{x}$. Hence

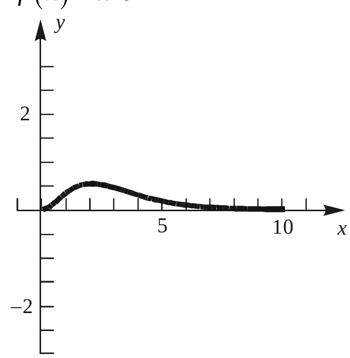
$$\lim_{x\to\infty}\sec^{-1}x=\lim_{x\to\infty}\cos^{-1}\frac{1}{x}=1$$

15. $f(x)=xe^{-x}$



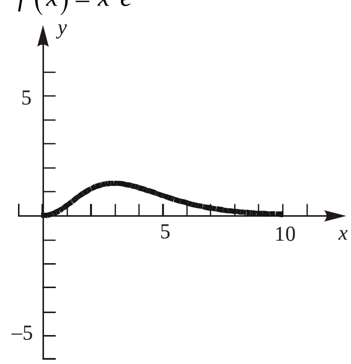
We would conjecture $\lim\limits_{x\to\infty}xe^{-x}=0$.

16. $f(x)=x^2e^{-x}$



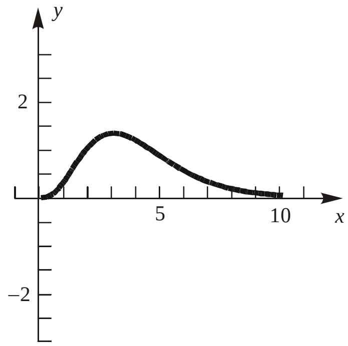
We would conjecture $\lim\limits_{x\to\infty}x^2e^{-x}=0$.

17. $f(x)=x^3e^{-x}$



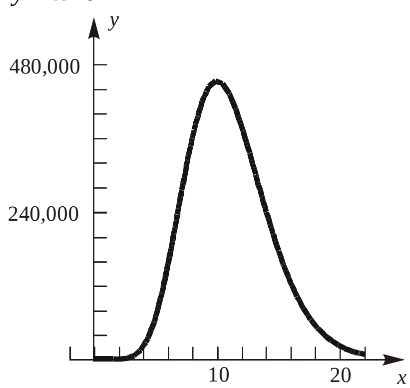
We would conjecture $\lim\limits_{x\to\infty}x^3e^{-x}=0$.

18. $f(x) = x^4 e^{-x}$



We would conjecture $\lim_{x\to\infty} x^{10}e^{-x} = 0$.

19. $y = x^{10}e^{-x}$



We would conjecture $\lim_{x\to\infty} x^2 e^{-x} = 0$.

20. Based on the results from problems 15-19, we would conjecture

$\lim_{x\to\infty} x^n e^{-x} = 0$

21. $\int_0^a e^{-x}\,dx = \left[-e^{-x}\right]_0^a = 1 - e^{-a}$

a	1	2	4	8	16
$1-e^{-a}$	0.632	0.865	0.982	0.9997	0.9999+

22. $\int_0^a xe^{-x^2}\,dx \underset{\substack{u=-x^2 \\ du=-2x\,dx}}{=} -\frac{1}{2}\left[e^{-x^2}\right] = 1 - \frac{e^{-a^2}}{2}$

a	$1-\frac{1}{2e^{a^2}}$
1	0.81606028
2	0.93233236
4	0.999999944
8	$1-(8.02\times10^{-29})$
16	1

23. $\int_0^a \frac{x}{1+x^2}\,dx \underset{\substack{u=x^2 \\ du=2x\,dx}}{=} \frac{1}{2}\left[\ln(1+x^2)\right]_0^a = \ln\left(\sqrt{1+a^2}\right)$

a	1	2	4	8	16
$\ln\left(\sqrt{1+a^2}\right)$	0.3466	0.8047	1.4166	2.0872	2.7745

24. $\int_0^a \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^a = \ln(1+a)$

a	1	2	4	8	16
$\ln(1+a)$	0.6931	1.0986	1.6094	2.1972	2.8332

25. $\int_1^a \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^a = 1 - \frac{1}{a}$

a	2	4	8	16
$1-\frac{1}{a}$	0.5	0.75	0.875	0.9375

26. $\int_1^a \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^a = \frac{1}{2}\left[1 - \frac{1}{a^2}\right]$

a	2	4	8	16
$\frac{1}{2}\left[1-\frac{1}{a^2}\right]$	0.375	0.46875	0.4921875	0.498046875

27. $\int_a^4 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_a^4 = 4 - 2\sqrt{a}$

a	1	$1/2$	$1/4$	$1/8$	$1/16$
$4-2\sqrt{a}$	2	2.58579	3	3.29289	3.5

28. $\int_a^4 \frac{1}{x}\,dx = \left[\ln x\right]_a^4 = \ln\frac{4}{a}$

a	1	$1/2$	$1/4$	$1/8$	$1/16$
$\ln\frac{4}{a}$	1.38629	2.07944	2.77259	3.46574	4.15888

CHAPTER 8 Indeterminate Forms and Improper Integrals

8.1 Concepts Review

1. $\lim_{x\to a} f(x)$; $\lim_{x\to a} g(x)$

2. $\dfrac{f'(x)}{g'(x)}$

3. $\sec^2 x$; 1; $\lim_{x\to 0} \cos x \neq 0$

4. Cauchy's Mean Value

Problem Set 8.1

1. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2x-\sin x}{x} = \lim_{x\to 0}\frac{2-\cos x}{1} = 1$$

2. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\pi/2}\frac{\cos x}{\pi/2-x} = \lim_{x\to\pi/2}\frac{-\sin x}{-1} = 1$$

3. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x-\sin 2x}{\tan x} = \lim_{x\to 0}\frac{1-2\cos 2x}{\sec^2 x} = \frac{1-2}{1} = -1$$

4. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\tan^{-1} 3x}{\sin^{-1} x} = \lim_{x\to 0}\frac{\frac{3}{1+9x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{3}{1} = 3$$

5. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to -2}\frac{x^2+6x+8}{x^2-3x-10} = \lim_{x\to -2}\frac{2x+6}{2x-3}$$
$$= \frac{2}{-7} = -\frac{2}{7}$$

6. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x^3-3x^2+x}{x^3-2x} = \lim_{x\to 0}\frac{3x^2+6x+1}{3x^2-2} = \frac{1}{-2} = -\frac{1}{2}$$

7. The limit is not of the form $\frac{0}{0}$.

As $x\to 1^-$, $x^2-2x+2\to 1$, and $x^2-1\to 0^-$ so

$$\lim_{x\to 1^-}\frac{x^2-2x+2}{x^2+1} = -\infty$$

8. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1}\frac{\ln x^2}{x^2-1} = \lim_{x\to 1}\frac{\frac{1}{x^2}2x}{2x} = \lim_{x\to 1}\frac{1}{x^2} = 1$$

9. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\pi/2}\frac{\ln(\sin x)^3}{\pi/2-x} = \lim_{x\to\pi/2}\frac{\frac{1}{\sin^3 x}3\sin^2 x\cos x}{-1}$$
$$= \frac{0}{-1} = 0$$

10. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{2\sin x} = \lim_{x\to 0}\frac{e^x+e^{-x}}{2\cos x} = \frac{2}{2} = 1$$

11. The limit is of the form $\frac{0}{0}$.

$$\lim_{t\to 1}\frac{\sqrt{t}-t^2}{\ln t} = \lim_{t\to 1}\frac{\frac{1}{2\sqrt{t}}-2t}{\frac{1}{t}} = \frac{-\frac{3}{2}}{1} = -\frac{3}{2}$$

12. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^+}\frac{7^{\sqrt{x}}-1}{2^{\sqrt{x}}-1} = \lim_{x\to 0^+}\frac{\frac{7^{\sqrt{x}}\ln 7}{2\sqrt{x}}}{\frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}}} = \lim_{x\to 0^+}\frac{7^{\sqrt{x}}\ln 7}{2^{\sqrt{x}}\ln 2}$$
$$= \frac{\ln 7}{\ln 2} \approx 2.81$$

13. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\ln\cos 2x}{7x^2} = \lim_{x\to 0}\frac{\frac{-2\sin 2x}{\cos 2x}}{14x} = \lim_{x\to 0}\frac{-2\sin 2x}{14x\cos 2x}$$
$$= \lim_{x\to 0}\frac{-4\cos 2x}{14\cos 2x-28x\sin 2x} = \frac{-4}{14-0} = -\frac{2}{7}$$

14. The limit is of the form $\frac{0}{0}.$

$$\lim_{x\to 0^-}\frac{3\sin x}{\sqrt{-x}}=\lim_{x\to 0^-}\frac{3\cos x}{-\frac{1}{2\sqrt{-x}}}$$
$$=\lim_{x\to 0^-}-6\sqrt{-x}\cos x=0$$

15. The limit is of the form $\frac{0}{0}.$ (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to 0}\frac{\tan x-x}{\sin 2x-2x}=\lim_{x\to 0}\frac{\sec^2 x-1}{2\cos 2x-2}$$
$$=\lim_{x\to 0}\frac{2\sec^2 x\tan x}{-4\sin 2x}=\lim_{x\to 0}\frac{2\sec^4 x+4\sec^2 x\tan^2 x}{-8\cos 2x}$$
$$=\frac{2+0}{-8}=-\frac{1}{4}$$

16. The limit is of the form $\frac{0}{0}.$ (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^2\sin x}=\lim_{x\to 0}\frac{\cos x-\sec^2 x}{2x\sin x+x^2\cos x}$$
$$=\lim_{x\to 0}\frac{-\sin x-2\sec^2 x\tan x}{2\sin x+4x\cos x-x^2\sin x}$$
$$=\lim_{x\to 0}\frac{-\cos x-2\sec^4 x-4\sec^2 x\tan^2 x}{6\cos x-x^2\cos x-6x\sin x}$$
$$=\frac{-1-2-0}{6-0-0}=-\frac{1}{2}$$

17. The limit is of the form $\frac{0}{0}.$ (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+}\frac{x^2}{\sin x-x}=\lim_{x\to 0^+}\frac{2x}{\cos x-1}=\lim_{x\to 0^+}\frac{2}{-\sin x}$$

This limit is not of the form $\frac{0}{0}.$ As $x\to 0^+, 2\to 2,$ and $-\sin x\to 0^-,$ so $\lim_{x\to 0^+}\frac{2}{\sin x}=-\infty.$

18. The limit is of the form $\frac{0}{0}.$ (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{e^x-\ln(1+x)-1}{x^2}=\lim_{x\to 0}\frac{e^x-\frac{1}{1+x}}{2x}$$
$$=\lim_{x\to 0}\frac{e^x+\frac{1}{(1+x)^2}}{2}=\frac{1+1}{2}=1$$

19. The limit is of the form $\frac{0}{0}.$ (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\tan^{-1}x-x}{8x^3}=\lim_{x\to 0}\frac{\frac{1}{1+x^2}-1}{24x^2}=\lim_{x\to 0}\frac{\frac{-2x}{(1+x^2)^2}}{48x}$$
$$=\lim_{x\to 0}-\frac{1}{24(1+x^2)^2}=-\frac{1}{24}$$

20. The limit is of the form $\frac{0}{0}.$ (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\cosh x-1}{x^2}=\lim_{x\to 0}\frac{\sinh x}{2x}=\lim_{x\to 0}\frac{\cosh x}{2}=\frac{1}{2}$$

21. The limit is of the form $\frac{0}{0}.$ (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+}\frac{1-\cos x-x\sin x}{2-2\cos x-\sin^2 x}$$
$$=\lim_{x\to 0^+}\frac{-x\cos x}{2\sin x-2\cos x\sin s}$$
$$=\lim_{x\to 0^+}\frac{x\sin x-\cos x}{2\cos x-2\cos^2 x+2\sin^2 x}$$

This limit is not of the form $\frac{0}{0}.$

As $x\to 0^+$, $x\sin x-\cos x\to -1$ and $2\cos x-2\cos^2 x+2\sin^2 x\to 0^+$, so

$$\lim_{x\to 0^+}\frac{x\sin x-\cos x}{2\cos x-2\cos^2 x+2\sin^2 x}=-\infty$$

22. The limit is of the form $\frac{0}{0}.$

$$\lim_{x\to 0^-}\frac{\sin x+\tan x}{e^x+e^{-x}-2}=\lim_{x\to 0^-}\frac{\cos x+\sec^2 x}{e^x-e^{-x}}$$

This limit is not of the form $\frac{0}{0}.$

As $x\to 0^-$, $\cos x+\sec^2 x\to 2,$ and $e^x-e^{-x}\to 0^-,$ so $\lim_{x\to 0^-}\frac{\cos x+\sec^2 x}{e^x-e^{-x}}=-\infty.$

23. The limit is of the form $\frac{0}{0}.$

$$\lim_{x\to 0}\frac{\int_0^x\sqrt{1+\sin t}\,dt}{x}=\lim_{x\to 0}\sqrt{1+\sin x}=1$$

24. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^+}\frac{\int_0^x \sqrt{t}\cos t\,dt}{x^2}=\lim_{x\to 0^+}\frac{\sqrt{x}\cos x}{2x}$$

$$=\lim_{x\to 0^+}\frac{\cos x}{2\sqrt{x}}=\infty$$

25. It would not have helped us because we proved $\lim_{x\to 0}\frac{\sin x}{x}=1$ in order to find the derivative of $\sin x$.

26. Note that $\sin(1/0)$ is undefined (not zero), so l'Hôpital's Rule cannot be used.

As $x\to 0, \frac{1}{x}\to\infty$ and $\sin\left(\frac{1}{x}\right)$ oscillates rapidly between –1 and 1, so

$$\lim_{x\to 0}\left|\frac{x^2\sin\left(\frac{1}{x}\right)}{\tan x}\right|\le\lim_{x\to 0}\frac{x^2}{\tan x}.$$

$$\frac{x^2}{\tan x}=\frac{x^2\cos x}{\sin x}$$

$$\lim_{x\to 0}\frac{x^2\cos x}{\sin x}=\lim_{x\to 0}\left[\left(\frac{x}{\sin x}\right)x\cos x\right]=0.$$

Thus, $\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\tan x}=0$.

A table of values or graphing utility confirms this.

27. a. $\overline{OB}=\cos t, \overline{BC}=\sin t$ and $\overline{AB}=1-\cos t$, so the area of triangle ABC is $\frac{1}{2}\sin t(1-\cos t)$.

The area of the sector COA is $\frac{1}{2}t$ while the area of triangle COB is $\frac{1}{2}\cos t\sin t$, thus the area of the curved region ABC is $\frac{1}{2}(t-\cos t\sin t)$.

$$\lim_{t\to 0^+}\frac{\text{area of triangle } ABC}{\text{area of curved region } ABC}=\lim_{t\to 0^+}\frac{\frac{1}{2}\sin t(1-\cos t)}{\frac{1}{2}(t-\cos t\sin t)}$$

$$=\lim_{t\to 0^+}\frac{\sin t(1-\cos t)}{t-\cos t\sin t}=\lim_{t\to 0^+}\frac{\cos t-\cos^2 t+\sin^2 t}{1-\cos^2 t+\sin^2 t}=\lim_{t\to 0^+}\frac{4\sin t\cos t-\sin t}{4\cos t\sin t}=\lim_{t\to 0^+}\frac{4\cos t-1}{4\cos t}=\frac{3}{4}$$

(L’Hôpital’s Rule was applied twice.)

b. The area of the sector BOD is $\frac{1}{2}t\cos^2 t$, so the area of the curved region BCD is $\frac{1}{2}\cos t\sin t-\frac{1}{2}t\cos^2 t$.

$$\lim_{t\to 0^+}\frac{\text{area of curved region } BCD}{\text{area of curved region } ABC}=\lim_{t\to 0^+}\frac{\frac{1}{2}\cos t(\sin t-t\cos t)}{\frac{1}{2}(t-\cos t\sin t)}$$

$$=\lim_{t\to 0^+}\frac{\cos t(\sin t-t\cos t)}{t-\sin t\cos t}=\lim_{t\to 0^+}\frac{\sin t(2t\cos t-\sin t)}{1-\cos^2 t+\sin^2 t}=\lim_{t\to 0^+}\frac{2t(\cos^2 t-\sin^2 t)}{4\cos t\sin t}=\lim_{t\to 0^+}\frac{t(\cos^2 t-\sin^2 t)}{2\cos t\sin t}$$

$$=\lim_{t\to 0^+}\frac{\cos^2 t-4t\cos t\sin t-\sin^2 t}{2\cos^2 t-2\sin^2 t}=\frac{1-0-0}{2-0}=\frac{1}{2}$$

(L’Hôpital’s Rule was applied three times.)

28. a. Note that $\angle DOE$ has measure t radians. Thus the coordinates of E are (cos$t$, sin$t$). Also, slope $\overline{BC}$ = slope $\overline{CE}$. Thus,

$$\frac{0-y}{(1-t)-0}=\frac{\sin t-0}{\cos t-(1-t)}$$

$$-y=\frac{(1-t)\sin t}{\cos t+t-1}$$

$$y=\frac{(t-1)\sin t}{\cos t+t-1}$$

$$\lim_{t\to 0^+} y=\lim_{t\to 0^+}\frac{(t-1)\sin t}{\cos t+t-1}$$

This limit is of the form $\frac{0}{0}$.

$$\lim_{t\to 0^+}\frac{(t-1)\sin t}{\cos t+t-1}=\lim_{t\to 0^+}\frac{\sin t+(t-1)\cos t}{-\sin t+1}=\frac{0+(-1)(1)}{-0+1}=-1$$

b. Slope $\overline{AF}$ = slope $\overline{EF}$. Thus,

$$\frac{t}{1-x}=\frac{t-\sin t}{1-\cos t}$$

$$\frac{t(1-\cos t)}{t-\sin t}=1-x$$

$$x=1-\frac{t(1+\cos t)}{t-\sin t}$$

$$x=\frac{t\cos t-\sin t}{t-\sin t}$$

$$\lim_{t\to 0^+} x=\lim_{t\to 0^+}\frac{t\cos t-\sin t}{t-\sin t}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{t\to 0^+}\frac{t\cos t-\sin t}{t-\sin t}=\lim_{t\to 0^+}\frac{-t\sin t}{1-\cos t}$$

$$=\lim_{t\to 0^+}\frac{-\sin t-t\cos t}{\sin t}=\lim_{t\to 0^+}\frac{t\sin t-2\cos t}{\cos t}=\frac{0-2}{1}=-2$$

29. By l'Hôpital's Rule $\left(\frac{0}{0}\right)$, we have $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{e^x-1}{x}=\lim_{x\to 0^+}\frac{e^x}{1}=1$ and

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{e^x-1}{x}=\lim_{x\to 0^-}\frac{e^x}{1}=1 \quad \text{so we define} \quad f(0)=1.$$

30. By l'Hôpital's Rule $\left(\frac{0}{0}\right)$, we have $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{\ln x}{x-1}=\lim_{x\to 1^+}\frac{1/x}{1}=1$ and

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{\ln x}{x-1}=\lim_{x\to 1^-}\frac{1/x}{1}=1 \quad \text{so we define} \quad f(1)=1.$$

31. A should approach $4\pi b^2$, the surface area of a sphere of radius b.

$$\lim_{a\to b^+}\left[2\pi b^2+\frac{2\pi a^2 b\arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]=2\pi b^2+2\pi b\lim_{a\to b^+}\frac{a^2\arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}$$

Focusing on the limit, we have

$$\lim_{a\to b^+}\frac{a^2\arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}=\lim_{a\to b^+}\frac{2a\arcsin\frac{\sqrt{a^2-b^2}}{a}+a^2\left(\frac{b}{a\sqrt{a^2-b^2}}\right)}{\frac{a}{\sqrt{a^2-b^2}}}=\lim_{a\to b^+}\left(2\sqrt{a^2-b^2}\arcsin\frac{\sqrt{a^2-b^2}}{a}+b\right)=b.$$

Thus, $\lim_{a\to b^+}A=2\pi b^2+2\pi b(b)=4\pi b^2$.

32. In order for l'Hôpital's Rule to be of any use, $a(1)^4+b(1)^3+1=0$, so $b=-1-a$.

Using l'Hôpital's Rule,

$$\lim_{x\to 1}\frac{ax^4+bx^3+1}{(x-1)\sin\pi x}=\lim_{x\to 1}\frac{4ax^3+3bx^2}{\sin\pi x+\pi(x-1)\cos\pi x}$$

To use l'Hôpital's Rule here,

$4a(1)^3+3b(1)^2=0$, so $4a+3b=0$, hence $a=3$, $b=-4$.

$$\lim_{x\to 1}\frac{3x^4-4x^3+1}{(x-1)\sin\pi x}=\lim_{x\to 1}\frac{12x^3-12x^2}{\sin\pi x+\pi(x-1)\cos\pi x}=\lim_{x\to 1}\frac{36x^2-24x}{2\pi\cos\pi x-\pi^2(x-1)\sin\pi x}=\frac{12}{-2\pi}=-\frac{6}{\pi}$$

$a=3,\ b=-4,\ c=-\frac{6}{\pi}$

33. If $f'(a)$ and $g'(a)$ both exist, then f and g are both continuous at a. Thus, $\lim_{x\to a}f(x)=0=f(a)$ and $\lim_{x\to a}g(x)=0=g(a)$.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}$$

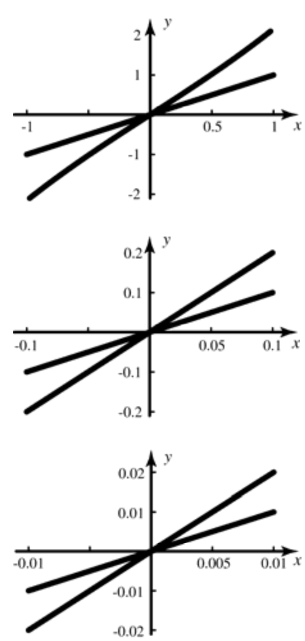
34. $\lim_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}}{x^4}=\frac{1}{24}$

35. $\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}-\frac{x^3}{6}}{x^4}=\frac{1}{24}$

36. $\lim_{x\to 0}\frac{1-\cos(x^2)}{x^3\sin x}=\frac{1}{2}$

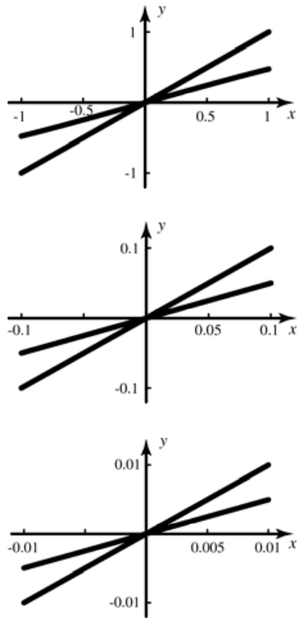
37. $\lim_{x\to 0}\frac{\tan x-x}{\arcsin x-x}=\lim_{x\to 0}\frac{\sec^2 x-1}{\frac{1}{\sqrt{1-x^2}}-1}=2$

38.



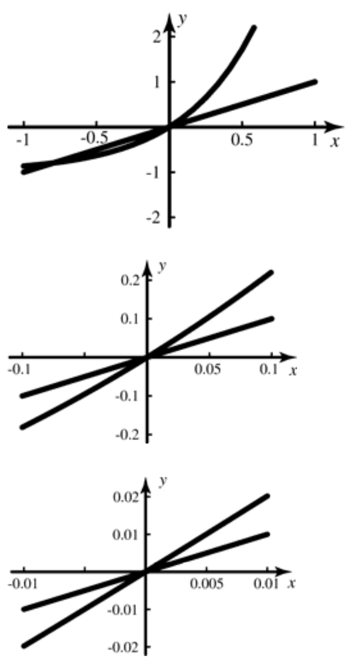
The slopes are approximately $0.02/0.01=2$ and $0.01/0.01=1$. The ratio of the slopes is therefore $2/1=2$, indicating that the limit of the ratio should be about 2. An application of l'Hopital's Rule confirms this.

39.



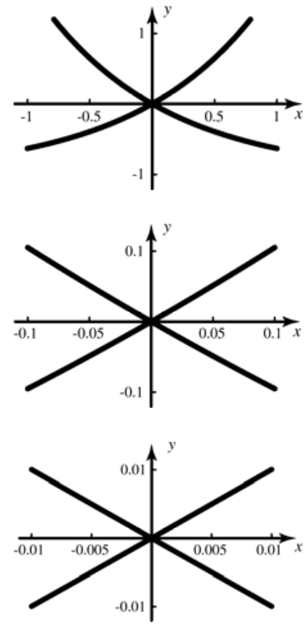
The slopes are approximately $0.005/0.01=1/2$ and $0.01/0.01=1$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

40.



The slopes are approximately $0.01/0.01=1$ and $0.02/0.01=2$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

41.



The slopes are approximately $0.01/0.01=1$ and $-0.01/0.01=1$. The ratio of the slopes is therefore $-1/1=-1$, indicating that the limit of the ratio should be about -1. An application of l'Hopital's Rule confirms this.

42. If f and g are locally linear at zero, then, since $\lim_{x\to0} f(x)=\lim_{x\to0} g(x)=0$, $f(x)\approx px$ and $g(x)\approx qx$, where $p=f'(0)$ and $q=g'(0)$. Then $f(x)/g(x)\approx px/px=p/q$ when x is near 0.

8.2 Concepts Review

1. $\dfrac{f'(x)}{g'(x)}$

2. $\lim_{x\to a} \dfrac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a} \dfrac{g(x)}{\frac{1}{f(x)}}$

3. $\infty - \infty, 0^\circ, \infty^\circ, 1^\infty$

4. $\ln x$

Problem Set 8.2

1. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln x^{1000}}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^{1000}} 1000x^{999}}{1}$$
$$= \lim_{x\to\infty} \frac{1000}{x} = 0$$

2. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\infty} \frac{(\ln x)^2}{2^x} = \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{2^x \ln 2}$$
$$= \lim_{x\to\infty} \frac{2\ln x}{x \cdot 2^x \ln 2} = \lim_{x\to\infty} \frac{2\left(\frac{1}{x}\right)}{2^x \ln 2(1 + x\ln 2)}$$
$$= \lim_{x\to\infty} \frac{2}{x \cdot 2^x \ln 2(1 + x\ln 2)} = 0$$

3. $\lim_{x\to\infty} \dfrac{x^{10000}}{e^x} = 0$ (See Example 2).

4. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to\infty} = \frac{3x}{\ln(100x + e^x)} = \lim_{x\to\infty} \frac{3}{\frac{1}{100x+e^x}(100 + e^x)}$$
$$= \lim_{x\to\infty} \frac{300x + 3e^x}{100 + e^x} = \lim_{x\to\infty} \frac{300 + 3e^x}{e^x}$$
$$= \lim_{x\to\infty} \frac{3e^x}{e^x} = 3$$

5. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\frac{\pi}{2}} \frac{3\sec x + 5}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{3\sec x \tan x}{\sec^2 x}$$
$$= \lim_{x\to\frac{\pi}{2}} \frac{3\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} 3\sin x = 3$$

6. The limit is of the form $\frac{-\infty}{-\infty}$.

$$\lim_{x\to 0^+} \frac{\ln \sin^2 x}{3\ln \tan x} = \lim_{x\to 0^+} \frac{\frac{1}{\sin^2 x} 2\sin x \cos x}{\frac{3}{\tan x} \sec^2 x}$$
$$= \lim_{x\to 0^+} \frac{2\cos^2 x}{3} = \frac{2}{3}$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln(\ln x^{1000})}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{\ln x^{1000}}\left(\frac{1}{x^{1000}} 1000x^{999}\right)}{\frac{1}{x}}$$
$$= \lim_{x\to\infty} \frac{1000}{x \ln x^{1000}} = 0$$

8. The limit is of the form $\frac{-\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\left(\frac{1}{2}\right)^-} \frac{\ln(4 - 8x)^2}{\tan \pi x} = \lim_{x\to\left(\frac{1}{2}\right)^-} \frac{\frac{1}{(4-8x)^2} 2(4 - 8x)(-8)}{\pi \sec^2 \pi x}$$
$$= \lim_{x\to\left(\frac{1}{2}\right)^-} \frac{-16\cos^2 \pi x}{\pi(4 - 8x)} = \lim_{x\to\left(\frac{1}{2}\right)^-} \frac{32\pi \cos \pi x \sin \pi x}{-8\pi}$$
$$= \lim_{x\to\left(\frac{1}{2}\right)^-} -4\cos \pi x \sin \pi x = 0$$

9. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\cot x}{\sqrt{-\ln x}} = \lim_{x\to 0^+} \frac{-\csc^2 x}{-\frac{1}{2x\sqrt{-\ln x}}}$$
$$= \lim_{x\to 0^+} \frac{2x\sqrt{-\ln x}}{\sin^2 x}$$
$$= \lim_{x\to 0^+} \left[\frac{2x}{\sin x} \csc x \sqrt{-\ln x}\right] = \infty$$

since $\lim_{x\to 0^+} \dfrac{x}{\sin x} = 1$ while $\lim_{x\to 0^+} \csc x = \infty$ and $\lim_{x\to 0^+} \sqrt{-\ln x} = \infty.$

10. The limit is of the form $\frac{\infty}{\infty}$, but the fraction can be simplified.

$$\lim_{x\to 0}\frac{2\csc^2 x}{\cot^2 x}=\lim_{x\to 0}\frac{2}{\cos^2 x}=\frac{2}{1^2}=2$$

11. $\lim_{x\to 0}(x\ln x^{1000})=\lim_{x\to 0}\frac{\ln x^{1000}}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0}\frac{\ln x^{1000}}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{1}{x^{1000}}1000x^{999}}{-\frac{1}{x^2}}$$

$$=\lim_{x\to 0}-1000x=0$$

12. $\lim_{x\to 0}3x^2\csc^2 x=\lim_{x\to 0}3\left(\frac{x}{\sin x}\right)^2=3$ since

$$\lim_{x\to 0}\frac{x}{\sin x}=1$$

13. $\lim_{x\to 0}(\csc^2 x-\cot^2 x)=\lim_{x\to 0}\frac{1-\cos^2 x}{\sin^2 x}$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2 x}=1$$

14. $\lim_{x\to\frac{\pi}{2}}(\tan x-\sec x)=\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}=\frac{0}{-1}=0$$

15. The limit is of the form 0^0.

Let $y=(3x)^{x^2}$, then $\ln y=x^2\ln 3x$

$$\lim_{x\to 0^+}x^2\ln 3x=\lim_{x\to 0^+}\frac{\ln 3x}{\frac{1}{x^2}}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln 3x}{\frac{1}{x^2}}=\lim_{x\to 0^+}\frac{\frac{1}{3x}\cdot 3}{-\frac{2}{x^3}}=\lim_{x\to 0^+}-\frac{x^2}{2}=0$$

$$\lim_{x\to 0^+}(3x)^{x^2}=\lim_{x\to 0^+}e^{\ln y}=1$$

16. The limit is of the form 1^∞.

Let $y=(\cos x)^{\csc x}$, then $\ln y=\csc x(\ln(\cos x))$

$$\lim_{x\to 0}\csc x(\ln(\cos x))=\lim_{x\to 0}\frac{\ln(\cos x)}{\sin x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\ln(\cos x)}{\sin x}=\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{\cos x}$$

$$=\lim_{x\to 0}-\frac{\sin x}{\cos^2 x}=-\frac{0}{1}=0$$

$$\lim_{x\to 0}(\cos x)^{\csc x}=\lim_{x\to 0}e^{\ln y}=1$$

17. The limit is of the form 0^∞, which is not an indeterminate form. $\lim_{x\to(\pi/2)^-}(5\cos x)^{\tan x}=0$

18. $\lim_{x\to 0}\left(\csc^2 x-\frac{1}{x^2}\right)^2=\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)^2=\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\sin^2 x}\right)^2$

Consider $\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}$. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule four times.)

$$\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{2x\sin^2 x+2x^2\sin x\cos x}=\lim_{x\to 0}\frac{x-\sin x\cos x}{x\sin^2 x+x^2\sin x\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x+\sin^2 x}{\sin^2 x+4x\sin x\cos x+x^2\cos^2 x-x^2\sin^2 x}=\lim_{x\to 0}\frac{4\sin x\cos x}{6x\cos x^2+6\cos x\sin x-4x^2\cos x\sin x-6x\sin^2 x}$$

$$=\lim_{x\to 0}\frac{4\cos^2 x-4\sin^2 x}{12\cos^2 x-4x^2\cos^2 x-32x\cos x\sin x-12\sin^2 x+4x^2\sin^2 x}=\frac{4}{12}=\frac{1}{3}$$

Thus, $\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\sin^2 x}\right)^2=\left(\frac{1}{3}\right)^2=\frac{1}{9}$

19. The limit is of the form 1^{∞}.

Let $y=(x+e^{x/3})^{3/x}$, then $\ln y=\frac{3}{x}\ln(x+e^{x/3})$.

$$\lim_{x\to 0}\frac{3}{x}\ln(x+e^{x/3})=\lim_{x\to 0}\frac{3\ln(x+e^{x/3})}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{3\ln(x+e^{x/3})}{x}=\lim_{x\to 0}\frac{\frac{3}{x+e^{x/3}}\left(1+\frac{1}{3}e^{x/3}\right)}{1}$$

$$=\lim_{x\to 0}\frac{3+e^{x/3}}{x+e^{x/3}}=\frac{4}{1}=4$$

$$\lim_{x\to 0}(x+e^{x/3})^{3/x}=\lim_{x\to 0}e^{\ln y}=e^4$$

20. The limit is of the form $(-1)^0$.
The limit does not exist.

21. The limit is of the form 1^0, which is not an indeterminate form.

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\cos x}=1$$

22. The limit is of the form ∞^{∞}, which is not an indeterminate form.

$$\lim_{x\to\infty}x^x=\infty$$

23. The limit is of the form ∞^0. Let

$y=x^{1/x}$, then $\ln y=\frac{1}{x}\ln x$.

$$\lim_{x\to\infty}\frac{1}{x}\ln x=\lim_{x\to\infty}\frac{\ln x}{x}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0$$

$$\lim_{x\to\infty}x^{1/x}=\lim_{x\to\infty}e^{\ln y}=1$$

24. The limit is of the form 1^{∞}.

Let $y=(\cos x)^{1/x^2}$, then $\ln y=\frac{1}{x^2}\ln(\cos x)$.

$$\lim_{x\to 0}\frac{1}{x^2}\ln(\cos x)=\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}$$

The limit is of the form $\frac{0}{0}$.

(Apply l'Hôpital's rule twice.)

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}=\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{2x}=\lim_{x\to 0}\frac{-\tan x}{2x}$$

$$=\lim_{x\to 0}\frac{-\sec^2 x}{2}=\frac{-1}{2}=-\frac{1}{2}$$

$$\lim_{x\to 0}(\cos x)^{1/x^2}=\lim_{x\to 0}e^{\ln y}=e^{-1/2}=\frac{1}{\sqrt{e}}$$

25. The limit is of the form 0^{∞}, which is not an indeterminate form.

$$\lim_{x\to 0^+}(\tan x)^{2/x}=0$$

26. The limit is of the form $\infty+\infty$, which is not an indeterminate form.

$$\lim_{x\to -\infty}(e^{-x}-x)=\lim_{x\to\infty}(e^x+x)=\infty$$

27. The limit is of the form 0^0. Let

$y=(\sin x)^x$, then $\ln y=x\ln(\sin x)$.

$$\lim_{x\to 0^+}x\ln(\sin x)=\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cos x}{-\frac{1}{x^2}}$$

$$=\lim_{x\to 0^+}\left[\frac{x}{\sin x}(-x\cos x)\right]=1\cdot 0=0$$

$$\lim_{x\to 0^+}(\sin x)^x=\lim_{x\to 0^+}e^{\ln y}=1$$

28. The limit is of the form 1^{∞}. Let

$y=(\cos x-\sin x)^{1/x}$, then $\ln y=\frac{1}{x}\ln(\cos x-\sin x)$.

$$\lim_{x\to 0}\frac{1}{x}\ln(\cos x-\sin x)=\lim_{x\to 0}\frac{\ln(\cos x-\sin x)}{x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\cos x-\sin x}(-\sin x-\cos x)}{1}$$

$$=\lim_{x\to 0}\frac{-\sin x-\cos x}{\cos x-\sin x}=-1$$

$$\lim_{x\to 0}(\cos x-\sin x)^{1/x}=\lim_{x\to 0}e^{\ln y}=e^{-1}$$

29. The limit is of the form $\infty - \infty$.

$$\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

30. The limit is of the form 1^∞.

Let $y = \left(1+\frac{1}{x}\right)^x$, then $\ln y = x\ln\left(1+\frac{1}{x}\right)$.

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^1 = e$$

31. The limit is of the form 3^∞, which is not an indeterminate form.

$$\lim_{x\to 0^+}(1+2e^x)^{1/x} = \infty$$

32. The limit is of the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{1}{x-1} - \frac{x}{\ln x}\right) = \lim_{x\to 1}\frac{\ln x - x^2 + x}{(x-1)\ln x}$$

The limit is of the form $\frac{0}{0}$.

Apply l'Hôpital's Rule twice.

$$\lim_{x\to 1}\frac{\ln x - x^2 + x}{(x-1)\ln x} = \lim_{x\to 1}\frac{\frac{1}{x} - 2x + 1}{\ln x + \frac{x-1}{x}}$$

$$= \lim_{x\to 1}\frac{1-2x^2+x}{x\ln x + x - 1} = \lim_{x\to 1}\frac{-4x+1}{\ln x + 2} = \frac{-3}{2} = -\frac{3}{2}$$

33. The limit is of the form 1^∞.

Let $y = (\cos x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(\cos x)$.

$$\lim_{x\to 0}\frac{1}{x}\ln(\cos x) = \lim_{x\to 0}\frac{\ln(\cos x)}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x} = \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{1} = \lim_{x\to 0} -\frac{\sin x}{\cos x} = 0$$

$$\lim_{x\to 0}(\cos x)^{1/x} = \lim_{x\to 0} e^{\ln y} = 1$$

34. The limit is of the form $0\cdot -\infty$.

$$\lim_{x\to 0^+}(x^{1/2}\ln x) = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

35. Since cos x oscillates between –1 and 1 as $x\to\infty$, this limit is not of an indeterminate form previously seen.

Let $y = e^{\cos x}$, then $\ln y = (\cos x)\ln e = \cos x$

$\lim_{x\to\infty}\cos x$ does not exist, so $\lim_{x\to\infty} e^{\cos x}$ does not exist.

36. The limit is of the form $\infty - \infty$.

$$\lim_{x\to\infty}[\ln(x+1) - \ln(x-1)] = \lim_{x\to\infty}\ln\frac{x+1}{x-1}$$

$$\lim_{x\to\infty}\frac{x+1}{x-1} = \lim_{x\to\infty}\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = 1, \text{ so } \lim_{x\to\infty}\ln\frac{x+1}{x-1} = 0$$

37. The limit is of the form $\frac{0}{-\infty}$, which is not an indeterminate form.

$$\lim_{x\to 0^+}\frac{x}{\ln x} = 0$$

38. The limit is of the form $-\infty\cdot\infty$, which is not an indeterminate form.

$$\lim_{x\to 0^+}(\ln x\cot x) = -\infty$$

39. $\sqrt{1+e^{-t}} > 1$ for all t, so
$\int_1^x \sqrt{1+e^{-t}}\,dt > \int_1^x dt = x-1$.

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\int_1^x \sqrt{1+e^{-t}}\,dt}{x} = \lim_{x\to\infty} \frac{\sqrt{1+e^{-x}}}{1} = 1$$

40. This limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1^+} \frac{\int_1^x \sin t\,dt}{x-1} = \lim_{x\to 1^+} \frac{\sin x}{1} = \sin(1)$$

41. a. Let $y = \sqrt[n]{a}$, then $\ln y = \frac{1}{n}\ln a$.

$$\lim_{n\to\infty} \frac{1}{n}\ln a = 0$$
$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} e^{\ln y} = 1$$

b. The limit is of the form ∞^0.

Let $y = \sqrt[n]{n}$, then $\ln y = \frac{1}{n}\ln n$.

$$\lim_{n\to\infty} \frac{1}{n}\ln n = \lim_{n\to\infty} \frac{\ln n}{n}$$

This limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$
$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\ln y} = 1$$

c. $$\lim_{n\to\infty} n\left(\sqrt[n]{a}-1\right) = \lim_{n\to\infty} \frac{\sqrt[n]{a}-1}{\frac{1}{n}}$$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n\to\infty} \sqrt[n]{a} = 1$ by part a.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a}-1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}\sqrt[n]{a}\ln a}{-\frac{1}{n^2}}$$
$$= \lim_{n\to\infty} \sqrt[n]{a}\ln a = \ln a$$

d. $$\lim_{n\to\infty} n\left(\sqrt[n]{n}-1\right) = \lim_{n\to\infty} \frac{\sqrt[n]{n}-1}{\frac{1}{n}}$$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n\to\infty} \sqrt[n]{n} = 1$ by part b.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n}-1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\left(\frac{1}{n^2}\right)(1-\ln n)}{-\frac{1}{n^2}}$$
$$= \lim_{n\to\infty} \sqrt[n]{n}(\ln n - 1) = \infty$$

42. a. The limit is of the form 0^0.

Let $y = x^x$, then $\ln y = x\ln x$.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$
$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

b. The limit is of the form 1^0, since

$\lim_{x\to 0^+} x^x = 1$ by part a.

Let $y = (x^x)^x$, then $\ln y = x\ln(x^x)$.

$$\lim_{x\to 0^+} x\ln(x^x) = 0$$
$$\lim_{x\to 0^+} (x^x)^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

Note that 1^0 is not an indeterminate form.

c. The limit is of the form 0^1, since

$\lim_{x\to 0^+} x^x = 1$ by part a.

Let $y = x^{(x^x)}$, then $\ln y = x^x \ln x$

$$\lim_{x\to 0^+} x^x \ln x = -\infty$$
$$\lim_{x\to 0^+} x^{(x^x)} = \lim_{x\to 0^+} e^{\ln y} = 0$$

Note that 0^1 is not an indeterminate form.

d. The limit is of the form 1^0, since

$\lim_{x\to 0^+}(x^x)^x = 1$ by part b.

Let $y = ((x^x)^x)^x$, then $\ln y = x\ln((x^x)^x)$.

$$\lim_{x\to 0^+} x\ln((x^x)^x) = 0$$

$$\lim_{x\to 0^+}((x^x)^x)^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

Note that 1^0 is not an indeterminate form.

e. The limit is of the form 0^0, since

$\lim_{x\to 0^+}(x^{(x^x)}) = 0$ by part c.

Let $y = x^{(x^{(x^x)})}$, then $\ln y = x^{(x^x)}\ln x$.

$$\lim_{x\to 0^+} x^{(x^x)}\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^{(x^x)}}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^{(x^x)}}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-x^{(x^x)}\left[x^x(\ln x+1)\ln x+\frac{x^x}{x}\right]}{(x^{(x^x)})^2}}$$

$$= \lim_{x\to 0^+}\frac{-x^{(x^x)}}{x^x x(\ln x)^2 + x^x x\ln x + x^x}$$

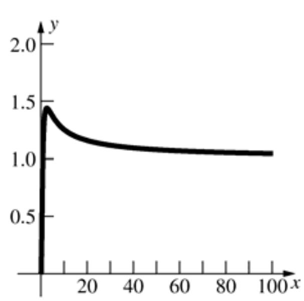
$$= \frac{0}{1\cdot 0+1\cdot 0+1} = 0$$

Note: $\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+}\frac{(\ln x)^2}{\frac{1}{x}}$

$$= \lim_{x\to 0^+}\frac{\frac{2}{x}\ln x}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -2x\ln x = 0$$

$$\lim_{x\to 0^+} x^{(x^{(x^x)})} = \lim_{x\to 0^+} e^{\ln y} = 1$$

43.



$$\ln y = \frac{\ln x}{x}$$

$$\lim_{x\to 0^+}\frac{\ln x}{x} = -\infty, \text{ so } \lim_{x\to 0^+} x^{1/x} = \lim_{x\to 0^+} e^{\ln y} = 0$$

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0, \text{ so } \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = 1$$

$$y = x^{1/x} = e^{\frac{1}{x}\ln x}$$

$$y' = \left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)e^{\frac{1}{x}\ln x}$$

$y' = 0$ when $x = e$.

y is maximum at $x = e$ since $y' > 0$ on $(0, e)$ and $y' < 0$ on (e, ∞). When $x = e$, $y = e^{1/e}$.

44. a. The limit is of the form $(1+1)^\infty = 2^\infty$, which is not an indeterminate form.

$$\lim_{x\to 0^+}(1^x + 2^x)^{1/x} = \infty$$

b. The limit is of the form $(1+1)^{-\infty} = 2^{-\infty}$, which is not an indeterminate form.

$$\lim_{x\to 0^-}(1^x + 2^x)^{1/x} = 0$$

c. The limit is of the form ∞^0.

Let $y = (1^x + 2^x)^{1/x}$, then

$$\ln y = \frac{1}{x}\ln(1^x + 2^x)$$

$$\lim_{x\to\infty}\frac{1}{x}\ln(1^x + 2^x) = \lim_{x\to\infty}\frac{\ln(1^x + 2^x)}{x}$$

The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\infty}\frac{\ln(1^x + 2^x)}{x} = \lim_{x\to\infty}\frac{\frac{1}{1^x+2^x}(1^x\ln 1 + 2^x\ln 2)}{1}$$

$$= \lim_{x\to\infty}\frac{2^x\ln 2}{1^x + 2^x} = \lim_{x\to\infty}\frac{2^x(\ln 2)^2}{1^x\ln 1 + 2^x\ln 2} = \ln 2$$

$$\lim_{x\to\infty}(1^x + 2^x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2$$

d. The limit is of the form 1^0, since $1^x = 1$ for all x. This is not an indeterminate form.

$$\lim_{x\to -\infty}(1^x + 2^x)^{1/x} = 1$$

45. $\lim_{n\to\infty}\frac{1^k+2^k+\cdots+n^k}{n^{k+1}}$

$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^k+\left(\frac{2}{n}\right)^k+\cdots\left(\frac{n}{n}\right)^k\right]$

$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\left(\frac{i}{n}\right)^k$

The summation has the form of a Reimann sum for $f(x)=x^k$ on the interval $[0,1]$ using a regular partition and evaluating the function at each right endpoint. Thus, $\Delta x_i=\frac{1}{n}$, $\overline{x}_i=\frac{i}{n}$, and $f(\overline{x}_i)=\left(\frac{i}{n}\right)^k$. Therefore,

$$\lim_{n\to\infty}\frac{1^k+2^k+\cdots+n^k}{n^{k+1}}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\left(\frac{i}{n}\right)^k$$

$$=\int_0^1 x^k\,dx=\left[\frac{1}{k+1}x^{k+1}\right]_0^1$$

$$=\frac{1}{k+1}$$

46. Let $y=\left(\sum_{i=1}^{n}c_i x_i{}^t\right)^{1/t}$, then $\ln y=\frac{1}{t}\ln\left(\sum_{i=1}^{n}c_i x_i{}^t\right)$.

$$\lim_{t\to 0^+}\frac{1}{t}\ln\left(\sum_{i=1}^{n}c_i x_i{}^t\right)=\lim_{t\to 0^+}\frac{\ln\left(\sum_{i=1}^{n}c_i x_i{}^t\right)}{t}$$

The limit is of the form $\frac{0}{0}$, since $\sum_{i=1}^{n}c_i=1$.

$$\lim_{t\to 0^+}\frac{\ln\left(\sum_{i=1}^{n}c_i x_i{}^t\right)}{t}=\lim_{t\to 0^+}\frac{1}{\sum_{i=1}^{n}c_i x_i{}^t}\sum_{i=1}^{n}c_i x_i{}^t\ln x_i$$

$$=\sum_{i=1}^{n}c_i\ln x_i=\sum_{i=1}^{n}\ln x_i{}^{c_i}$$

$$\lim_{t\to 0^+}\left(\sum_{i=1}^{n}c_i x_i{}^t\right)^{1/t}=\lim_{t\to 0^+}e^{\ln y}$$

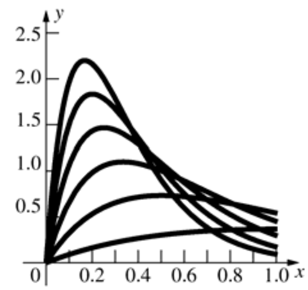
$$=e^{\sum_{i=1}^{n}\ln x_i{}^{c_i}}=x_1{}^{c_1}x_2{}^{c_2}\ldots x_n{}^{c_n}=\prod_{i=1}^{n}x_i{}^{c_i}$$

47. a. $\lim_{t\to 0^+}\left(\frac{1}{2}2^t+\frac{1}{2}5^t\right)^{1/t}=\sqrt{2}\sqrt{5}\approx 3.162$

b. $\lim_{t\to 0^+}\left(\frac{1}{5}2^t+\frac{4}{5}5^t\right)^{1/t}=\sqrt[5]{2}\cdot\sqrt[5]{5^4}\approx 4.163$

c. $\lim_{t\to 0^+}\left(\frac{1}{10}2^t+\frac{9}{10}5^t\right)^{1/t}=\sqrt[10]{2}\cdot\sqrt[10]{5^9}\approx 4.562$

48. a.



b. $n^2xe^{-nx}=\frac{n^2x}{e^{nx}}$, so the limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty}\frac{n^2x}{e^{nx}}=\lim_{n\to\infty}\frac{2nx}{xe^{nx}}$$

This limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty}\frac{2nx}{xe^{nx}}=\lim_{n\to\infty}\frac{2x}{x^2e^{nx}}=0$$

c. $\int_0^1 xe^{-x}dx=\left[-xe^{-x}-e^{-x}\right]_0^1=1-\frac{2}{e}$

$\int_0^1 4xe^{-2x}dx=\left[-2xe^{-2x}-e^{-2x}\right]_0^1=1-\frac{3}{e^2}$

$\int_0^1 9xe^{-3x}dx=\left[-3xe^{-3x}-e^{-3x}\right]_0^1=1-\frac{4}{e^3}$

$\int_0^1 16xe^{-4x}dx=\left[-4xe^{-4x}-e^{-4x}\right]_0^1=1-\frac{5}{e^4}$

$\int_0^1 25xe^{-5x}=\left[-5xe^{-5x}-e^{-5x}\right]_0^1=1-\frac{6}{e^5}$

$\int_0^1 36e^{-6x}dx=\left[-6xe^{-6x}-e^{-6x}\right]_0^1=1-\frac{7}{e^6}$

d. Guess: $\lim_{n\to\infty}\int_0^1 n^2xe^{-nx}dx=1$

$$\int_0^1 n^2xe^{-nx}dx=\left[-nxe^{-nx}-e^{-nx}\right]_0^1$$

$$=-(n+1)e^{-n}+1=1-\frac{n+1}{e^n}$$

$$\lim_{n\to\infty}\int_0^1 n^2xe^{-nx}dx=\lim_{n\to\infty}\left(1-\frac{n+1}{e^n}\right)$$

$=1-\lim_{n\to\infty}\frac{n+1}{e^n}$ if this last limit exists. The limit is of the form $\frac{\infty}{\infty}$.

$\lim_{n\to\infty}\frac{n+1}{e^n}=\lim_{n\to\infty}\frac{1}{e^n}=0,$ so

$\lim_{n\to\infty}\int_0^1 n^2xe^{-nx}dx=1$.

49. Note $f(x) > 0$ on $[0, \infty)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{x^{25}}{e^x} + \frac{x^3}{e^x} + \left(\frac{2}{e}\right)^x\right) = 0$$

Therefore there is no absolute minimum.

$$f'(x) = (25x^{24} + 3x^2 + 2^x \ln 2)e^{-x}$$
$$-(x^{25} + x^3 + 2^x)e^{-x}$$
$$= (-x^{25} + 25x^{24} - x^3 + 3x^2 - 2^x + 2^x \ln 2)e^{-x}$$

Solve for x when $f'(x) = 0$. Using a numerical method, $x \approx 25$.
A graph using a computer algebra system verifies that an absolute maximum occurs at about $x = 25$.

8.3 Concepts Review

1. converge

2. $\lim_{b\to\infty}\int_0^b \cos x\,dx$

3. $\int_{-\infty}^0 f(x)dx; \int_0^\infty f(x)dx$

4. $p > 1$

Problem Set 8.3

In this section and the chapter review, it is understood that $[g(x)]_a^\infty$ means $\lim_{b\to\infty}[g(x)]_a^b$ and likewise for similar expressions.

1. $\int_{100}^\infty e^x dx = \left[e^x\right]_{100}^\infty = \infty - e^{100} = \infty$
The integral diverges.

2. $\int_{-\infty}^{5}\frac{dx}{x^4} = \left[-\frac{1}{3x^3}\right]_{-\infty}^{-5} = -\frac{1}{3(-125)} - 0 = \frac{1}{375}$

3. $\int_1^\infty 2xe^{-x^2} dx = \left[-e^{-x^2}\right]_1^\infty = 0 - (-e^{-1}) = \frac{1}{e}$

4. $\int_{-\infty}^1 e^{4x} dx = \left[\frac{1}{4}e^{4x}\right]_{-\infty}^1 = \frac{1}{4}e^4 - 0 = \frac{1}{4}e^4$

5. $\int_9^\infty \frac{x\,dx}{\sqrt{1+x^2}} = \left[\sqrt{1+x^2}\right]_9^\infty = \infty - \sqrt{82} = \infty$
The integral diverges.

6. $\int_1^\infty \frac{dx}{\sqrt{\pi x}} = \left[2\sqrt{\frac{x}{\pi}}\right]_1^\infty = \infty - \frac{2}{\sqrt{\pi}} = \infty$
The integral diverges.

7. $\int_1^\infty \frac{dx}{x^{1.00001}} = \left[-\frac{1}{0.00001x^{0.00001}}\right]_1^\infty$
$= 0 - \left(-\frac{1}{0.00001}\right) = \frac{1}{0.00001} = 100,000$

8. $\int_{10}^\infty \frac{x}{1+x^2}dx = \frac{1}{2}\left[\ln(1+x^2)\right]_{10}^\infty$
$= \infty - \frac{1}{2}\ln|101| = \infty$
The integral diverges.

9. $\int_1^\infty \frac{dx}{x^{0.99999}} = \left[\frac{x^{0.00001}}{0.00001}\right]_1^\infty = \infty - 100,000 = \infty$
The integral diverges.

10. $\int_1^\infty \frac{x}{(1+x^2)^2}dx = \left[-\frac{1}{2(1+x^2)}\right]_1^\infty$
$= 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$

11. $\int_e^\infty \frac{1}{x\ln x}dx = [\ln(\ln x)]_e^\infty = \infty - 0 = \infty$
The integral diverges.

12. $\int_e^\infty \frac{\ln x}{x}dx = \left[\frac{1}{2}(\ln x)^2\right]_e^\infty = \infty - \frac{1}{2} = \infty$
The integral diverges.

13. Let $u = \ln x,\ du = \frac{1}{x}dx, dv = \frac{1}{x^2}dx, v = -\frac{1}{x}.$

$$\int_2^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_2^b \frac{\ln x}{x^2}dx$$
$$= \lim_{b\to\infty}\left[-\frac{\ln x}{x}\right]_2^b + \lim_{b\to\infty}\int_2^b \frac{1}{x^2}dx$$
$$= \lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^b = \frac{\ln 2 + 1}{2}$$

14. $\int_1^\infty xe^{-x}dx$
$u = x, du = dx$
$dv = e^{-x}dx, v = -e^{-x}$
$$\int_1^\infty xe^{-x}d = \left[-xe^{-x}\right]_1^\infty + \int_1^\infty e^{-x}dx$$
$$= \left[-xe^{-x} - e^{-x}\right]_1^\infty = 0 - 0 - (-e^{-1} - e^{-1}) = \frac{2}{e}$$

15. $\int_{-\infty}^1 \frac{dx}{(2x-3)^3} = \left[-\frac{1}{4(2x-3)^2}\right]_{-\infty}^1$
$= -\frac{1}{4} - (-0) = -\frac{1}{4}$

16. $\int_4^{\infty} \frac{dx}{(\pi - x)^{2/3}} = \left[-3(\pi - x)^{1/3}\right]_4^{\infty} = \infty + 3\sqrt[3]{\pi - 4} = \infty$

The integral diverges.

17. $\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \int_{-\infty}^{0} \frac{x}{\sqrt{x^2+9}} dx + \int_{0}^{\infty} \frac{x}{\sqrt{x^2+9}} dx = \left[\sqrt{x^2+9}\right]_{-\infty}^{0} + \left[\sqrt{x^2+9}\right]_{0}^{\infty} = (3-\infty)+(\infty-3)$

The integral diverges since both $\int_{-\infty}^{0} \frac{x}{\sqrt{x^2+9}} dx$ and $\int_{0}^{\infty} \frac{x}{\sqrt{x^2+9}} dx$ diverge.

18. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+16)^2} = \int_{-\infty}^{0} \frac{dx}{(x^2+16)^2} + \int_{0}^{\infty} \frac{dx}{(x^2+16)^2}$

$\int \frac{dx}{(x^2+16)^2} = \frac{1}{128} \tan^{-1} \frac{x}{4} + \frac{x}{32(x^2+16)}$ by using the substitution $x = 4 \tan \theta$.

$\int_{-\infty}^{0} \frac{dx}{(x^2+16)^2} = \left[\frac{1}{128} \tan^{-1} \frac{x}{4} + \frac{x}{32(x^2+16)}\right]_{-\infty}^{0} = 0 - \left[\frac{1}{128}\left(-\frac{\pi}{2}\right) + 0\right] = \frac{\pi}{256}$

$\int_{0}^{\infty} \frac{dx}{(x^2+16)^2} = \left[\frac{1}{128} \tan^{-1} \frac{x}{4} + \frac{x}{32(x^2+16)}\right]_{0}^{\infty} = \frac{1}{128}\left(\frac{\pi}{2}\right) + 0 - (0) = \frac{\pi}{256}$

$\int_{-\infty}^{\infty} \frac{dx}{(x^2+16)^2} = \frac{\pi}{256} + \frac{\pi}{256} = \frac{\pi}{128}$

19. $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10} dx = \int_{-\infty}^{\infty} \frac{1}{(x+1)^2+9} dx = \int_{-\infty}^{0} \frac{1}{(x+1)^2+9} dx + \int_{0}^{\infty} \frac{1}{(x+1)^2+9} dx$

$\int \frac{1}{(x+1)^2+9} dx = \frac{1}{3} \tan^{-1} \frac{x+1}{3}$ by using the substitution $x + 1 = 3 \tan \theta$.

$\int_{-\infty}^{0} \frac{1}{(x+1)^2+9} dx = \left[\frac{1}{3} \tan^{-1} \frac{x+1}{3}\right]_{-\infty}^{0} = \frac{1}{3} \tan^{-1} \frac{1}{3} - \frac{1}{3}\left(-\frac{\pi}{2}\right) = \frac{1}{6}\left(\pi + 2 \tan^{-1} \frac{1}{3}\right)$

$\int_{0}^{\infty} \frac{1}{(x+1)^2+9} dx = \left[\frac{1}{3} \tan^{-1} \frac{x+1}{3}\right]_{0}^{\infty} = \frac{1}{3}\left(\frac{\pi}{2}\right) - \frac{1}{3} \tan^{-1} \frac{1}{3} = \frac{1}{6}\left(\pi - 2 \tan^{-1} \frac{1}{3}\right)$

$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10} dx = \frac{1}{6}\left(\pi + 2 \tan^{-1} \frac{1}{3}\right) + \frac{1}{6}\left(\pi - 2 \tan^{-1} \frac{1}{3}\right) = \frac{\pi}{3}$

20. $\int_{-\infty}^{\infty} \frac{x}{e^{2|x|}} dx = \int_{-\infty}^{0} \frac{x}{e^{-2x}} dx + \int_{0}^{\infty} \frac{x}{e^{2x}} dx$

For $\int_{-\infty}^{0} \frac{x}{e^{-2x}} dx = \int_{-\infty}^{0} xe^{2x} dx$, use $u = x, du = dx, dv = e^{2x}dx, v = \frac{1}{2}e^{2x}$.

$\int_{-\infty}^{0} xe^{2x} dx = \left[\frac{1}{2}xe^{2x}\right]_{-\infty}^{0} - \frac{1}{2}\int_{-\infty}^{0} e^{2x} dx = \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_{-\infty}^{0} = 0 - \frac{1}{4} - (0) = -\frac{1}{4}$

For $\int_{0}^{\infty} \frac{x}{e^{2x}} dx = \int_{0}^{\infty} xe^{-2x} dx$, use $u = x, du = dx, dv = e^{-2x}dx, v = -\frac{1}{2}e^{-2x}$.

$\int_{0}^{\infty} xe^{-2x} dx = \left[-\frac{1}{2}xe^{-2x}\right]_{0}^{\infty} + \frac{1}{2}\int_{0}^{\infty} e^{-2x} dx = \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_{0}^{\infty} = 0 - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

$\int_{-\infty}^{\infty} \frac{x}{e^{2|x|}} dx = -\frac{1}{4} + \frac{1}{4} = 0$

21. $\int_{-\infty}^{\infty} \operatorname{sech} x\,dx = \int_{-\infty}^{0} \operatorname{sech} x\,dx = \int_{0}^{\infty} \operatorname{sech} x\,dx$

$= [\tan^{-1}(\sinh x)]_{-\infty}^{0} + [\tan^{-1}(\sinh x)]_{0}^{\infty}$

$= \left[0 - \left(-\frac{\pi}{2}\right)\right] + \left[\frac{\pi}{2} - 0\right] = \pi$

22. $\int_{1}^{\infty} \operatorname{csch} x\,dx = \int_{1}^{\infty} \frac{1}{\sinh x}dx = \int_{1}^{\infty} \frac{2}{e^{x} - e^{-x}}dx$

$= \int_{1}^{\infty} \frac{2e^{x}}{e^{2x} - 1}dx$

Let $u = e^{x}, du = e^{x}dx$.

$\int_{1}^{\infty} \frac{2e^{x}}{e^{2x} - 1}dx = \int_{e}^{\infty} \frac{2}{u^{2} - 1}du = \int_{e}^{\infty}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$

$= [\ln(u-1) - \ln(u+1)]_{e}^{\infty} = \left[\ln\frac{u-1}{u+1}\right]_{e}^{\infty}$

$= 0 - \ln\frac{e-1}{e+1} \approx 0.7719$

$\left(\lim_{b\to\infty} \ln\frac{b-1}{b+1} = 0 \text{ since } \lim_{b\to\infty}\frac{b-1}{b+1} = 1\right)$

23. $\int_{0}^{\infty} e^{-x}\cos x\,dx = \left[\frac{1}{2e^{x}}(\sin x - \cos x)\right]_{0}^{\infty}$

$= 0 - \frac{1}{2}(0-1) = \frac{1}{2}$

(Use Formula 68 with $a = -1$ and $b = 1$.)

24. $\int_{0}^{\infty} e^{-x}\sin x\,dx = \left[-\frac{1}{2e^{x}}(\cos x + \sin x)\right]_{0}^{\infty}$

$= 0 + \frac{1}{2}(1+0) = \frac{1}{2}$

(Use Formula 67 with $a = -1$ and $b = 1$.)

25. The area is given by

$\int_{1}^{\infty} \frac{2}{4x^{2}-1}dx = \int_{1}^{\infty}\left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)dx$

$= \frac{1}{2}\left[\ln|2x-1| - \ln|2x+1|\right]_{1}^{\infty} = \frac{1}{2}\left[\ln\left|\frac{2x-1}{2x+1}\right|\right]_{1}^{\infty}$

$= \frac{1}{2}\left(0 - \ln\left(\frac{1}{3}\right)\right) = \frac{1}{2}\ln 3$

Note:. $\lim_{x\to\infty} \ln = \left|\frac{2x-1}{2x+1}\right| = 0$ since

$\lim_{x\to\infty}\left(\frac{2x-1}{2x+1}\right) = 1 \cdot$

26. The area is

$\int_{1}^{\infty} \frac{1}{x^{2}+x}dx = \int_{1}^{\infty}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$

$= \left[\ln|x| - \ln|x+1|\right]_{1}^{\infty} = \left[\ln\left|\frac{x}{x+1}\right|\right]_{1}^{\infty} = 0 - \ln\frac{1}{2} = \ln 2$

.

27. The integral would take the form

$k\int_{3960}^{\infty}\frac{1}{x}dx = [k\ln x]_{3960}^{\infty} = \infty$

which would make it impossible to send anything out of the earth's gravitational field.

28. At $x = 1080$ mi, $F = 165$, so

$k = 165(1080)^{2} \approx 1.925 \times 10^{8}$. So the work done in mi-lb is

$1.925\times10^{8}\int_{1080}^{\infty}\frac{1}{x^{2}}dx = 1.925\times10^{8}\left[-x^{-1}\right]_{1080}^{\infty}$

$= \frac{1.925\times10^{8}}{1080} \approx 1.782\times10^{5}$ mi-lb.

29. $FP = \int_{0}^{\infty} e^{-rt} f(t)\,dt = \int_{0}^{\infty} 100{,}000e^{-0.08t}$

$= \left[-\frac{1}{0.08}100{,}000e^{-0.08t}\right]_{0}^{\infty} = 1{,}250{,}000$

The present value is \$1,250,000.

30. $FP = \int_{0}^{\infty} e^{-0.08t}(100{,}000 + 1000t)dt$

$= \left[-1{,}250{,}000e^{-0.08t} - 12{,}500te^{-0.08t} - 156{,}250e^{-0.08t}\right]_{0}^{\infty} = 1{,}406{,}250$

The present value is \$1,406,250.

31. **a.**

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} 0\,dx + \int_{a}^{b}\frac{1}{b-a}dx + \int_{b}^{\infty} 0\,dx$

$= 0 + \frac{1}{b-a}[x]_{a}^{b} + 0 = \frac{1}{b-a}(b-a)$

b.

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$
$$= \int_{-\infty}^{a} x \cdot 0\, dx + \int_{a}^{b} x \frac{1}{b-a}\, dx + \int_{b}^{\infty} x \cdot 0\, dx$$
$$= 0 + \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b + 0$$
$$= \frac{b^2 - a^2}{2(b-a)}$$
$$= \frac{(b+a)(b-a)}{2(b-a)}$$
$$= \frac{a+b}{2}$$
$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, dx$$
$$= \int_{-\infty}^{a} (x-\mu)^2 \cdot 0\, dx + \int_{a}^{b} (x-\mu)^2 \frac{1}{b-a}\, dx + \int_{b}^{\infty} (x-\mu)^2 \cdot 0\, dx$$
$$= 0 + \frac{1}{b-a}\left[\frac{(x-\mu)^3}{3}\right]_a^b + 0$$
$$= \frac{1}{b-a}\frac{(b-\mu)^3 - (a-\mu)^3}{3}$$
$$= \frac{1}{b-a}\frac{b^3 - 3b^2\mu + 3b\mu^2 - a^3 + 3a^2\mu - 3a\mu^2}{3}$$

Next, substitute $\mu = (a+b)/2$ to obtain

$$\sigma^2 = \frac{1}{3(b-a)}\left[\tfrac{1}{4}b^3 - \tfrac{3}{4}b^2 a + \tfrac{3}{4}ba^2 - \tfrac{1}{4}a^3\right]$$
$$= \frac{1}{12(b-a)}(b-a)^3$$
$$= \frac{(b-a)^2}{12}$$

c.

$$P(X < 2) = \int_{-\infty}^{2} f(x)\, dx$$
$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{2} \frac{1}{10-0}\, dx$$
$$= \frac{2}{10} = \frac{1}{5}$$

32. **a.**

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\, dx$$

In the second integral, let $u = (x/\theta)^\beta$. Then, $du = (\beta/\theta)(t/\theta)^{\beta-1}\, dt$. When $x = 0, u = 0$ and when $x \to \infty, u \to \infty$. Thus,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{\infty} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\, dx$$
$$= \int_{0}^{\infty} e^{-u}\, du = \left[-e^{-u}\right]_0^\infty = -0 + e^0 = 1$$

b. $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} \frac{\beta}{\theta} x \left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)}\,dx\partial$

$= \frac{2}{3}\int_0^{\infty} x^2 e^{-(x/3)^2}\,dx = \frac{3}{2}\sqrt{\pi}$

$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{0} (x-\mu)^2 \cdot 0\,dx + \frac{2}{9}\int_0^{\infty} (x-\mu)^2 xe^{-(x^2/9)}\,dx$

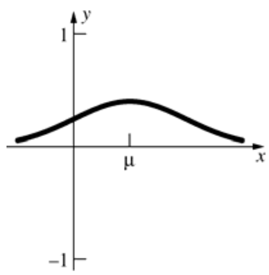
$= \frac{3}{2}\sqrt{\pi} - \mu = \frac{3}{2}\sqrt{\pi} - \frac{3}{2}\sqrt{\pi} = 0$

c. The probability of being less than 2 is

$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^2 \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^{\beta}}\,dx = 0 + \left[-e^{-(x/\theta)^{\beta}}\right]_0^2$

$= 1 - e^{-(2/\theta)^{\beta}} = 1 - e^{-(2/3)^2} \approx 0.359$

33.



$f'(x) = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} + \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$= \left(\frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} - \frac{1}{\sigma^3\sqrt{2\pi}}\right) e^{-(x-\mu)^2/2\sigma^2} =$

$\frac{1}{\sigma^5\sqrt{2\pi}}[(x-\mu)^2 - \sigma^2] e^{-(x-\mu)^2/2\sigma^2}$

$f''(x) = 0$ when $(x-\mu)^2 = \sigma^2$ so $x = \mu \pm \sigma$ and the distance from μ to each inflection point is σ.

34. **a.** $\int_{-\infty}^{\infty} f(x)dx = \int_M^{\infty} \frac{CM^k}{x^{k+1}}dx = CM^k\left[-\frac{1}{kx^k}\right]_M^{\infty} = CM^k\left(0 + \frac{1}{kM^k}\right) = \frac{C}{k}$. Thus, $\frac{C}{k} = 1$ when $C = k$.

b. $\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_M^{\infty} x\frac{kM^k}{x^{k+1}}dx = kM^k\int_M^{\infty}\frac{1}{x^k}dx = kM^k\left(\lim_{b\to\infty}\int_M^b \frac{1}{x^k}dx\right)$

This integral converges when $k > 1$.

When $k > 1$, $\mu = kM^k\left(\lim_{b\to\infty}\left[-\frac{1}{(k-1)x^{k-1}}\right]_M^b\right) = kM^k\left(-0 + \frac{1}{(k-1)M^{k-1}}\right) = \frac{kM}{k-1}$

The mean is finite only when $k > 1$.

c. Since the mean is finite only when $k > 1$, the variance is only defined when $k > 1$.

$$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_M^{\infty}\left(x-\frac{kM}{k-1}\right)^2 \frac{kM^k}{x^{k+1}}dx = kM^k\int_M^{\infty}\left(x^2-\frac{2kM}{k-1}x+\frac{k^2M^2}{(k-1)^2}\right)\frac{1}{x^{k+1}}dx$$

$$= kM^k\int_M^{\infty}\frac{1}{x^{k-1}}dx - \frac{2k^2M^{k+1}}{k-1}\int_M^{\infty}\frac{1}{x^k}dx + \frac{k^3M^{k+2}}{(k-1)^2}\int_M^{\infty}\frac{1}{x^{k+1}}dx$$

The first integral converges only when $k - 1 > 1$ or $k > 2$. The second integral converges only when $k > 1$, which is taken care of by requiring $k > 2$.

$$\sigma^2 = kM^k\left[-\frac{1}{(k-2)x^{k-2}}\right]_M^{\infty} - \frac{2k^2M^{k+1}}{k-1}\left[-\frac{1}{(k-1)x^{k-1}}\right]_M^{\infty} + \frac{k^3M^{k+2}}{(k-1)^2}\left[-\frac{1}{kx^k}\right]_M^{\infty}$$

$$= kM^k\left(-0+\frac{1}{(k-2)M^{k-2}}\right) - \frac{2k^2M^{k+1}}{k-1}\left(-0+\frac{1}{(k-1)M^{k-1}}\right) + \frac{k^3M^{k+2}}{(k-1)^2}\left(-0+\frac{1}{kM^k}\right)$$

$$= \frac{kM^2}{k-2} - \frac{2k^2M^2}{(k-1)^2} + \frac{k^2M^2}{(k-1)^2}$$

$$= kM^2\left(\frac{1}{k-2}-\frac{k}{(k-1)^2}\right) = kM^2\left(\frac{k^2-2k+1-k^2+2k}{(k-2)(k-1)^2}\right) = \frac{kM^2}{(k-2)(k-1)^2}$$

35. We use the results from problem 34:

a. To have a probability density function (34 a.) we need $C = k$; so $C = 3$. Also, $\mu = \frac{kM}{k-1}$ (34 b.) and since, in our problem, $\mu = 20{,}000$ and $k = 3$, we have

$$20000 = \frac{3}{2}M \text{ or } M = \frac{4\times10^4}{3}.$$

b. By 34 c., $\sigma^2 = \dfrac{kM^2}{(k-2)(k-1)^2}$ so that

$$\sigma^2 = \frac{3}{4}\left(\frac{4\times10^4}{3}\right)^2 = \frac{4\times10^8}{3}$$

c. $$\int_{10^5}^{\infty} f(x)\,dx = \left(\frac{4\times10^4}{3}\right)^3 \lim_{t\to\infty}\int_{10^5}^{t}\frac{3}{x^4}dx =$$

$$-\left(\frac{4\times10^4}{3}\right)^3 \lim_{t\to\infty}\left[\frac{1}{x^3}\right]_{10^5}^{t}$$

$$= \left(\frac{4\times10^4}{3}\right)^3 \lim_{t\to\infty}\left[\frac{1}{10^{15}}-\frac{1}{t^3}\right] = \frac{64}{27\times10^3}$$

$$\approx 0.0024$$

Thus $6/25$ of one percent earn over \$100,000.

36. $$u = Ar\int_a^{\infty}\frac{dx}{(r^2+x^2)^{3/2}}$$

$$= \frac{A}{r}\left[\frac{x}{\sqrt{r^2+x^2}}\right]_a^{\infty} = \frac{A}{r}\left(1-\frac{a}{\sqrt{r^2+a^2}}\right)$$

Note that $\int \frac{dx}{(r^2+x^2)^{3/2}} = \frac{x}{r^2\sqrt{r^2+x^2}}$ by using the substitution $x = r\tan\theta$.

37. a. $$\int_{-\infty}^{\infty}\sin x\,dx = \int_{-\infty}^{0}\sin x\,dx + \int_0^{\infty}\sin x\,dx$$

$$= \lim_{a\to\infty}[-\cos x]_0^a + \lim_{a\to-\infty}[-\cos x]_a^0$$

Both do not converge since $-\cos x$ is oscillating between –1 and 1, so the integral diverges.

b. $$\lim_{a\to\infty}\int_{-a}^{a}\sin x\,dx = \lim_{a\to\infty}[-\cos x]_{-a}^{a}$$

$$= \lim_{a\to\infty}[-\cos a+\cos(-a)]$$

$$= \lim_{a\to\infty}[-\cos a+\cos a] = \lim_{a\to\infty} 0 = 0$$

38. a. The total mass of the wire is $\int_0^{\infty}\frac{1}{1+x^2}dx = \frac{\pi}{2}$ from Example 4.

b. $\int_0^{\infty}\frac{x}{1+x^2}dx = \left[\frac{1}{2}\ln\left|1+x^2\right|\right]_0^{\infty}$ which diverges. Thus, the wire does not have a center of mass.

39. For example, the region under the curve $y = \frac{1}{x}$ to the right of $x = 1$.
Rotated about the x-axis the volume is $\pi \int_1^{\infty} \frac{1}{x^2} dx = \pi$. Rotated about the y-axis, the volume is $2\pi \int_1^{\infty} x \cdot \frac{1}{x} dx$ which diverges.

40. a. Suppose $\lim_{x \to \infty} f(x) = M \neq 0,$ so the limit exists but is non-zero. Since $\lim_{x \to \infty} f(x) = M,$ there is some $N > 0$ such that when $x \geq N,$ $|f(x) - M| \leq \frac{M}{2},$ or

$$M - \frac{M}{2} \leq f(x) \leq M + \frac{M}{2}$$

Since $f(x)$ is nonnegative, $M > 0$, thus $\frac{M}{2} > 0$ and

$$\int_0^{\infty} f(x)dx = \int_0^N f(x)dx + \int_N^{\infty} f(x)dx$$

$$\geq \int_0^N f(x)dx + \int_N^{\infty} \frac{M}{2} dx = \int_0^N f(x)dx + \left[\frac{Mx}{2} \right]_N^{\infty} = \infty$$

so the integral diverges. Thus, if the limit exists, it must be 0.

b. For example, let $f(x)$ be given by

$$f(x) = \begin{cases} 2n^2x - 2n^3 + 1 & \text{if } n - \frac{1}{2n^2} \leq x \leq n \\ -2n^2x + 2n^3 + 1 & \text{if } n < x \leq n + \frac{1}{2n^2} \\ 0 & \text{otherwise} \end{cases}$$

for every positive integer n.

$$f\left(n - \frac{1}{2n^2}\right) = 2n^2\left(n - \frac{1}{2n^2}\right) - 2n^3 + 1$$

$$= 2n^3 - 1 - 2n^3 + 1 = 0$$

$$f(n) = 2n^2(n) - 2n^3 + 1 = 1$$

$$\lim_{x \to n^+} f(n) = \lim_{x \to n^+} (-2n^2x + 2n^3 + 1) = 1 = f(n)$$

$$f\left(n + \frac{1}{2n^2}\right) = -2n^2\left(n + \frac{1}{2n^2}\right) + 2n^3 + 1$$

$$= -2n^3 - 1 + 2n^3 + 1 = 0$$

Thus, f is continuous at $n - \frac{1}{2n^2}$, n, and $n + \frac{1}{2n^2}$.

Note that the intervals $\left[n, n + \frac{1}{2n^2}\right]$ and $\left[n + 1 - \frac{1}{2(n+1)^2}, n + 1\right]$ will never overlap since $\frac{1}{2n^2} \leq \frac{1}{2}$ and $\frac{1}{2(n+1)^2} \leq \frac{1}{8}$.

The graph of f consists of a series of isosceles triangles, each of height 1, vertices at $\left(n - \frac{1}{2n^2}, 0\right)$, $(n, 1)$, and $\left(n + \frac{1}{2n^2}, 0\right)$, based on the x-axis, and centered over each integer n.

$\lim_{x \to \infty} f(x)$ does not exist, since $f(x)$ will be 1 at each integer, but 0 between the triangles.

Each triangle has area

$$\frac{1}{2}bh = \frac{1}{2}\left[n + \frac{1}{2n^2} - \left(n - \frac{1}{2n^2}\right)\right](1)$$

$$= \frac{1}{2}\left(\frac{1}{n^2}\right) = \frac{1}{2n^2}$$

$\int_0^{\infty} f(x)dx$ is the area in all of the triangles, thus

$$\int_0^{\infty} f(x)dx = \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \frac{1}{2} + \frac{1}{2}\int_1^{\infty} \frac{1}{x^2} dx$$

$$= \frac{1}{2} + \frac{1}{2}\left[-\frac{1}{x}\right]_1^{\infty} = \frac{1}{2} + \frac{1}{2}(-0 + 1) = 1$$

(By viewing $\sum_{n=2}^{\infty} \frac{1}{n^2}$ as a lower Riemann sum for $\frac{1}{x^2}$)

Thus, $\int_0^{\infty} f(x)dx$ converges, although $\lim_{x \to \infty} f(x)$ does not exist.

41. $\int_1^{100} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^{100} = 0.99$

$\int_1^{100} \frac{1}{x^{1.1}}dx = \left[-\frac{1}{0.1x^{0.1}}\right]_1^{100} \approx 3.69$

$\int_1^{100} \frac{1}{x^{1.01}}dx = \left[-\frac{1}{0.01x^{0.01}}\right]_1^{100} \approx 4.50$

$\int_1^{100} \frac{1}{x}dx = [\ln x]_1^{100} = \ln 100 \approx 4.61$

$\int_1^{100} \frac{1}{x^{0.99}}dx = \left[\frac{x^{0.01}}{0.01}\right]_1^{100} \approx 4.71$

42. $\int_0^{10} \frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\left[\tan^{-1} x\right]_0^{10}$

$\approx \frac{1.4711}{\pi} \approx 0.468$

$\int_0^{50} \frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\left[\tan^{-1} x\right]_0^{50}$

$\approx \frac{1.5508}{\pi} \approx 0.494$

$\int_0^{100} \frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}\left[\tan^{-1} x\right]_0^{100}$

$\approx \frac{1.5608}{\pi} \approx 0.497$

43. $\int_0^1 \frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.3413$

$\int_0^2 \frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.4772$

$\int_0^3 \frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.4987$

$\int_0^4 \frac{1}{\sqrt{2\pi}}\exp(-0.5x^2)dx \approx 0.5000$

8.4 Concepts Review

1. unbounded

2. 2

3. $\lim_{b\to 4^-} \int_0^b \frac{1}{\sqrt{4-x}}dx$

4. $p < 1$

Problem Set 8.4

1. $\int_1^3 \frac{dx}{(x-1)^{1/3}} = \lim_{b\to 1^+}\left[\frac{3(x-1)^{2/3}}{2}\right]_b^3$

$= \frac{3}{2}\sqrt[3]{2^2} - \lim_{b\to 1^+}\frac{3(b-1)^{2/3}}{2} = \frac{3}{\sqrt[3]{2}} - 0 = \frac{3}{\sqrt[3]{2}}$

2. $\int_1^3 \frac{dx}{(x-1)^{4/3}} = \lim_{b\to 1^+}\left[-\frac{3}{(x-1)^{1/3}}\right]_b^3$

$= -\frac{3}{\sqrt[3]{2}} + \lim_{b\to 1^+}\frac{3}{(x-1)^{1/3}} = -\frac{3}{\sqrt[3]{2}} + \infty$

The integral diverges.

3. $\int_3^{10} \frac{dx}{\sqrt{x-3}} = \lim_{b\to 3^+}\left[2\sqrt{x-3}\right]_b^{10}$

$= 2\sqrt{7} - \lim_{b\to 3^+} 2\sqrt{b-3} = 2\sqrt{7}$

4. $\int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{b\to 9^-}\left[-2\sqrt{9-x}\right]_0^b$

$= \lim_{b\to 9^-} -2\sqrt{9-b} + 2\sqrt{9} = 6$

5. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-}\left[\sin^{-1} x\right]_0^b$

$= \lim_{b\to 1^-} \sin^{-1} b - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

6. $\int_{100}^{\infty} \frac{x}{\sqrt{1+x^2}}dx = \lim_{b\to\infty}\left[\sqrt{1+x^2}\right]_{100}^b$

$= \lim_{b\to\infty} \sqrt{1+b^2} + \sqrt{10,001} = \infty$

The integral diverges.

7. $\int_{-1}^3 \frac{1}{x^3}dx = \lim_{b\to 0^-}\int_{-1}^b \frac{1}{x^3}dx + \lim_{b\to 0^+}\int_b^3 \frac{1}{x^3}dx$

$= \lim_{b\to 0^-}\left[-\frac{1}{2x^2}\right]_{-1}^b + \lim_{b\to 0^+}\left[-\frac{1}{2x^2}\right]_b^3$

$= \left(\lim_{b\to 0^-} -\frac{1}{2b^2} + \frac{1}{2}\right) + \left(-\frac{1}{18} + \lim_{b\to 0^+}\frac{1}{2b^2}\right)$

$= \left(-\infty + \frac{1}{2}\right) + \left(-\frac{1}{8} + \infty\right)$

The integral diverges.

8. $\int_5^{-5} \frac{1}{x^{2/3}}dx = \lim_{b\to 0^+}\int_5^b \frac{1}{x^{2/3}}dx + \lim_{b\to 0^-}\int_b^{-5}\frac{1}{x^{2/3}}dx$

$= \lim_{b\to 0^+}\left[3x^{1/3}\right]_5^b + \lim_{b\to 0^-}\left[3x^{1/3}\right]_b^{-5}$

$= \lim_{b\to 0^+} 3b^{1/3} - 3\sqrt[3]{5} + 3\sqrt[3]{-5} - \lim_{b\to 0^-} 3b^{1/3}$

$= 0 - 3\sqrt[3]{5} + 3\sqrt[3]{5} - 0 = 3\sqrt[3]{-5} - 3\sqrt[3]{5} = -6\sqrt[3]{5}$

9. $\int_{-1}^{128} x^{-5/7}dx$

$= \lim_{b\to 0^-}\int_{-1}^b x^{-5/7}dx + \lim_{b\to 0^+}\int_b^{128} x^{-5/7}dx$

$= \lim_{b\to 0^-}\left[\frac{7}{2}x^{2/7}\right]_{-1}^b + \lim_{b\to 0^+}\left[\frac{7}{2}x^{2/7}\right]_b^{128}$

$= \lim_{b\to 0^-}\frac{7}{2}b^{2/7} - \frac{7}{2}(-1)^{2/7} + \frac{7}{2}(128)^{2/7} - \lim_{b\to 0^+}\frac{7}{2}b^{2/7}$

$= 0 - \frac{7}{2} + \frac{7}{2}(4) - 0 = \frac{21}{2}$

10. $\int_0^1 \frac{x}{\sqrt[3]{1-x^2}}dx = \lim_{b\to 1^-}\int_0^b \frac{x}{\sqrt[3]{1-x^2}}dx$

$= \lim_{b\to 1^-}\left[-\frac{3}{4}(1-x^2)^{2/3}\right]_0^b$

$= \lim_{b\to 1^-} -\frac{3}{4}(1-b^2)^{2/3} + \frac{3}{4} = -0 + \frac{3}{4} = \frac{3}{4}$

11. $\int_0^4 \frac{dx}{(2-3x)^{1/3}} = \lim_{b\to \frac{2}{3}^-}\int_0^b \frac{dx}{(2-3x)^{1/3}} + \lim_{b\to \frac{2}{3}^+}\int_b^4 \frac{dx}{(2-3x)^{1/3}} = \lim_{b\to \frac{2}{3}^-}\left[-\frac{1}{2}(2-3x)^{2/3}\right]_0^b + \lim_{b\to \frac{2}{3}^+}\left[-\frac{1}{2}(2-3x)^{2/3}\right]_b^4$

$= \lim_{b\to \frac{2}{3}^-} -\frac{1}{2}(2-3b)^{2/3} + \frac{1}{2}(2)^{2/3} - \frac{1}{2}(-10)^{2/3} + \lim_{b\to \frac{2}{3}^+}\frac{1}{2}(2-3b)^{2/3}$

$= 0 + \frac{1}{2}2^{2/3} - \frac{1}{2}10^{2/3} + 0 = \frac{1}{2}(2^{2/3} - 10^{2/3})$

12. $\int_{\sqrt{5}}^{\sqrt{8}} \frac{x}{(16-2x^2)^{2/3}}dx = \lim_{b\to\sqrt{8}^-}\left[-\frac{3}{4}(16-2x^2)^{1/3}\right]_{\sqrt{5}}^b = \lim_{b\to\sqrt{8}^-} -\frac{3}{4}(16-2b^2)^{1/3} + \frac{3}{4}\sqrt[3]{6} = \frac{3}{4}\sqrt[3]{6}$

13. $\int_0^{-4} \frac{x}{16-2x^2}dx = \lim_{b\to -\sqrt{8}^+}\int_0^b \frac{x}{16-2x^2}dx + \lim_{b\to -\sqrt{8}^-}\int_b^{-4}\frac{x}{16-2x^2}dx$

$= \lim_{b\to -\sqrt{8}^+}\left[-\frac{1}{4}\ln\left|16-2x^2\right|\right]_0^b + \lim_{b\to -\sqrt{8}^-}\left[-\frac{1}{4}\ln\left|16-2x^2\right|\right]_b^{-4}$

$= \lim_{b\to -\sqrt{8}^+} -\frac{1}{4}\ln\left|16-2b^2\right| + \frac{1}{4}\ln 16 - \frac{1}{4}\ln 16 + \lim_{b\to -\sqrt{8}^-}\frac{1}{4}\ln\left|16-2b^2\right|$

$= \left[-(-\infty) + \frac{1}{4}\ln 16\right] + \left[-\frac{1}{4}\ln 16 + (-\infty)\right]$

The integral diverges.

14. $\int_0^3 \frac{x}{\sqrt{9-x^2}}dx = \lim_{b\to 3^-}\left[-\sqrt{9-x^2}\right]_0^b = \lim_{b\to 3^-} -\sqrt{9-b^2} + \sqrt{9} = 3$

15. $\int_{-2}^{-1}\frac{dx}{(x+1)^{4/3}} = \lim_{b\to -1^-}\left[-\frac{3}{(x+1)^{1/3}}\right]_{-2}^b = \lim_{b\to -1^-} -\frac{3}{(b+1)^{1/3}} + \frac{3}{(-1)^{1/3}} = -(-\infty) - 3$

The integral diverges.

16. Note that $\int \frac{dx}{x^2+x-2} = \int \frac{dx}{(x-1)(x+2)} = \int \left[\frac{1}{3(x-1)} - \frac{1}{3(x+2)}\right] dx$ by using a partial fraction decomposition.

$$\int_0^3 \frac{dx}{x^2+x-2} = \lim_{b\to 1^-} \int_0^b \frac{dx}{x^2+x-2} + \lim_{b\to 1^+} \int_b^3 \frac{dx}{x^2+x-2}$$

$$= \lim_{b\to 1^-}\left[\frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2|\right]_0^b + \lim_{b\to 1^+}\left[\frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2|\right]_b^3$$

$$= \lim_{b\to 1^-}\left[\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\right]_0^b + \lim_{b\to 1^+}\left[\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\right]_b^3 = \lim_{b\to 1^-} \frac{1}{3}\ln\left|\frac{b-1}{b+2}\right| - \frac{1}{3}\ln\frac{1}{2} + \frac{1}{3}\ln\frac{2}{5} - \lim_{b\to 1^+}\frac{1}{3}\ln\left|\frac{b-1}{b+2}\right|$$

$$= \left(-\infty - \frac{1}{3}\ln\frac{1}{2}\right) + \left(\frac{1}{3}\ln\frac{2}{5} + \infty\right)$$

The integral diverges.

17. Note that $\frac{1}{x^3-x^2-x+1} = \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}$

$$\int_0^3 \frac{dx}{x^3-x^2-x+1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x^3-x^2-x+1} + \lim_{b\to 1^+}\int_b^3 \frac{dx}{x^3-x^2-x+1}$$

$$= \lim_{b\to 1^-}\left[-\frac{1}{2(x-1)} - \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1|\right]_0^b + \lim_{b\to 1^+}\left[-\frac{1}{2(x-1)} - \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1|\right]_b^3$$

$$\lim_{b\to 1^-}\left[\left(-\frac{1}{2(b-1)} + \frac{1}{4}\ln\left|\frac{b+1}{b-1}\right|\right) + \left(-\frac{1}{2} + 0\right)\right] + \lim_{b\to 1^+}\left[-\frac{1}{4} + \frac{1}{4}\ln 2 - \left(-\frac{1}{2(b-1)} + \frac{1}{4}\ln\left|\frac{b+1}{b-1}\right|\right)\right]$$

$$= \left(\infty + \infty - \frac{1}{2}\right) + \left(-\frac{1}{4} + \frac{1}{4}\ln 2 + \infty - \infty\right)$$

The integral diverges.

18. Note that $\frac{x^{1/3}}{x^{2/3}-9} = \frac{1}{x^{1/3}} + \frac{9}{x^{1/3}(x^{2/3}-9)}$.

$$\int_0^{27} \frac{x^{1/3}}{x^{2/3}-9}dx = \lim_{b\to 27^-}\left[\frac{3}{2}x^{2/3} + \frac{27}{2}\ln\left|x^{2/3}-9\right|\right]_0^b = \lim_{b\to 27^-}\left(\frac{3}{2}b^{2/3} + \frac{27}{2}\ln\left|b^{2/3}-9\right|\right) - \left(0 + \frac{27}{2}\ln 9\right)$$

$$= \frac{27}{2} - \infty - \frac{27}{2}\ln 9$$

The integral diverges.

19. $\int_0^{\pi/4} \tan 2x\,dx = \lim_{b\to \frac{\pi}{4}^-}\left[-\frac{1}{2}\ln|\cos 2x|\right]_0^b$

$$= \lim_{b\to \frac{\pi}{4}^-} -\frac{1}{2}\ln|\cos 2b| + \frac{1}{2}\ln 1 = -(-\infty) + 0$$

The integral diverges.

20. $\int_0^{\pi/2} \csc x\,dx = \lim_{b\to 0^+}\left[\ln|\csc x - \cot x|\right]_b^{\pi/2}$

$$= \ln|1-0| - \lim_{b\to 0^+}\ln|\csc b - \cot b|$$

$$= 0 - \lim_{b\to 0^+}\ln\left|\frac{1-\cos b}{\sin b}\right|$$

$\lim_{b\to 0^+}\frac{1-\cos b}{\sin b}$ is of the form $\frac{0}{0}$.

$$\lim_{b\to 0^+}\frac{1-\cos b}{\sin b} = \lim_{b\to 0^+}\frac{\sin b}{\cos b} = \frac{0}{1} = 0$$

Thus, $\lim_{b\to 0^+}\ln\left|\frac{1-\cos b}{\sin b}\right| = -\infty$ and the integral diverges.

21. $\int_0^{\pi/2} \frac{\sin x}{1-\cos x} dx = \lim_{b\to 0^+} \left[\ln|1-\cos x|\right]_b^{\pi/2}$

$= \ln 1 - \lim_{b\to 0^+} \ln|1-\cos b| = 0-(-\infty)$

The integral diverges.

22. $\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}} dx = \lim_{b\to 0^+} \left[\frac{3}{2}\sin^{2/3} x\right]_b^{\pi/2}$

$= \frac{3}{2}(1)^{2/3} - \frac{3}{2}(0)^{2/3} = \frac{3}{2}$

23. $\int_0^{\pi/2} \tan^2 x \sec^2 x\, dx = \lim_{b\to \frac{\pi}{2}^-} \left[\frac{1}{3}\tan^3 x\right]_0^b$

$= \lim_{b\to \frac{\pi}{2}^-} \frac{1}{3}\tan^3 b - \frac{1}{3}(0)^3 = \infty$

The integral diverges.

24. $\int_0^{\pi/4} \frac{\sec^2 x}{(\tan x-1)^2} dx = \lim_{b\to \frac{\pi}{4}^-} \left[-\frac{1}{\tan x-1}\right]_0^b$

$= \lim_{b\to \frac{\pi}{4}^-} -\frac{1}{\tan b-1} + \frac{1}{0-1} = -(-\infty)-1$

The integral diverges.

25. Since $\frac{1-\cos x}{2} = \sin^2 \frac{x}{2}$,

$\frac{1}{\cos x-1} = -\frac{1}{2}\csc^2 \frac{x}{2}$.

$\int_0^{\pi} \frac{dx}{\cos x-1} = \lim_{b\to 0^+} \left[\cot \frac{x}{2}\right]_b^{\pi}$

$= \cot\frac{\pi}{2} - \lim_{b\to 0^+} \cot\frac{b}{2} = 0-\infty$

The integral diverges.

26. $\int_{-3}^{-1} \frac{dx}{x\sqrt{\ln(-x)}} = \lim_{b\to -1^-} \left[2\sqrt{\ln(-x)}\right]_{-3}^b$

$= \lim_{b\to -1^-} 2\sqrt{\ln(-b)} - 2\sqrt{\ln 3} = 0-2\sqrt{\ln 3}$

$= -2\sqrt{\ln 3}$

27. $\int_0^{\ln 3} \frac{e^x dx}{\sqrt{e^x-1}} = \lim_{b\to 0^+} \left[2\sqrt{e^x-1}\right]_b^{\ln 3}$

$= 2\sqrt{3-1} - \lim_{b\to 0^+} 2\sqrt{e^b-1} = 2\sqrt{2}-0 = 2\sqrt{2}$

28. Note that $\sqrt{4x-x^2} = \sqrt{4-(x^2-4x+4)} = \sqrt{2^2-(x-2)^2}$. (by completing the square)

$\int_2^4 \frac{dx}{\sqrt{4x-x^2}} = \lim_{b\to 4^-} \int_2^b \frac{dx}{\sqrt{4x-x^2}} = \lim_{b\to 4^-} \left[\sin^{-1}\frac{x-2}{2}\right]_2^b = \lim_{b\to 4^-} \sin^{-1}\frac{b-2}{2} - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

29. $\int_1^e \frac{dx}{x\ln x} = \lim_{b\to 1^+} [\ln(\ln x)]_b^e = \ln(\ln e) - \lim_{b\to 1^+} \ln(\ln b) = \ln 1 - \ln 0 = 0+\infty$

The integral diverges.

30. $\int_1^{10} \frac{dx}{x\ln^{100} x} = \lim_{b\to 1^+} \left[-\frac{1}{99\ln^{99} x}\right]_b^{10} = -\frac{1}{99\ln^{99} 10} + \lim_{b\to 1^+} \frac{1}{99\ln^{99} b} = -\frac{1}{99\ln^{99} 10} + \infty$

The integral diverges.

31. $\int_{2c}^{4c} \frac{dx}{\sqrt{x^2-4c^2}} = \lim_{b\to 2c^+} \left[\ln\left|x+\sqrt{x^2-4c^2}\right|\right]_b^{4c} = \ln\left[(4+2\sqrt{3})c\right] - \lim_{b\to 2c^+} \ln\left|b+\sqrt{b^2-4c^2}\right|$

$= \ln\left[(4+2\sqrt{3})c\right] - \ln 2c = \ln(2+\sqrt{3})$

32. $\int_c^{2c} \frac{x\,dx}{\sqrt{x^2+xc-2c^2}} = \int_c^{2c} \frac{x\,dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}} = \int_c^{2c} \frac{\left(x+\frac{c}{2}\right)dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}} - \frac{c}{2}\int_0^{2c} \frac{dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}}$

$= \lim_{b\to c^+}\left[\sqrt{x^2+xc-2c^2} - \frac{c}{2}\ln\left|x+\frac{c}{2}+\sqrt{x^2+xc-2c^2}\right|\right]_b^{2c}$

$= \sqrt{4c^2} - \frac{c}{2}\ln\left|\frac{5c}{2}+\sqrt{4c^2}\right| - \lim_{b\to c^+}\left[\sqrt{b^2+bc-2c^2} - \frac{c}{2}\ln\left|b+\frac{c}{2}+\sqrt{b^2+bc-2c^2}\right|\right]$

$= 2c - \frac{c}{2}\ln\frac{9c}{2} - \left(0-\frac{c}{2}\ln\left|\frac{3c}{2}+0\right|\right) = 2c - \frac{c}{2}\ln\frac{9c}{2} + \frac{c}{2}\ln\frac{3c}{2} = 2c - \frac{c}{2}\ln 3$

33. For $0 < c < 1$, $\frac{1}{\sqrt{x}(1+x)}$ is continuous. Let $u = \frac{1}{1+x}, du = -\frac{1}{(1+x)^2}dx$.

$dv = \frac{1}{\sqrt{x}}dx, v = 2\sqrt{x}$.

$\int_c^1 \frac{1}{\sqrt{x}(1+x)}dx = \left[\frac{2\sqrt{x}}{1+x}\right]_c^1 + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2} = \frac{2}{2} - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2} = 1 - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2}$

Thus, $\lim_{c\to 0}\int_c^1 \frac{1}{\sqrt{x}(1+x)}dx = \lim_{c\to 0}\left[1-\frac{2\sqrt{c}}{1+c}+2\int_c^1\frac{\sqrt{x}dx}{(1+x)^2}\right] = 1-0+2\int_0^1\frac{\sqrt{x}dx}{(1+x)^2}$

This last integral is a proper integral.

34. Let $u = \frac{1}{\sqrt{1+x}}, du = -\frac{1}{2(1+x)^{3/2}}dx$

$dv = \frac{1}{\sqrt{x}}dx, v = 2\sqrt{x}$.

For $0 < c < 1$, $\int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \left[\frac{2\sqrt{x}}{\sqrt{1+x}}\right]_c^1 + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx = \frac{2\sqrt{1}}{\sqrt{2}} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx$

Thus, $\int_0^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c\to 0}\int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c\to 0}\left[\sqrt{2} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx\right] = \sqrt{2} - 0 + \int_0^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx$

This is a proper integral.

35. $\int_{-3}^3 \frac{x}{\sqrt{9-x^2}}dx = \int_{-3}^0 \frac{x}{\sqrt{9-x^2}}dx + \int_0^3 \frac{x}{\sqrt{9-x^2}}dx = \lim_{b\to -3^+}\left[-\sqrt{9-x^2}\right]_b^0 + \lim_{b\to 3^-}\left[-\sqrt{9-x^2}\right]_0^b$

$= -\sqrt{9} + \lim_{b\to -3^+}\sqrt{9-b^2} - \lim_{b\to 3^-}\sqrt{9-b^2} + \sqrt{9} = -3 + 0 - 0 + 3 = 0$

36. $\int_{-3}^3 \frac{x}{9-x^2}dx = \int_{-3}^0 \frac{x}{9-x^2}dx + \int_0^3 \frac{x}{9-x^2}dx = \lim_{b\to 3^+}\left[-\frac{1}{2}\ln\left|9-x^2\right|\right]_b^0 + \lim_{b\to 3^-}\left[-\frac{1}{2}\ln\left|9-x^2\right|\right]_0^b$

$= -\ln 3 + \lim_{b\to -3^+}\frac{1}{2}\ln\left|9-b^2\right| - \lim_{b\to 3^-}\frac{1}{2}\ln\left|9-b^2\right| + \ln 3 = (-\ln 3 - \infty) + (\infty + \ln 3)$

The integral diverges.

37. $\int_{-4}^{4}\frac{1}{16-x^2}dx=\int_{-4}^{0}\frac{1}{16-x^2}dx+\int_{0}^{4}\frac{1}{16-x^2}dx=\lim_{b\to -4^+}\left[\frac{1}{8}\ln\left|\frac{x+4}{x-4}\right|\right]_b^0+\lim_{b\to 4^-}\left[\frac{1}{8}\ln\left|\frac{x+4}{x-4}\right|\right]_0^b$

$=\frac{1}{8}\ln 1-\lim_{b\to -4^+}\frac{1}{8}\ln\left|\frac{b+4}{b-4}\right|+\lim_{b\to 4^-}\frac{1}{8}\ln\left|\frac{b+4}{b-4}\right|-\frac{1}{8}\ln 1=(0+\infty)+(\infty-0)$

The integral diverges.

38. $\int_{-1}^{1}\frac{1}{x\sqrt{-\ln|x|}}dx=\int_{-1}^{-1/2}\frac{1}{x\sqrt{-\ln|x|}}dx+\int_{-1/2}^{0}\frac{1}{x\sqrt{-\ln|x|}}dx+\int_{0}^{1/2}\frac{1}{x\sqrt{-\ln|x|}}dx+\int_{1/2}^{1}\frac{1}{x\sqrt{-\ln|x|}}dx$

$=\lim_{b\to -1^+}\left[-2\sqrt{-\ln|x|}\right]_b^{-1/2}+\lim_{b\to 0^-}\left[-2\sqrt{-\ln|x|}\right]_{-1/2}^{b}+\lim_{b\to 0^+}\left[-2\sqrt{-\ln|x|}\right]_b^{1/2}+\lim_{b\to 1^-}\left[-2\sqrt{-\ln|x|}\right]_{1/2}^{b}$

$=(-2\sqrt{\ln 2}+0)+(-\infty+2\sqrt{\ln 2})+(-2\sqrt{\ln 2}+\infty)+(0+2\sqrt{\ln 2})$

The integral diverges.

39. $\int_0^\infty \frac{1}{x^p}dx=\int_0^1\frac{1}{x^p}dx+\int_1^\infty\frac{1}{x^p}dx$

If $p>1$, $\int_0^1\frac{1}{x^p}dx=\left[\frac{1}{-p+1}x^{-p+1}\right]_0^1$ diverges

since $\lim_{x\to 0^+}x^{-p+1}=\infty$.

If $p<1$ and $p\neq 0$, $\int_1^\infty\frac{1}{x^p}dx=\left[\frac{1}{-p+1}x^{-p+1}\right]_1^\infty$

diverges since $\lim_{x\to\infty}x^{-p+1}=\infty$.

If $p=0$, $\int_0^\infty dx=\infty$.

If $p=1$, both $\int_0^1\frac{1}{x}dx$ and $\int_1^\infty\frac{1}{x}dx$ diverge.

40. $\int_0^\infty f(x)dx$

$=\lim_{b\to 1^-}\int_0^b f(x)dx+\lim_{b\to 1^+}\int_b^c f(x)dx+\lim_{b\to\infty}\int_c^b f(x)dx$

where $1<c<\infty$.

41. $\int_0^8(x-8)^{-2/3}dx=\lim_{b\to 8^-}\left[3(x-8)^{1/3}\right]_0^b$

$=3(0)-3(-2)=6$

42. $\int_0^1\left(\frac{1}{x}-\frac{1}{x^3+x}\right)dx$

$=\lim_{b\to 0^-}\int_b^1\frac{x}{x^2+1}dx=\lim_{b\to 0^-}\left[\frac{1}{2}\ln\left|x^2+1\right|\right]_b^1$

$=\frac{1}{2}\ln 2-\lim_{b\to 0^-}\frac{1}{2}\ln\left|b^2+1\right|=\frac{1}{2}\ln 2$

43. a. $\int_0^1 x^{-2/3}dx=\lim_{b\to 0^+}\left[3x^{1/3}\right]_b^1=3$

b. $V=\pi\int_0^1 x^{-4/3}dx=\lim_{b\to 0^+}\pi\left[-3x^{-1/3}\right]_b^1$

$=-3\pi+3\pi\lim_{b\to 0}b^{-1/3}$

The limit tends to infinity as $b\to 0$, so the volume is infinite.

44. Since $\ln x<0$ for $0<x<1$, $b>1$

$\int_0^b\ln x\,dx=\lim_{c\to 0^-}\int_c^1\ln x\,dx+\int_1^b\ln x\,dx$

$=\lim_{c\to 0^+}\left[x\ln x-x\right]_c^1+\left[x\ln x-x\right]_1^b$

$=-1-\lim_{c\to 0^+}(c\ln c-c)+b\ln b-b+1$

$=b\ln b-b$

Thus, $b\ln b-b=0$ when $b=e$.

45. $\int_0^1\frac{\sin x}{x}dx$ is not an improper integral since

$\frac{\sin x}{x}$ is bounded in the interval $0\le x\le 1$.

46. For $x\ge 1$, $\frac{1}{1+x^4}<1$ so $\frac{1}{x^4(1+x^4)}<\frac{1}{x^4}$.

$\int_1^\infty\frac{1}{x^4}dx=\lim_{b\to\infty}\left[-\frac{1}{3x^3}\right]_1^b=-\lim_{b\to\infty}\frac{1}{3b^3}+\frac{1}{3}$

$=-0+\frac{1}{3}=\frac{1}{3}$

Thus, by the Comparison Test $\int_1^\infty\frac{1}{x^4(1+x^4)}dx$ converges.

47. For $x \geq 1$, $x^2 \geq x$ so $-x^2 \leq -x$, thus

$e^{-x^2} \leq e^{-x}$.

$$\int_1^\infty e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_1^b = -\lim_{b\to\infty}\frac{1}{e^b} + e^{-1}$$

$$= -0 + \frac{1}{e} = \frac{1}{e}$$

Thus, by the Comparison Test, $\int_1^\infty e^{-x^2}dx$ converges.

48. Since $\sqrt{x+2} - 1 \leq \sqrt{x+2}$ we know that

$\frac{1}{\sqrt{x+2}-1} \geq \frac{1}{\sqrt{x+2}}$. Consider $\int_0^\infty \frac{1}{\sqrt{x+2}}dx$

$$\int_2^\infty \frac{1}{\sqrt{x+2}}dx = \lim_{b\to\infty}\int_2^b \frac{1}{\sqrt{x+2}}dx$$

$$= \lim_{b\to\infty}\left[2\sqrt{x+2}\right]_2^\infty = \lim_{b\to\infty} 2\left(\sqrt{b+2} - 2\right) = \infty$$

Thus, by the Comparison Test of Problem 46, we conclude that $\int_0^\infty \frac{1}{\sqrt{x+2}}dx$ diverges.

49. Since $x^2 \ln(x+1) \geq x^2$, we know that

$\frac{1}{x^2\ln(x+1)} \leq \frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^\infty = 1$

we can apply the Comparison Test of Problem 46 to conclude that $\int_1^\infty \frac{1}{x^2\ln(x+1)}dx$ converges.

50. If $0 \leq f(x) \leq g(x)$ on $[a, b]$ and either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ or $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = \infty$, then the convergence of $\int_a^b g(x)dx$ implies the convergence of $\int_a^b f(x)dx$ and the divergence of $\int_a^b f(x)dx$ implies the divergence of $\int_a^b g(x)dx$.

51. a. From Example 2 of Section 8.2, $\lim_{x\to\infty}\frac{x^a}{e^x} = 0$ for a any positive real number.

Thus $\lim_{x\to\infty}\frac{x^{n+1}}{e^x} = 0$ for any positive real number n, hence there is a number M such that $0 < \frac{x^{n+1}}{e^x} \leq 1$ for $x \geq M$. Divide the inequality by x^2 to get that $0 < \frac{x^{n-1}}{e^x} \leq \frac{1}{x^2}$ for $x \geq M$.

b. $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = -\lim_{b\to\infty}\frac{1}{b} + \frac{1}{1}$

$= -0 + 1 = 1$

$$\int_1^\infty x^{n-1}e^{-x}dx = \int_1^M x^{n-1}e^{-x}dx + \int_M^\infty x^{n-1}e^{-x}dx$$

$$\leq \int_1^M x^{n-1}e^{-x}dx + \int_1^\infty \frac{1}{x^2}dx$$

$$= 1 + \int_1^M x^{n-1}e^{-x}dx$$

by part a and Problem 46. The remaining integral is finite, so $\int_1^\infty x^{n-1}e^{-x}dx$ converges.

52. $\int_0^1 e^{-x}dx = \left[-e^{-x}\right]_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}$, so the integral converges when $n = 1$. For $0 \leq x \leq 1$, $0 \leq x^{n-1} \leq 1$ for $n > 1$. Thus, $\frac{x^{n-1}}{e^x} = x^{n-1}e^{-x} \leq e^{-x}$. By the comparison test from Problem 50, $\int_0^1 x^{n-1}e^{-x}dx$ converges.

53. a. $\Gamma(1) = \int_0^\infty x^0 e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1$

b. $\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx$

Let $u = x^n, dv = e^{-x}dx$,

$du = nx^{n-1}dx, v = -e^{-x}$.

$$\Gamma(n+1) = [-x^n e^{-x}]_0^\infty + \int_0^\infty nx^{n-1}e^{-x}dx$$

$$= 0 + n\int_0^\infty x^{n-1}e^{-x}dx = n\Gamma(n)$$

c. From parts a and b,

$\Gamma(1) = 1, \Gamma(2) = 1\cdot\Gamma(1) = 1$,

$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$.

Suppose $\Gamma(n) = (n-1)!$, then by part b,

$\Gamma(n+1) = n\Gamma(n) = n[(n-1)!] = n!$.

54. $n = 1,\ \int_0^{\infty} e^{-x}dx = 1 = 0! = (1-1)!$

$n = 2,\ \int_0^{\infty} xe^{-x}dx = 1 = 1! = (2-1)!$

$n = 3,\ \int_0^{\infty} x^2e^{-x}dx = 2 = 2! = (3-1)!$

$n = 4,\ \int_0^{\infty} x^3e^{-x}dx = 6 = 3! = (4-1)!$

$n = 5,\ \int_0^{\infty} x^4e^{-x}dx = 24 = 4! = (5-1)!$

55. a. $\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} Cx^{\alpha-1}e^{-\beta x}dx$

Let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta}dy$.

$$\int_0^{\infty} Cx^{\alpha-1}e^{-\beta x}dx = \int_0^{\infty} C\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y}\frac{1}{\beta}dy = \frac{C}{\beta^{\alpha}}\int_0^{\infty} y^{\alpha-1}e^{-y}dy = C\beta^{-\alpha}\Gamma(\alpha)$$

$C\beta^{-\alpha}\Gamma(\alpha) = 1$ when $C = \frac{1}{\beta^{-\alpha}\Gamma(\alpha)} = \frac{\beta^{\alpha}}{\Gamma(\alpha)}$.

b. $\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha}e^{-\beta x}dx$

Let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta}dy$.

$$\mu = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha} e^{-y}\frac{1}{\beta}dy = \frac{1}{\beta\Gamma(\alpha)}\int_0^{\infty} y^{\alpha}e^{-y}dy = \frac{1}{\beta\Gamma(\alpha)}\Gamma(\alpha+1) = \frac{1}{\beta\Gamma(\alpha)}\alpha\Gamma(\alpha) = \frac{\alpha}{\beta}$$

(Recall that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for $\alpha > 0$.)

c. $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_0^{\infty}\left(x-\frac{\alpha}{\beta}\right)^2 \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(x^2 - \frac{2\alpha}{\beta}x + \frac{\alpha^2}{\beta^2}\right)x^{\alpha-1}e^{-\beta x}dx$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha+1}e^{-\beta x}dx - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha}e^{-\beta x}dx + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1}e^{-\beta x}dx$$

In all three integrals, let $y = \beta x$, so $x = \frac{y}{\beta}$ and $dx = \frac{1}{\beta}dy$.

$$\sigma^2 = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha+1} e^{-y}\frac{1}{\beta}dy - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha} e^{-y}\frac{1}{\beta}dy + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y}\frac{1}{\beta}dy$$

$$= \frac{1}{\beta^2\Gamma(\alpha)}\int_0^{\infty} y^{\alpha+1}e^{-y}dy - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\int_0^{\infty} y^{\alpha}e^{-y}dy + \frac{\alpha^2}{\beta^2\Gamma(\alpha)}\int_0^{\infty} y^{\alpha-1}e^{-y}dy$$

$$= \frac{1}{\beta^2\Gamma(\alpha)}\Gamma(\alpha+2) - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\Gamma(\alpha+1) + \frac{\alpha^2}{\beta^2\Gamma(\alpha)}\Gamma(\alpha) = \frac{1}{\beta^2\Gamma(\alpha)}(\alpha+1)\alpha\Gamma(\alpha) - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\alpha\Gamma(\alpha) + \frac{\alpha^2}{\beta^2}$$

$$= \frac{\alpha^2+\alpha}{\beta^2} - \frac{2\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

56. a. $L\{t^{\alpha}\}(s) = \int_0^{\infty} t^{\alpha} e^{-st}\,dt$

Let $t = \frac{x}{s}$, so $dt = \frac{1}{s}dx$, then

$$\int_0^{\infty} t^{\alpha} e^{-st}\,dt = \int_0^{\infty} \left(\frac{x}{s}\right)^{\alpha} e^{-x}\frac{1}{s}\,dx = \int_0^{\infty} \frac{1}{s^{\alpha+1}} x^{\alpha} e^{-x}\,dx = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

If $s \le 0$, $t^{\alpha}e^{-st} \to \infty$ as $t \to \infty$, so the integral does not converge. Thus, the transform is defined only when $s > 0$.

b. $L\{e^{\alpha t}\}(s) = \int_0^{\infty} e^{\alpha t} e^{-st}\,dt = \int_0^{\infty} e^{(\alpha - s)t}\,dt = \left[\frac{1}{\alpha - s} e^{(\alpha - s)t}\right]_0^{\infty} = \frac{1}{\alpha - s}\left[\lim_{b\to\infty} e^{(\alpha - s)b} - 1\right]$

$$\lim_{b\to\infty} e^{(\alpha - s)b} = \begin{cases} \infty & \text{if } \alpha > s \\ 0 & \text{if } s > \alpha \end{cases}$$

Thus, $L\{e^{\alpha t}\}(s) = \frac{-1}{\alpha - s} = \frac{1}{s - \alpha}$ when $s > \alpha$. (When $s \le \alpha$, the integral does not converge.)

c. $L\{\sin(\alpha t)\}(s) = \int_0^{\infty} \sin(\alpha t) e^{-st}\,dt$

Let $I = \int_0^{\infty} \sin(\alpha t) e^{-st}\,dt$ and use integration by parts with $u = \sin(\alpha t)$, $du = \alpha\cos(\alpha t)dt$,

$dv = e^{-st}dt$, and $v = -\frac{1}{s}e^{-st}$.

Then $I = \left[-\frac{1}{s}\sin(\alpha t)e^{-st}\right]_0^{\infty} + \frac{\alpha}{s}\int_0^{\infty}\cos(\alpha t)e^{-st}\,dt$

Use integration by parts on this integral with

$u = \cos(\alpha t)$, $du = -\alpha\sin(\alpha t)dt$, $dv = e^{-st}dt$, and $v = -\frac{1}{s}e^{-st}$.

$$I = \left[-\frac{1}{s}\sin(\alpha t)e^{-st}\right]_0^{\infty} + \frac{\alpha}{s}\left(\left[-\frac{1}{s}\cos(\alpha t)e^{-st}\right]_0^{\infty} - \frac{\alpha}{s}\int_0^{\infty}\sin(\alpha t)e^{-st}\,dt\right)$$

$$= -\frac{1}{s}\left[e^{-st}\left(\sin(\alpha t) + \frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^{\infty} - \frac{\alpha^2}{s^2}I$$

Thus,

$$I\left(1 + \frac{\alpha^2}{s^2}\right) = -\frac{1}{s}\left[e^{-st}\left(\sin(\alpha t) + \frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^{\infty}$$

$$I = -\frac{1}{s\left(1 + \frac{\alpha^2}{s^2}\right)}\left[e^{-st}\left(\sin(\alpha t) + \frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^{\infty} = -\frac{s}{s^2 + \alpha^2}\left[\lim_{b\to\infty} e^{-sb}\left(\sin(\alpha b) + \frac{\alpha}{s}\cos(\alpha b)\right) - \frac{\alpha}{s}\right]$$

$$\lim_{b\to\infty} e^{-sb}\left(\sin(\alpha b) + \frac{\alpha}{s}\cos(\alpha b)\right) = \begin{cases} 0 & \text{if } s > 0 \\ \infty & \text{if } s \le 0 \end{cases}$$

Thus, $I = \frac{\alpha}{s^2 + \alpha^2}$ when $s > 0$.

57. a. The integral is the area between the curve $y^2 = \frac{1-x}{x}$ and the x-axis from $x = 0$ to $x = 1$.

$y^2 = \frac{1-x}{x}$; $xy^2 = 1-x$; $x(y^2+1) = 1$

$x = \frac{1}{y^2+1}$

As $x \to 0$, $y = \sqrt{\frac{1-x}{x}} \to \infty$, while when $x = 1$, $y = \sqrt{\frac{1-1}{1}} = 0$, thus the area is

$\int_0^\infty \frac{1}{y^2+1} dy = \lim_{b\to\infty} [\tan^{-1} y]_0^b$

$= \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2}$

b. The integral is the area between the curve $y^2 = \frac{1+x}{1-x}$ and the x-axis from $x = -1$ to $x = 1$.

$y^2 = \frac{1+x}{1-x}$; $y^2 - xy^2 = 1+x$; $y^2 - 1 = x(y^2+1)$;

$x = \frac{y^2-1}{y^2+1}$

When $x = -1$, $y = \sqrt{\frac{1+(-1)}{1-(-1)}} = \sqrt{\frac{0}{2}} = 0$, while as $x \to 1$, $y = \sqrt{\frac{1+x}{1-x}} \to \infty$.

The area in question is the area to the right of the curve $y = \sqrt{\frac{1+x}{1-x}}$ and to the left of the line $x = 1$. Thus, the area is

$\int_0^\infty \left(1 - \frac{y^2-1}{y^2+1}\right) dy = \int_0^\infty \frac{2}{y^2+1} dy$

$= \lim_{b\to\infty} \left[2\tan^{-1} y\right]_0^b$

$\lim_{b\to\infty} 2\tan^{-1} b - 2\tan^{-1} 0 = 2\left(\frac{\pi}{2}\right) = \pi$

58. For $0 < x < 1$, $x^p > x^q$ so $2x^p > x^p + x^q$ and $\frac{1}{x^p + x^q} > \frac{1}{2x^p}$. For $1 < x$, $x^q > x^p$ so $2x^q > x^p + x^q$ and $\frac{1}{x^p + x^q} > \frac{1}{2x^q}$.

$\int_0^\infty \frac{1}{x^p + x^q} dx = \int_0^1 \frac{1}{x^p + x^q} dx + \int_1^\infty \frac{1}{x^p + x^q} dx$

Both of these integrals must converge.

$\int_0^1 \frac{1}{x^p + x^q} dx > \int_0^1 \frac{1}{2x^p} dx = \frac{1}{2}\int_0^1 \frac{1}{x^p} dx$ which converges if and only if $p < 1$.

$\int_1^\infty \frac{1}{x^p + x^q} dx > \int_1^\infty \frac{1}{2x^q} dx = \frac{1}{2}\int_1^\infty \frac{1}{x^q} dx$ which converges if and only if $q > 1$. Thus, $0 < p < 1$ and $1 < q$.

8.5 Chapter Review

Concepts Test

1. True: See Example 2 of Section 8.2.

2. True: Use l'Hôpital's Rule.

3. False: $\lim_{x\to\infty} \frac{1000x^4 + 1000}{0.001x^4 + 1} = \frac{1000}{0.001} = 10^6$

4. False: $\lim_{x\to\infty} xe^{-1/x} = \infty$ since $e^{-1/x} \to 1$ and $x \to \infty$ as $x \to \infty$.

5. False: For example, if $f(x) = x$ and $g(x) = e^x$, $\lim_{x\to\infty} \frac{x}{e^x} = 0$.

6. False: See Example 7 of Section 8.2.

7. True: Take the inner limit first.

8. True: Raising a small number to a large exponent results in an even smaller number.

9. True: Since $\lim_{x\to a} f(x) = -1 \neq 0$, it serves only to affect the sign of the limit of the product.

10. False: Consider $f(x)=(x-a)^2$ and $g(x)=\dfrac{1}{(x-a)^2}$, then $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\infty$, while $\lim\limits_{x\to a}[f(x)g(x)]=1.$

11. False: Consider $f(x)=3x^2$ and $g(x)=x^2+1$, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{3x^2}{x^2+1}$$

$$=\lim_{x\to\infty}\frac{3}{1+\frac{1}{x^2}}=3,\text{ but}$$

$$\lim_{x\to\infty}[f(x)-3g(x)]$$

$$=\lim_{x\to\infty}[3x^2-3(x^2+1)]$$

$$=\lim_{x\to\infty}[-3]=-3$$

12. True: As $x\to a$, $f(x)\to 2$ while $\dfrac{1}{|g(x)|}\to\infty.$

13. True: See Example 7 of Section 8.2.

14. True: Let $y=[1+f(x)]^{1/f(x)}$, then $\ln y=\dfrac{1}{f(x)}\ln[1+f(x)].$

$$\lim_{x\to a}\frac{1}{f(x)}\ln[1+f(x)]=\lim_{x\to a}\frac{\ln[1+f(x)]}{f(x)}$$

This limit is of the form $\dfrac{0}{0}.$

$$\lim_{x\to a}\frac{\ln[1+f(x)]}{f(x)}=\lim_{x\to a}\frac{\frac{1}{1+f(x)}f'(x)}{f'(x)}$$

$$=\lim_{x\to a}\frac{1}{1+f(x)}=1$$

$$\lim_{x\to a}[1+f(x)]^{1/f(x)}=\lim_{x\to a}e^{\ln y}=e^1=e$$

15. True: Use repeated applications of l'Hôpital's Rule.

16. True: $e^0=1$ and $p(0)$ is the constant term.

17. False: Consider $f(x)=3x^2+x+1$ and $g(x)=4x^3+2x+3;\ \ f'(x)=6x+1$

$g'(x)=12x^2+2,$ and so

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{6x+1}{12x^2+2}=\frac{1}{2}\text{ while}$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{3x^2+x+1}{4x^3+2x+3}=\frac{1}{3}$$

18. False: $p>1$. See Example 4 of Section 8.4.

19. True: $\displaystyle\int_0^\infty\frac{1}{x^p}dx=\int_0^1\frac{1}{x^p}dx+\int_1^\infty\frac{1}{x^p}dx;$

$\displaystyle\int_0^1\frac{1}{x^p}dx$ diverges for $p\ge 1$ and

$\displaystyle\int_1^\infty\frac{1}{x^p}dx$ diverges for $p\le 1.$

20. False: Consider $\displaystyle\int_0^\infty\frac{1}{x+1}dx$.

21. True: $\displaystyle\int_{-\infty}^\infty f(x)dx=\int_{-\infty}^0 f(x)dx+\int_0^\infty f(x)dx$

If f is an even function, then $f(-x)=f(x)$ so

$$\int_{-\infty}^0 f(x)dx=\int_0^\infty f(x)dx.$$

Thus, both integrals making up $\displaystyle\int_{-\infty}^\infty f(x)dx$ converge so their sum converges.

22. False: See Problem 37 of Section 8.3.

23. True: $\displaystyle\int_0^\infty f'(x)dx=\lim_{b\to\infty}\int_0^b f'(x)dx$

$$=\lim_{b\to\infty}[f(x)]_0^b=\lim_{b\to\infty}f(b)-f(0)$$

$= 0 - f(0) = -f(0).$

$f(0)$ must exist and be finite since $f'(x)$ is continuous on $[0,\infty)$.

24. True: $\displaystyle\int_0^\infty f(x)dx\le\int_0^\infty e^{-x}dx=\lim_{b\to\infty}[-e^{-x}]_0^b$

$=\lim\limits_{b\to\infty}-e^{-b}+1=1,$ so $\displaystyle\int_0^\infty f(x)dx$ must converge.

25. False: The integrand is bounded on the interval $\left[0,\dfrac{\pi}{4}\right]$.

Sample Test Problems

1. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{4x}{\tan x}=\lim_{x\to 0}\frac{4}{\sec^2 x}=4$$

2. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\tan 2x}{\sin 3x}=\lim_{x\to 0}\frac{2\sec^2 2x}{3\cos 3x}=\frac{2}{3}$$

3. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\sin x-\tan x}{\frac{1}{3}x^2}=\lim_{x\to 0}\frac{\cos x-\sec^2 x}{\frac{2}{3}x}$$

$$=\lim_{x\to 0}\frac{-\sin x-2\sec x(\sec x\tan x)}{\frac{2}{3}}=0$$

4. $\lim_{x\to 0}\frac{\cos x}{x^2}=\infty$ (L'Hôpital's Rule does not apply since cos(0) = 1.)

5. $\lim_{x\to 0}2x\cot x=\lim_{x\to 0}\frac{2x\cos x}{\sin x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2x\cos x}{\sin x}=\lim_{x\to 0}\frac{2\cos x-2x\sin x}{\cos x}$$

$$=\frac{2-0}{1}=2$$

6. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 1^-}\frac{\ln(1-x)}{\cot \pi x}=\lim_{x\to 1^-}\frac{-\frac{1}{1-x}}{-\pi\csc^2 \pi x}$$

$$=\lim_{x\to 1^-}\frac{\sin^2 \pi x}{\pi(1-x)}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1^-}\frac{\sin^2 \pi x}{\pi(1-x)}=\lim_{x\to 1^-}\frac{2\pi\sin \pi x\cos \pi x}{-\pi}=0$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{t\to\infty}\frac{\ln t}{t^2}=\lim_{t\to\infty}\frac{\frac{1}{t}}{2t}=\lim_{t\to\infty}\frac{1}{2t^2}=0$$

8. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{2x^3}{\ln x}=\lim_{x\to\infty}\frac{6x^2}{\frac{1}{x}}=\lim_{x\to\infty}6x^3=\infty$$

9. As $x\to 0, \sin x\to 0$, and $\frac{1}{x}\to\infty$. A number less than 1, raised to a large power, is a very small number $\left(\left(\frac{1}{2}\right)^{32}=2.328\times 10^{-10}\right)$ so

$$\lim_{x\to 0^+}(\sin x)^{1/x}=0.$$

10. $\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$$

11. The limit is of the form 0^0.

Let $y=x^x$, then $\ln y=x\ln x$.

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$$

$$\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{\ln y}=1$$

12. The limit is of the form 1^∞.

Let $y=(1+\sin x)^{2/x}$, then $\ln y=\frac{2}{x}\ln(1+\sin x)$.

$$\lim_{x\to 0}\frac{2}{x}\ln(1+\sin x)=\lim_{x\to 0}\frac{2\ln(1+\sin x)}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2\ln(1+\sin x)}{x}=\lim_{x\to 0}\frac{\frac{2}{1+\sin x}\cos x}{1}$$

$$=\lim_{x\to 0}\frac{2\cos x}{1+\sin x}=\frac{2}{1}=2$$

$$\lim_{x\to 0}(1+\sin x)^{2/x}=\lim_{x\to 0}e^{\ln y}=e^2$$

13. $\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}}$

The limit is of the form $\frac{\infty}{\infty}$.

$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$

14. The limit is of the form ∞^0.

Let $y = t^{1/t}$, then $\ln y = \frac{1}{t}\ln t$.

$\lim_{t\to\infty} \frac{1}{t}\ln t = \lim_{t\to\infty} \frac{\ln t}{t}$

The limit is of the form $\frac{\infty}{\infty}$.

$\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = \lim_{t\to\infty} \frac{1}{t} = 0$

$\lim_{t\to\infty} t^{1/t} = \lim_{t\to\infty} e^{\ln y} = 1$

15. $\lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0^+} \frac{x - \sin x}{x\sin x}$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$\lim_{x\to 0^+} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0^+} \frac{1-\cos x}{\sin x + x\cos x}$

$= \lim_{x\to 0^+} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$

16. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule three times.)

$\lim_{x\to\frac{\pi}{2}} \frac{\tan 3x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{3\sec^2 3x}{\sec^2 x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{3\cos^2 x}{\cos^2 3x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x\sin x}{\cos 3x\sin 3x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x - \sin^2 x}{3(\cos^2 3x - \sin^2 3x)} = -\frac{1}{3(0-1)} = \frac{1}{3}$

17. The limit is of the form 1^∞.

Let $y = (\sin x)^{\tan x}$, then $\ln y = \tan x \ln(\sin x)$.

$\lim_{x\to\frac{\pi}{2}} \tan x\ln(\sin x) = \lim_{x\to\frac{\pi}{2}} \frac{\sin x\ln(\sin x)}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$\lim_{x\to\frac{\pi}{2}} \frac{\sin x\ln(\sin x)}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x\ln(\sin x) + \frac{\sin x}{\sin x}\cos x}{\sin x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x(1+\ln(\sin x))}{\sin x} = \frac{0}{1} = 0$

$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = \lim_{x\to\frac{\pi}{2}} e^{\ln y} = 1$

18. $\lim_{x\to\frac{\pi}{2}} \left(x\tan x - \frac{\pi}{2}\sec x\right) = \lim_{x\to\frac{\pi}{2}} \frac{x\sin x - \frac{\pi}{2}}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$\lim_{x\to\frac{\pi}{2}} \frac{x\sin x - \frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\sin x + x\cos x}{\sin x} = \frac{1}{1} = 1$

19. $\int_0^\infty \frac{dx}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_0^\infty = 0 + 1 = 1$

20. $\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

21. $\int_{-\infty}^1 e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_{-\infty}^1 = \frac{1}{2}e^2 - 0 = \frac{1}{2}e^2$

22. $\int_{-1}^1 \frac{dx}{1-x} = \lim_{b\to 1}[-\ln(1-x)]_{-1}^b$

$= -\lim_{b\to 1}\ln(1-b) + \ln 2 = \infty$

The integral diverges.

23. $\int_0^\infty \frac{dx}{x+1} = [\ln(x+1)]_0^\infty = \infty - 0 = \infty$

The integral diverges.

24. $\int_{\frac{1}{2}}^{2} \frac{dx}{x(\ln x)^{1/5}} = \lim_{b\to 1^-} \int_{\frac{1}{2}}^{b} \frac{dx}{x(\ln x)^{1/5}} + \lim_{b\to 1^+} \int_{b}^{2} \frac{dx}{x(\ln x)^{1/5}} = \lim_{b\to 1^-}\left[\frac{5}{4}(\ln x)^{4/5}\right]_{\frac{1}{2}}^{b} + \lim_{b\to 1^+}\left[\frac{5}{4}(\ln x)^{4/5}\right]_{b}^{2}$

$= \left(\frac{5}{4}(0) - \frac{5}{4}\left(\ln\frac{1}{2}\right)^{4/5}\right) + \left(\frac{5}{4}(\ln 2)^{4/5} - \frac{5}{4}(0)\right) = \frac{5}{4}(\ln 2)^{4/5} - \frac{5}{4}\left(\ln\frac{1}{2}\right)^{4/5} = \frac{5}{4}[(\ln 2)^{4/5} - (-\ln 2)^{4/5}]$

$= \frac{5}{4}[(\ln 2)^{4/5} - (\ln 2)^{4/5}] = 0$

25. $\int_{1}^{\infty} \frac{dx}{x^2+x^4} = \int_{1}^{\infty}\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)dx = \left[-\frac{1}{x} - \tan^{-1} x\right]_{1}^{\infty} = 0 - \frac{\pi}{2} + 1 + \tan^{-1} 1 = 1 + \frac{\pi}{4} - \frac{\pi}{2} = 1 - \frac{\pi}{4}$

26. $\int_{-\infty}^{1} \frac{dx}{(2-x)^2} = \left[\frac{1}{2-x}\right]_{-\infty}^{1} = \frac{1}{1} - 0 = 1$

27. $\int_{-2}^{0} \frac{dx}{2x+3} = \lim_{b\to -\frac{3}{2}^-} \int_{-2}^{b} \frac{dx}{2x+3} + \lim_{b\to -\frac{3}{2}^+} \int_{b}^{0} \frac{dx}{2x+3} = \lim_{b\to -\frac{3}{2}^-}\left[\frac{1}{2}\ln|2x+3|\right]_{-2}^{b} + \lim_{b\to -\frac{3}{2}^+}\left[\frac{1}{2}\ln|2x+3|\right]_{b}^{0}$

$= \left(\lim_{b\to -\frac{3}{2}^-} \frac{1}{2}\ln|2b+3| - \frac{1}{2}(0)\right) + \left(\frac{1}{2}\ln 3 - \lim_{b\to -\frac{3}{2}^+} \frac{1}{2}\ln|2b+3|\right) = (-\infty) + \left(\frac{1}{2}\ln 3 + \infty\right)$

The integral diverges.

28. $\int_{1}^{4} \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^+} [2\sqrt{x-1}]_{b}^{4} = 2\sqrt{3} - \lim_{b\to 1^+} 2\sqrt{x-1} = 2\sqrt{3} - 0 = 2\sqrt{3}$

29. $\int_{2}^{\infty} \frac{dx}{x(\ln x)^2} = \left[-\frac{1}{\ln x}\right]_{2}^{\infty} = -0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$

30. $\int_{0}^{\infty} \frac{dx}{e^{x/2}} = \left[-\frac{2}{e^{x/2}}\right]_{0}^{\infty} = -0 + \frac{2}{1} = 2$

31. $\int_{3}^{5} \frac{dx}{(4-x)^{2/3}} = \lim_{b\to 4^-} \int_{3}^{b} \frac{dx}{(4-x)^{2/3}} + \lim_{b\to 4^+} \int_{b}^{5} \frac{dx}{(4-x)^{2/3}} = \lim_{b\to 4^-}\left[-3(4-x)^{1/3}\right]_{3}^{b} + \lim_{b\to 4^+}\left[-3(4-x)^{1/3}\right]_{b}^{5}$

$= \lim_{b\to 4^-} -3(4-b)^{1/3} + 3(1)^{1/3} - 3(-1)^{1/3} + \lim_{b\to 4^+} 3(4-b)^{1/3} = 0 + 3 + 3 + 0 = 6$

32. $\int_{2}^{\infty} xe^{-x^2}\, dx = \left[-\frac{1}{2}e^{-x^2}\right]_{2}^{\infty} = 0 + \frac{1}{2}e^{-4} = \frac{1}{2}e^{-4}$

33. $\int_{-\infty}^{\infty} \frac{x}{x^2+1}dx = \int_{-\infty}^{0} \frac{x}{x^2+1}dx + \int_{0}^{\infty} \frac{x}{x^2+1}dx$

$= \frac{1}{2}\left[\ln(x^2+1)\right]_{-\infty}^{0} + \frac{1}{2}\left[\ln(x^2+1)\right]_{0}^{\infty} =$ The integral diverges.

$(0+\infty) + (\infty - 0)$

34. $\int_{-\infty}^{\infty} \frac{x}{1+x^4}dx = \int_{-\infty}^{0} \frac{x}{1+x^4}dx + \int_{0}^{\infty} \frac{x}{1+x^4}dx = \left[\frac{1}{2}\tan^{-1} x^2\right]_{-\infty}^{0} + \left[\frac{1}{2}\tan^{-1} x^2\right]_{0}^{\infty}$

$= \frac{1}{2}\tan^{-1} 0 - \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\tan^{-1} 0 = 0 - \frac{\pi}{4} + \frac{\pi}{4} - 0 = 0$

35. $\frac{e^x}{e^{2x}+1}=\frac{e^x}{(e^x)^2+1}$

Let $u=e^x, du=e^x dx$

$\int_0^{\infty}\frac{e^x}{e^{2x}+1}dx=\int_1^{\infty}\frac{1}{u^2+1}du=\left[\tan^{-1}u\right]_1^{\infty}=\frac{\pi}{2}-\tan^{-1}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

36. Let $u=x^3, du=3x^2 dx$

$\int_{-\infty}^{\infty}x^2e^{-x^3}dx=\int_{-\infty}^{\infty}\frac{1}{3}e^{-u}du=\frac{1}{3}\int_{-\infty}^{0}e^{-u}du+\frac{1}{3}\int_0^{\infty}e^{-u}du=\frac{1}{3}\left[-e^{-u}\right]_{-\infty}^{0}+\frac{1}{3}\left[-e^{-u}\right]_0^{\infty}=\frac{1}{3}(-1+\infty)+\frac{1}{3}(-0+1)$

The integral diverges.

37. $\int_{-3}^{3}\frac{x}{\sqrt{9-x^2}}dx=0$

See Problem 35 in Section 8.4.

38. let $u=\ln(\cos x)$, then $du=\frac{1}{\cos x}\cdot -\sin x\,dx=-\tan x\,dx$

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\tan x}{(\ln\cos x)^2}dx=\int_{\ln\frac{1}{2}}^{-\infty}-\frac{1}{u^2}du=\int_{-\infty}^{\ln\frac{1}{2}}\frac{1}{u^2}du=\left[-\frac{1}{u}\right]_{-\infty}^{\ln\frac{1}{2}}=-\frac{1}{\ln\frac{1}{2}}+0=\frac{1}{\ln 2}$

39. For $p\neq 1, p\neq 0$, $\int_1^{\infty}\frac{1}{x^p}dx=\left[-\frac{1}{(p-1)x^{p-1}}\right]_1^{\infty}=\lim_{b\to\infty}\frac{1}{(1-p)b^{p-1}}+\frac{1}{p-1}$

$\lim_{b\to\infty}\frac{1}{b^{p-1}}=0$ when $p-1>0$ or $p>1$, and $\lim_{b\to\infty}\frac{1}{b^{p-1}}=\infty$ when $p<1, p\neq 0$.

When $p=1$, $\int_1^{\infty}\frac{1}{x}dx=[\ln x]_1^{\infty}=\infty-0$. The integral diverges.

When $p=0$, $\int_1^{\infty}1dx=[x]_1^{\infty}=\infty-1$. The integral diverges.

$\int_1^{\infty}\frac{1}{x^p}dx$ converges when $p>1$ and diverges when $p\le 1$.

40. For $p\neq 1, p\neq 0$, $\int_0^1\frac{1}{x^p}dx=\left[-\frac{1}{(p-1)x^{p-1}}\right]_0^1=\frac{1}{1-p}+\lim_{b\to 0}\frac{1}{(p-1)b^{p-1}}$

$\lim_{b\to 0}\frac{1}{b^{p-1}}$ converges when $p-1<0$ or $p<1$.

When $p=1$, $\int_0^1\frac{1}{x}dx=[\ln x]_0^1=0-\lim_{b\to 0^+}\ln b=\infty$. The integral diverges.

When $p=0$, $\int_0^1 1dx=[x]_0^1=1-0=1$

$\int_0^1\frac{1}{x^p}dx$ converges when $p<1$ and diverges when $1\le p$.

41. For $x\ge 1$, $x^6+x>x^6$, so $\sqrt{x^6+x}>\sqrt{x^6}=x^3$ and $\frac{1}{\sqrt{x^6+x}}<\frac{1}{x^3}$. Hence, $\int_1^{\infty}\frac{1}{\sqrt{x^6+x}}dx<\int_1^{\infty}\frac{1}{x^3}dx$ which converges since $3>1$ (see Problem 39). Thus $\int_1^{\infty}\frac{1}{\sqrt{x^6+x}}dx$ converges.

42. For $x > 1$, $\ln x < e^x$, so $\frac{\ln x}{e^x} < 1$ and

$\frac{\ln x}{e^{2x}} = \frac{\ln x}{(e^x)^2} < \frac{1}{e^x}$.

Hence,

$\int_1^{\infty} \frac{\ln x}{e^{2x}} dx < \int_1^{\infty} e^{-x} dx = [-e^{-x}]_1^{\infty} = -0 + e^{-1} = \frac{1}{e}$.

Thus, $\int_1^{\infty} \frac{\ln x}{e^{2x}} dx$ converges.

43. For $x > 3$, $\ln x > 1$, so $\frac{\ln x}{x} > \frac{1}{x}$. Hence,

$\int_3^{\infty} \frac{\ln x}{x} dx > \int_3^{\infty} \frac{1}{x} dx = [\ln x]_3^{\infty} = \infty - \ln 3$.

The integral diverges, thus $\int_3^{\infty} \frac{\ln x}{x} dx$ also diverges.

44. For $x \geq 1$, $\ln x < x$, so $\frac{\ln x}{x} < 1$ and $\frac{\ln x}{x^3} < \frac{1}{x^2}$.

Hence,

$\int_1^{\infty} \frac{\ln x}{x^3} dx < \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{\infty} = -0 + 1 = 1$.

Thus, $\int_1^{\infty} \frac{\ln x}{x^3} dx$ converges.

Review and Preview Problems

1. Original: If $x > 0$, then $x^2 > 0$ (AT)

Converse: If $x^2 > 0$, then $x > 0$

Contrapositive: If $x^2 \leq 0$, then $x \leq 0$ (AT)

2. Original: If $x^2 > 0$, then $x > 0$

Converse: If $x > 0$, then $x^2 > 0$ (AT)

Contrapositive: If $x \leq 0$, then $x^2 \leq 0$

3. Original:
f differentiable at $c \Rightarrow f$ continuous at c (AT)
Converse:
f continuous at $c \Rightarrow f$ differentiable at c
Contrapositive:
f discontinuous at $c \Rightarrow f$ non-differentiable at c (AT)

4. Original:
f continuous at $c \Rightarrow f$ differentiable at c
Converse:
f differentiable at $c \Rightarrow f$ continuous at c (AT)
Contrapositive:
f non-differentiable at $c \Rightarrow f$ discontinuous at c

5. Original:
f right continuous at $c \Rightarrow f$ continuous at c
Converse:
f continuous at $c \Rightarrow f$ right continuous at c (AT)
Contrapositive:
f discontinuous at $c \Rightarrow f$ not right continuous at c

6. Original: $f'(x) \equiv 0 \Rightarrow f(x) = c$ (AT)
Converse: $f(x) = c \Rightarrow f'(x) \equiv 0$ (AT)
Contrapositive: $f(x) \neq c \Rightarrow f'(x) \not\equiv 0$ (AT)

7. Original: $f(x) = x^2 \Rightarrow f'(x) = 2x$ (AT)
Converse: $f'(x) = 2x \Rightarrow f(x) = x^2$
(Could have $f(x) = x^2 + 3$)
Contrapositive: $f'(x) \neq 2x \Rightarrow f(x) \neq x^2$ (AT)

8. Original: $a < b \Rightarrow a^2 < b^2$
Converse: $a^2 < b^2 \Rightarrow a < b$
Contrapositive: $a^2 \geq b^2 \Rightarrow a \geq b$

9. $1 + \frac{1}{2} + \frac{1}{4} = \frac{4}{4} + \frac{2}{4} + \frac{1}{4} = \frac{7}{4}$

10. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} =$
$\frac{32}{32} + \frac{16}{32} + \frac{8}{32} + \frac{4}{32} + \frac{2}{32} + \frac{1}{32} = \frac{63}{32}$

11. $\sum_{i=1}^{4} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$

12. $\sum_{k=1}^{4} \frac{(-1)^k}{2^k} = \frac{-1}{2} + \frac{1}{4} + \frac{-1}{8} + \frac{1}{16} =$
$\frac{-8+4-2+1}{16} = \frac{-5}{16}$

13. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$:

$\lim_{x \to \infty} \frac{x}{2x+1} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2}$

14. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$ twice:

$$\lim_{n\to\infty}\frac{n^2}{2n^2+1}=\lim_{n\to\infty}\frac{2n}{4n}=\frac{2}{4}=\frac{1}{2}$$

15. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$ twice:

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$

16. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$ twice:

$$\lim_{n\to\infty}\frac{n^2}{e^n}=\lim_{n\to\infty}\frac{2n}{e^n}=\lim_{n\to\infty}\frac{2}{e^n}=0$$

17. $\int_1^\infty \frac{1}{x}dx=\lim_{t\to\infty}\int_1^t \frac{1}{x}dx=$

$\lim_{t\to\infty}\left[\ln x\right]_1^t=\lim_{t\to\infty}\left[\ln t\right]=\infty$

Integral does not converge.

18. $\int_1^\infty \frac{1}{x^2}dx=\lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx=$

$\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t=\lim_{t\to\infty}\left[1-\frac{1}{t}\right]=1$

Integral converges.

19. $\int_1^\infty \frac{1}{x^{1.001}}dx=\lim_{t\to\infty}\int_1^t \frac{1}{x^{1.001}}dx=$

$\lim_{t\to\infty}\left[-\frac{1000}{x^{0.001}}\right]_1^t=\lim_{t\to\infty}\left[1000-\frac{1000}{t^{0.001}}\right]=1000$

Integral converges.

20. $\int_1^\infty \frac{x}{x^2+1}dx=\lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}dx=$

$u=x^2+1$
$du=2x\,dx$

$\frac{1}{2}\lim_{t\to\infty}\int_2^{t^2+1}\frac{1}{u}du=\infty$

Integral does not converge (see problem 17).

21. $\int_1^\infty \frac{x}{x^2+1}dx=\lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}dx=\frac{1}{2}\ln\left(x^2+1\right)\Big|_1^\infty=\infty$

Integral does not converge.

22. $\int_2^\infty \frac{1}{x(\ln x)^2}dx=\lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}dx=$

$u=\ln x$
$du=\frac{1}{x}dx$

$\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u^2}du=\lim_{t\to\infty}\left[-\frac{1}{u}\right]_{\ln 2}^{\ln t}=$

$\lim_{t\to\infty}\left[\frac{1}{\ln 2}-\frac{1}{\ln t}\right]=\frac{1}{\ln 2}\approx 1.443$

Integral converges.

CHAPTER 9 Infinite Series

9.1 Concepts Review

1. a sequence

2. $\lim_{n\to\infty} a_n$ exists (finite sense)

3. bounded above

4. –1; 1

Problem Set 9.1

1. $a_1 = \frac{1}{2}, a_2 = \frac{2}{5}, a_3 = \frac{3}{8}, a_4 = \frac{4}{11}, a_5 = \frac{5}{14}$

$\lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3};$

converges

2. $a_1 = \frac{5}{2}, a_2 = \frac{8}{3}, a_3 = \frac{11}{4}, a_4 = \frac{14}{5}, a_5 = \frac{17}{6}$

$\lim_{n\to\infty} \frac{3n+2}{n+1} = \lim_{n\to\infty} \frac{3+\frac{2}{n}}{1+\frac{1}{n}} = 3;$

converges

3. $a_1 = \frac{6}{3} = 2, a_2 = \frac{18}{9} = 2, a_3 = \frac{38}{17},$

$a_4 = \frac{66}{27} = \frac{22}{9}, a_5 = \frac{102}{39} = \frac{34}{13}$

$\lim_{n\to\infty} \frac{4n^2+2}{n^2+3n-1} = \lim_{n\to\infty} \frac{4+\frac{2}{n^2}}{1+\frac{3}{n}-\frac{1}{n^2}} = 4;$

converges

4. $a_1 = 5, a_2 = \frac{14}{3}, a_3 = \frac{29}{5}, a_4 = \frac{50}{7}, a_5 = \frac{77}{9}$

$\lim_{n\to\infty} \frac{3n^2+2}{2n-1} = \lim_{n\to\infty} \frac{3n+\frac{2}{n}}{2-\frac{1}{n}} = \infty;$

diverges

5. $a_1 = \frac{7}{8}, a_2 = \frac{26}{27}, a_3 = \frac{63}{64}, \ a_4 = \frac{124}{125}, a_5 = \frac{215}{216}$

$\lim_{n\to\infty} \frac{n^3+3n^2+3n}{(n+1)^3} = \lim_{n\to\infty} \frac{n^3+3n^2+3n}{n^3+3n^2+3n+1}$

$= \lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}} = 1$

6. $a_1 = \frac{\sqrt{5}}{3}, a_2 = \frac{\sqrt{14}}{5}, a_3 = \frac{\sqrt{29}}{7},$

$a_4 = \frac{\sqrt{50}}{9} = \frac{5\sqrt{2}}{9}, a_5 = \frac{\sqrt{77}}{11}$

$\lim_{n\to\infty} \frac{\sqrt{3n^2+2}}{2n+1} = \lim_{n\to\infty} \frac{\sqrt{3+\frac{2}{n^2}}}{2+\frac{1}{n}} = \frac{\sqrt{3}}{2};$

converges

7. $a_1 = -\frac{1}{3}, a_2 = \frac{2}{4} = \frac{1}{2}, a_3 = -\frac{3}{5}, a_4 = \frac{4}{6} = \frac{2}{3},$

$a_5 = -\frac{5}{7}$

$\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1,$ but since it alternates between positive and negative, the sequence diverges.

8. $a_1 = -1, a_2 = \frac{2}{3}, a_3 = -\frac{3}{5}, a_4 = \frac{4}{7}, a_5 = -\frac{5}{9}$

$\cos(n\pi) = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$

$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2},$ but since $\cos(n\pi)$ alternates between 1 and –1, the sequence diverges.

9. $a_1 = -1, a_2 = \frac{1}{2}, a_3 = -\frac{1}{3}, a_4 = \frac{1}{4}, a_5 = -\frac{1}{5}$

$\cos(n\pi) = (-1)^n$, so $-\frac{1}{n} \le \frac{\cos(n\pi)}{n} \le \frac{1}{n}$.

$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, so by the Squeeze Theorem, the sequence converges to 0.

10. $a_1 = e^{-1}\sin 1 \approx 0.3096, a_2 = e^{-2}\sin 2 \approx 0.1231,$

$a_3 = e^{-3}\sin 3 \approx 0.0070, a_4 = e^{-4}\sin 4 \approx -0.0139,$

$a_5 = e^{-5}\sin 5 \approx -0.0065$

$-1 \le \sin n \le 1$ for all n, so

$-e^{-n} \le e^{-n}\sin n \le e^{-n}$.

$\lim_{n\to\infty} -e^{-n} = \lim_{n\to\infty} e^{-n} = 0$, so by the Squeeze Theorem, the sequence converges to 0.

11. $a_1 = \frac{e^2}{3} \approx 2.4630, a_2 = \frac{e^4}{9} \approx 6.0665,$

$a_3 = \frac{e^6}{17} \approx 23.7311, a_4 = \frac{e^8}{27} \approx 110.4059,$

$a_5 = \frac{e^{10}}{39} \approx 564.7812$

Consider

$\lim_{x\to\infty} \frac{e^{2x}}{x^2+3x-1} = \lim_{x\to\infty} \frac{2e^{2x}}{2x+3} = \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty$

by using l'Hôpital's Rule twice. The sequence diverges.

12. $a_1 = \frac{e^2}{4} \approx 1.8473, a_2 = \frac{e^4}{16} \approx 3.4124,$

$a_3 = \frac{e^6}{64} \approx 6.3036, a_4 = \frac{e^8}{256} \approx 11.6444,$

$a_5 = \frac{e^{10}}{1024} \approx 21.510$

$\frac{e^{2n}}{4^n} = \left(\frac{e^2}{4}\right)^n, \frac{e^2}{4} > 1$ so the sequence diverges.

13. $a_1 = -\frac{\pi}{5} \approx -0.6283, a_2 = \frac{\pi^2}{25} \approx 0.3948,$

$a_3 = -\frac{\pi^3}{125} \approx -0.2481, a_4 = \frac{\pi^4}{625} \approx 0.1559,$

$a_5 = -\frac{\pi^5}{3125} \approx -0.0979$

$\frac{(-\pi)^n}{5^n} = \left(-\frac{\pi}{5}\right)^n, -1 < -\frac{\pi}{5} < 1,$ thus the sequence converges to 0.

14. $a_1 = \frac{1}{4} + \sqrt{3} \approx 1.9821, a_2 = \frac{1}{16} + 3 = 3.0625,$

$a_3 = \frac{1}{64} + 3\sqrt{3} \approx 5.2118, a_4 = \frac{1}{256} + 9 \approx 9.0039,$

$a_5 = \frac{1}{1024} + 9\sqrt{3} \approx 15.589$

$\left(\frac{1}{4}\right)^n$ converges to 0 since $-1 < \frac{1}{4} < 1$.

$3^{n/2} = \left(\sqrt{3}\right)^n$ diverges since $\sqrt{3} \approx 1.732 > 1$.

Thus, the sum diverges.

15. $a_1 = 2.99, a_2 = 2.9801, a_3 \approx 2.9703,$

$a_4 \approx 2.9606, a_5 \approx 2.9510$

$(0.99)^n$ converges to 0 since $-1 < 0.99 < 1$, thus $2 + (0.99)^n$ converges to 2.

16. $a_1 = \frac{1}{e} \approx 0.3679, a_2 = \frac{2^{100}}{e^2} \approx 1.72\times 10^{29},$

$a_3 = \frac{3^{100}}{e^3} \approx 2.57\times 10^{46}, a_4 = \frac{4^{100}}{e^4} \approx 2.94\times 10^{58},$

$a_5 = \frac{5^{100}}{e^5} \approx 5.32\times 10^{67}$

Consider $\lim_{x\to\infty} \frac{x^{100}}{e^x}$. By Example 2 of Section 8.2, $\lim_{x\to\infty} \frac{x^{100}}{e^x} = 0$. Thus, $\lim_{n\to\infty} \frac{n^{100}}{e^n} = 0$; converges

17. $a_1 = \frac{\ln 1}{\sqrt{1}} = 0, a_2 = \frac{\ln 2}{\sqrt{2}} \approx 0.4901,$

$a_3 = \frac{\ln 3}{\sqrt{3}} \approx 0.6343, a_4 = \frac{\ln 4}{2} \approx 0.6931,$

$a_5 = \frac{\ln 5}{\sqrt{5}} \approx 0.7198$

Consider $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$ by using l'Hôpital's Rule. Thus, $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$; converges.

18. $a_1 = \frac{\ln 1}{\sqrt{2}} = 0, a_2 = \frac{\ln \frac{1}{2}}{2} \approx -0.3466,$

$a_3 = \frac{\ln \frac{1}{3}}{\sqrt{6}} \approx -0.4485, a_4 = \frac{\ln \frac{1}{4}}{2\sqrt{2}} \approx -0.4901,$

$a_5 = \frac{\ln \frac{1}{5}}{\sqrt{10}} \approx -0.5089$

Consider $\lim_{x\to\infty} \frac{\ln \frac{1}{x}}{\sqrt{2x}} = \lim_{x\to\infty} \frac{-\ln x}{\sqrt{2x}} = \lim_{x\to\infty} \frac{-\frac{1}{x}}{\frac{1}{\sqrt{2x}}}$

$= \lim_{x\to\infty} -\frac{\sqrt{2}}{\sqrt{x}} = 0$ by using l'Hôpital's Rule. Thus,

$\lim_{n\to\infty} \frac{\ln \frac{1}{n}}{\sqrt{2n}} = 0;$ converges

19. $a_1 = \left(1+\frac{2}{1}\right)^{1/2} = \sqrt{3} \approx 1.7321,$

$a_2 = \left(1+\frac{2}{2}\right)^{2/2} = 2,$

$a_3 = \left(1+\frac{2}{3}\right)^{3/2} = \left(\frac{5}{3}\right)^{3/2} \approx 2.1517,$

$a_4 = \left(1+\frac{2}{4}\right)^{4/2} = \left(\frac{3}{2}\right)^{2} = \frac{9}{4},$

$a_5 = \left(1+\frac{2}{5}\right)^{5/2} = \left(\frac{7}{5}\right)^{5/2} \approx 2.3191$

Let $\frac{2}{n} = h,$ then as $n \to \infty, h \to 0$ and

$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n/2} = \lim_{h\to 0} (1+h)^{1/h} = e$ by Theorem 6.5A; converges

20. $a_1 = 2^{1/2} \approx 1.4142, a_2 = 4^{1/4} = 2^{1/2} \approx 1.4142,$

$a_3 = 6^{1/6} \approx 1.3480, a_4 = 8^{1/8} = 2^{3/8} \approx 1.2968,$

$a_5 = 10^{1/10} \approx 1.2589$

Consider $\lim_{x\to\infty} (2x)^{1/2x}$. This limit is of the form ∞^0. Let $y = (2x)^{1/2x}$, then $\ln y = \frac{1}{2x} \ln 2x.$

$\lim_{x\to\infty} \frac{1}{2x} \ln 2x = \lim_{x\to\infty} \frac{\ln 2x}{2x}$

This limit is of the form $\frac{\infty}{\infty}$.

$\lim_{x\to\infty} \frac{\ln 2x}{2x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2} = \lim_{x\to\infty} \frac{1}{2x} = 0$

$\lim_{x\to\infty} (2x)^{1/2x} = \lim_{x\to\infty} e^{\ln y} = 1$

Thus $\lim_{n\to\infty} (2n)^{1/2n} = 1;$ converges

21. $a_n = \frac{n}{n+1}$ or $a_n = 1 - \frac{1}{n+1};$

$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1;$ converges

22. $a_n = \frac{n}{2^{n+1}}$

Consider $\frac{x}{2^x}$. Now, $\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$

by l'Hôpital's Rule. Thus, $\lim_{n\to\infty} \frac{n}{2^{n+1}} = 0;$

converges

23. $a_n = (-1)^n \frac{n}{2n-1}; \lim_{n\to\infty} \frac{n}{2n-1}$

$= \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}$, but due to $(-1)^n$, the terms of the sequence alternate between positive and negative, so the sequence diverges.

24. $a_n = \frac{1}{1 - \frac{n-1}{n}} = n;$

$\lim_{n\to\infty} n = \infty$; diverges

25. $a_n = \frac{n}{n^2 - (n-1)^2} = \frac{n}{n^2 - (n^2 - 2n + 1)} = \frac{n}{2n-1};$

$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2};$ converges

26. $a_n = \dfrac{n}{(n+1)-\frac{1}{n+1}} = \dfrac{n(n+1)}{(n+1)^2-1} = \dfrac{n^2+n}{n^2+2n}$;

$\lim_{n\to\infty}\dfrac{n^2+n}{n^2+2n} = \lim_{n\to\infty}\dfrac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1$; converges

27. $a_n = n\sin\dfrac{1}{n}$; $\lim_{n\to\infty} n\sin\dfrac{1}{n} = \lim_{n\to\infty}\dfrac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$ since

$\lim_{x\to 0}\dfrac{\sin x}{x} = 1$; converges

28. $a_n = (-1)^n\dfrac{n^2}{3^n}$;

$\lim_{n\to\infty}\dfrac{n^2}{3^n} = \lim_{n\to\infty}\dfrac{2n}{3^n\ln 3} = \lim_{n\to\infty}\dfrac{2}{3^n(\ln 3)^2} = 0$

by using l'Hôpital's Rule twice; converges

29. $a_n = \dfrac{2^n}{n^2}$;

$\lim_{n\to\infty}\dfrac{2^n}{n^2} = \lim_{n\to\infty}\dfrac{2^n\ln 2}{2n} = \lim_{n\to\infty}\dfrac{2^n(\ln 2)^2}{2} = \infty$;

diverges

30. $a_n = \dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{n+1-n}{n(n+1)} = \dfrac{1}{n(n+1)}$;

$\lim_{n\to\infty}\dfrac{1}{n(n+1)} = 0$; converges

31. $a_1 = \dfrac{1}{2}, a_2 = \dfrac{5}{4}, a_3 = \dfrac{9}{8}, a_4 = \dfrac{13}{16}$

a_n is positive for all n, and $a_{n+1} < a_n$ for all $n \geq 2$ since $a_{n+1} - a_n = -\dfrac{4n-7}{2^{n+1}}$, so $\{a_n\}$ converges to a limit $L \geq 0$.

32. $a_1 = \dfrac{1}{2}; a_2 = \dfrac{7}{6}; a_3 = \dfrac{17}{12}; a_4 = \dfrac{31}{20}$

$a_n = \dfrac{2n^2-1}{n^2+n} < 2$ for all n, and $a_n < a_{n+1}$ for all n since $a_{n+1} - a_n = \dfrac{2}{n^2+2n}$, so $\{a_n\}$ converges to a limit $L \leq 2$.

33. $a_2 = \dfrac{3}{4}; a_3 = \left(\dfrac{3}{4}\right)\left(\dfrac{8}{9}\right) = \dfrac{2}{3}$;

$a_4 = \left(\dfrac{3}{4}\right)\left(\dfrac{8}{9}\right)\left(\dfrac{15}{16}\right) = \dfrac{5}{8}$;

$a_5 = \left(\dfrac{3}{4}\right)\left(\dfrac{8}{9}\right)\left(\dfrac{15}{16}\right)\left(\dfrac{24}{25}\right) = \dfrac{3}{5}$

$a_n > 0$ for all n and $a_{n+1} < a_n$ since

$a_{n+1} = a_n\left(1 - \dfrac{1}{(n+1)^2}\right)$ and $1 - \dfrac{1}{(n+1)^2} < 1$, so $\{a_n\}$ converges to a limit $L \geq 0$.

34. $a_1 = 1; a_2 = \dfrac{3}{2}; a_3 = \dfrac{5}{3}; a_4 = \dfrac{41}{24}$

$a_n < 2$ for all n since

$$1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \leq \frac{1}{2^0} + \frac{1}{2^1} + \cdots + \frac{1}{2^{n+1}} < \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2$$

the sum never reaches 2. $a_n < a_{n+1}$ since each term is the previous term plus a positive quantity, so $\{a_n\}$ converges to a limit $L \leq 2$.

35. $a_1 = 1, a_2 = 1 + \dfrac{1}{2}(1) = \dfrac{3}{2}, a_3 = 1 + \dfrac{1}{2}\left(\dfrac{3}{2}\right) = \dfrac{7}{4}$,

$a_4 = 1 + \dfrac{1}{2}\left(\dfrac{7}{4}\right) = \dfrac{15}{8}$

Suppose that $1 < a_n < 2$, then $\dfrac{1}{2} < \dfrac{1}{2}a_n < 1$, so $\dfrac{3}{2} < 1 + \dfrac{1}{2}a_n < 2$, or $\dfrac{3}{2} < a_{n+1} < 2$. Thus, since $1 < a_2 < 2$, every subsequent term is between $\dfrac{3}{2}$ and 2.

$a_n < 2$ thus $\dfrac{1}{2}a_n < 1$, so $a_n < 1 + \dfrac{1}{2}a_n = a_{n+1}$ and the sequence is nondecreasing, so $\{a_n\}$ converges to a limit $L \leq 2$.

36. $a_1 = 2, a_2 = \frac{1}{2}\left(2+\frac{2}{2}\right) = \frac{3}{2},$

$a_3 = \frac{1}{2}\left(\frac{3}{2}+\frac{4}{3}\right) = \frac{17}{12}, a_4 = \frac{1}{2}\left(\frac{17}{12}+\frac{24}{17}\right) = \frac{577}{408}$

Suppose $a_n > \sqrt{2}$, and consider

$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) > \sqrt{2} \Leftrightarrow a_n + \frac{2}{a_n} > 2\sqrt{2} \Leftrightarrow$

$a_n^2 + 2 > 2\sqrt{2}a_n \Leftrightarrow a_n^2 - 2\sqrt{2}a_n + 2 > 0 \Leftrightarrow$

$\left(a_n - \sqrt{2}\right)^2 > 0$, which is always true. Hence, $a_n > \sqrt{2}$ for all *n*. Also,

$a_{n+1} \le a_n \Leftrightarrow \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \le a_n$

$\Leftrightarrow \frac{1}{a_n} \le \frac{1}{2}a_n \Leftrightarrow \sqrt{2} \le a_n$

which is true. Hence, $\sqrt{2} < a_{n+1} \le a_n$ and the series converges to a limit $L \ge \sqrt{2}$.

37.

n	u_n
1	1.73205
2	2.17533
3	2.27493
4	2.29672
5	2.30146
6	2.30249
7	2.30271
8	2.30276
9	2.30277
10	2.30278
11	2.30278

$\lim_{n\to\infty} u_n \approx 2.3028$

38. Suppose that $0 < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$, then

$3 < 3 + u_n < \frac{1}{2}\left(7+\sqrt{13}\right)$ and

$\sqrt{3} < \sqrt{3+u_n} = u_{n+1} < \sqrt{\frac{1}{2}\left(7+\sqrt{13}\right)} = \frac{1}{2}\left(1+\sqrt{13}\right)$

$\left(\sqrt{\frac{1}{2}\left(7+\sqrt{13}\right)} = \frac{1}{2}\left(1+\sqrt{13}\right)\right.$ can be seen by squaring both sides of the equality and noting that both sides are positive.) Hence, since

$0 < u_1 = \sqrt{3} \approx 1.73 < \frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.3028,$

$\sqrt{3} < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$ for all *n*; $\{u_n\}$ is bounded above.

$u_{n+1} = \sqrt{3+u_n} > u_n$ if $3 + u_n > u_n^2$ or $u_n^2 - u_n - 3 < 0$. $u_n^2 - u_n - 3 = 0$ when $u_n = \frac{1}{2}\left(1 \pm \sqrt{13}\right)$, thus $u_{n+1} > u_n$ if $\frac{1}{2}\left(1-\sqrt{13}\right) < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$, $\frac{1}{2}\left(1-\sqrt{13}\right) < 0$ and $0 < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$ for all *n*, as shown above, so $\{u_n\}$ is increasing. Hence, by Theorem D, $\{u_n\}$ converges.

39. If $u = \lim_{n\to\infty} u_n$, then $u = \sqrt{3+u}$ or $u^2 = 3+u$;

$u^2 - u - 3 = 0$ when $u = \frac{1}{2}\left(1 \pm \sqrt{13}\right)$ so

$u = \frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.3028$ since $u > 0$ and

$\frac{1}{2}\left(1-\sqrt{13}\right) < 0.$

40. If $a = \lim_{n\to\infty} a_n$ where $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, then

$a = \frac{1}{2}\left(a + \frac{2}{a}\right)$ or $2a^2 = a^2 + 2$; $a^2 = 2$ when

$a = \pm\sqrt{2}$, so $a = \sqrt{2}$, since $a > 0$.

41.

n	u_n
1	0
2	1
3	1.1
4	1.11053
5	1.11165
6	1.11177
7	1.11178
8	1.11178

$\lim_{n\to\infty} u_n \approx 1.1118$

42. Since 1.1 > 1, $1.1^a > 1.1^b$ if $a > b$. Thus, since $u_3 = 1.1 > 1 = u_2$, $u_4 = 1.1^{1.1} > 1.1^1 = u_3$. Suppose that $u_n < u_{n+1}$ for all $n \le N$. Then $u_{N+1} = 1.1^{u_N} > 1.1^{u_{N-1}} = u_N$, since $u_N > u_{N-1}$ by the induction hypothesis. Thus, u_n is increasing.

$1.1^{u_n} < 2$ if and only if $u_n \ln 1.1 < \ln 2$;

$u_n < \frac{\ln 2}{\ln 1.1} \approx 7.3$. Thus, unless $u_n > 7.3$, $u_{n+1} = 1.1^{u_n} < 2$. This means that $\{u_n\}$ is bounded above by 2, since $u_1 = 0$.

43. As $n \to \infty, \frac{k}{n} \to 0;$ using $\Delta x = \frac{1}{n},$ an equivalent definite integral is
$\int_0^1 \sin x\, dx = [-\cos x]_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1$
≈ 0.4597

44. As $n \to \infty, \frac{k}{n} \to 0;$ using $\Delta x = \frac{1}{n},$ an equivalent definite integral is
$\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

45. $\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1};$
$\frac{1}{n+1} < \varepsilon$ is the same as $\frac{1}{\varepsilon} < n+1.$ For any given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon} - 1$ then
$n \ge N \Rightarrow \left|\frac{n}{n+1} - 1\right| < \varepsilon.$

46. For $n > 0$, $\left|\frac{n}{n^2+1}\right| = \frac{n}{n^2+1}.$ $\frac{n}{n^2+1} < \varepsilon$ is the same as $\frac{n^2+1}{n} = n + \frac{1}{n} > \frac{1}{\varepsilon}.$
Since $n + \frac{1}{n} > n$, it suffices to take $n > \frac{1}{\varepsilon}$. So for any given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon},$ then
$n \ge N \Rightarrow \left|\frac{n}{n^2+1}\right| < \varepsilon.$

47. Recall that every rational number can be written as either a terminating or a repeating decimal. Thus if the sequence 1, 1.4, 1.41, 1.414, … has a limit within the rational numbers, the terms of the sequence would eventually either repeat or terminate, which they do not since they are the decimal approximations to $\sqrt{2},$ which is irrational. Within the real numbers, the least upper bound is $\sqrt{2}$.

48. Suppose that $\{a_n\}$ is a nondecreasing sequence, and U is an upper bound for $\{a_n\},$ so $S = \{a_n : n \in \mathbb{N}\}$ is bounded above. By the completeness property, S has a least upper bound, which we call A. Then $A \le U$ by definition and $a_n \le A$ for all n. Suppose that $\lim_{n\to\infty} a_n \ne A,$ i.e., that $\{a_n\}$ either does not converge, or does not converge to A. Then there is some $\varepsilon > 0$ such that $A - a_n > \varepsilon$ for all n, since if $A - a_N \le \varepsilon,$ $A - a_n \le \varepsilon$ for $n \ge N$ since $\{a_n\}$ is nondecreasing and $a_n \le A$ for all n. However, if $A - a_n > \varepsilon$ for all $n,$ $a_n < A - \frac{\varepsilon}{2} < A$ for all $n,$ which contradicts A being the least upper bound for the set S. For the second part of Theorem D, suppose that $\{a_n\}$ is a nonincreasing sequence, and L is a lower bound for $\{a_n\}.$ Then $\{-a_n\}$ is a nondecreasing sequence and $-L$ is an upper bound for $\{-a_n\}.$ By what was just proven, $\{-a_n\}$ converges to a limit $A \le -L$, so $\{a_n\}$ converges to a limit $B = -A \ge L$.

49. If $\{b_n\}$ is bounded, there are numbers N and M with $N \le |b_n| \le M$ for all n. Then
$|a_n N| \le |a_n b_n| \le |a_n M|.$
$\lim_{n\to\infty} |a_n N| = |N| \lim_{n\to\infty} |a_n| = 0$ and
$\lim_{n\to\infty} |a_n M| = |M| \lim_{n\to\infty} |a_n| = 0,$ so $\lim_{n\to\infty} |a_n b_n| = 0$
by the Squeeze Theorem, and by Theorem C, $\lim_{n\to\infty} a_n b_n = 0.$

50. Suppose $\{a_n + b_n\}$ converges. Then, by Theorem A
$\lim_{n\to\infty} [(a_n + b_n) - a_n] = \lim_{n\to\infty} (a_n + b_n) - \lim_{n\to\infty} a_n.$
But since $(a_n + b_n) - a_n = b_n,$ this would mean that $\{b_n\}$ converges. Thus $\{a_n + b_n\}$ diverges.

51. No. Consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Both $\{a_n\}$ and $\{b_n\}$ diverge, but
$a_n + b_n = (-1)^n + (-1)^{n+1} = (-1)^n(1 + (-1)) = 0$ so $\{a_n + b_n\}$ converges.

52. **a.** $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8,$
$f_7 = 13, f_8 = 21, f_9 = 34, f_{10} = 55$

b. Using the formula,
$$f_1 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = 1$$
$$f_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{1+2\sqrt{5}+5-(1-2\sqrt{5}+5)}{4}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{4\sqrt{5}}{4}\right] = 1.$$

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n}=\lim_{n\to\infty}\frac{\phi^{n+1}-(-1)^{n+1}\phi^{-n-1}}{\phi^n-(-1)^n\phi^{-n}}$$

$$=\lim_{n\to\infty}\frac{\phi^{n+1}-\frac{(-1)^{n+1}}{\phi^{n+1}}}{\phi^n-\frac{(-1)^n}{\phi^n}}=\lim_{n\to\infty}\frac{\phi-\frac{(-1)^{n+1}}{\phi^{2n+1}}}{1-\frac{(-1)^n}{\phi^{2n}}}=\phi$$

c. $\phi^2-\phi-1=\left[\frac{1}{2}\left(1+\sqrt{5}\right)\right]^2-\frac{1}{2}\left(1+\sqrt{5}\right)-1$

$=\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)-1=0$

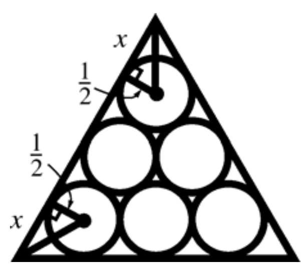
Therefore ϕ satisfies $x^2-x-1=0$.
Using the Quadratic Formula on $x^2-x-1=0$ yields

$x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$.

$\phi=\frac{1+\sqrt{5}}{2}$;

$-\frac{1}{\phi}=-\frac{2}{1+\sqrt{5}}=-\frac{2(1-\sqrt{5})}{1-5}=\frac{1-\sqrt{5}}{2}$

53.



From the figure shown, the sides of the triangle have length $n - 1 + 2x$. The small right triangles marked are 30-60-90 right triangles, so $x=\frac{\sqrt{3}}{2}$; thus the sides of the large triangle have lengths

$n-1+\sqrt{3}$ and $B_n=\frac{\sqrt{3}}{4}\left(n-1+\sqrt{3}\right)^2$

$=\frac{\sqrt{3}}{4}\left(n^2+2\sqrt{3}n-2n-2\sqrt{3}+4\right)$ while

$A_n=\frac{n(n+1)}{2}\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{8}(n^2+n)$

$$\lim_{n\to\infty}\frac{A_n}{B_n}=\lim_{n\to\infty}\frac{\frac{\pi}{8}(n^2+n)}{\frac{\sqrt{3}}{4}(n^2+2\sqrt{3}n-2n-2\sqrt{3}+4)}$$

$$=\lim_{n\to\infty}\frac{\pi\left(1+\frac{1}{n}\right)}{2\sqrt{3}\left(1+\frac{2\sqrt{3}}{n}-\frac{2}{n}-\frac{2\sqrt{3}}{n^2}+\frac{4}{n^2}\right)}=\frac{\pi}{2\sqrt{3}}$$

54. Let $f(x)=\left(1+\frac{1}{x}\right)^x$.

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0^+}(1+x)^{1/x}=e$, so

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$.

55. Let $f(x)=\left(1+\frac{1}{2x}\right)^x$.

$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^x=\lim_{x\to 0^+}\left(1+\frac{x}{2}\right)^{1/x}$

$=\lim_{x\to 0^+}\left[\left(1+\frac{x}{2}\right)^{2/x}\right]^{1/2}=e^{1/2}$, so

$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n=e^{1/2}$.

56. Let $f(x)=\left(1+\frac{1}{x^2}\right)^x$.

$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x=\lim_{x\to\infty}\left(1+\left(\frac{1}{x}\right)^2\right)^{\frac{1}{1/x}}$

Using the fact that $\lim_{x\to\infty}f(x)=\lim_{x\to 0^+}f\left(\frac{1}{x}\right)$, we can write

$\lim_{x\to\infty}\left(1+\left(\frac{1}{x}\right)^2\right)^{\frac{1}{1/x}}=\lim_{x\to 0^+}\left(1+x^2\right)^{1/x}$ which leads to the indeterminate form 1^∞.

Let $y=\left(1+x^2\right)^{1/x}$. Then,

$\ln y=\ln\left(1+x^2\right)^{1/x}$

$\ln y=\frac{1}{x}\ln\left(1+x^2\right)$

$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln\left(1+x^2\right)}{x}=\lim_{x\to 0^+}\frac{\frac{2x}{1+x^2}}{1}$

$=\lim_{x\to 0^+}\frac{2x}{1+x^2}=0$

This gives us

$\lim_{x\to 0^+}\ln y=0$

$\ln\left(\lim_{x\to 0^+}y\right)=0$

$\lim_{x\to 0^+}y=e^0=1$ or $\lim_{x\to 0^+}\left(1+x^2\right)^{1/x}=1$

Thus, $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n=1$.

57. Let $f(x)=\left(\frac{x-1}{x+1}\right)^x$.

Using the fact that $\lim_{x\to\infty} f(x)=\lim_{x\to 0^+} f\left(\frac{1}{x}\right)$, we can write

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x=\lim_{x\to 0^+}\left(\frac{\frac{1}{x}-1}{\frac{1}{x}+1}\right)^{1/x}=\lim_{x\to 0^+}\left(\frac{\frac{1-x}{x}}{\frac{1+x}{x}}\right)^{1/x}$$

$=\lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x}$ which leads to the indeterminate form 1^∞.

Let $y=\left(\frac{1-x}{1+x}\right)^{1/x}$. Then,

$$\ln y=\ln\left(\frac{1-x}{1+x}\right)^{1/x}$$

$$\ln y=\frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)$$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)$$

$$\ln\left[\lim_{x\to 0^+} y\right]=\lim_{x\to 0^+}\frac{\ln\left(\frac{1-x}{1+x}\right)}{x}$$

$$=\lim_{x\to 0^+}\frac{-2}{1-x^2}\quad\text{(l'Hopital's Rule)}$$

-2

This gives us,

$$\ln\left[\lim_{x\to 0^+} y\right]=-2$$

$$\lim_{x\to 0^+} y=e^{-2}\quad\text{or}\quad\lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x}=e^{-2}$$

Thus, $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n=e^{-2}$.

58. Let $f(x)=\left(\frac{2+x^2}{3+x^2}\right)^x$.

Using the fact that $\lim_{x\to\infty} f(x)=\lim_{x\to 0^+} f\left(\frac{1}{x}\right)$, we can write

$$\lim_{x\to\infty}\left(\frac{2+x^2}{3+x^2}\right)^x=\lim_{x\to 0^+}\left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x}$$

$=\lim_{x\to 0^+}\left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x}=\lim_{x\to 0^+}\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x}$ which leads to the indeterminate form 1^∞.

Let $y=\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x}$. Then,

$$\ln y=\ln\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x}$$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{1}{x}\ln\left(\frac{2x^2+1}{3x^2+1}\right)$$

$$\ln\left[\lim_{x\to 0^+} y\right]=\lim_{x\to 0^+}\frac{\ln\left(\frac{2x^2+1}{3x^2+1}\right)}{x}$$

$$=\lim_{x\to 0^+}\left[\frac{4x}{2x^2+1}-\frac{6x}{3x^2+1}\right]\quad\text{(l'Hopital's Rule)}$$

$=0$

This gives us,

$$\ln\left[\lim_{x\to 0^+} y\right]=0$$

$$\lim_{x\to 0^+} y=e^0=1\quad\text{or}\quad\lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x}=1$$

Thus,

$$\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^n=1.$$

59. Let $f(x)=\left(\frac{2+x^2}{3+x^2}\right)^{x^2}$

Using the fact that $\lim_{x\to\infty} f(x)=\lim_{x\to 0^+} f\left(\frac{1}{x}\right)$, we can write

$$\lim_{x\to\infty}\left(\frac{2+x^2}{3+x^2}\right)^{x^2} = \lim_{x\to 0^+}\left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x^2}$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x^2} = \lim_{x\to 0^+}\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2} \text{ which}$$

leads to the indeterminate form 1^∞.

Let $y=\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2}$. Then,

$$\ln y=\ln\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2}$$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{1}{x^2}\ln\left(\frac{2x^2+1}{3x^2+1}\right)$$

$$\ln\left[\lim_{x\to 0^+} y\right]=\lim_{x\to 0^+}\frac{\ln\left(\frac{2x^2+1}{3x^2+1}\right)}{x^2}$$

$$=\lim_{x\to 0^+}\left[\frac{-1}{\left(2x^2+1\right)\left(3x^2+1\right)}\right] \quad \text{(l'Hopital's Rule)}$$

$=-1$

This gives us,

$$\ln\left[\lim_{x\to 0^+} y\right]=-1$$

$$\lim_{x\to 0^+} y=e^{-1} \quad \text{or} \quad \lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x^2}=e^{-1}$$

Thus,

$$\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^{n^2}=e^{-1}.$$

9.2 Concepts Review

1. an infinite series

2. $a_1+a_2+\ldots+a_n$

3. $|r|<1; \frac{a}{1-r}$

4. diverges

Problem Set 9.2

1. $\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)^k=\frac{1}{7}+\frac{1}{7}\cdot\frac{1}{7}+\frac{1}{7}\left(\frac{1}{7}\right)^2+\ldots$; a geometric series with $a=\frac{1}{7}, r=\frac{1}{7}$; $S=\frac{\frac{1}{7}}{1-\frac{1}{7}}=\frac{\frac{1}{7}}{\frac{6}{7}}=\frac{1}{6}$

2. $\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^{-k-2}=\left(-\frac{1}{4}\right)^{-3}+\left(-\frac{1}{4}\right)^{-4}+\left(-\frac{1}{4}\right)^{-5}+\ldots$

$=(-4)^3+(-4)^4+(-4)^5+\ldots$; a geometric series with

$a=(-4)^3, r=-4$; $|r|=4>1$ so the series diverges.

3. $\sum_{k=0}^{\infty}2\left(\frac{1}{4}\right)^k=2+2\cdot\frac{1}{4}+2\left(\frac{1}{4}\right)^2+\ldots$; a geometric series with a = 2, $r=\frac{1}{4}$; $S=\frac{2}{1-\frac{1}{4}}=\frac{2}{\frac{3}{4}}=\frac{8}{3}$.

$\sum_{k=0}^{\infty}3\left(-\frac{1}{5}\right)^k=3-3\cdot\frac{1}{5}+3\left(\frac{1}{5}\right)^2-\ldots$; a geometric series with a = 3, $r=-\frac{1}{5}$;

$$S=\frac{3}{1-\left(-\frac{1}{5}\right)}=\frac{3}{\frac{6}{5}}=\frac{5}{2}$$

Thus, by Theorem B,

$$\sum_{k=0}^{\infty}\left[2\left(\frac{1}{4}\right)^k+3\left(-\frac{1}{5}\right)^k\right]=\frac{8}{3}+\frac{5}{2}=\frac{31}{6}$$

4. $\sum_{k=1}^{\infty}5\left(\frac{1}{2}\right)^k=\frac{5}{2}+\frac{5}{2}\cdot\frac{1}{2}+\frac{5}{2}\left(\frac{1}{2}\right)^2+\ldots$; a geometric series with $a=\frac{5}{2}, r=\frac{1}{2}$; $S=\frac{\frac{5}{2}}{1-\frac{1}{2}}=\frac{\frac{5}{2}}{\frac{1}{2}}=5.$

$\sum_{k=1}^{\infty}3\left(\frac{1}{7}\right)^{k+1}=\frac{3}{49}+\frac{3}{49}\cdot\frac{1}{7}+\frac{3}{49}\left(\frac{1}{7}\right)^2+\ldots$; a geometric series with $a=\frac{3}{49}, r=\frac{1}{7}$;

$$S=\frac{\frac{3}{49}}{1-\frac{1}{7}}=\frac{\frac{3}{49}}{\frac{6}{7}}=\frac{1}{14}$$

Thus, by Theorem B,

$$\sum_{k=1}^{\infty}\left[2\left(\frac{1}{4}\right)^k-3\left(\frac{1}{7}\right)^{k+1}\right]=5-\frac{1}{14}=\frac{69}{14}.$$

5. $\displaystyle\sum_{k=1}^{\infty}\frac{k-5}{k+2}=-\frac{4}{3}-\frac{3}{4}-\frac{2}{5}-\frac{1}{6}+0+\frac{1}{8}+\frac{2}{9}+\ldots;$

$\displaystyle\lim_{k\to\infty}\frac{k-5}{k+2}=\lim_{k\to\infty}\frac{1-\frac{5}{k}}{1+\frac{2}{k}}=1\neq 0;$ the series diverges.

6. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{9}{8}\right)^k=\frac{9}{8}+\frac{9}{8}\cdot\frac{9}{8}+\frac{9}{8}\left(\frac{9}{8}\right)^2+\ldots;$ a geometric series with $a=\frac{9}{8}, r=\frac{9}{8}; \left|\frac{9}{8}\right|>1,$ so the series diverges.

7. $\displaystyle\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right)=\left(\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{3}\right)+\ldots;$

$\displaystyle S_n=\left(\frac{1}{2}-1\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n-2}\right)+\left(\frac{1}{n}-\frac{1}{n-1}\right)=-1+\frac{1}{n};$

$\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}-1+\frac{1}{n}=-1,$ so $\displaystyle\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right)=-1$

8. $\displaystyle\sum_{k=1}^{\infty}\frac{3}{k}=3\sum_{k=1}^{\infty}\frac{1}{k}$ which diverges since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

9. $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{100^k}=\frac{1}{100}+\frac{2}{10{,}000}+\frac{6}{1{,}000{,}000}+\ldots$

Consider $\{a_n\}$, where $a_{n+1}=\frac{n+1}{100}a_n, a_1=\frac{1}{100}$. $a_n>0$ for all n, and for $n>99$, $a_{n+1}>a_n$, so the sequence is eventually an increasing sequence, hence $\lim_{n\to\infty}a_n\neq 0$. The sequence can also be described by

$a_n=\frac{n!}{100^n}$, hence $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{100^k}$ diverges.

10. $\displaystyle\sum_{k=1}^{\infty}\frac{2}{(k+2)k}=\frac{2}{3}+\frac{2}{8}+\frac{2}{15}+\ldots=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right)=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots$

$\displaystyle S_n=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)$

$\displaystyle=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}=\frac{3}{2}-\frac{2n+3}{n^2+3n+2}$

$\displaystyle\lim_{n\to\infty}S_n=\frac{3}{2}-\lim_{n\to\infty}\frac{2n+3}{n^2+3n+2}=\frac{3}{2}-\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}}=\frac{3}{2},$ so $\displaystyle\sum_{k=1}^{\infty}\frac{2}{(k+2)k}=\frac{3}{2}.$

11. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{e}{\pi}\right)^{k+1}=\left(\frac{e}{\pi}\right)^2+\left(\frac{e}{\pi}\right)^2\cdot\frac{e}{\pi}+\left(\frac{e}{\pi}\right)^2\left(\frac{e}{\pi}\right)^2+\ldots;$ a geometric series with $a=\left(\frac{e}{\pi}\right)^2, r=\frac{e}{\pi}<1;$

$\displaystyle S=\frac{\left(\frac{e}{\pi}\right)^2}{1-\frac{e}{\pi}}=\frac{\left(\frac{e}{\pi}\right)^2}{\frac{\pi-e}{\pi}}=\frac{e^2}{\pi(\pi-e)}\approx 5.5562$

12. $\displaystyle\sum_{k=1}^{\infty}\frac{4^{k+1}}{7^{k-1}}=\frac{16}{1}+16\cdot\frac{4}{7}+16\left(\frac{4}{7}\right)^2+\ldots;$ a geometric series with $a=16$, $r=\frac{4}{7}<1;$ $S=\frac{16}{1-\frac{4}{7}}=\frac{16}{\frac{3}{7}}=\frac{112}{3}$

13. $\sum_{k=2}^{\infty}\left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right)=\left(\frac{3}{1}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{3}{9}\right)+\left(\frac{3}{9}-\frac{3}{16}\right)+\dots;$

$S_n=\left(3-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{3}{16}\right)+\dots+\left(\frac{3}{(n-2)^2}-\frac{3}{(n-1)^2}\right)+\left(\frac{3}{(n-1)^2}-\frac{3}{n^2}\right)$

$=3-\frac{3}{n^2};\ \lim_{n\to\infty}S_n=3-\lim_{n\to\infty}\frac{3}{n^2}=3,$ so

$\sum_{k=2}^{\infty}\left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right)=3.$

14. $\sum_{k=6}^{\infty}\frac{2}{k-5}=\frac{2}{1}+\frac{2}{2}+\frac{2}{3}+\frac{2}{4}+\dots$

$=2\sum_{k=1}^{\infty}\frac{1}{k}$ which diverges since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

15. $0.22222\ldots=\sum_{k=1}^{\infty}\frac{2}{10}\left(\frac{1}{10}\right)^{k-1}$

$=\frac{\frac{2}{10}}{1-\frac{1}{10}}=\frac{2}{9}$

16. $0.21212121\ldots=\sum_{k=1}^{\infty}\frac{21}{100}\left(\frac{1}{100}\right)^{k-1}$

$=\frac{\frac{21}{100}}{1-\frac{1}{100}}=\frac{21}{99}=\frac{7}{33}$

17. $0.013013013...=\sum_{k=1}^{\infty}\frac{13}{1000}\left(\frac{1}{1000}\right)^{k-1}$

$=\frac{\frac{13}{1000}}{1-\frac{1}{1000}}=\frac{13}{999}$

18. $0.125125125...=\sum_{k=1}^{\infty}\frac{125}{1000}\left(\frac{1}{1000}\right)^{k-1}$

$=\frac{\frac{125}{1000}}{1-\frac{1}{1000}}=\frac{125}{999}$

19. $0.4999...=\frac{4}{10}+\sum_{k=1}^{\infty}\frac{9}{100}\left(\frac{1}{10}\right)^{k-1}$

$=\frac{4}{10}+\frac{\frac{9}{100}}{1-\frac{1}{10}}=\frac{1}{2}$

20. $0.36717171...=\frac{36}{100}+\sum_{k=1}^{\infty}\frac{71}{10,000}\left(\frac{1}{100}\right)^{k-1}$

$=\frac{36}{100}+\frac{\frac{71}{10,000}}{1-\frac{1}{100}}=\frac{727}{1980}$

21. Let $s = 1 - r$, so $r = 1 - s$. Since $0 < r < 2$, $-1 < 1 - r < 1$, so

$|s|<1,$ and $\sum_{k=0}^{\infty}r(1-r)^k=\sum_{k=0}^{\infty}(1-s)s^k$

$=\sum_{k=1}^{\infty}(1-s)s^{k-1}=\frac{1-s}{1-s}=1$

22. $\sum_{k=0}^{\infty}(-1)^k x^k=\sum_{k=0}^{\infty}(-x)^k=\sum_{k=1}^{\infty}(-x)^{k-1};$

if $-1 < x < 1$ then

$-1 < -x < 1$ so $|x|<1$;

$\sum_{k=1}^{\infty}(-x)^{k-1}=\frac{1}{1-(-x)}=\frac{1}{1+x}$

23. $\ln \dfrac{k}{k+1} = \ln k - \ln(k+1)$

$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \ldots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) \quad = \ln 1 - \ln(n+1) = -\ln(n+1)$

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} -\ln(n+1) = -\infty$, thus $\sum\limits_{k=1}^{\infty} \ln \dfrac{k}{k+1}$ diverges.

24. $\ln\left(1 - \dfrac{1}{k^2}\right) = \ln \dfrac{k^2 - 1}{k^2} = \ln(k^2 - 1) - \ln k^2 \quad = \ln[(k+1)(k-1)] - \ln k^2 = \ln(k+1) + \ln(k-1) - 2\ln k$

$S_n = (\ln 3 + \ln 1 - 2\ln 2) + (\ln 4 + \ln 2 - 2\ln 3) + (\ln 5 + \ln 3 - 2\ln 4) + \ldots$

$+(\ln n + \ln(n-2) - 2\ln(n-1)) + (\ln(n+1) + \ln(n-1) - 2\ln n)$

$= -\ln 2 + \ln(n+1) - \ln n \; = -\ln 2 + \ln \dfrac{n+1}{n}$

$\lim\limits_{n\to\infty} S_n = -\ln 2 + \lim\limits_{n\to\infty} \ln \dfrac{n+1}{n} \quad = -\ln 2 + \ln\left(\lim\limits_{n\to\infty} \dfrac{n+1}{n}\right) = -\ln 2 + \ln 1 = -\ln 2$

25. The ball drops 100 feet, rebounds up $100\left(\frac{2}{3}\right)$ feet, drops $100\left(\frac{2}{3}\right)$ feet, rebounds up $100\left(\frac{2}{3}\right)^2$ feet, drops $100\left(\frac{2}{3}\right)^2$, etc. The total distance it travels is

$100 + 200\left(\frac{2}{3}\right) + 200\left(\frac{2}{3}\right)^2 + 200\left(\frac{2}{3}\right)^3 + \ldots \quad = -100 + 200 + 200\left(\frac{2}{3}\right) + 200\left(\frac{2}{3}\right)^2 + 200\left(\frac{2}{3}\right)^3 + \ldots$

$= -100 + \sum\limits_{k=1}^{\infty} 200\left(\frac{2}{3}\right)^{k-1} \quad = -100 + \dfrac{200}{1 - \frac{2}{3}} = 500$ feet

26. Each gets $\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right) + \ldots = \sum\limits_{k=1}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^{k-1} \quad = \dfrac{\frac{1}{4}}{1-\frac{1}{4}} = \dfrac{1}{3}$

(This can be seen intuitively, since the size of the leftover piece is approaching 0, and each person gets the same amount.)

27. \$1 billion + 75% of \$1 billion + 75% of 75% of \$1 billion + ... = $\sum\limits_{k=1}^{\infty} (\$1 \text{ billion})0.75^{k-1} = \dfrac{\$1 \text{ billion}}{1 - 0.75} = \$4$ billion

28. $\sum\limits_{k=1}^{\infty} \$1 \text{ billion } (0.90)^{k-1} \; = \dfrac{\$1 \text{ billion}}{1 - 0.90} \; = \$10$ billion

29. As the midpoints of the sides of a square are connected, a new square is formed. The new square has sides $\frac{1}{\sqrt{2}}$ times the sides of the old square. Thus, the new square has area $\frac{1}{2}$ the area of the old square. Then in the next step, $\frac{1}{8}$ of each new square is shaded.

Area = $\frac{1}{8}\cdot 1 + \frac{1}{8}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{4} + \ldots = \sum\limits_{k=1}^{\infty} \frac{1}{8}\left(\frac{1}{2}\right)^{k-1} \quad = \dfrac{\frac{1}{8}}{1-\frac{1}{2}} = \dfrac{1}{4}$

The area will be $\frac{1}{4}$.

30. $\frac{1}{9}+\frac{1}{9}\left(\frac{8}{9}\right)+\frac{1}{9}\left(\frac{8}{9}\cdot\frac{8}{9}\right)+\cdots=\sum_{k=1}^{\infty}\frac{1}{9}\left(\frac{8}{9}\right)^{k-1}=\frac{\frac{1}{9}}{1-\frac{8}{9}}=1$; the whole square will be painted.

31. $\frac{3}{4}+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right)+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right)\left(\frac{1}{4}\cdot\frac{1}{4}\right)+\ldots=\sum_{k=1}^{\infty}\frac{3}{4}\left(\frac{1}{16}\right)^{k-1}=\frac{\frac{3}{4}}{1-\frac{1}{16}}=\frac{4}{5}$

The original does not need to be equilateral since each smaller triangle will have $\frac{1}{4}$ area of the previous larger triangle.

32. Ratio of inscribed circle to triangle is $\frac{\pi}{3\sqrt{3}}$, so $\sum_{k=1}^{\infty}\frac{\pi}{3\sqrt{3}}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^{k-1}=\frac{\left(\frac{\pi}{4\sqrt{3}}\right)}{1-\frac{1}{4}}=\frac{\pi}{3\sqrt{3}}$

(This can be seen intuitively, since *every* small triangle has a circle inscribed in it.)

33. **a.** We first note that, at each stage, the number of sides is four times the number in the previous stage and the length of each side is one-third the length in the previous stage. Summarizing:

Stage	# of sides	length/ side (in.)	perimeter p_n
0	3	9	27
1	3(4)	$9\left(\frac{1}{3}\right)$	36
⋮	⋮	⋮	⋮
n	$3(4^n)$	$9\left(\frac{1}{3^n}\right)$	$27\left(\frac{4}{3}\right)^n$

The perimeter of the Koch snowflake is $\lim_{n\to\infty} p_n=\lim_{n\to\infty}27\left(\frac{4}{3}\right)^n$ which is infinite since $\frac{4}{3}>1$.

b. We note the following:

1. The area of an equilateral triangle of side s is $\frac{\sqrt{3}}{4}s^2$

2. The number of new triangles added at each stage is equal to the number of sides the figure had at the previous stage and

3. the area of each new triangle at a given stage is $\frac{\sqrt{3}}{4}(\text{side length at that stage})^2$. Using results from part a. we can summarize:

Stage	Additional triangles (col 2, part *a.*)	Area of each new Δ (see col 3, part *a.*)	Additional area, A_n
0	original	$\frac{\sqrt{3}}{4}\left(9^2\right)$	$\frac{\sqrt{3}}{4}\left(9^2\right)$
1	3	$\frac{\sqrt{3}}{4}\left(3^2\right)$	36
⋮	⋮	⋮	⋮
n	$3\left(4^{n-1}\right)$	$\frac{\sqrt{3}}{4}\left(\frac{9}{3^n}\right)^2$	$3\sqrt{3}\left(\frac{4}{9}\right)^{n-2}$

Thus the area of the Koch snowflake is

$$\sum_{n=0}^{\infty} A_n = \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + \sum_{n=1}^{\infty} 3\sqrt{3}\left(\frac{4}{9}\right)^{n-1}$$
$$= \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + \left(\frac{3\sqrt{3}}{\left(1-\frac{4}{9}\right)}\right)$$
$$= \frac{81\sqrt{3}}{4} + \frac{1}{3}\left(\frac{81\sqrt{3}}{4}\right) + \frac{4}{15}\left(\frac{81\sqrt{3}}{4}\right) = \frac{8}{5}\left(\frac{81\sqrt{3}}{4}\right)$$

Note: By generalizing the above argument it can be shown that, no matter what the size of the original equilateral triangle, the area of the Koch snowflake constructed from it will be $\frac{8}{5}$ times the area of the original triangle.

34. We note the following:

1. Each triangle contains the angles $90, \theta, 90-\theta$ 2. The height of each triangle will be the hypotenuse of the succeeding triangle. Summarizing:

# triangle	base	height	area A_n
1	$h\cos\theta$	$h\sin\theta$	$\frac{1}{2}h^2\sin\theta\cos\theta$
2	$h\sin\theta\cos\theta$	$h\sin^2\theta$	$\frac{1}{2}h^2\sin^3\theta\cos\theta$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$h(\sin^{n-1}\theta)\cos\theta$	$h\sin^n\theta$	$\frac{1}{2}h^2\sin^{2n-1}\theta\cos\theta$

Thus the total area of the small triangles is $A = \sum_{n=1}^{\infty} A_n = \frac{h^2}{2}\left(\frac{\cos\theta}{\sin\theta}\right)\sum_{n=2}^{\infty}(\sin^2\theta)^{n-1}$

Now consider the infinite geometric series $S = \sum_{n=1}^{\infty}(\sin^2\theta)^{n-1} = \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$

then: $\sum_{n=2}^{\infty}(\sin^2\theta)^{n-1} = S-1 = \frac{1}{\cos^2\theta} - 1 = \frac{\sin^2\theta}{\cos^2\theta}$ Therefore: $A = \frac{h^2}{2}\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{\sin^2\theta}{\cos^2\theta}\right) = \frac{h^2}{2}\tan\theta$

In $\triangle ABC$, height = h and base = $h\tan\theta$; thus the area of $\triangle ABC = \frac{1}{2}(h\tan\theta)h = \frac{h^2}{2}\tan\theta$, the same as A.

35. Both Achilles and the tortoise will have moved.

$$100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \ldots = \sum_{k=1}^{\infty} 100\left(\frac{1}{10}\right)^{k-1}$$
$$= \frac{100}{1-\frac{1}{10}} = 111\frac{1}{9} \text{ yards}$$

Also, one can see this by the following reasoning. In the time it takes the tortoise to run $\frac{d}{10}$ yards, Achilles will run d yards. Solve $d = 100 + \frac{d}{10}. d = \frac{1000}{9} = 111\frac{1}{9}$ yards

36. a. Say Trot and Tom start from the left, Joel from the right. Trot and Joel run towards each other at 30 mph. Since they are 60 miles apart they will meet in 2 hours. Trot will have run 40 miles and Tom will have run 20 miles, so they will be 20 miles apart. Trot and Tom will now be approaching each other at 30 mph, so they will meet after 2/3 hour. Trot will have run another 40/3 miles and will be 80/3 miles from the left. Joel will have run another 20/3 miles and will be at 100/3 miles from the left, so they will be 20/3 miles apart. They will meet after 2/9 hour, during which Trot will have run 40/9 miles, etc. So Trot runs

$$40+\frac{40}{3}+\frac{40}{9}+\cdots=\sum_{k=1}^{\infty}40\left(\frac{1}{3}\right)^{k-1}=\frac{40}{1-\frac{1}{3}}$$

$= 60$ miles.

b. Tom and Joel are approaching each other at 20 mph. They are 60 miles apart, so they will meet in 3 hours. Trot is running at 20 mph during that entire time, so he runs 60 miles.

37. Note that:

1. If we let t_n be the probability that Peter wins on his *n*th flip, then the total probability that Peter wins is $T=\sum_{n=1}^{\infty}t_n$

2. The probability that neither man wins in their first *k* flips is $\left(\frac{2}{3}\cdot\frac{2}{3}\right)^k=\left(\frac{4}{9}\right)^k$.

3. The probability that Peter wins on his *n*th flip requires that (i) he gets a head on the *n*th flip, and (ii) neither he nor Paul gets a head on their previous *n*-1 flips. Thus:

$$t_n=\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^{n-1} \text{ and } T=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^{n-1}=$$

$$\left(\frac{1/3}{(1-4/9)}\right)=\frac{1}{3}\cdot\frac{9}{5}=\frac{3}{5}$$

38. In this case (see problem 37),

$$t_n=p\left[(1-p)^2\right]^{n-1} \text{ so}$$

$$T=\sum_{n=1}^{\infty}p\left[(1-p)^2\right]^{n-1}=\frac{p}{\left(1-(1-p)^2\right)}$$

$$=\frac{p}{\left(2p-p^2\right)}=\frac{1}{2-p}$$

39. Let X = *number of rolls needed to get first 6* For X to equal n, two things must occur:

1. Mary must get a non-6 (probability = $\frac{5}{6}$) on each of her first *n*-1 rolls, and

2. Mary rolls a 6 (probability = $\frac{1}{6}$) on the *n*th roll. Thus,

$$\Pr(X=n)=\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) \text{ and}$$

$$\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)=\frac{1/6}{\left(1-5/6\right)}=1$$

40. $$EV(X)=\sum_{n=1}^{\infty}n\cdot\Pr(X=n)=\sum_{n=1}^{\infty}n\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{n-1}$$

$$=\frac{1}{6}\left(\frac{1}{p}\right)\sum_{n=1}^{\infty}n\cdot p^n=\frac{1}{6}\left(\frac{6}{5}\right)\left(\frac{5/6}{(1-5/6)^2}\right)$$

$$=\frac{1}{6}\left(\frac{6}{5}\right)\left(\frac{5}{6}\right)\left(\frac{36}{1}\right)=6$$

41. (Proof by contradiction) Assume $\sum_{k=1}^{\infty}ca_k$ converges, and $c \neq 0$. Then $\frac{1}{c}$ is defined, so $\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{\infty}\frac{1}{c}ca_k=\frac{1}{c}\sum_{k=1}^{\infty}ca_k$ would also converge, by Theorem B(i).

42. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots=\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{k}\right)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

43. a. The top block is supported *exactly* at its center of mass. The location of the center of mass of the top *n* blocks is the average of the locations of their individual centers of mass, so the *n*th block moves the center of mass left by $\frac{1}{n}$ of the location of its center of mass, that is, $\frac{1}{n}\cdot\frac{1}{2}$ or $\frac{1}{2n}$ to the left. But this is exactly how far the (*n* + 1)st block underneath it is offset.

b. Since $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$, which diverges, there is no limit to how far the top block can protrude.

44. $N = 31$; $S_{31} \approx 4.0272$ and $S_{30} \approx 3.9950$.

45. (Proof by contradiction) Assume $\sum_{k=1}^{\infty}(a_k + b_k)$ converges. Since $\sum_{k=1}^{\infty} b_k$ converges, so would $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k + b_k) + (-1)\sum_{k=1}^{\infty} b_k$, by Theorem B(ii).

46. (Answers may vary). $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(-1)\frac{1}{n}$ both diverge, but $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n}\right)$ converges to 0.

47. Taking vertical strips, the area is
$1\cdot 1 + 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + \cdots = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}$.
Taking horizontal strips, the area is
$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \cdots = \sum_{k=1}^{\infty}\frac{k}{2^k}$.

a. $\sum_{k=1}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1} = \frac{1}{1-\frac{1}{2}} = 2$

b. The moment about $x = 0$ is
$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \cdot (1)k = \sum_{k=0}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\frac{k}{2^k} = 2.$
$\bar{x} = \frac{\text{moment}}{\text{area}} = \frac{2}{2} = 1$

48. If $\sum_{k=1}^{\infty} kr^k$ converges, so will $r\sum_{k=1}^{\infty} kr^k$, by Theorem B.
$rS = r\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} kr^{k+1} = \sum_{k=2}^{\infty}(k-1)r^k$ while
$S = \sum_{k=1}^{\infty} kr^k = r + \sum_{k=2}^{\infty} kr^k$ so
$S - rS = r + \sum_{k=2}^{\infty} kr^k - \sum_{k=2}^{\infty}(k-1)r^k$
$= r + \sum_{k=2}^{\infty}[k-(k-1)]r^k = r + \sum_{k=2}^{\infty} r^k = \sum_{k=1}^{\infty} r^k$
Since $|r| < 1$, $\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$, thus
$S = \frac{1}{1-r}\sum_{k=1}^{\infty} r^k = \frac{r}{(1-r)^2}$.

49. a. $A = \sum_{n=0}^{\infty} Ce^{-nkt} = \sum_{n=1}^{\infty} C\left(\frac{1}{e^{kt}}\right)^{n-1}$
$= \frac{C}{1-\frac{1}{e^{kt}}} = \frac{Ce^{kt}}{e^{kt}-1}$

b. $\frac{1}{2} = e^{-kt} = e^{-6k} \Rightarrow k = \frac{\ln 2}{6} \Rightarrow A = \frac{4}{3}C$;
if $C = 2$ mg, then $A = \frac{8}{3}$ mg.

50. Using partial fractions, $\frac{2^k}{(2^{k+1}-1)(2^k-1)} = \frac{1}{2^k-1} - \frac{1}{2^{k+1}-1}$

$$S_n = \left(\frac{1}{2^1-1} - \frac{1}{2^2-1}\right) + \left(\frac{1}{2^2-1} - \frac{1}{2^3-1}\right) + \ldots + \left(\frac{1}{2^{n-1}-1} - \frac{1}{2^n-1}\right) + \left(\frac{1}{2^n-1} - \frac{1}{2^{n+1}-1}\right)$$
$$= \frac{1}{2-1} - \frac{1}{2^{n+1}-1} = 1 - \frac{1}{2^{n+1}-1}$$
$$\lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty}\frac{1}{2^{n+1}-1} = 1 - 0 = 1$$

51. $\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}} = \frac{f_{k+2} - f_k}{f_k f_{k+1} f_{k+2}} = \frac{1}{f_k f_{k+2}}$

since $f_{k+2} = f_{k+1} + f_k$. Thus,

$\sum_{k=1}^{\infty} \frac{1}{f_k f_{k+2}} = \sum_{k=1}^{\infty} \left(\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}} \right)$ and

$S_n = \left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \cdots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) + \left(\frac{1}{f_n f_{n+1}} - \frac{1}{f_{n+1} f_{n+2}} \right)$

$= \frac{1}{f_1 f_2} - \frac{1}{f_{n+1} f_{n+2}} = \frac{1}{1 \cdot 1} - \frac{1}{f_{n+1} f_{n+2}} = 1 - \frac{1}{f_{n+1} f_{n+2}}$

The terms of the Fibonacci sequence increase without bound, so

$\lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty} \frac{1}{f_{n+1} f_{n+2}} = 1 - 0 = 1$

9.3 Concepts Review

1. bounded above

2. $f(k)$; continuous; positive; nonincreasing

3. convergence or divergence

4. $p > 1$

Problem Set 9.3

1. $\frac{1}{x+3}$ is continuous, positive, and nonincreasing on $[0, \infty)$.

$\int_0^{\infty} \frac{1}{x+3} dx = \left[\ln|x+3| \right]_0^{\infty} = \infty - \ln 3 = \infty$

The series diverges.

2. $\frac{3}{2x-3}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

$\int_2^{\infty} \frac{3}{2x-3} dx = \left[\frac{3}{2} \ln|2x-3| \right]_2^{\infty} = \infty - \frac{3}{2} \ln 1 = \infty$

The series diverges.

3. $\frac{x}{x^2+3}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

$\int_2^{\infty} \frac{x}{x^2+3} dx = \left[\frac{1}{2} \ln\left|x^2+3\right| \right]_2^{\infty} = \infty - \frac{1}{2} \ln 7 = \infty$

The series diverges.

4. $\frac{3}{2x^2+1}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{3}{2x^2+1} dx = \left[\frac{3}{\sqrt{2}} \tan^{-1} \sqrt{2}x \right]_1^{\infty}$

$= \frac{3}{\sqrt{2}} \left(\frac{\pi}{2} - \tan^{-1} \sqrt{2} \right) < \infty$

The series converges.

5. $\frac{2}{\sqrt{x+2}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{2}{\sqrt{x+2}} dx = \left[4\sqrt{x+2} \right]_1^{\infty} = \infty - 4\sqrt{3} = \infty$

Thus $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k+2}}$ diverges, hence

$\sum_{k=1}^{\infty} \frac{-2}{\sqrt{k+2}} = -\sum_{k=1}^{\infty} \frac{2}{\sqrt{k+2}}$ also diverges.

6. $\frac{3}{(x+2)^2}$ is continuous, positive, and nonincreasing on $[100, \infty)$.

$\int_{100}^{\infty} \frac{3}{(x+2)^2} dx = \left[-\frac{3}{x+2} \right]_{100}^{\infty} = 0 + \frac{3}{102} = \frac{3}{102} < \infty$

The series converges.

7. $\frac{7}{4x+2}$ is continuous, positive, and nonincreasing on. $[2,\infty)$

$$\int_2^\infty \frac{7}{4x+2}dx = \left[\frac{7}{4}\ln|4x+2|\right]_2^\infty = \infty - \frac{7}{4}\ln 10 = \infty$$

The series diverges.

8. $\frac{x^2}{e^x}$ is continuous, positive, and nonincreasing $[2,\infty)$. Using integration by parts twice, with $u = x^i$, $i = 1, 2$ and $dv = e^{-x}dx$,

$$\int_2^\infty x^2e^{-x}dx = [-x^2e^{-x}]_2^\infty + 2\int_2^\infty xe^{-x}dx$$
$$= [-x^2e^{-x}]_2^\infty + 2\left([-xe^{-x}]_2^\infty + \int_2^\infty e^{-x}dx\right)$$
$$= [-x^2e^{-x} - 2xe^{-x} - 2e^{-x}]_2^\infty$$
$$= 0 + 4e^{-2} + 4e^{-2} + 2e^{-2} = 10e^{-2} < \infty$$

The series converges.

9. $\frac{3}{(4+3x)^{7/6}}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^\infty \frac{3}{(4+3x)^{7/6}}dx = \left[-\frac{6}{(4+3x)^{1/6}}\right]_1^\infty$$
$$= 0 + \frac{6}{7^{1/6}} = 6\cdot 7^{-1/6} < \infty$$

The series converges.

10. $\frac{1000x^2}{1+x^3}$ is continuous, positive, and nonincreasing on $[2,\infty)$.

$$\int_2^\infty \frac{1000x^2}{1+x^3}dx = \left[\frac{1000}{3}\ln\left|1+x^3\right|\right]_2^\infty$$
$$= \infty - \frac{1000}{3}\ln 9 = \infty$$

The series diverges.

11. xe^{-3x^2} is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^\infty xe^{-3x^2}dx = \left[-\frac{1}{6}e^{-3x^2}\right]_1^\infty = 0 + \frac{1}{6}e^{-3}$$
$$= \frac{1}{6e^3} < \infty$$

The series converges.

12. $\frac{1000}{x(\ln x)^2}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$$\int_5^\infty \frac{1000}{x(\ln x)^2}dx = \left[-\frac{1000}{\ln x}\right]_5^\infty = 0 + \frac{1000}{\ln 5}$$
$$= \frac{1000}{\ln 5} < \infty$$

The series converges.

13. $\lim_{k\to\infty}\frac{k^2+1}{k^2+5} = \lim_{k\to\infty}\frac{1+\frac{1}{k^2}}{1+\frac{5}{k^2}} = 1 \neq 0,$ so the series diverges.

14. $\sum_{k=1}^\infty \left(\frac{3}{\pi}\right)^k = \sum_{k=1}^\infty \frac{3}{\pi}\left(\frac{3}{\pi}\right)^{k-1}$; a geometric series with $a = \frac{3}{\pi}, r = \frac{3}{\pi};$ $\left|\frac{3}{\pi}\right| < 1$ so the series converges.

15. $\sum_{k=1}^\infty \left(\frac{1}{2}\right)^k$ is a geometric series with $r = \frac{1}{2}; \left|\frac{1}{2}\right| < 1$ so the series converges.

In $\sum_{k=1}^\infty \frac{k-1}{2k+1}, \lim_{k\to\infty}\frac{k-1}{2k+1} = \lim_{k\to\infty}\frac{1-\frac{1}{k}}{2+\frac{1}{k}} = \frac{1}{2} \neq 0,$ so the series diverges. Thus, the sum of the series diverges.

16. $\frac{1}{x^2}$ is continuous, positive, and nonincreasing on $[1,\infty)$. $\int_1^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^\infty = 0 + 1 = 1 < \infty,$ so $\sum_{k=1}^\infty \frac{1}{k^2}$ converges.

$\sum_{k=1}^\infty \frac{1}{2^k} = \sum_{k=1}^\infty \left(\frac{1}{2}\right)^k$; a geometric series with $r = \frac{1}{2}; \left|\frac{1}{2}\right| < 1,$ so the series converges. Thus, the sum of the series converges.

17. $\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 1 & k = 4j+1 \\ -1 & k = 4j+3, \\ 0 & k \text{ is even} \end{cases}$

where j is any nonnegative integer.

Thus $\lim_{k\to\infty}\left|\sin\left(\frac{k\pi}{2}\right)\right|$ does not exist, hence

$\lim_{k\to\infty}\left|\sin\left(\frac{k\pi}{2}\right)\right| \neq 0$ and the series diverges.

18. As $k \to \infty, \frac{1}{k} \to 0$. Let $y = \frac{1}{k}$, then

$\lim_{k\to\infty} k\sin\frac{1}{k} = \lim_{y\to 0}\frac{1}{y}\sin y = \lim_{y\to 0}\frac{\sin y}{y} = 1 \neq 0$, so the series diverges.

19. $x^2e^{-x^3}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$\int_1^\infty x^2e^{-x^3}dx = \left[-\frac{1}{3}e^{-x^3}\right]_1^\infty = 0 + \frac{1}{3}e^{-1} < \infty$, so the series converges.

20. $S_n = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n-1}\right) = 1-\frac{1}{n-1}$

$\lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty}\frac{1}{n-1} = 1-0 = 1$

The series converges to 1.

21. $\frac{\tan^{-1}x}{1+x^2}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$\int_1^\infty \frac{\tan^{-1}x}{1+x^2}dx = \left[\frac{1}{2}(\tan^{-1}x)^2\right]_1^\infty$

$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{32} < \infty$, so the series converges.

22. $\frac{1}{1+4x^2}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$\int_1^\infty \frac{1}{1+4x^2}dx = \left[\frac{1}{2}\tan^{-1}(2x)\right]_1^\infty$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\tan^{-1}2 < \infty$,

so the series converges.

23. $\frac{x}{e^x}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{k}{e^k} \le \int_5^\infty \frac{x}{e^x}dx = [-xe^{-x}]_5^\infty + \int_5^\infty e^{-x}dx$

$= [-xe^{-x} - e^{-x}]_5^\infty = 0 + 5e^{-5} + e^{-5} = 6e^{-5}$

≈ 0.0404

24. $\frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{1}{k\sqrt{k}} \le \int_5^\infty \frac{1}{x^{3/2}}dx = \left[-\frac{2}{\sqrt{x}}\right]_5^\infty = 0 + \frac{2}{\sqrt{5}}$

≈ 0.8944

25. $\frac{1}{1+x^2}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{1}{1+k^2} \le \int_5^\infty \frac{1}{1+x^2}dx = [\tan^{-1}x]_5^\infty$

$= \frac{\pi}{2} - \tan^{-1}5 \approx 0.1974$

26. $\frac{1}{x(x+1)}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{1}{k(k+1)} \le \int_5^\infty \frac{1}{x(x+1)}dx = \int_5^\infty\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$

$= \left[\ln|x| - \ln|x+1|\right]_5^\infty = \left[\ln\left|\frac{x}{x+1}\right|\right]_5^\infty = 0 - \ln\frac{5}{6}$

$= \ln\frac{6}{5} \approx 0.1823$

27. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \int_n^{\infty} \frac{1}{x^2} dx = \lim_{A\to\infty} \int_n^A \frac{1}{x^2} dx =$

$\lim_{A\to\infty} \left[\frac{1}{n} - \frac{1}{A}\right] = \frac{1}{n}$

$\frac{1}{n} < 0.0002 \Rightarrow n > 5000$

28. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{k^3} < \int_n^{\infty} \frac{1}{x^3} dx = \lim_{A\to\infty} \int_n^A \frac{1}{x^3} dx =$

$\lim_{A\to\infty} \left[\frac{1}{2n^2} - \frac{1}{2A^2}\right] = \frac{1}{2n^2}$

$\frac{1}{2n^2} < 0.0002 \Rightarrow n > \frac{1}{\sqrt{0.0004}} = 50$

29. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{1+k^2} < \int_n^{\infty} \frac{1}{1+x^2} dx$

$= \lim_{A\to\infty} \int_n^A \frac{1}{1+x^2} dx = \lim_{A\to\infty} \left[\tan^{-1} A - \tan^{-1} n\right]$

$= \frac{\pi}{2} - \tan^{-1} n$

$\frac{\pi}{2} - \tan^{-1} n < 0.0002 \Rightarrow \tan^{-1} n > \frac{\pi}{2} - 0.0002$

$\Rightarrow n > \tan\left(\frac{\pi}{2} - 0.0002\right) \approx 5000$

30. $E_n = \sum_{k=n+1}^{\infty} \frac{k}{e^{k^2}} < \underbrace{\int_n^{\infty} \frac{x}{e^{x^2}} dx}_{\substack{u = x^2 \\ du = 2x\,dx}} =$

$\lim_{A\to\infty} \frac{1}{2} \int_{n^2}^{A} \frac{1}{e^u} du =$

$\left(-\frac{1}{2}\right) \lim_{A\to\infty} \left[\frac{1}{e^A} - \frac{1}{e^{n^2}}\right] = \frac{1}{2e^{n^2}}$

$\frac{1}{2e^{n^2}} < 0.0002 \Rightarrow n > \sqrt{\ln \frac{1}{0.0004}}$

$\approx 2.797 \quad \therefore n > 2$

31. $E_n = \sum_{k=n+1}^{\infty} \frac{k}{1+k^4} < \underbrace{\int_n^{\infty} \frac{x}{1+x^4} dx}_{\substack{u=x^2 \\ du=2xdx}} = \lim_{A\to\infty} \frac{1}{2} \int_{n^2}^{A} \frac{du}{1+u^2}$

$= \frac{1}{2} \lim_{A\to\infty} \left[\tan^{-1} A - \tan^{-1} n^2\right] = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-2}\left(n^2\right)\right]$

$\frac{1}{2}\left[\frac{\pi}{2} - \tan^{-2}\left(n^2\right)\right] < 0.0002$

$\Rightarrow \frac{\pi}{2} - \tan^{-2}\left(n^2\right) < 0.0004 \Rightarrow \tan^{-1}\left(n^2\right) > 1.5703963$

$\Rightarrow n > \sqrt{\tan\left(1.5703963\right)} \approx 50$

32. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} < \int_n^{\infty} \frac{1}{x(x+1)} dx =$

$\lim_{A\to\infty} \int_n^A \left(\frac{1}{x} - \frac{1}{x+1}\right) dx =$

$\lim_{A\to\infty} \left[\ln\left(\frac{A}{A+1}\right) - \ln\left(\frac{n}{n+1}\right)\right] = 0 - \ln\left(\frac{n}{n+1}\right) =$

$\ln\left(\frac{n+1}{n}\right)$

$\ln\left(\frac{n+1}{n}\right) < 0.0002 \Rightarrow 1 + \frac{1}{n} < e^{0.0002} \approx 1.0002$

$\Rightarrow n > \frac{1}{0.0002} = 5000$

33. Consider $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx.$ Let $u = \ln x$,

$du = \frac{1}{x} dx.$

$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\infty} \frac{1}{u^p} du$ which converges for $p > 1$.

34. $\frac{1}{x \ln x \ln(\ln x)}$ is continuous, positive, and nonincreasing on $[3, \infty)$.

$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx$

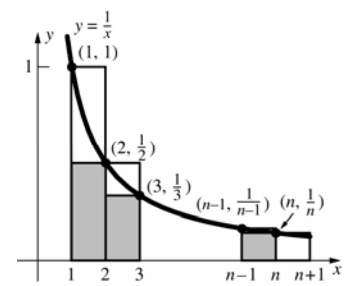
Let $u = \ln(\ln x)$, $du = \frac{1}{x \ln x} dx.$

$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx = \int_{\ln(\ln 3)}^{\infty} \frac{1}{u} du = [\ln u]_{\ln(\ln 3)}^{\infty}$

$= \infty - \ln(\ln(\ln 3)) = \infty$

The series diverges.

35.



The upper rectangles, which extend to $n + 1$ on the right, have area $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. These rectangles are above the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n + 1$. Thus,

$\int_1^{n+1} \frac{1}{x} dx = [\ln x]_1^{n+1} = \ln(n+1) - \ln 1 = \ln(n+1)$

$< 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$

The lower (shaded) rectangles have area $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$ These rectangles lie below the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n$. Thus

$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \int_1^n \frac{1}{x} dx = \ln n,$ so

$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + \ln n.$

36. From Problem 35, B_n is the area of the region within the upper rectangles but above the curve $y = \frac{1}{x}$. Each time n is incremented by 1, the added area is a positive amount, thus B_n is increasing.

From the inequalities in Problem 35,

$$0 < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n+1) < 1 + \ln n - \ln(n+1)$$

$$= 1 + \ln \frac{n}{n+1}$$

Since $\frac{n}{n+1} < 1, \ln \frac{n}{n+1} < 0,$ thus $B_n < 1$ for all n, and B_n is bounded by 1.

37. $\{B_n\}$ is a nondecreasing sequence that is bounded above, thus by the Monotonic Sequence Theorem (Theorem D of Section 9.1), $\lim_{n\to\infty} B_n$ exists. The rationality of γ is a famous unsolved problem.

38. From Problem 35, $\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k} < 1 + \ln n,$ thus

$$\ln(10,000,001) \approx 16.1181 < \sum_{k=1}^{10,000,000} \frac{1}{k}$$

$$< 1 + \ln(10,000,000) \approx 17.1181$$

39. $\gamma + \ln(n+1) > 20 \Rightarrow \ln(n+1) > 20 - \gamma \approx 19.4228$

$\Rightarrow n + 1 > e^{19.4228} \approx 272,404,867$

$\Rightarrow n > 272,404,866$

40. **a.** Each time n is incremented by 1, a positive amount of area is added.

b. The leftmost rectangle has area $1 \cdot f(1) = f(1).$ If each shaded region to the right of $x = 2$ is shifted until it is in the leftmost rectangle, there will be no overlap of the shaded area, since the top of each rectangle is at the bottom of the shaded region to the left. Thus, the total shaded area is less than or equal to the area of the leftmost rectangle, or $B_n \le f(1)$.

c. By parts a and b, $\{B_n\}$ is a nondecreasing sequence that is bounded above, so $\lim_{n\to\infty} B_n$ exists.

d. Let $f(x) = \frac{1}{x},$ then

$\int_1^{n+1} f(x)dx = \int_1^{n+1} \frac{1}{x} dx = \ln(n+1)$ and

$\lim_{n\to\infty} B_n = \gamma$ as defined in Problem 37.

41. Every time n is incremented by 1, a positive amount of area is added, thus $\{A_n\}$ is an increasing sequence. Each curved region has horizontal width 1, and can be moved into the heavily outlined triangle without any overlap. This can be done by shifting the nth shaded region, which goes from $(n, f(n))$ to $(n + 1, f(n + 1))$, as follows:

shift $(n + 1, f(n + 1))$ to $(2, f(2))$ and $(n, f(n))$ to $(1, f(2)–[f(n + 1) – f(n)])$.

The slope of the line forming the bottom of the shaded region between $x = n$ and $x = n + 1$ is

$$\frac{f(n+1) - f(n)}{(n+1) - n} = f(n+1) - f(n) > 0$$

since f is increasing.

By the Mean Value Theorem,

$f(n+1) - f(n) = f'(c)$ for some c in $(n, n + 1)$.

Since f is concave down, $n < c < n + 1$ means that $f'(c) < f'(b)$ for all b in $[1, n]$. Thus, the nth shaded region will not overlap any other shaded region when shifted into the heavily outlined triangle. Thus, the area of all of the shaded regions is less than or equal to the area of the heavily outlined triangle, so $\lim_{n\to\infty} A_n$ exists.

42. $\ln x$ is continuous, increasing, and concave down on $[1,\infty)$, so the conditions of Problem 41 are met.

a. See the figure in the text for Problem 41. The area under the curve from $x = 1$ to $x = n$ is $\int_1^n \ln x\,dx$ and the area of the nth trapezoid is $\frac{\ln n + \ln(n+1)}{2}$, thus $A_n = \int_1^n \ln x\,dx - \left[\frac{\ln 1 + \ln 2}{2} + \ldots + \frac{\ln(n-1)+\ln(n)}{2}\right]$.

Using integration by parts with $u = \ln x,\ du = \frac{1}{x}dx,\ dv = dx,\ v = x$

$$\int_1^n \ln x\,dx = [x\ln x]_1^n - \int_1^n dx = [x\ln x - x]_1^n \ = n\ln n - n - (\ln 1 - 1) = n\ln n - n + 1$$

The sum of the areas of the n trapezoids is

$$\frac{\ln 1+\ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \ldots + \frac{\ln(n-2)+\ln(n-1)}{2} + \frac{\ln(n-1)+\ln(n)}{2} = \frac{2\ln 2 + 2\ln 3 + \ldots + 2\ln(n-1) + \ln n}{2}$$

$$= \ln 2 + \ln 3 + \ldots + \ln n - \frac{\ln n}{2} \ = \ln(2\cdot 3\cdot \ldots \cdot n) - \frac{\ln n}{2} = \ln n! - \ln\sqrt{n}$$

Thus, $A_n = n\ln n - n + 1 - \left(\ln n! - \ln\sqrt{n}\right) \ = n\ln n - n + 1 - \ln n! + \ln\sqrt{n} \ \ = \ln n^n - \ln e^n + 1 - \ln n! + \ln\sqrt{n}$

$$= \ln\left(\frac{n}{e}\right)^n + 1 + \ln\frac{\sqrt{n}}{n!} = 1 + \ln\left[\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!}\right]$$

b. By Problem 41, $\lim_{n\to\infty} A_n$ exists, hence part a says that $\lim_{n\to\infty}\left[1 + \ln\left[\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!}\right]\right]$ exists.

$$\lim_{n\to\infty}\left[1 + \ln\left[\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!}\right]\right] = 1 + \lim_{n\to\infty} \ln\left[\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!}\right] = 1 + \ln\left[\lim_{n\to\infty}\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!}\right]$$

Since the limit exists, $\lim_{n\to\infty}\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!} = m.$ m cannot be 0 since $\lim_{x\to 0^+} \ln x = -\infty.$

Thus, $\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\frac{\left(\frac{n}{e}\right)^n\sqrt{n}}{n!}} = \frac{1}{\lim_{n\to\infty}\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!}} = \frac{1}{m}$, i.e., the limit exists.

c. From part b, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, thus, $15! \approx \sqrt{30\pi}\left(\frac{15}{e}\right)^{15} \approx 1.3004\times 10^{12}$

The exact value is $15! = 1,307,674,368,000$.

43. (Refer to *fig 2* in the text). Let $b_k = \int_k^{k+1} f(x)\,dx$; then from *fig 2*, it is clear that $a_k \ge b_k$ for $k = 1, 2, \ldots, n, \ldots$

Therefore $\sum_{k=n+1}^{t} a_k \ge \sum_{k=n+1}^{t} b_k = \int_{n+1}^{t} f(x)\,dx$ so that

$$E_n = \sum_{k=n+1}^{\infty} a_k = \lim_{t\to\infty}\sum_{k=n+1}^{t} a_k \ge \lim_{t\to\infty}\int_{n+1}^{t} f(x)\,dx = \int_{n+1}^{\infty} f(x)\,dx\ .$$

9.4 Concepts Review

1. $0 \le a_k \le b_k$

2. $\lim\limits_{k\to\infty} \frac{a_k}{b_k}$

3. $\rho < 1$; $\rho > 1$; $\rho = 1$

4. Ratio; Limit Comparison

Problem Set 9.4

1. $a_n = \frac{n}{n^2+2n+3}; b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+3} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{3}{n^2}} = 1;$$

$0 < 1 < \infty$

$\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

2. $a_n = \frac{3n+1}{n^3-4}; b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^3+n^2}{n^3-4} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1-\frac{4}{n^3}} = 3;$$

$0 < 3 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

3. $a_n = \frac{1}{n\sqrt{n+1}} = \frac{1}{\sqrt{n^3+n^2}};\quad b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+n^2}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+n^2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1;\ 0 < 1 < \infty$$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

4. $a_n = \frac{\sqrt{2n+1}}{n^2}; b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{2n+1}}{n^2} = \lim_{n\to\infty} \sqrt{\frac{2n^4+n^3}{n^4}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{2+\frac{1}{n}}{1}} = \sqrt{2}; 0 < \sqrt{2} < \infty$$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

5. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{8^{n+1} n!}{(n+1)!8^n} = \lim_{n\to\infty} \frac{8}{n+1} = 0 < 1$$

The series converges.

6. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1} n^5}{(n+1)^5 5^n} = \lim_{n\to\infty} \frac{5n^5}{(n+1)^5}$$

$$= \lim_{n\to\infty} \frac{5n^5}{n^5+5n^4+10n^3+10n^2+5n+1}$$

$$= \lim_{n\to\infty} \frac{5}{1+\frac{5}{n}+\frac{10}{n^2}+\frac{10}{n^3}+\frac{5}{n^4}+\frac{1}{n^5}} = 5 > 1$$

The series diverges.

7. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!n^{100}}{(n+1)^{100} n!} = \lim_{n\to\infty} \frac{n^{100}}{(n+1)^{99}}$$

$$= \lim_{n\to\infty} \frac{n}{\left(\frac{n+1}{n}\right)^{99}} = \infty \text{ since } \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{99} = 1$$

The series diverges.

8. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\left(\frac{1}{3}\right)^{n+1}}{n\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \frac{n+1}{3n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3} = \frac{1}{3} < 1$$

The series converges.

9. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3(2n)!}{(2n+2)!n^3}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{(2n+2)(2n+1)n^3} = \lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{4n^5+6n^4+2n^3}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{3}{n^3}+\frac{3}{n^4}+\frac{1}{n^5}}{4+\frac{6}{n}+\frac{2}{n^2}} = 0 < 1$$

The series converges.

10. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(3^{n+1}+n+1)n!}{(n+1)!(3^n+n)}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}+n+1}{(3^n+n)(n+1)} = \lim_{n\to\infty} \frac{3^{n+1}+n+1}{n3^n+3^n+n^2+n}$$

$$= \lim_{n\to\infty} \frac{3+\frac{n}{3^n}+\frac{1}{3^n}}{n+1+\frac{n^2}{3^n}+\frac{n}{3^n}} = 0 < \infty \text{ since } \lim_{n\to\infty} \frac{n}{3^n} = 0$$

and $\lim\limits_{n\to\infty} \frac{n^2}{3^n} = 0$ which can be seen by using l'Hôpital's Rule. The series converges.

11. $\lim\limits_{n\to\infty} \frac{n}{n+200} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{200}{n}} = 1 \neq 0$

The series diverges; nth-Term Test

12. $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)!(5+n)}{(6+n)n!}$

$= \lim\limits_{n\to\infty} \frac{(n+1)(5+n)}{6+n} = \lim\limits_{n\to\infty} \frac{n^2+6n+5}{6+n}$

$= \lim\limits_{n\to\infty} \frac{n+6+\frac{5}{n}}{\frac{6}{n}+1} = \infty > 1$

The series diverges; Ratio Test

13. $a_n = \frac{n+3}{n^2\sqrt{n}}; b_n = \frac{1}{n^{3/2}}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n^{5/2}+3n^{3/2}}{n^{5/2}} = \lim\limits_{n\to\infty} \frac{1+\frac{3}{n}}{1} = 1;$

$0 < 1 < \infty$. $\sum\limits_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

14. $a_n = \frac{\sqrt{n+1}}{n^2+1}; b_n = \frac{1}{n^{3/2}}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n^{3/2}\sqrt{n+1}}{n^2+1} = \lim\limits_{n\to\infty} \frac{\sqrt{n^4+n^3}}{n^2+1}$

$= \lim\limits_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{1+\frac{1}{n^2}} = 1; 0 < 1 < \infty.$

$\sum\limits_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

15. $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)^2 n!}{(n+1)!n^2} = \lim\limits_{n\to\infty} \frac{n^2+2n+1}{(n+1)n^2}$

$= \lim\limits_{n\to\infty} \frac{n^2+2n+1}{n^3+n^2} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^3}}{1+\frac{1}{n}} = 0 < 1$

The series converges; Ratio Test

16. $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\ln(n+1)2^n}{2^{n+1}\ln n} = \lim\limits_{n\to\infty} \frac{\ln(n+1)}{2\ln n}$

Using l'Hôpital's Rule,

$\lim\limits_{n\to\infty} \frac{\ln(n+1)}{2\ln n} = \lim\limits_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{2}{n}} = \lim\limits_{n\to\infty} \frac{n}{2(n+1)}$

$= \lim\limits_{n\to\infty} \frac{1}{2+\frac{2}{n}} = \frac{1}{2} < 1.$

The series converges; Ratio Test

17. $a_n = \frac{4n^3+3n}{n^5-4n^2+1}; b_n = \frac{1}{n^2}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{4n^5+3n^3}{n^5-4n^2+1} = \lim\limits_{n\to\infty} \frac{4+\frac{3}{n^2}}{1-\frac{4}{n^3}+\frac{1}{n^5}} = 4;$

$0 < 4 < \infty$

$\sum\limits_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

18. $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{[(n+1)^2+1]3^n}{3^{n+1}(n^2+1)}$

$= \lim\limits_{n\to\infty} \frac{n^2+2n+2}{3n^2+3} = \lim\limits_{n\to\infty} \frac{1+\frac{2}{n}+\frac{2}{n^2}}{3+\frac{3}{n^2}} = \frac{1}{3} < 1$

The series converges; Ratio Test

19. $a_n = \frac{1}{n(n+1)} = \frac{1}{n^2+n}; b_n = \frac{1}{n^2}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n^2}{n^2+n} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1;$

$0 < 1 < \infty$

$\sum\limits_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

20. $a_n = \frac{n}{(n+1)^2} = \frac{n}{n^2+2n+1}; b_n = \frac{1}{n}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1;$

$0 < 1 < \infty$

$\sum\limits_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ diverges; Limit Comparison Test

21. $a_n = \frac{n+1}{n(n+2)(n+3)} = \frac{n+1}{n^3+5n^2+6n}; b_n = \frac{1}{n^2}$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n^3+n^2}{n^3+5n^2+6n} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{5}{n}+\frac{6}{n^2}} = 1;$

$0 < 1 < \infty$

$\sum\limits_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

22. $a_n = \dfrac{n}{n^2+1}; b_n = \dfrac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = 1; 0 < 1 < \infty$$

$\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges;
Limit Comparison Test

23. $a_n = \dfrac{n}{3^n}; \lim_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty}\dfrac{(n+1)3^n}{3^{n+1}n}$

$$= \lim_{n\to\infty}\frac{n+1}{3n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{3} = \frac{1}{3} < 1$$

The series converges; Ratio Test

24. $a_n = \dfrac{3^n}{n!};$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{3^{n+1}n!}{(n+1)!3^n} = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1$$

The series converges; Ratio Test

25. $a_n = \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}; \dfrac{1}{x^{3/2}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty}\frac{1}{x^{3/2}}dx = \left[-\frac{2}{\sqrt{x}}\right]_1^{\infty} = 0 + 2 = 2 < \infty$$

The series converges; Integral Test

26. $a_n = \dfrac{\ln n}{n^2}; \dfrac{\ln x}{x^2}$ is continuous, positive, and nonincreasing on $[2,\infty)$. Use integration by parts with $u = \ln x$ and $dv = \dfrac{1}{x^2}dx$ for

$$\int_2^{\infty}\frac{\ln x}{x^2}dx = \left[-\frac{\ln x}{x}\right]_2^{\infty} + \int_2^{\infty}\frac{1}{x^2}dx$$

$$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^{\infty} = 0 + \frac{\ln 2}{2} + \frac{1}{2} < \infty$$

$\left(\lim_{x\to\infty}\dfrac{\ln x}{x} = 0 \text{ by l'Hôpital's Rule.}\right)$

The series converges; Integral Test

27. $0 \le \sin^2 n \le 1$ for all n, so

$2 \le 2 + \sin^2 n \le 3 \Rightarrow \dfrac{1}{2} \ge \dfrac{1}{2+\sin^2 n} \ge \dfrac{1}{3}$ for all n.

Thus, $\lim_{n\to\infty}\dfrac{1}{2+\sin^2 n} \ne 0$ and the series diverges; nth-Term Test

28. $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{5(3^n+1)}{(3^{n+1}+1)5} = \lim_{n\to\infty}\frac{1+\frac{1}{3^n}}{3+\frac{1}{3^n}}$$

$$= \frac{1}{3} < 1$$

The series converges; Ratio Test

29. $-1 \le \cos n \le 1$ for all n, so

$3 \le 4 + \cos n \le 5 \Rightarrow \dfrac{3}{n^3} \le \dfrac{4+\cos n}{n^3} \le \dfrac{5}{n^3}$ for all n.

$\sum_{n=1}^{\infty}\dfrac{5}{n^3}$ converges $\Rightarrow \sum_{n=1}^{\infty}\dfrac{4+\cos n}{n^3}$ converges;
Comparison Test

30. $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{5^{2n+2}n!}{(n+1)!5^{2n}} = \lim_{n\to\infty}\frac{25}{n+1} = 0 < 1$$

The series converges; Ratio Test

31. $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}(2n)!}{(2n+2)!n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+2)(2n+1)n^n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{2(n+1)(2n+1)n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{2(2n+1)n^n} = \lim_{n\to\infty}\left[\frac{1}{4n+2}\left(\frac{n+1}{n}\right)^n\right]$$

$$= \left[\lim_{n\to\infty}\frac{1}{4n+2}\right]\left[\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\right] = 0\cdot e = 0 < 1$$

(The limits can be separated since both limits exist.) The series converges; Ratio Test

32. Let $y = \left(1-\dfrac{1}{x}\right)^x; \ln y = x\ln\left(1-\dfrac{1}{x}\right)$

$$\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{\frac{1/x^2}{\left(1-\frac{1}{x}\right)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{1}{\left(1-\frac{1}{x}\right)} = -1$$

Thus $\lim_{x\to\infty} y = e^{-1}$, so $\lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n = e^{-1}$.

The series diverges; nth-Term Test

33. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(4^{n+1}+n+1)n!}{(n+1)!(4^n+n)}$

$=\lim_{n\to\infty}\frac{4^{n+1}+n+1}{(n+1)(4^n+n)}=\lim_{n\to\infty}\frac{4+\frac{n}{4^n}+\frac{1}{4^n}}{(n+1)\left(1+\frac{n}{4^n}\right)}$

$=\lim_{n\to\infty}\frac{4+\frac{n}{4^n}+\frac{1}{4^n}}{1+n+\frac{n}{4^n}+\frac{n^2}{4^n}}=0$

since $\lim_{n\to\infty}\frac{n^2}{4^n}=0,\ \lim_{n\to\infty}\frac{n}{4^n}=0$, and

$\lim_{n\to\infty}\frac{1}{4^n}=0.$ The series converges; Ratio Test

34. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)(2+n5^n)}{[2+(n+1)5^{n+1}]n}$

$=\lim_{n\to\infty}\frac{2n+n^2 5^n+2+n5^n}{2n+n^2 5^{n+1}+n5^{n+1}}$

$=\lim_{n\to\infty}\frac{\frac{2}{n5^n}+1+\frac{2}{n^2 5^n}+\frac{1}{n}}{\frac{2}{n5^n}+5+\frac{5}{n}}=\frac{1}{5}<1$

The series converge; Ratio Test

35. Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n=0.$ Thus, there is some positive integer N such that $0<a_n<1$ for all $n \ge N$. $a_n<1\Rightarrow a_n{}^2<a_n$, thus $\sum_{n=N}^{\infty} a_n{}^2<\sum_{n=N}^{\infty} a_n$. Hence $\sum_{n=N}^{\infty} a_n{}^2$ converges, and $\sum a_n{}^2$ also converges, since adding a finite number of terms does not affect the convergence or divergence of a series.

36. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges by Example 7, thus

$\lim_{n\to\infty}\frac{n!}{n^n}=0$ by the nth-Term Test.

37. If $\lim_{n\to\infty}\frac{a_n}{b_n}=0$ then there is some positive integer N such that $0\le\frac{a_n}{b_n}<1$ for all $n \ge N$. Thus, for $n \ge N$, $a_n<b_n$. By the Comparison Test, since $\sum_{n=N}^{\infty} b_n$ converges, $\sum_{n=N}^{\infty} a_n$ also converges. Thus, $\sum a_n$ converges since adding a finite number of terms will not affect the convergence or divergence of a series.

38. If $\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$ then there is some positive integer N such that $\frac{a_n}{b_n}>1$ for all $n \ge N$. Thus, for $n \ge N$, $a_n>b_n$ and by the Comparison Test, since $\sum_{n=N}^{\infty} b_n$ diverges, $\sum_{n=N}^{\infty} a_n$ also diverges. Thus, $\sum a_n$ diverges since adding a finite number of terms will not affect the convergence or divergence of a series.

39. If $\lim_{n\to\infty} na_n=1$ then there is some positive integer N such that $a_n\ge 0$ for all $n\ge N$, Let $b_n=\frac{1}{n}$, so $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty} na_n=1<\infty.$

Since $\sum_{n=N}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=N}^{\infty} a_n$ diverges by the Limit Comparison Test.
Thus $\sum a_n$ diverges since adding a finite number of terms will not affect the convergence or divergence of a series.

40. Consider $f(x)=x-\ln(1+x)$, then

$f'(x)=1-\frac{1}{1+x}=\frac{x}{1+x}>0$ on $(0,\ \infty)$.

$f(0) = 0 - \ln 1 = 0$, so since $f(x)$ is increasing, $f(x) > 0$ on $(0,\ \infty)$, i.e., $x > \ln(1 + x)$ for $x > 0$. Thus, since a_n is a series of positive terms, $\sum\ln(1+a_n)<\sum a_n$, hence if $\sum a_n$ converges, $\sum\ln(1+a_n)$ also converges.

41. Suppose that $\lim_{n\to\infty}(a_n)^{1/n}=R$ where $a_n>0$.
If $R < 1$, there is some number r with $R < r < 1$ and some positive integer N such that $\left|(a_n)^{1/n}-R\right|<r-R$ for all $n \ge N$. Thus,

$R-r<(a_n)^{1/n}-R<r-R$ or

$-r<(a_n)^{1/n}<r<1.$ Since $a_n>0$,

$0<(a_n)^{1/n}<r$ and $0<a_n<r^n$ for all $n \ge N$.

Thus, $\sum_{n=N}^{\infty} a_n<\sum_{n=N}^{\infty} r^n$, which converges since $|r|<1.$ Thus, $\sum_{n=N}^{\infty} a_n$ converges so $\sum a_n$ also converges.
If $R > 1$, there is some number r with $1 < r < R$ and some positive integer N such that

$\left|(a_n)^{1/n} - R\right| < R - r$ for all $n \ge N$. Thus,

$r - R < (a_n)^{1/n} - R < R - r$ or

$r < (a_n)^{1/n} < 2R - r$ for all $n \ge N$. Hence

$r^n < a_n$ for all $n \ge N$, so $\sum_{n=N}^{\infty} r^n < \sum_{n=N}^{\infty} a_n$, and

since $\sum_{n=N}^{\infty} r^n$ diverges ($r > 1$), $\sum_{n=N}^{\infty} a_n$ also

diverges, so $\sum a_n$ diverges.

42. **a.** $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{1}{\ln n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{1}{\ln n}$

$= 0 < 1$

The series converges.

b. $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n}{3n+2}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty}\frac{n}{3n+2} = \lim_{n\to\infty}\frac{1}{3+\frac{2}{n}} = \frac{1}{3} < 1$

The series converges.

c. $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{1}{2}+\frac{1}{n}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{n}\right) = \frac{1}{2} < 1$

The series converges.

43. **a.** $\ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \ldots$

$+(\ln n - \ln(n-1)) + (\ln(n+1) - \ln n)$

$= -\ln 1 + \ln(n+1) = \ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\ln(n+1) = \infty$

Since the partial sums are unbounded, the series diverges.

b. $\ln\frac{(n+1)^2}{n(n+2)} = 2\ln(n+1) - \ln n - \ln(n+2)$

$S_n = (2\ln 2 - \ln 1 - \ln 3) + (2\ln 3 - \ln 2 - \ln 4)$

$+(2\ln 4 - \ln 3 - \ln 5) + \ldots$

$+(2\ln n - \ln(n-1) - \ln(n+1))$

$+(2\ln(n+1) - \ln n - \ln(n+2))$

$= \ln 2 - \ln 1 + \ln(n+1) - \ln(n+2)$

$= \ln 2 + \ln\frac{n+1}{n+2}$

$\lim_{n\to\infty} S_n = \ln 2 + \lim_{n\to\infty}\ln\frac{n+1}{n+2} = \ln 2$

Since the partial sums converge, the series converges.

c. $\left(\frac{1}{\ln x}\right)^{\ln x}$ is continuous, positive, and nonincreasing on $[2, \infty)$, thus $\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}\right)^{\ln n}$ converges if and only if $\int_2^{\infty}\left(\frac{1}{\ln x}\right)^{\ln x} dx$ converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$$\int_2^{\infty}\left(\frac{1}{\ln x}\right)^{\ln x} dx = \int_{\ln 2}^{\infty}\left(\frac{1}{u}\right)^u e^u du = \int_{\ln 2}^{\infty}\left(\frac{e}{u}\right)^u du$$

This integral converges if and only if the associated series, $\sum_{n=1}^{\infty}\left(\frac{e}{n}\right)^n$ converges. With $a_n = \left(\frac{e}{n}\right)^n$, the Root Test (Problem 41) gives $\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{e}{n}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty}\frac{e}{n} = 0 < 1$

Thus, $\sum_{n=1}^{\infty}\left(\frac{e}{n}\right)^n$ converges, so $\int_{\ln 2}^{\infty}\left(\frac{e}{u}\right)^u du$ converges, whereby $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ converges.

d. $\left(\frac{1}{\ln(\ln x)}\right)^{\ln x}$ is continuous, positive, and nonincreasing on $[3, \infty)$, thus $\sum_{n=3}^{\infty}\left(\frac{1}{\ln(\ln n)}\right)^{\ln n}$ converges if and only if $\int_3^{\infty}\left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx$ converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$$\int_3^{\infty}\left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx$$

$$= \int_{\ln 3}^{\infty}\left(\frac{1}{\ln u}\right)^u e^u du = \int_{\ln 3}^{\infty}\left(\frac{e}{\ln u}\right)^u du.$$

This integral converges if and only if the

associated series, $\sum_{n=2}^{\infty}\left(\frac{e}{\ln n}\right)^n$ converges.

With $a_n=\left(\frac{e}{\ln n}\right)^n$, the Root Test (Problem 41) gives

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{e}{\ln n}\right)^n\right]^{1/n}.$$

$$=\lim_{n\to\infty}\frac{e}{\ln n}=0<1$$

Thus, $\sum_{n=2}^{\infty}\left(\frac{e}{\ln n}\right)^n$ converges, so

$\int_{\ln 3}^{\infty}\left(\frac{e}{\ln u}\right)^u du$ converges, whereby

$\sum_{n=3}^{\infty}\frac{1}{(\ln(\ln n))^{\ln n}}$ converges.

e. $a_n=1/n;\ b_n=1/(\ln n)^4$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1/n}{1/(\ln n)^4}=\lim_{n\to\infty}\frac{(\ln n)^4}{n}$$

$$=\lim_{n\to\infty}\frac{4(\ln n)^3(1/n)}{1}=\lim_{n\to\infty}\frac{4(\ln n)^3}{n}$$

$$=\lim_{n\to\infty}\frac{12(\ln n)^2(1/n)}{1}=\lim_{n\to\infty}\frac{12(\ln n)^2}{n}$$

$$=\lim_{n\to\infty}\frac{24(\ln n)}{n}=\lim_{n\to\infty}\frac{24(1/n)}{1}$$

$$=\lim_{n\to\infty}\frac{24}{n}=0$$

$\sum_{n=2}^{\infty}\frac{1}{n}$ diverges $\Rightarrow \sum_{n=2}^{\infty}\frac{1}{(\ln n)^4}$ diverges

f. $\left(\frac{\ln x}{x}\right)^2$ is continuous, positive, and nonincreasing on $[3,\infty)$. Using integration by parts twice,

$$\int_3^{\infty}\left(\frac{\ln x}{x}\right)^2 dx=\left[-\frac{(\ln x)^2}{x}\right]_3^{\infty}+\int_3^{\infty}\frac{2\ln x}{x^2}dx$$

$$=\left[-\frac{(\ln x)^2}{x}\right]_3^{\infty}+\left[-\frac{2\ln x}{x}\right]_3^{\infty}+\int_3^{\infty}\frac{2}{x^2}dx$$

$$=\left[-\frac{(\ln x)^2}{x}-\frac{2\ln x}{x}-\frac{2}{x}\right]_3^{\infty}\approx 1.8<\infty$$

Thus, $\sum_{n=3}^{\infty}\left(\frac{\ln x}{x}\right)^2$ converges.

44. The degree of $p(n)$ must be at least 2 less than the degree of $q(n)$. If $p(n)$ and $q(n)$ have the same degree, r, then

$p(n)=c_rn^r+c_{r-1}n^{r-1}+\ldots+c_1n+c_0$ and

$q(n)=d_rn^r+d_{r-1}n^{r-1}+\ldots+d_1n+d_0$ where $c_r, d_r\neq 0$ and

$$\lim_{n\to\infty}\frac{p(n)}{q(n)}=\lim_{n\to\infty}\frac{c_rn^r+c_{r-1}n^{r-1}+\ldots+c_1n+c_0}{d_rn^r+d_{r-1}n^{r-1}+\ldots+d_1n+d_0}=\lim_{n\to\infty}\frac{c_r+\frac{c_{r-1}}{n}+\ldots+\frac{c_1}{n^{r-1}}+\frac{c_0}{n^r}}{d_r+\frac{d_{r-1}}{n}+\ldots+\frac{d_1}{n^{r-1}}+\frac{d_0}{n^r}}=\frac{c_r}{d_r}\neq 0.$$

Thus, the series diverges by the nth-Term Test. If the degree of $p(n)$ is r and the degree of $q(n)$ is s, then the Limit Comparison Test with $a_n=\frac{p(n)}{q(n)}, b_n=\frac{1}{n^{s-r}}$ will give $\lim_{n\to\infty}\frac{a_n}{b_n}=L$ with $0<L<\infty$, since $\frac{a_n}{b_n}=\frac{n^{s-r}p(n)}{q(n)}$ and the degrees of $n^{s-r}p(n)$ and $q(n)$ are the same, similar to the previous case. Since $0 < L < \infty$, a_n and b_n either both converge or both diverge.

If $s \geq r+2$, then $s-r \geq 2$ so $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{s-r}}\leq\sum_{n=1}^{\infty}\frac{1}{n^2}$. Thus $\sum_{n=1}^{\infty}b_n$, and hence $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}$ converges.

If $s<r+2$, then $s-r\leq 1$ so $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{s-r}}\geq\sum_{n=1}^{\infty}\frac{1}{n}$. Thus $\sum_{n=1}^{\infty}b_n$, and hence $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}$ diverges.

45. Let $a_n = \frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}\right)$ and $b_n = \frac{1}{n^p}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}\right) = \sum_{n=1}^{\infty}\frac{1}{n^p}$ which converges if $p > 1$. Thus, by the Limit Comparison Test, if $\sum_{n=1}^{\infty} b_n$ converges for $p > 1$, so does $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)$ also converges. For $p \le 1$, since $1+\frac{1}{2^p}+\ldots+\frac{1}{n^p} > 1$, $\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right) > \frac{1}{n^p}$. Hence, since $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges for $p \le 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right)$ also diverges. The series converges for $p > 1$ and diverges for $p \le 1$.

46. a. Let $a_n = \sin^2\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n^2\sin^2\left(\frac{1}{n}\right) = \lim_{u\to 0^+}\left(\frac{1}{u}\right)^2\sin^2 u = \lim_{u\to 0^+}\left(\frac{\sin u}{u}\right)^2 = 1$ using the substitution $u = \frac{1}{n}$. Since $0 < 1 < \infty$, both $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ converge.

b. Let $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{n\sin\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n}\right)}$

$= \lim_{u\to 0}\frac{\left(\frac{1}{u}\right)\sin u}{\cos u} = \lim_{u\to 0}\left(\frac{\sin u}{u}\cos u\right) = 1$ using the substitution $u = \frac{1}{n}$. Since $0 < 1 < \infty$, both $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$ diverge.

c. $\sum_{n=1}^{\infty}\sqrt{n}\left(1-\cos\frac{1}{n}\right) = \sum_{n=1}^{\infty}\sqrt{n}\left(1-\cos\frac{1}{n}\right)\left(\frac{1+\cos\frac{1}{n}}{1+\cos\frac{1}{n}}\right) = \sum_{n=1}^{\infty}\frac{\sqrt{n}\left(1-\cos^2\frac{1}{n}\right)}{1+\cos\frac{1}{n}} = \sum_{n=1}^{\infty}\frac{\sqrt{n}\sin^2\frac{1}{n}}{1+\cos\frac{1}{n}} < \sum_{n=1}^{\infty}\sqrt{n}\sin^2\frac{1}{n}$

Let $a_n = \sqrt{n}\sin^2\frac{1}{n}$ and $b_n = \frac{1}{n^{3/2}}$.

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n^2\sin^2\frac{1}{n} = \lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^2 = 1$, since $\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{u\to 0^+}\frac{\sin u}{u} = 1$ with $u = \frac{1}{n}$.

Thus, by the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\sqrt{n}\sin^2\frac{1}{n}$ converges, and hence, $\sum_{n=1}^{\infty}\sqrt{n}\left(1-\cos\frac{1}{n}\right)$ converges by the Comparison Test.

9.5 Concepts Review

1. $\lim_{n\to\infty} a_n = 0$

2. absolutely; conditionally

3. the alternating harmonic series

4. rearranged

Problem Set 9.5

1. $a_n = \frac{2}{3n+1}; \frac{2}{3n+1} > \frac{2}{3n+4}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{2}{3n+1} = 0.$ $S_9 \approx 0.363$. The error made by using S_9 is not more than $a_{10} \approx 0.065$.

2. $a_n = \frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} = \frac{1}{\sqrt{n}} = 0.$ $S_9 \approx 0.76695$. The error made by using S_9 is not more than $a_{10} \approx 0.31623$.

3. $a_n = \frac{1}{\ln(n+1)}; \frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0.$ $S_9 \approx 1.137$. The error made by using S_9 is not more than $a_{10} \approx 0.417$.

4. $a_n = \frac{n}{n^2+1}; \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{n}{n^2+1} = 0.$ $S_9 \approx 0.32153$. The error made by using S_9 is not more than $a_{10} \approx 0.09901$.

5. $a_n = \frac{\ln n}{n}; \frac{\ln n}{n} > \frac{\ln(n+1)}{n+1}$ is equivalent to $\ln \frac{n^{n+1}}{(n+1)^n} > 0$ or $\frac{n^{n+1}}{(n+1)^n} > 1$ which is true for $n > 2$. $S_9 \approx -0.041$. The error made by using S_9 is not more than $a_{10} \approx 0.230$.

6. $a_n = \frac{\ln n}{\sqrt{n}}; \frac{\ln n}{\sqrt{n}} > \frac{\ln(n+1)}{\sqrt{n+1}}$ for $n \geq 7$, so $a_n > a_{n+1}$ for $n \geq 7$. $S_9 \approx 0.17199$. The error made by using S_9 is not more than $a_{10} \approx 0.72814$.

7. $\frac{|u_{n+1}|}{|u_n|} = \frac{\left|\left(-\frac{3}{4}\right)^{n+1}\right|}{\left|\left(-\frac{3}{4}\right)^{n}\right|} = \frac{3}{4} < 1$, so the series converges absolutely.

8. $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which converges since $\frac{3}{2} > 1$, thus $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\sqrt{n}}$ converges absolutely.

9. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n}$; $\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$, so the series converges absolutely.

10. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \frac{(n+1)^2}{en^2}$; $\lim_{n\to\infty} \frac{(n+1)^2}{en^2} = \frac{1}{e} \approx 0.36788 < 1$, so the series converges absolutely.

11. $n(n+1) = n^2 + n > n^2$ for all $n > 0$, thus $\frac{1}{n(n+1)} < \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges since 2 > 1, thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$ converges absolutely.

12. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1}$; $\lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$, so the series converges absolutely.

13. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which converges since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. The series is conditionally convergent since $\frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

14. $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{5n^{1.1}} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges since 1.1 > 1. The series is absolutely convergent.

15. $\lim_{n\to\infty} \frac{n}{10n+1} = \frac{1}{10} \neq 0$. Thus the sequence of partial sums does not converge; the series diverges.

16. $\frac{n}{10n^{1.1}+1} > \frac{n+1}{10(n+1)^{1.1}+1}$, so $a_n > a_{n+1}$;

$\lim_{n\to\infty} \frac{n}{10n^{1.1}+1} = \lim_{n\to\infty} \frac{1}{10n^{0.1}+\frac{1}{n}} = 0$. The alternating series converges.

Let $a_n = \frac{n}{10n^{1.1}+1}$ and $b_n = \frac{1}{n^{0.1}}$. Then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{1.1}}{10n^{1.1}+1} = \frac{1}{10}$; $0 < \frac{1}{10} < \infty$; so

both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, since $\sum_{n=1}^{\infty} \frac{1}{n^{0.1}}$ diverges. The series is conditionally convergent.

17. $\lim_{n\to\infty} \frac{1}{n\ln n} = 0$; $\frac{1}{n\ln n} > \frac{1}{(n+1)\ln(n+1)}$ is equivalent to $(n+1)^{n+1} > n^n$ which is true for all $n > 0$ so $a_n > a_{n+1}$. The alternating series converges.

$\sum_{n=2}^{\infty} |u_n| = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$; $\frac{1}{x\ln x}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

Using $u = \ln x$, $du = \frac{1}{x}dx$,

$\int_2^{\infty} \frac{1}{x\ln x}dx = \int_{\ln 2}^{\infty} \frac{1}{u}du = \left[\ln|u|\right]_{\ln 2}^{\infty} = \infty$. Thus,

$\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges and $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\ln n}$ is conditionally convergent.

18. $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n(1+\sqrt{n})} \le \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

which converges since $\frac{3}{2} > 1$. The series is absolutely convergent.

19. $$\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}} = \frac{(n+1)^4}{2n^4};$$

$\lim_{n\to\infty} \frac{(n+1)^4}{2n^4} = \frac{1}{2} < 1$.

The series is absolutely convergent.

20. $a_n = \frac{1}{\sqrt{n^2-1}}$; $\frac{1}{\sqrt{n^2-1}} > \frac{1}{\sqrt{n^2+2n}}$, so

$a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0$, hence the alternating series converges.

Let $b_n = \frac{1}{n}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = 1$;

$0 < 1 < \infty$.

Thus, since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$ also diverges. The series converges conditionally.

21. $a_n = \frac{n}{n^2+1}$; $\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$ is equivalent to

$n^2+n-1 > 0$, which is true for $n > 1$, so

$a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, hence the alternating series converges. Let $b_n = \frac{1}{n}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$; $0 < 1 < \infty$. Thus, since

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also diverges. The series is conditionally convergent.

22. $a_n = \frac{n-1}{n}$; $\lim_{n\to\infty} \frac{n-1}{n} = 1 \neq 0$

The series is divergent.

23. $\cos n\pi = (-1)^n = \frac{1}{(-1)}(-1)^{n+1}$ so the series is

$-1\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, -1 times the alternating harmonic series. The series is conditionally convergent.

24. $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}=1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\cdots,$ since

$\sin\frac{n\pi}{2}=\begin{cases}0 & n \text{ even}\\ (-1)^{\frac{n-1}{2}} & n \text{ odd}\end{cases}.$

Thus, $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)^2}.$

$\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$(2n-1)^2>n^2$ for $n > 1$, thus

$\sum_{n=2}^{\infty}\frac{1}{(2n-1)^2}<\sum_{n=2}^{\infty}\frac{1}{n^2}$, which converges since $2 > 1$. The series is absolutely convergent.

25. $|\sin n|\le 1$ for all n, so

$\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{|\sin n|}{n\sqrt{n}}\le\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which converges since $\frac{3}{2}>1.$ Thus the series is absolutely convergent.

26. $n\sin\left(\frac{1}{n}\right)=\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$. As $n\to\infty, \frac{1}{n}\to 0$ and

$\lim_{k\to 0}\frac{\sin k}{k}=1,$ so $\lim_{n\to\infty}n\sin\left(\frac{1}{n}\right)=1.$ The series diverges.

27. $a_n=\frac{1}{\sqrt{n(n+1)}};\ \frac{1}{\sqrt{n(n+1)}}>\frac{1}{\sqrt{(n+1)(n+2)}}$ and

$\lim_{n\to\infty}\frac{1}{\sqrt{n(n+1)}}=0$ so the alternating series converges.

Let $b_n=\frac{1}{n},$ then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1;$

$0 < 1 < \infty$.

Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ also diverges.

The series is conditionally convergent.

28. $a_n=\frac{1}{\sqrt{n+1}+\sqrt{n}};$

$\frac{1}{\sqrt{n+1}+\sqrt{n}}>\frac{1}{\sqrt{n+2}+\sqrt{n+1}},$ so $a_n>a_{n+1};$

$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0.$ The alternating series converges.

Let $b_n=\frac{1}{\sqrt{n}},$ then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2};$

$0<\frac{1}{2}<\infty.$ Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges $\left(\frac{1}{2}<1\right),$ $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$ also diverges. The series is conditionally convergent.

29. $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{n^2}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n^2};\ \lim_{n\to\infty}\frac{3^{n+1}}{n^2}\ne 0,$ so the series diverges.

30. $a_n=\sin\frac{\pi}{n};$ for $n\ \ge\ 2$, $\sin\frac{\pi}{n}>0$ and

$\sin\frac{\pi}{n}>\sin\frac{\pi}{n+1},$ so $a_n>a_{n+1};\ \lim_{n\to\infty}\sin\frac{\pi}{n}=0.$

The alternating series converges.

We have $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}=\lim_{n\to 0}\frac{\sin n}{n}=1$.

The series $\sum_{n=1}^{\infty}\sin\frac{\pi}{n}$ and $\sum_{n=1}^{\infty}\frac{\pi}{n}$ either both converge or both diverge. Since $\sum_{n=1}^{\infty}\frac{\pi}{n}=\pi\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, it follows that $\sum_{n=1}^{\infty}\sin\frac{\pi}{n}$ is divergent.

The series is conditionally convergent.

31. Suppose $\sum|a_n|$ converges. Thus, $\sum 2|a_n|$ converges, so $\sum\left(|a_n|+a_n\right)$ converges since $0\le|a_n|+a_n\le 2|a_n|.$ By the linearity of convergent series, $\sum a_n=\sum\left(|a_n|+a_n\right)-\sum|a_n|$ converges, which is a contradiction.

32. Let $\sum a_n=\sum(-1)^{n+1}\frac{1}{\sqrt{n}}=\sum b_n$. $\sum a_n$ and $\sum b_n$ both converge, but $\sum a_n b_n=\sum\frac{1}{n}$ diverges.

33. The positive-term series is

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots=\sum_{n=1}^{\infty}\frac{1}{2n-1}.$$

$\sum_{n=1}^{\infty}\frac{1}{2n-1}>\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges since the harmonic series diverges.

Thus, $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges.

The negative-term series is

$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\ldots=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges, since the harmonic series diverges.

34. If the positive terms and negative terms both formed convergent series then the series would be absolutely convergent. If one series was convergent and the other was divergent, the sum, which is the original series, would be divergent

35. a. $1+\frac{1}{3}\approx 1.33$

b. $1+\frac{1}{3}-\frac{1}{2}\approx 0.833$

c. $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\approx 1.38$

$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}\approx 1.13$

36. $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}-\frac{1}{6}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}-\frac{1}{8}+\frac{1}{25}+\frac{1}{27}+\frac{1}{29}+\frac{1}{31}$

$S_{20}\approx 1.3265$

37. Written response. Consider the partial sum of the positive terms of the series, and the partial sum of the negative terms. If both partial sums were bounded, the series would be absolutely convergent. Therefore, at least one of the partial sums must sum to ∞ (or $-\infty$). If the series of positive terms summed to ∞ and the series of negative terms summed to a finite number, the original series would not be convergent (similarly for the series of negative terms). Therefore, the positive terms sum to ∞ and the negative terms sum to $-\infty$. We can then rearrange the terms to make the original series sum to any value we wish.

38. Possible answer: take several positive terms, add one negative term, then add positive terms whose sum is at least one greater than the negative term previously added. Add another negative term, then add positive terms whose sum is at least one greater that the negative term just added. Continue in this manner and the resulting series will diverge.

39. Consider $1-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{9}+\ldots$

It is clear that $\lim_{n\to\infty}a_n=0.$ Pairing successive terms, we obtain $\frac{1}{n}-\frac{1}{n^2}=\frac{n-1}{n^2}>0$ for $n>1$.

Let $a_n=\frac{n-1}{n^2}$ and $b_n=\frac{1}{n}$. Then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2-n}{n^2}=1;\ 0<1<\infty$.

Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$ also diverges.

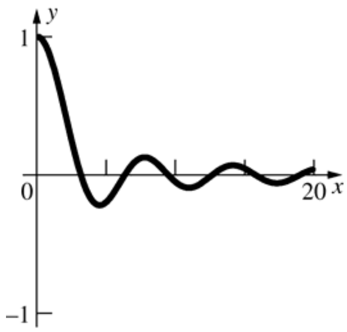
40. $\frac{1}{\sqrt{n}-1}-\frac{1}{\sqrt{n}+1}=\frac{2}{n-1}$, so

$$\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\cdots=\sum_{n=2}^{\infty}\frac{2}{n-1}$$

$=2\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges.

41. Note that $(a_k + b_k)^2 \geq 0$ and $(a_k - b_k)^2 \geq 0$ for all k. Thus, $a_k^2 \pm 2a_k b_k + b_k^2 \geq 0$, or $a_k^2 + b_k^2 \geq \pm 2a_k b_k$ for all k, and $a_k^2 + b_k^2 \geq 2|a_k b_k|$. Since $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ both converge, $\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$ also converges, and by the Comparison Test, $\sum_{k=1}^{\infty} 2|a_k b_k|$ converges. Hence, $\sum_{k=1}^{\infty}|a_k b_k| = \frac{1}{2}\sum_{k=1}^{\infty} 2|a_k b_k|$ converges, i.e., $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

42.



$\int_0^{\infty} \frac{\sin x}{x} dx$ gives the area of the region above the x-axis minus the area of the region below.

Note that

$$\int_{2k\pi}^{(2k+1)\pi}\left(\frac{\sin x}{x} + \frac{\sin(x+\pi)}{x+\pi}\right)dx = \int_{2k\pi}^{(2k+1)\pi}\frac{\sin x}{x}dx + \int_{2k\pi}^{(2k+1)\pi}\frac{\sin(x+\pi)}{x+\pi}dx$$

$$= \int_{2k\pi}^{(2k+1)\pi}\frac{\sin x}{x}dx + \int_{(2k+1)\pi}^{(2k+2)\pi}\frac{\sin u}{u}du = \int_{2k\pi}^{(2k+2)\pi}\frac{\sin x}{x}dx$$

by using the substitution $u = x + \pi$, then changing the variable of integration back to x.

Thus, $\int_0^{\infty}\frac{\sin x}{x}dx = \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\left(\frac{\sin x}{x} + \frac{\sin(x+\pi)}{x+\pi}\right)dx = \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\frac{(x+\pi)\sin x + x\sin(x+\pi)}{x(x+\pi)}dx$

$$= \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\frac{x\sin x + \pi\sin x - x\sin x}{x(x+\pi)}dx = \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\frac{\pi\sin x}{x(x+\pi)}dx.$$

For $k > 0$, on $[2k\pi, (2k+1)\pi]$ $0 \leq \sin x \leq 1$ while $0 < \frac{\pi}{x(x+\pi)} \leq \frac{\pi}{2k\pi(2k\pi+\pi)} = \frac{1}{(4k^2+2k)\pi}$.

Thus, $0 \leq \int_{2k\pi}^{(2k+1)\pi}\frac{\pi\sin x}{x(x+\pi)}dx \leq \frac{1}{(4k^2+2k)\pi}\int_{2k\pi}^{(2k+1)\pi} dx = \frac{1}{4k^2+2k}$.

Hence, $\int_{2\pi}^{\infty}\frac{\sin x}{x}dx \leq \sum_{k=1}^{\infty}\frac{1}{4k^2+2k} \leq \sum_{k=1}^{\infty}\frac{1}{4k^2}$ which converges.

Adding $\int_0^{2\pi}\frac{\sin x}{x}dx$ will not affect the convergence, so $\int_0^{\infty}\frac{\sin x}{x}dx$ converges.

43. Consider the graph of $\frac{|\sin x|}{x}$ on the interval $[k\pi, (k+1)\pi]$.

Note that for $k\pi + \frac{\pi}{6} \leq x \leq k\pi + \frac{5\pi}{6}$, $\frac{1}{2} \leq |\sin x|$ while $\frac{1}{\left(k+\frac{5}{6}\right)\pi} \leq \frac{1}{x}$. Thus on $\left[\left(k+\frac{1}{6}\right)\pi, \left(k+\frac{5}{6}\right)\pi\right]$

$$\frac{1}{2\left(k+\frac{5}{6}\right)\pi} = \frac{1}{\left(2k+\frac{5}{3}\right)\pi} \leq \frac{|\sin x|}{x}, \text{ so } \int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}dx \geq \int_{(k+1/6)\pi}^{(k+5/6)\pi}\frac{|\sin x|}{x}dx \geq \frac{1}{\left(2k+\frac{5}{3}\right)\pi}\int_{(k+1/6)\pi}^{(k+5/6)\pi}dx = \frac{1}{3k+\frac{5}{2}}.$$

Hence, $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{\infty} \frac{1}{3k+\frac{5}{2}}$. Let $a_k = \frac{1}{3k+\frac{5}{2}}$ and $b_k = \frac{1}{k}$.

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k}{3k+\frac{5}{2}} = \lim_{k\to\infty} \frac{1}{3+\frac{5}{2k}} = \frac{1}{3}$; $0 < \frac{1}{3} < \infty$. Thus, since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{3k+\frac{5}{2}}$ also diverges. Hence, $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx$ also diverges and adding $\int_{0}^{\pi} \frac{|\sin x|}{x} dx$ will not affect its divergence.

44. Recall that a straight line is the shortest distance between two points. Note that $\sin\frac{\pi}{x} = 1$ when $x = \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$ and $\sin\frac{\pi}{x} = -1$ when $x = \frac{2}{3}, \frac{2}{7}, \frac{2}{11}, \ldots$. Thus, for $n \geq 1$, the curve $y = x\sin\frac{\pi}{x}$ goes from $\left(\frac{2}{4n+1}, \frac{2}{4n+1}\right)$ to $\left(\frac{2}{4n+3}, -\frac{2}{4n+3}\right)$. The distance between these two points is

$$\sqrt{\left(\frac{2}{4n+1} - \frac{2}{4n+3}\right)^2 + \left(\frac{2}{4n+1} + \frac{2}{4n+3}\right)^2} = \sqrt{2\left(\frac{2}{4n+1}\right)^2 + 2\left(\frac{2}{4n+3}\right)^2}$$

$$= \frac{2\sqrt{2(4n+3)^2 + 2(4n+1)^2}}{(4n+1)(4n+3)} = \frac{2\sqrt{64n^2 + 64n + 20}}{16n^2 + 16n + 3} = \frac{4\sqrt{16n^2 + 16n + 5}}{16n^2 + 16n + 3}$$

The length of $x\sin\frac{\pi}{x}$ on (0, 1] is greater than $\sum_{n=1}^{\infty} \frac{4\sqrt{16n^2+16n+5}}{16n^2+16n+3}$ because this sum does not even take into account the distances from $\left(\frac{2}{4n+3}, -\frac{2}{4n+3}\right)$ to $\left(\frac{2}{4(n+1)+1}, \frac{2}{4(n+1)+1}\right)$ which are still shorter than the lengths along the curve.

Let $a_n = \frac{4\sqrt{16n^2+16n+5}}{16n^2+16n+3}$ and $b_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n\sqrt{16n^2+16n+5}}{16n^2+16n+3} = \lim_{n\to\infty} \frac{4\sqrt{16n^4+16n^3+5n^2}}{16n^2+16n+3} = \lim_{n\to\infty} \frac{4\sqrt{16+\frac{16}{n}+\frac{5}{n^2}}}{16+\frac{16}{n}+\frac{3}{n^2}}$

$= \frac{16}{16} = 1; 0 < 1 < \infty$

Thus, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{4\sqrt{16n^2+16n+5}}{16n^2+16n+3}$ also diverges.

Since the length of the graph is greater than $\sum_{n=1}^{\infty} a_n$, the length of the graph is infinite.

45. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right]\left(\frac{1}{n}\right)$

This is a Riemann sum for the function $f(x) = \frac{1}{x}$ from $x = 1$ to 2 where $\Delta x = \frac{1}{n}$.

$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{1+\frac{k}{n}}\left(\frac{1}{n}\right)\right] = \int_1^2 \frac{1}{x} dx = \ln 2$

9.6 Concepts Review

1. power series

2. where it converges

3. interval; endpoints

4. (–1, 1)

Problem Set 9.6

1. $\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}; \rho = \lim_{n\to\infty} \left| \frac{(n-1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n} \right| = 0$. Series converges for all x.

2. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}; \rho = \lim_{n\to\infty} \left| \frac{3^n x^{n+1}}{3^{n+1} x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \right| = \frac{|x|}{3}$; convergence on $(-3,3)$.

   For $x = 3,\ a_n = 1$ and the series diverges.

   For $x = -3,\ a_n = (-1)^n$ and the series diverges.

   Series converges on $(-3,3)$

3. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}; \rho = \lim_{n\to\infty} \left| \frac{n^2 x^{n+1}}{(n+1)^2 x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{(1+2/n + 1/n^2)} \right| = |x|$; convergence on $(-1,1)$.

   For $x = 1,\ a_n = \frac{1}{n^2}$ (p-series, p=2) and the series converges.

   For $x = -1,\ a_n = \frac{(-1)^n}{n^2}$ (alternating p-series, p=2) and the series converges. by the Absolute Convergence Test.

   Series converges on $[-1,1]$

4. $\sum_{n=1}^{\infty} nx^n; \rho = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = \lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right) x \right| = |x|$; convergence on $(-1,1)$.

   For $x = 1,\ a_n = n$ and the series diverges.

   For $x = -1,\ a_n = (-1)^n n$ and the series diverges ($\lim_{n\to\infty} (-1)^n n \neq 0$)

   Series converges on $(-1,1)$

5. This is the alternating series for problem 3; thus it converges on $[-1,1]$ by the Absolute Convergence Test.

6. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}; \rho = \lim_{n\to\infty} \left| \frac{nx^{n+1}}{(n+1)x^n} \right| = \lim_{n\to\infty} \left| x\left(\frac{n}{n+1}\right) \right| = |x|$; convergence on $(-1,1)$.

   For $x = 1,\ a_n = \frac{(-1)^n}{n}$ (Alternating Harmonic Series) and the series converges.

   For $x = -1,\ a_n = \frac{1}{n}$ (Harmonic Series) and the series diverges.

   Series converges on $(-1,1]$

7. Let $u = x - 2$; then, from problem 6, the series converges when $u \in (-1,1]$; that is when $x \in (1,3]$.

8. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n!}$; $\rho = \lim_{n\to\infty} \left| \frac{n!(x+1)^{n+1}}{(n+1)!(x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{x+1}{n+1} \right| = 0$. Series converges for all x.

9. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)}$; $\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)(n+2)} \div \frac{x^n}{n(n+1)} \right| = \lim_{n\to\infty} |x| \left| \frac{n}{n+2} \right| = |x|$

When $x = 1$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$ which converges absolutely by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

When $x = -1$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n(n+1)}$

$= \sum_{n=1}^{\infty} (-1) \frac{1}{n(n+1)} = (-1) \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ which converges since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

The series converges on $-1 \le x \le 1$.

10. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right| = \lim_{n\to\infty} |x| \left| \frac{1}{n+1} \right| = 0$

The series converges for all x.

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$; $\rho = \lim_{n\to\infty} \left| \frac{x^{2n+1}}{(2n+1)!} \div \frac{x^{2n-1}}{(2n-1)!} \right| = \lim_{n\to\infty} \left| x^2 \right| \left| \frac{1}{2n(2n+1)} \right| = 0$

The series converges for all x.

12. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$; $\rho = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \div \frac{x^{2n}}{(2n)!} \right|$

$= \lim_{n\to\infty} \left| x^2 \right| \left| \frac{1}{(2n+2)(2n+1)} \right| = 0$

The series converges for all x.

13. $\sum_{n=1}^{\infty} nx^n$; $\rho = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right|$

$= \lim_{n\to\infty} |x| \left| \frac{n+1}{n} \right| = |x|$

When $x = 1$, the series is $\sum_{n=1}^{\infty} n$ which clearly diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty} n(-1)^n$; $a_n = n$;

$\lim_{n\to\infty} a_n \ne 0$, thus the series diverges.

The series converges on $-1 < x < 1$.

14. $\sum_{n=1}^{\infty} n^2 x^n$; $\rho = \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right|$

$= \lim_{n\to\infty} |x| \left| \frac{(n+1)^2}{n^2} \right| = |x|$

When $x = 1$, the series is $\sum_{n=1}^{\infty} n^2$ which clearly diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty} n^2 (-1)^n$;

$a_n = n^2$; $\lim_{n\to\infty} a_n \ne 0$, thus the series diverges.

The series converges on $-1 < x < 1$.

15. $1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$; $\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \div \frac{x^n}{n} \right|$

$= \lim_{n\to\infty} |x| \left| \frac{n}{n+1} \right| = |x|$

When $x = 1$, the series is $1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, which is 1 added to the alternating harmonic series multiplied by –1, which converges.

When $x = -1$, the series is

$1 + \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n} = 1 + \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

The series converges on $-1 < x \le 1$.

16. $1+\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$; $\rho=\lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\div\frac{x^n}{\sqrt{n}}\right|$

$=\lim_{n\to\infty}|x|\left|\sqrt{\frac{n}{n+1}}\right|=|x|$

When $x = 1$, the series is $1+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges since $\frac{1}{2}<1$.

When $x = –1$, the series is $1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$;

$a_n=\frac{1}{\sqrt{n}}$; $\frac{1}{\sqrt{n}}>\frac{1}{\sqrt{n+1}}$, so $a_n>a_{n+1}$ and

$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, so the series converges.

The series converges on $-1 \le x < 1$.

17. $1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n(n+2)}$;

$\rho=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)(n+3)}\div\frac{x^n}{n(n+2)}\right|$

$=\lim_{n\to\infty}|x|\left|\frac{n^2+2n}{n^2+4n+3}\right|=|x|$

When $x = 1$ the series is $1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(n+2)}$ which converges absolutely by comparison with the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

When $x = –1$, the series is

$1+\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n}{n(n+2)}=1+\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ which converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

The series converges on $-1 \le x \le 1$.

18. $\sum_{n=1}^{\infty}\frac{x^n}{(n+1)^2-1}$;

$\rho=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)^2-1}\div\frac{x^n}{(n+1)^2-1}\right|$

$=\lim_{n\to\infty}|x|\left|\frac{n^2+2n}{n^2+4n+3}\right|=|x|$

When $x = 1$, the series is

$\sum_{n=1}^{\infty}\frac{1}{(n+1)^2-1}=\sum_{n=1}^{\infty}\frac{1}{n^2+2n}=\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ which converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

When $x = –1$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2-1}$ which converges absolutely by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

The series converges on $-1 \le x \le 1$.

19. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n}$; $\rho=\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\div\frac{x^n}{2^n}\right|=\lim_{n\to\infty}\left|\frac{x}{2}\right|$

$=\left|\frac{x}{2}\right|$; $\left|\frac{x}{2}\right|<1$ when $-2<x<2$.

When $x = 2$, the series is $\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{2^n}=\sum_{n=0}^{\infty}(-1)^n$ which diverges.

When $x = –2$, the series is

$\sum_{n=0}^{\infty}(-1)^n\frac{(-2)^n}{2^n}=\sum_{n=0}^{\infty}(-1)^n(-1)^n=\sum_{n=0}^{\infty}1$ which diverges. The series converges on $-2<x<2$.

20. $\sum_{n=0}^{\infty}2^n x^n$; $\rho=\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right|=\lim_{n\to\infty}|2x|=|2x|$;

$|2x|<1$ when $-\frac{1}{2}<x<\frac{1}{2}$.

When $x=\frac{1}{2}$, the series is $\sum_{n=0}^{\infty}2^n\left(\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}1$ which diverges.

When $x=-\frac{1}{2}$, the series is

$\sum_{n=0}^{\infty}2^n\left(-\frac{1}{2}\right)^n=\sum_{n=0}^{\infty}(-1)^n$ which diverges.

The series converges on $-\frac{1}{2}<x<\frac{1}{2}$.

21. $\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$; $\rho=\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{(n+1)!}\div\frac{2^n x^n}{n!}\right|$

$=\lim_{n\to\infty}|2x|\left|\frac{1}{n+1}\right|=0$.

The series converges for all x.

22. $\displaystyle\sum_{n=1}^{\infty}\frac{nx^n}{n+1};\ \rho=\lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{n+2}\div\frac{nx^n}{n+1}\right|$

$\displaystyle=\lim_{n\to\infty}|x|\left|\frac{n^2+2n+1}{n^2+2n}\right|=|x|$

When $x = 1$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n+1}$ which diverges since $\displaystyle\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0.$

When $x = –1$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{n(-1)^n}{n+1}$ which diverges since $\displaystyle\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0.$

The series converges on $–1 < x < 1$.

23. $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{n};\ \rho=\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\div\frac{(x-1)^n}{n}\right|$

$\displaystyle=\lim_{n\to\infty}|x-1|\left|\frac{n}{n+1}\right|=|x-1|$; $|x-1|<1$ when $0 < x < 2$.

When $x = 0$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges.

When $x = 2$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges.

The series converges on $0 \le x < 2$.

24. $\displaystyle\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!};\ \rho=\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)!}\div\frac{(x+2)^n}{n!}\right|$

$\displaystyle=\lim_{n\to\infty}|x+2|\left|\frac{1}{n+1}\right|=0$

The series converges for all x.

25. $\displaystyle\sum_{n=0}^{\infty}\frac{(x+1)^n}{2^n};\ \rho=\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}}\div\frac{(x+1)^n}{2^n}\right|$

$\displaystyle=\lim_{n\to\infty}\left|\frac{x+1}{2}\right|=\left|\frac{x+1}{2}\right|$; $\displaystyle\left|\frac{x+1}{2}\right|<1$ when $–3 < x < 1$.

When $x = –3$, the series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{2^n}=\sum_{n=0}^{\infty}(-1)^n$ which diverges.

When $x = 1$, the series is $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{2^n}=\sum_{n=0}^{\infty}1$ which diverges.

The series converges on $–3 < x < 1$.

26. $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2};\ \rho=\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2}\div\frac{(x-2)^n}{n^2}\right|$

$\displaystyle=\lim_{n\to\infty}|x-2|\left|\frac{n^2}{(n+1)^2}\right|=|x-2|$; $|x-2|<1$ when $1 < x < 3$.

When $x = 1$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ which converges absolutely since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

When $x = 3$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges. The series converges on $1 \le x \le 3$.

27. $\displaystyle\sum_{n=1}^{\infty}\frac{(x+5)^n}{n(n+1)};\ \rho=\lim_{n\to\infty}\left|\frac{(x+5)^{n+1}}{(n+1)(n+2)}\div\frac{(x+5)^n}{n(n+1)}\right|$

$\displaystyle=\lim_{n\to\infty}|x+5|\left|\frac{n}{n+2}\right|=|x+5|$; $|x+5|<1$ when $–6 < x < –4$.

When $x = –4$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ which converges by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

When $x = –6$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}$ which converges absolutely since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges.

The series converges on $–6 \le x \le –4$.

28. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n(x+3)^n;\ \rho=\lim_{n\to\infty}\left|\frac{(n+1)(x+3)^{n+1}}{n(x+3)^n}\right|$

$\displaystyle=\lim_{n\to\infty}|x+3|\left|\frac{n+1}{n}\right|=|x+3|$; $|x+3|<1$ when $–4 < x < –2$.

When $x = –2$, the series is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n$ which diverges since $\displaystyle\lim_{n\to\infty}n\neq 0.$

When $x = –4$, the series is

$\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n(-1)^n=\sum_{n=1}^{\infty}-n$, which diverges.

The series converges on $–4 < x < –2$.

29. If for some x_0, $\displaystyle\lim_{n\to\infty}\frac{x_0^n}{n!}\neq 0$, then $\displaystyle\sum\frac{x_0^n}{n!}$ could not converge.

30. For any number k, since
$k - n < k - n + 1 < \ldots < k - 2 < k - 1 < k$,
$|(k-1)(k-2)\ldots(k-n)| < k^n$, thus

$$\lim_{n\to\infty}\left|\frac{k(k-1)(k-2)\ldots(k-n)}{n!}x^n\right| < \lim_{n\to\infty}\left|\frac{k^{n+1}}{n!}x^n\right|$$

$= |k|\lim_{n\to\infty}\left|\frac{k^n}{n!}x^n\right|$. Since $-1 < x < 1$, $\lim_{n\to\infty} x^n = 0$,

and by Problem 21, $\lim_{n\to\infty}\left|\frac{k^n}{n!}\right| = 0$, hence

$$\lim_{n\to\infty}\frac{k(k-1)(k-2)\ldots(k-n)}{n!}x^n = 0.$$

31. The Absolute Ratio Test gives

$$\rho = \lim_{n\to\infty}\left|\frac{(n+1)!x^{2n+3}}{1\cdot 3\cdot 5\cdots(2n+1)} \div \frac{n!x^{2n+1}}{1\cdot 3\cdot 5\cdots(2n-1)}\right|$$

$= \lim_{n\to\infty}|x^2|\left|\frac{n+1}{2n+1}\right| = \left|\frac{x^2}{2}\right|$; $\left|\frac{x^2}{2}\right| < 1$ when
$|x| < \sqrt{2}.$

The radius of convergence is $\sqrt{2}$.

32. Using the Absolute Ratio Test,

$$\lim_{n\to\infty}\left|\frac{(pn+p)!}{((n+1)!)^p}x^{n+1} \div \frac{(pn)!}{(n!)^p}x^n\right|$$

$$= \lim_{n\to\infty}|x|\left|\frac{(pn+p)(pn+p-1)\ldots(pn+p-(p-1))}{(n+1)^p}\right|$$

$$= \lim_{n\to\infty}|x|\left|p\left(p-\frac{1}{n+1}\right)\left(p-\frac{2}{n+1}\right)\ldots\left(p-\frac{p-1}{n+1}\right)\right|$$

$$= |x|\,p^p$$

The radius of convergence is p^{-p}.

33. This is a geometric series, so it converges for $|x-3| < 1$, $2 < x < 4$. For these values of x, the series converges to $\frac{1}{1-(x-3)} = \frac{1}{4-x}$.

34. $\sum_{n=0}^{\infty} a_n(x-3)^n$ converges on an interval of the form $(3 - a, 3 + a)$, where $a \geq 0$. If the series converges at $x = -1$, then $3 - a \leq -1$, or $a \geq 4$, since $x = -1$ could be an endpoint where the series converges. If $a \geq 4$, then $3 + a \geq 7$ so the series will converge at $x = 6$. The series may not converge at $x = 7$, since $x = 7$ may be an endpoint of the convergence intervals, where the series might or might not converge.

35. a. $\rho = \lim_{n\to\infty}\left|\frac{(3x+1)^{n+1}}{(n+1)\cdot 2^{n+1}} \div \frac{(3x+1)^n}{n\cdot 2^n}\right| = \lim_{n\to\infty}|3x+1|\left|\frac{n}{2n+2}\right| = \frac{1}{2}|3x+1|$; $\frac{1}{2}|3x+1| < 1$ when $-1 < x < \frac{1}{3}$.

When $x = -1$, the series is $\sum_{n=1}^{\infty}\frac{(-2)^n}{n\cdot 2^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, which converges.

When $x = \frac{1}{3}$, the series is $\sum_{n=1}^{\infty}\frac{2^n}{n\cdot 2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which diverges. The series converges on $-1 \le x < \frac{1}{3}$.

b. $\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2x-3)^{n+1}}{4^{n+1}\sqrt{n+1}} \div \frac{(-1)^n(2x-3)^n}{4^n\sqrt{n}}\right| = \lim_{n\to\infty}|2x-3|\left|\frac{\sqrt{n}}{4\sqrt{n+1}}\right| = \frac{1}{4}|2x-3|$;

$\frac{1}{4}|2x-3| < 1$ when $-\frac{1}{2} < x < \frac{7}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{(-4)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges since $\frac{1}{2} < 1$.

When $x = \frac{7}{2}$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$; $a_n = \frac{1}{\sqrt{n}}$; $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$;

$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, so $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ converges. The series converges on $-\frac{1}{2} < x \le \frac{7}{2}$.

36. From Problem 52 of Section 9.1,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{1}{2^n\sqrt{5}}\left[\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right]$$

$$\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}\sqrt{5}}\left[\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right] \div \frac{x^n}{2^n\sqrt{5}}\left[\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right]\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{2}\right|\left|\frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{x}{2}\right|\left|\frac{1+\sqrt{5} - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n\left(1-\sqrt{5}\right)}{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}\right|$$

$$= \left|\frac{1+\sqrt{5}}{2}x\right|; \left|\frac{1+\sqrt{5}}{2}x\right| < 1 \text{ when } -\frac{2}{1+\sqrt{5}} < x < \frac{2}{1+\sqrt{5}}.$$

$$\left(\text{Note that } \lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n = 0 \text{ since } \left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| < 1.\right)$$

$$R = \frac{2}{1+\sqrt{5}} \approx 0.618$$

37. If $a_{n+3} = a_n$, then $a_0 = a_3 = a_6 = a_{3n}, a_1 = a_4 = a_7 = a_{3n+1}$, and $a_2 = a_5 = a_8 = a_{3n+2}$. Thus,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_0 x^3 + a_1 x^4 + a_2 x^5 + \cdots = (a_0 + a_1 x + a_2 x^2)(1 + x^3 + x^6 + \cdots)$$

$$= (a_0 + a_1 x + a_2 x^2)\sum_{n=0}^{\infty} x^{3n} = (a_0 + a_1 x + a_2 x^2)\sum_{n=0}^{\infty}(x^3)^n.$$

$a_0 + a_1 x + a_2 x^2$ is a polynomial, which will converge for all *x*.

$\sum_{n=0}^{\infty}(x^3)^n$ is a geometric series which, converges for $\left|x^3\right| < 1$, or, equivalently, $|x| < 1$.

Since $\sum_{n=0}^{\infty}(x^3)^n = \frac{1}{1-x^3}$ for $|x| < 1$, $S(x) = \frac{a_0 + a_1 x + a_2 x^2}{1-x^3}$ for $|x| < 1$.

38. If $a_n = a_{n+p}$, then $a_0 = a_p = a_{2p} = a_{np}, a_1 = a_{p+1} = a_{2p+1} = a_{np+1}$, etc. Thus,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_{p-1}x^{p-1} + a_0 x^p + a_1 x^{p+1} + \cdots + a_{p-1}x^{2p-1} + \cdots$$

$$= (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})(1 + x^p + x^{2p} + \cdots) = (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})\sum_{n=0}^{\infty} x^{np}$$

$a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}$ is a polynomial, which will converge for all *x*.

$\sum_{n=0}^{\infty} x^{np} = \sum_{n=0}^{\infty}(x^p)^n$ is a geometric series which converges for $\left|x^p\right| < 1$, or, equivalently, $|x| < 1$.

Since $\sum_{n=0}^{\infty}(x^p)^n = \frac{1}{1-x^p}$ for $|x| < 1$, $S(x) = (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})\left(\frac{1}{1-x^p}\right)$ for $|x| < 1$.

9.7 Concepts Review

1. integrated; interior

2. $-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}$

3. $1+x^2+\frac{x^4}{2}+\frac{x^6}{6}$

4. $1+x+\frac{3x^2}{2}+\frac{x^3}{3}+\frac{3x^4}{4}$

Problem Set 9.7

1. From the geometric series for $\frac{1}{1-x}$ with x replaced by $-x$, we get
$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots$,
radius of convergence 1.

2. $\frac{d}{dx}\left(\frac{1}{1+x}\right)=-\frac{1}{(1+x)^2}$
$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots$ so
$\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+5x^4-\cdots$;
radius of convergence 1.

3. $\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2}$; $\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right)=\frac{2}{(1-x)^3}$,
so $\frac{1}{(1-x)^3}$ is $\frac{1}{2}$ of the second derivative of $\frac{1}{1-x}$. Thus,
$\frac{1}{(1-x)^3}=1+3x+6x^2+10x^3+\cdots$;
radius of convergence 1.

4. Using the result of Problem 2,
$\frac{x}{(1+x)^2}=x-2x^2+3x^3-4x^4+5x^5-\cdots$;
radius of convergence 1.

5. From the geometric series for $\frac{1}{1-x}$ with x replaced by $\frac{3}{2}x$, we get
$\frac{1}{2-3x}=\frac{1}{2}+\frac{3x}{4}+\frac{9x^2}{8}+\frac{27x^3}{16}+\cdots$;
radius of convergence $\frac{2}{3}$.

6. $\frac{1}{3+2x}=\frac{1}{3}\left(\frac{1}{1+\frac{2}{3}x}\right)$. Since
$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots$,
$\frac{1}{3}\left(\frac{1}{1+\frac{2}{3}x}\right)=\frac{1}{3}-\frac{2x}{9}+\frac{4x^2}{27}-\frac{8x^3}{81}+\frac{16x^4}{243}-\cdots$;
radius of convergence $\frac{3}{2}$.

7. From the geometric series for $\frac{1}{1-x}$ with x replaced by x^4, we get
$\frac{x^2}{1-x^4}=x^2+x^6+x^{10}+x^{14}+\cdots$;
radius of convergence 1.

8. $\frac{x^3}{2-x^3}=\frac{x^3}{2}\left(\frac{1}{1-\frac{x^3}{2}}\right)=\frac{x^3}{2}+\frac{x^6}{4}+\frac{x^9}{8}+\frac{x^{12}}{16}+\cdots$
for $\left|\frac{x^3}{2}\right|<1$ or $-\sqrt[3]{2}<x<\sqrt[3]{2}$.

9. From the geometric series for $\ln(1+x)$ with x replaced by t, we get
$\int_0^x \ln(1+t)dt=\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{12}-\frac{x^5}{20}+\ldots$;
radius of convergence 1.

10. $\int_0^x \tan^{-1} t\,dt=\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{30}-\frac{x^8}{56}+\ldots$;
radius of convergence 1.

11. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, -1 < x \le 1$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots, -1 \le x < 1$

$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$

$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$; radius of convergence 1.

12. If $M = \frac{1+x}{1-x}$, then $M - Mx = 1 + x$;

$M - 1 = (M + 1)x$; $x = \frac{M-1}{M+1}$.

$\left|\frac{M-1}{M+1}\right| < 1$ is equivalent to $-M - 1 < M - 1 < M + 1$ or $0 < 2M < 2M + 2$ which is true for $M > 0$. Thus, the natural logarithm of any positive number can be found by using the series from Problem 11. For $M = 8$, $x = \frac{7}{9}$, so

$$\ln 8 = 2\left(\frac{7}{9}\right) + \frac{2}{3}\left(\frac{7}{9}\right)^3 + \frac{2}{5}\left(\frac{7}{9}\right)^5 + \frac{2}{7}\left(\frac{7}{9}\right)^7 + \frac{2}{9}\left(\frac{7}{9}\right)^9 + \frac{2}{11}\left(\frac{7}{9}\right)^{11} + \cdots$$

$\approx 1.55556 + 0.31367 + 0.11385 + 0.04919 + 0.02315 + 0.01146 + 0.00586 + 0.00307 + 0.00164 + 0.00089 + 0.00049 + 0.00027 + 0.00015 + 0.00008 \approx 2.079$

13. Substitute $-x$ for x in the series for e^x to get:

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$.

14. $xe^{x^2} = x\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \frac{x^9}{4!} + \cdots$

15. Add the result of Problem 13 to the series for e^x to get:

$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \cdots$.

16. $e^{2x} - 1 - 2x = -1 - 2x + \left(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots\right) = \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \frac{32x^5}{5!} + \cdots$

17. $e^{-x} \cdot \frac{1}{1-x} = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)(1 + x + x^2 + \cdots) = 1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^4}{8} + \frac{11x^5}{30} + \cdots$

18. $e^x \tan^{-1} x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = x + x^2 + \frac{x^3}{6} - \frac{x^4}{6} + \frac{3x^5}{40} + \cdots$

19. $\frac{\tan^{-1} x}{e^x} = e^{-x} \tan^{-1} x = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = x - x^2 + \frac{x^3}{6} + \frac{x^4}{6} + \frac{3x^5}{40} + \cdots$

20. $\frac{e^x}{1 + \ln(1+x)} = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots} = 1 + x^2 - \frac{7x^3}{6} + \frac{47x^4}{24} - \frac{46x^5}{15} + \cdots$

21. $(\tan^{-1} x)(1+x^2+x^4) = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)(1+x^2+x^4) = x + \frac{2x^3}{3} + \frac{13x^5}{15} - \frac{29x^7}{105} + \cdots$

22. $\frac{\tan^{-1} x}{1+x^2+x^4} = \frac{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}{1+x^2+x^4} = x - \frac{4x^3}{3} + \frac{8x^5}{15} + \frac{23x^7}{35} - \cdots$

23. The series representation of $\frac{e^x}{1+x}$ is $1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \frac{11x^5}{30} + \cdots$, so $\int_0^x \frac{e^t}{1+t} dt = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5 - \ldots .$

24. The series representation of $\frac{\tan^{-1} x}{x}$ is $1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots$, so $\int_0^x \frac{\tan^{-1} t}{t} dt = x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \cdots$

25. **a.** $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$, so $\frac{x}{1+x} = x - x^2 + x^3 - x^4 + x^5 - \cdots .$

b. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$, so $\frac{e^x - (1+x)}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots .$

c. $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, so $-\ln(1-2x) = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots .$

26. **a.** Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$, $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots .$

b. Again using $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$, $\frac{1}{1-\cos x} - 1 = \cos x + \cos^2 x + \cos^3 x + \cdots .$

c. $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$, so $\ln(1-x^2) = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \cdots$, and

$-\frac{1}{2}\ln(1-x^2) = \ln\frac{1}{\sqrt{1-x^2}} = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \cdots .$

27. Differentiating the series for $\frac{1}{1-x}$ yields $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ multiplying this series by x gives

$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$, hence $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $-1 < x < 1$.

28. Differentiating the series for $\frac{1}{x-1}$ twice yields $\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots$. Multiplying this series by x

gives $\frac{2x}{(1-x)^3} = 2x + 3\cdot 2x^2 + 4\cdot 3x^3 + 5\cdot 4x^4 + \cdots$, hence $\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$ for $-1 < x < 1$.

29. **a.** $\tan^{-1}(e^x - 1) = (e^x - 1) - \frac{(e^x-1)^3}{3} + \frac{(e^x-1)^5}{5} - \cdots = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \frac{1}{3}\left(x + \frac{x^2}{2!} + \cdots\right)^3 + \cdots$

$= x + \frac{x^2}{2} - \frac{x^3}{6} - \cdots$

b. $e^{e^x-1}=1+(e^x-1)+\frac{(e^x-1)^2}{2!}+\frac{(e^x-1)^3}{3!}+\cdots$

$=1+\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)+\frac{1}{2!}\left(x+\frac{x^2}{2!}+\cdots\right)^2+\frac{1}{3!}\left(x+\frac{x^2}{2!}+\cdots\right)^3$

$=1+\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)+\frac{1}{2!}\left(x^2+2\frac{x^3}{2!}+\cdots\right)+\frac{1}{3!}\left(x^3+3\frac{x^4}{2!}+\cdots\right)=1+x+x^2+\frac{5x^3}{6}+\cdots$

30. $f(x)=a_0+a_1x+a_2x^2+\cdots=b_0+b_1x+b_2x^2+\cdots;$
$f(0)=a_0=b_0$, so $a_0=b_0$.
$f'(x)=a_1+2a_2x+3a_3x^2+\cdots=b_1+2b_2x+3b_3x^3+\cdots;$
$f'(0)=a_1=b_1$, so $a_1=b_1$.
The nth derivative of $f(x)$ is
$f^{(n)}(x)=n!a_n+(n+1)!a_{n+1}x+\frac{(n+2)!}{2}a_{n+2}x^2+\cdots=n!b_n+(n+1)!b_{n+1}x+\frac{(n+2)!}{2}b_{n+2}x^2+\cdots;$
$f^{(n)}(0)=n!a_n=n!b_n$, so $a_n=b_n$.

31. $\frac{x}{x^2-3x+2}=\frac{x}{(x-2)(x-1)}=\frac{2}{x-2}-\frac{1}{x-1}=-\frac{1}{1-\frac{x}{2}}+\frac{1}{1-x}=-\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots\right)+\left(1+x+x^2+x^3+\cdots\right)$

$=\frac{x}{2}+\frac{3x^2}{4}+\frac{7x^3}{8}+\cdots=\sum_{n=1}^{\infty}\frac{(2^n-1)x^n}{2^n}$

32. $y''=-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\cdots=-y$, so $y''+y=0$. It is clear that $y(0)=0$ and $y'(0)=1$. Both the sine and cosine functions satisfy $y''+y=0$, however, only the sine function satisfies the given initial conditions. Thus, $y=\sin x$.

33. $F(x)-xF(x)-x^2F(x)=(f_0+f_1x+f_2x^2+f_3x^3+\cdots)-(f_0x+f_1x^2+f_2x^3+\cdots)-(f_0x^2+f_1x^3+f_2x^4+\cdots)$

$=f_0+(f_1-f_0)x+(f_2-f_1-f_0)x^2+(f_3-f_2-f_1)x^3+\cdots$

$=f_0+(f_1-f_0)x+\sum_{n=2}^{\infty}(f_n-f_{n-1}-f_{n-2})x^n=0+x+\sum_{n=0}^{\infty}(f_{n+2}-f_{n+1}-f_n)x^{n+2}$

Since $f_{n+2}=f_{n+1}+f_n$, $f_{n+2}-f_{n+1}-f_n=0$. Thus $F(x)-xF(x)-x^2F(x)=x$.

$F(x)=\frac{x}{1-x-x^2}$

34. $y(x)=\frac{f_0}{0!}+\frac{f_1}{1!}x+\frac{f_2}{2!}x^2+\frac{f_3}{3!}x^3+\frac{f_4}{4!}x^4+\cdots;\quad y'(x)=\frac{f_1}{0!}+\frac{f_2}{1!}x+\frac{f_3}{2!}x^2+\frac{f_4}{3!}x^3+\cdots;$

$y''(x)=\frac{f_2}{0!}+\frac{f_3}{1!}x+\frac{f_4}{2!}x^2+\cdots$

(Recall that 0! = 1.)

$y''(x)-y'(x)-y(x)=\left(\frac{f_2}{0!}+\frac{f_3}{1!}x+\frac{f_4}{2!}x^2+\cdots\right)-\left(\frac{f_1}{0!}+\frac{f_2}{1!}x+\frac{f_3}{2!}x^2+\cdots\right)-\left(\frac{f_0}{0!}+\frac{f_1}{1!}x+\frac{f_2}{2!}x^2+\cdots\right)$

$=\frac{1}{0!}(f_2-f_1-f_0)+\frac{1}{1!}(f_3-f_2-f_1)x+\frac{1}{2!}(f_4-f_3-f_2)x^2+\cdots$

$=\sum_{n=0}^{\infty}\frac{1}{n!}(f_{n+2}-f_{n+1}-f_n)x^n=0$ since $f_{n+2}=f_{n+1}+f_n$ for all $n\geq 0$.

35. $\pi \approx 16\left(\frac{1}{5}-\frac{1}{375}+\frac{1}{15,625}-\frac{1}{546,875}+\frac{1}{17,578,125}\right)-4\left(\frac{1}{239}\right) \approx 3.14159$

36. For any positive integer $k \le n$, both $\frac{n!}{k}$ and $\frac{n!}{k!}$ are positive integers. Thus, since $q < n$, $n!e = \frac{n!p}{q}$ is a positive integer and $M = n!e - n! - n! - \frac{n!}{2!} - \frac{n!}{3!} - \cdots - \frac{n!}{n!}$ is also an integer. M is positive since

$$e - 1 - 1 - \frac{1}{2!} - \cdots - \frac{1}{n!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$M < \frac{1}{n}$ contradicts that M is a positive integer since for $n \ge 1$, $\frac{1}{n} \le 1$ and there are no positive integers less than 1.

9.8 Concepts Review

1. $\frac{f^{(k)}(0)}{k!}$

2. $\lim_{n\to\infty} R_n(x) = 0$

3. $-\infty$; ∞

4. $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$

Problem Set 9.8

1. $\tan x = \frac{\sin x}{\cos x} = \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots} = x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$

2. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots}{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots} = x-\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$

3. $e^x \sin x = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right) = x+x^2+\frac{x^3}{3}-\frac{x^5}{30}-\ldots$

4. $e^{-x}\cos x = \left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right) = 1-x+\frac{x^3}{3}-\frac{x^4}{6}+\frac{x^5}{30}+\ldots$

5. $\cos x \ln(1+x) = \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right) = x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{3x^5}{40}-\ldots$

6. $(\sin x)\sqrt{1+x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots\right) = x + \frac{x^2}{2} - \frac{7x^3}{24} - \frac{x^4}{48} - \frac{19x^5}{1920} + \ldots,\ -1 < x < 1$

7. $e^x + x + \sin x = x + \left(1 + x + \frac{x^2}{2!} + \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = 1 + 3x + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{2x^5}{5!} + \ldots$

8. $\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$, so $\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \ldots$

9. $\frac{1}{1-x}\cosh x = (1 + x + x^2 + x^3 + \ldots)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) = 1 + x + \frac{3x^2}{2} + \frac{3x^3}{2} + \frac{37x^4}{24} + \frac{37x^5}{24} + \ldots,\ -1 < x < 1$

10. $\frac{-\ln(1+x)}{1+x} = \frac{-\ln(1+x)}{1-(-x)} = \left(-x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \ldots\right)(1 - x + x^2 - x^3 + x^4 - \ldots)$

$= -x + \frac{3x^2}{2} - \frac{11x^3}{6} + \frac{25x^4}{12} - \frac{137x^5}{60} + \ldots,\ -1 < x < 1$

11. $\frac{1}{1+x+x^2} = \frac{1}{1+x+x^2}\cdot\frac{1-x}{1-x} = \frac{1}{1-x^3}(1-x) = (1-x)\sum_{n=0}^{\infty} x^{3n} = 1 - x + x^3 - x^4 + \cdots,\ |x| < 1$

12. $\frac{1}{1-\sin x} = 1 + \sin x + (\sin x)^2 + (\sin x)^3 + \ldots$

$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \ldots\right)^2 + \left(x - \frac{x^3}{3!} + \ldots\right)^3 + \left(x - \frac{x^3}{3!} + \ldots\right)^4 + \left(x - \frac{x^3}{3!} + \ldots\right)^5 + \cdots$

$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) + \left(x^2 - 2\frac{x^4}{3!} + \ldots\right) + \left(x^3 - 3\frac{x^5}{3!} + \ldots\right) + (x^4 - \ldots) + (x^5 - \ldots)$

$= 1 + x + x^2 + \frac{5x^3}{6} + \frac{2x^4}{3} + \frac{61x^5}{120} + \ldots,\ |x| < \frac{\pi}{2}.$

13. $\sin^3 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^2\left(x - \frac{x}{3!} + \frac{x^5}{5!} - \ldots\right) = \left(x^2 - 2\frac{x^4}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = x^3 - \frac{x^5}{2} + \ldots$

14. $x(\sin 2x + \sin 3x) = x\left[\left(2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \ldots\right) + \left(3x - \frac{27x^3}{3!} + \frac{243x^5}{5!} - \ldots\right)\right] = x\left(5x - \frac{35x^3}{3!} + \ldots\right) = 5x^2 - \frac{35x^4}{3!} + \ldots$

15. $x\sec(x^2) + \sin x = \frac{x}{\cos(x^2)} + \sin x = \frac{x}{1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \ldots} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

$= \left(x + \frac{x^5}{2} + \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = 2x - \frac{x^3}{3!} + \frac{61x^5}{120} + \ldots$

16. $\frac{\cos x}{\sqrt{1+x}} = (\cos x)(1+x)^{-1/2} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\left(1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \ldots\right)$

$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \frac{49x^4}{384} - \frac{85x^5}{768} + \ldots,\ -1 < x < 1$

17. $(1+x)^{3/2} = 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \ldots,\ -1 < x < 1$

18. $(1-x^2)^{2/3}=[1+(-x^2)]^{2/3}=1+\frac{2}{3}(-x^2)-\frac{1}{9}(-x^2)^2+\frac{4}{81}(-x^2)^3+\ldots=1-\frac{2x^2}{3}-\frac{x^4}{9}-\ldots,\ -1<-x^2<1$ or $-1<x<1$

19. $f^{(n)}(x)=e^x$ for all n. $f(1)=f'(1)=f''(1)=f'''(1)=e$

$$e^x\approx e+e(x-1)+\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3$$

20. $f\left(\frac{\pi}{6}\right)=\frac{1}{2};\ f'\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2};\ f''\left(\frac{\pi}{6}\right)=-\frac{1}{2};\ f'''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2};\ \sin x\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$

21. $f\left(\frac{\pi}{3}\right)=\frac{1}{2};\ f'\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2};\ f''\left(\frac{\pi}{3}\right)=-\frac{1}{2};\ f'''\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2};\ \cos x\approx\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3$

22. $f\left(\frac{\pi}{4}\right)=1;\ f'\left(\frac{\pi}{4}\right)=2;\ f''\left(\frac{\pi}{4}\right)=4;\ f'''\left(\frac{\pi}{4}\right)=16$

$$\tan x\approx1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3$$

23. $f(1)=3;\ f'(1)=2+3=5;$

$f''(1)=2+6=8;\ f'''(1)=6$

$1+x^2+x^3=3+5(x-1)+4(x-1)^2+(x-1)^3$

This is exact since $f^{(n)}(x)=0$ for $n\geq 4$.

24. $f(-1)=2+1+3+1=7;$

$f'(-1)=-1-6-3=-10;$

$f''(-1)=6+6=12;\ f'''(1)=-6$

$2-x+3x^2-x^3=7-10(x+1)+6(x+1)^2-(x+1)^3$

This is exact since $f^{(n)}(x)=0$ for $n\geq 4$.

25. The derivative of an even function is an odd function and the derivative of an odd function is an even function. (Problem 50 of Section 3.2).

Since $f(x)=\sum a_nx^n$ is an even function, $f'(x)$ is an odd function, so $f''(x)$ is an even function, hence $f'''(x)$ is an odd function, etc.

Thus $f^{(n)}(x)$ is an even function when n is even and an odd function when n is odd.

By the Uniqueness Theorem, if $f(x)=\sum a_nx^n$, then $a_n=\frac{f^{(n)}(0)}{n!}$. If $g(x)$ is an odd function, $g(0)=0$, thence $a_n=0$ for all odd n since $f^{(n)}(x)$ is an odd function for odd n.

26. Let $f(x)=\sum a_nx^n$ be an odd function ($f(-x)=-f(x)$) for x in $(-R, R)$. Then $a_n=0$ if n is even.

The derivative of an even function is an odd function and the derivative of an odd function is an even function (Problem 50 of Section 3.2).

Since $f(x)=\sum a_nx^n$ is an odd function, $f'(x)$ is an even function, so $f''(x)$ is an odd function, hence $f'''(x)$ is an even function, etc. Thus, $f^{(n)}(x)$ is an even function when n is odd and an odd function when n is even.

By the Uniqueness Theorem, if $f(x)=\sum a_nx^n$, then $a_n=\frac{f^{(n)}(0)}{n!}$. If $g(x)$ is an odd function, $g(0)=0$, hence $a_n=0$ for all even n since $f^{(n)}(x)$ is an odd function for all even n.

27. $\frac{1}{\sqrt{1-t^2}}=[1+(-t^2)]^{-1/2}$

$=1-\frac{1}{2}(-t^2)+\frac{3}{8}(-t^2)^2-\frac{5}{16}(-t^2)^3+\cdots$

$=1+\frac{t^2}{2}+\frac{3t^4}{8}+\frac{5t^6}{16}+\cdots$

Thus, $\sin^{-1}x=\int_0^x\frac{1}{\sqrt{1-t^2}}dt$

$=\int_0^x\left(1+\frac{t^2}{2}+\frac{3t^4}{8}+\frac{5t^6}{16}+\cdots\right)dt$

$=\left[t+\frac{t^3}{6}+\frac{3t^5}{40}+\frac{5t^7}{112}+\cdots\right]_0^x$

$=x+\frac{x^3}{6}+\frac{3x^5}{40}+\frac{5x^7}{112}+\cdots$

28. $\dfrac{1}{\sqrt{1+t^2}}=(1+t^2)^{-1/2}$

$=1-\dfrac{1}{2}t^2+\dfrac{3}{8}(t^2)^2-\dfrac{5}{16}(t^2)^3+\cdots$

$=1-\dfrac{t^2}{2}+\dfrac{3t^4}{8}-\dfrac{5t^6}{16}+\cdots$

Thus, $\sinh^{-1}(x)=\int_0^x \dfrac{1}{\sqrt{1+t^2}}dt$

$=\int_0^x\left(1-\dfrac{t^2}{2}+\dfrac{3t^4}{8}-\dfrac{5t^6}{16}+\cdots\right)dt$

$=\left[t-\dfrac{t^3}{6}+\dfrac{3t^5}{40}-\dfrac{5t^7}{112}+\cdots\right]_0^x$

$=x-\dfrac{x^3}{6}+\dfrac{3x^5}{40}-\dfrac{5x^7}{112}+\cdots$

29. $\cos(x^2)=1-\dfrac{x^4}{2!}+\dfrac{x^8}{4!}-\dfrac{x^{12}}{6!}+\dfrac{x^{16}}{8!}-\cdots$

$\int_0^1\cos(x^2)dx=\int_0^1\left(1-\dfrac{x^4}{2!}+\dfrac{x^8}{4!}-\dfrac{x^{12}}{6!}+\dfrac{x^{16}}{8!}-\cdots\right)dx$

$=\left[x-\dfrac{x^5}{10}+\dfrac{x^9}{216}-\dfrac{x^{13}}{9360}+\dfrac{x^{17}}{685,440}-\cdots\right]_0^1$

$=1-\dfrac{1}{10}+\dfrac{1}{216}-\dfrac{1}{9360}+\dfrac{1}{685,440}-\cdots\approx 0.90452$

30. $\sin\sqrt{x}=\sqrt{x}-\dfrac{x^{3/2}}{3!}+\dfrac{x^{5/2}}{5!}-\dfrac{x^{7/2}}{7!}+\dfrac{x^{9/2}}{9!}-\ldots$

$\int_0^{0.5}\sin\sqrt{x}dx=\int_0^{0.5}\left(\sqrt{x}-\dfrac{x^{3/2}}{3!}+\dfrac{x^{5/2}}{5!}-\dfrac{x^{7/2}}{7!}+\dfrac{x^{9/2}}{9!}-\ldots\right)dx$

$=\left[\dfrac{2}{3}x^{3/2}-\dfrac{2}{5}\dfrac{x^{5/2}}{3!}+\dfrac{2}{7}\dfrac{x^{7/2}}{5!}-\dfrac{2}{9}\dfrac{x^{9/2}}{7!}+\dfrac{2}{11}\dfrac{x^{11/2}}{9!}-\ldots\right]_0^{0.5}$

$=\dfrac{2}{3}(0.5)^{3/2}-\dfrac{1}{15}(0.5)^{5/2}+\dfrac{1}{420}(0.5)^{7/2}-\dfrac{1}{22,680}(0.5)^{9/2}+\dfrac{1}{1,995,840}(0.5)^{11/2}-\ldots\approx 0.22413$

31. $\dfrac{1}{x}=\dfrac{1}{1-(1-x)}=1+(1-x)+(1-x)^2+(1-x)^3+\cdots=1-(x-1)+(x-1)^2-(x-1)^3+\cdots$

for $-1<1-x<1$, or $0<x<2$.

32. $(1+x)^{1/2}=1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dfrac{1}{16}x^3-\dfrac{5}{128}x^4+\dfrac{7}{256}x^5-\ldots$

$(1-x)^{1/2}=1-\dfrac{1}{2}x-\dfrac{1}{8}x^2-\dfrac{1}{16}x^2-\dfrac{5}{128}x^4-\dfrac{7}{256}x^5-\ldots$

so $f(x)=2-\dfrac{1}{4}x^2-\dfrac{5}{64}x^4-\ldots=\sum_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}x^n$

Note that $f^{(n)}(0)=0$ when n is odd.

Thus, $\dfrac{f^{(4)}(0)}{4!}=-\dfrac{5}{64}$ and $\dfrac{f^{(51)}(0)}{51!}=0$, so $f^{(4)}(0)=-\dfrac{5}{64}4!=-\dfrac{15}{8}$ and $f^{(51)}(0)=0.$

33. a. $f(x)=1+(x+x^2)+\dfrac{(x+x^2)^2}{2!}+\dfrac{(x+x^2)^3}{3!}+\dfrac{(x+x^2)^4}{4!}+\ldots$

$=1+(x+x^2)+\dfrac{1}{2}(x^2+2x^3+x^4)+\dfrac{1}{6}(x^3+3x^4+3x^5+x^6)+\dfrac{1}{24}(x^4+4x^5+6x^6+4x^7+x^8)+\ldots$

$=1+x+\dfrac{3x^2}{2}+\dfrac{7x^3}{6}+\dfrac{25x^4}{24}+\ldots=\displaystyle\sum_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}x^n$

Thus $\dfrac{f^{(4)}(0)}{4!}=\dfrac{25}{24}$ so $f^{(4)}(0)=\dfrac{25}{24}4!=25.$

b. $f(x)=1+\sin x+\dfrac{\sin^2 x}{2!}+\dfrac{\sin^3 x}{3!}+\dfrac{\sin^4 x}{4!}+\ldots$

$=1+\left(x-\dfrac{x^3}{3!}+\ldots\right)+\dfrac{1}{2}\left(x-\dfrac{x^3}{3!}+\ldots\right)^2+\dfrac{1}{6}\left(x-\dfrac{x^3}{3!}+\ldots\right)^3+\dfrac{1}{24}\left(x-\dfrac{x^3}{3!}+\ldots\right)^4$

$=1+\left(x-\dfrac{x^3}{3!}+\ldots\right)+\dfrac{1}{2}\left(x^2-2\dfrac{x^4}{3!}+\ldots\right)+\dfrac{1}{6}(x^3-\ldots)+\dfrac{1}{24}(x^4-\ldots)\;=1+x+\dfrac{x^2}{2}-\dfrac{x^4}{8}-\ldots=\displaystyle\sum_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}x^n$

Thus, $\dfrac{f^{(4)}(0)}{4!}=-\dfrac{1}{8}$ so $f^{(4)}(0)=-\dfrac{1}{8}4!=-3.$

c. $e^{t^2}-1=-1+\left(1+t^2+\dfrac{t^4}{2!}+\dfrac{t^6}{3!}+\dfrac{t^8}{4!}+\ldots\right)=t^2+\dfrac{t^4}{2!}+\dfrac{t^6}{3!}+\dfrac{t^8}{4!}+\ldots$

so $\dfrac{e^{t^2}-1}{t^2}=1+\dfrac{t^2}{2}+\dfrac{t^4}{6}+\dfrac{t^6}{24}+\ldots$

$f(x)=\displaystyle\int_0^x\left(1+\dfrac{t^2}{2}+\dfrac{t^4}{6}+\dfrac{t^6}{24}+\ldots\right)dt=\left[t+\dfrac{t^3}{6}+\dfrac{t^5}{30}+\ldots\right]_0^x=x+\dfrac{x^3}{6}+\dfrac{x^5}{30}+\ldots=\sum_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}x^n$

Thus, $\dfrac{f^{(4)}(0)}{4!}=0$ so $f^{(4)}(0)=0.$

d. $e^{\cos x-1}=1+(\cos x-1)+\dfrac{(\cos x-1)^2}{2!}+\dfrac{(\cos x-1)^3}{3!}+\ldots$

$=1+\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots\right)+\dfrac{1}{2}\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots\right)^2+\dfrac{1}{6}\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots\right)^3+\ldots$

$=1+\left(-\dfrac{x^2}{2}+\dfrac{x^4}{24}-\ldots\right)+\dfrac{1}{2}\left(\dfrac{x^4}{4}-\ldots\right)+\dfrac{1}{6}\left(-\dfrac{x^6}{8}+\ldots\right)=1-\dfrac{x^2}{2}+\dfrac{x^4}{6}-\ldots$

Hence $f(x)=e-\dfrac{e}{2}x^2+\dfrac{e}{6}x^4-\ldots=\displaystyle\sum_{n=0}^{\infty}\dfrac{f^{(n)}(0)}{n!}x^n$

Thus, $\dfrac{f^{(4)}(0)}{4!}=\dfrac{e}{6}$ so $f^{(4)}(0)=\dfrac{e}{6}4!=4e.$

e. Observe that $\ln(\cos^2 x) = \ln(1-\sin^2 x)$.

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots$$

$$\ln(1-\sin^2 x) = -\sin^2 x - \frac{\sin^4 x}{2} - \frac{\sin^6 x}{3} - \cdots$$

$$= -\left(x^2 - \frac{x^4}{3} + \cdots\right) - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \cdots\right)^2 - \frac{1}{3}\left(x^2 - \frac{x^4}{3} + \cdots\right)^3$$

$$= -\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots\right) - \frac{1}{2}\left(x^4 - \frac{2x^6}{3} + \cdots\right) - \frac{1}{3}(x^6 - \cdots) = -x^2 - \frac{x^4}{6} - \frac{2x^6}{45} - \cdots$$

Hence $f(x) = -x^2 - \frac{1}{6}x^4 - \frac{2}{45}x^6 - \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$.

Thus, $\frac{f^{(4)}(0)}{4!} = -\frac{1}{6}$ so $f^{(4)}(0) = -\frac{1}{6}4! = -4$.

34. $\sec x = \frac{1}{\cos x} = a_0 + a_1 x + a_2 x^2 + \ldots$ so

$$1 = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right)$$

$$= a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 - \frac{a_1}{2}\right)x^3 + \left(a_4 - \frac{a_2}{2} + \frac{a_0}{24}\right)x^4 + \ldots.$$

Thus $a_0 = 1, a_1 = 0, a_2 - \frac{a_0}{2} = 0, a_3 - \frac{a_1}{2} = 0, a_4 - \frac{a_2}{2} + \frac{a_0}{24} = 0,$ so

$a_0 = 1, a_1 = 0, a_2 = \frac{1}{2}, a_3 = 0, a_4 = \frac{5}{24}$

and therefore $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots$

35. $\tanh x = \frac{\sinh x}{\cosh x} = a_0 + a_1 x + a_2 x^2 + \ldots$

so $\sinh x = \cosh x(a_0 + a_1 x + a_2 x^2 + \ldots)$

or $x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots\right)\left(a_0 + a_1 x + a_2 x^2 + \ldots\right)$

$$= a_0 + a_1 x + \left(a_2 + \frac{a_0}{2}\right)x^2 + \left(a_3 + \frac{a_1}{2}\right)x^3 + \left(a_4 + \frac{a_2}{2} + \frac{a_0}{24}\right)x^4 + \left(a_5 + \frac{a_3}{2} + \frac{a_1}{24}\right)x^5 + \ldots$$

Thus $a_0 = 0, a_1 = 1, a_2 + \frac{a_0}{2} = 0, a_3 + \frac{a_1}{2} = \frac{1}{6},$

$a_4 + \frac{a_2}{2} + \frac{a_0}{24} = 0, a_5 + \frac{a_3}{2} + \frac{a_1}{24} = \frac{1}{120},$ so

$a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3}, a_4 = 0, a_5 = \frac{2}{15}$ and therefore

$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \ldots$

36. $\operatorname{sech} x = \dfrac{1}{\cosh x} = a_0 + a_1 x + a_2 x^2 + \ldots$

so $1 = \cosh x(a_0 + a_1 x + a_2 x^2 + \ldots)$

or $1 = \left(1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \ldots\right)\left(a_0 + a_1 x + a_2 x^2 + \ldots\right)$

$= a_0 + a_1 x + \left(a_2 + \dfrac{a_0}{2}\right)x^2 + \left(a_3 + \dfrac{a_1}{2}\right)x^3 + \left(a_4 + \dfrac{a_2}{2} + \dfrac{a_0}{24}\right)x^4 + \left(a_5 + \dfrac{a_3}{2} + \dfrac{a_1}{24}\right)x^5 + \cdots$

Thus, $a_0 = 1, a_1 = 0, \left(a_2 + \dfrac{a_0}{2}\right) = 0, \left(a_3 + \dfrac{a_1}{2}\right) = 0, \left(a_4 + \dfrac{a_2}{2} + \dfrac{a_0}{24}\right) = 0, \left(a_5 + \dfrac{a_3}{2} + \dfrac{a_0}{24}\right) = 0$, so

$a_0 = 1,\ a_1 = 0,\ a_2 = -\dfrac{1}{2},\ a_3 = 0,\ a_4 = \dfrac{5}{24},\ a_5 = 0$ and therefore

$\operatorname{sech} x = 1 - \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 - \cdots$

37. **a.** First define $R_3(x)$ by

$$R_3(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \frac{f'''(a)}{3!}(x-a)^3$$

For any t in the interval $[a, x]$ we define

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - R_3(x)\frac{(x-t)^4}{(x-a)^4}$$

Next we differentiate with respect to t using the Product and Power Rules:

$$g'(t) = 0 - f'(t) - \left[-f'(t) + f''(t)(x-t)\right] - \frac{1}{2!}\left[-2f''(t)(x-t) + f'''(t)(x-t)^2\right]$$

$$-\frac{1}{3!}\left[-3f'''(t)(x-t)^2 + f^{(4)}(t)(x-t)^3\right] + R_3(x)\frac{4(x-t)^3}{(x-a)^4}$$

$$= -\frac{f^{(4)}(t)(x-t)^3}{3!} + 4R_3(x)\frac{(x-t)^3}{(x-a)^4}$$

Since $g(x) = 0$, $g(a) = R_3(x) - R_3(x) = 0$, and $g(t)$ is continuous on $[a, x]$, we can apply the Mean Value Theorem for Derivatives. There exists, therefore, a number c between a and x such that $g'(c) = 0$. Thus,

$$0 = g'(c) = -\frac{f^{(4)}(c)(x-c)^3}{3!} + 4R_3(x)\frac{(x-c)^3}{(x-a)^4}$$

which leads to:

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(x-a)^4$$

b. Like the previous part, first define $R_n(x)$ by

$$R_n(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For any t in the interval $[a, x]$ we define

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}$$

Next we differentiate with respect to t using the Product and Power Rules:

$$g'(t) = 0 - f'(t) - \left[-f'(t) + f''(t)(x-t)\right] - \frac{1}{2!}\left[-2f''(t)(x-t) + f'''(t)(x-t)^2\right] - \cdots$$

$$-\frac{1}{n!}\left[-nf^{(n)}(t)(x-t)^{n-1} + f^{(n+1)}(t)(x-t)^n\right] + R_n(x)\frac{(n+1)(x-t)^n}{(x-a)^{n+1}}$$

$$= -\frac{f^{(n+1)}(t)(x-t)^n}{n!} + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

Since $g(x) = 0$, $g(a) = R_n(x) - R_n(x) = 0$, and $g(t)$ is continuous on $[a, x]$, we can apply the Mean Value Theorem for Derivatives. There exists, therefore, a number c between a and x such that $g'(c) = 0$. Thus,

$$0 = g'(c) = -\frac{f^{(n+1)}(c)(x-c)^n}{n!} + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

which leads to:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

38. a. For $\sum_{n=1}^{\infty}\binom{p}{n}x^n$, $\rho = \lim_{n\to\infty}\left|\binom{p}{n+1}x^{n+1} \div \binom{p}{n}x^n\right| = \lim_{n\to\infty}|x|\left|\frac{p(p-1)\ldots(p-n+1)(p-n)}{(n+1)!} \div \frac{p(p-1)\ldots(p-n+1)}{n!}\right|$

$$= \lim_{n\to\infty}|x|\left|\frac{p-n}{n+1}\right| = |x|$$

Thus $f(x) = 1 + \sum_{n=1}^{\infty}\binom{p}{n}x^n$ converges for $|x| < 1$.

b. It is clear that $f(0) = 1$.

Since $f(x) = 1 + \sum_{n=1}^{\infty}\binom{p}{n}x^n$, $f'(x) = \sum_{n=1}^{\infty}n\binom{p}{n}x^{n-1}$ and

$$(x+1)f'(x) = \sum_{n=1}^{\infty}n(x+1)\binom{p}{n}x^{n-1} = \sum_{n=1}^{\infty}\left[nx^n\binom{p}{n} + n\binom{p}{n}x^{n-1}\right] = 1\cdot\binom{p}{1}x^0 + \sum_{n=1}^{\infty}\left[n\binom{p}{n} + (n+1)\binom{p}{n+1}\right]x^n$$

$$n\binom{p}{n} + (n+1)\binom{p}{n+1} = n\frac{p(p-1)\ldots(p-n+1)}{n!} + (n+1)\frac{p(p-1)\ldots(p-n+1)(p-n)}{(n+1)!}$$

$$= \frac{1}{n!}[np(p-1)\ldots(p-n+1) + p(p-1)\ldots(p-n+1)(p-n)] = \frac{p(p-1)\ldots(p-n+1)}{n!}[n+p-n] = \binom{p}{n}p$$

and since $\binom{p}{1} = p$, $(1+x)f'(x) = p + \sum_{n=1}^{\infty}p\binom{p}{n}x^n = pf(x)$.

c. Let $y = f(x)$, then the differential equation is $(1+x)y' = py$ or $\frac{y'}{y} = \frac{p}{1+x}$.

$$\int\frac{dy}{y} = \int\frac{p}{1+x}dx \Rightarrow \ln|y| = p\ln|1+x| + C_1 \text{ or } y = C(1+x)^p \text{ so } f(x) = C(1+x)^p.$$

Since $f(0) = C(1)^p = C$ and $f(0) = 1$, $C = 1$ and $f(x) = (1+x)^p$.

39. $f'(t)=\begin{cases}0 & \text{if } t<0\\ 4t^3 & \text{if } t\ge 0\end{cases}$,

$f''(t)=\begin{cases}0 & \text{if } t<0\\ 12t^2 & \text{if } t\ge 0\end{cases}$,

$f'''(t)=\begin{cases}0 & \text{if } t<0\\ 24t & \text{if } t\ge 0\end{cases}$,

$f^{(4)}(t)=\begin{cases}0 & \text{if } t<0\\ 24 & \text{if } t\ge 0\end{cases}$

$\lim_{t\to 0^+} f^{(4)}(t)=24$ while $\lim_{t\to 0^-} f^{(4)}(t)=0,$ thus $f^{(4)}(0)$ does not exist, and $f(t)$ cannot be represented by a Maclaurin series.

Suppose that $g(t)$ as described in the text is represented by a Maclaurin series, so

$g(t)=a_0+a_1t+a_2t^2+\ldots=\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^n$ for all t in $(-R, R)$ for some $R > 0$. It is clear that, for $t\le 0$, $g(t)$ is represented by $g(t)=0+0t+0t^2+\ldots.$ However, this will not represent $g(t)$ for any $t > 0$ since the car is moving for $t > 0$. Similarly, any series that represents $g(t)$ for $t > 0$ cannot be 0 everywhere, so it will not represent $g(t)$ for $t < 0$. Thus, $g(t)$ cannot be represented by a Maclaurin series.

40. **a.** $f'(0)=\lim_{h\to 0}\frac{e^{-1/h^2}}{h}=\lim_{h\to 0}\frac{\frac{1}{h}}{e^{1/h^2}}$

$=\lim_{h\to 0}\frac{he^{-1/h^2}}{2}=0$ (by l’Hôpital’s Rule)

b. $f'(x)=\begin{cases}2x^{-3}e^{-1/x^2} & x\ne 0\\ 0 & x=0\end{cases}$

so

$f''(0)=\lim_{h\to 0}\frac{2e^{-1/h^2}}{h^4}=\lim_{h\to 0}\frac{\frac{2}{h^4}}{e^{1/h^2}}=\lim_{h\to 0}\frac{\frac{4}{h^2}}{e^{1/h^2}}$

$=\lim_{h\to 0}\frac{4}{e^{1/h^2}}=0$ (by using l’Hôpital’s Rule twice)

c. If $f^{(n)}(0)=0$ for all n, then the Maclaurin series for $f(x)$ is 0.

d. No, $f(x)\ne 0$ for $x\ne 0$. It only represents $f(x)$ at $x = 0$.

e. Note that for any n and $x\ne 0$, $R(x)=e^{-1/x^2}$.

41. $\sin x=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots$

42. $\exp x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots$

43. $3\sin x-2\exp x=-2+x-x^2-\frac{5x^3}{6}-\cdots$

$3\sin x=3x-\frac{x^3}{2}+\frac{x^5}{40}-\frac{x^7}{1680}+\cdots$

$-2\exp x=-2-2x-x^2-\frac{x^3}{3}-\cdots$

Thus, $3\sin x-2\exp x=-2+x-x^2-\frac{5x^3}{6}-\cdots$

44. $\exp(x^2)=1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\cdots$

$\exp(x^2)=1+x^2+\frac{(x^2)^2}{2}+\frac{(x^2)^3}{6}+\cdots$

$=1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\cdots$

45. $\sin(\exp x - 1) = x + \frac{x^2}{2} - \frac{5x^4}{24} - \frac{23x^5}{120} - \cdots$

$\exp x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

$\sin(\exp x - 1) = \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\right) - \frac{1}{6}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)^3 + \frac{1}{120}\left(x + \frac{x^2}{2} + \ldots\right)^5 - \cdots$

$= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right) - \frac{1}{6}\left(x^3 + \frac{3x^4}{2} + \frac{5x^5}{4} + \cdots\right) + \frac{1}{120}(x^5 + \cdots) - \cdots$

$= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\right) - \left(\frac{x^3}{6} + \frac{x^4}{4} + \frac{5x^5}{24} + \ldots\right) + \left(\frac{x^5}{120} + \ldots\right) - \cdots = x + \frac{x^2}{2} - \frac{5x^4}{24} - \frac{23x^5}{120} - \cdots$

46. $\exp(\sin x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \cdots$

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$

$\exp(\sin x) = 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + \cdots\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{6} + \cdots\right)^4 + \cdots$

$= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \cdots\right) + \frac{1}{6}\left(x^3 - \frac{x^5}{2} + \cdots\right) + \frac{1}{24}\left(x^4 - \frac{2x^6}{3} + \cdots\right) + \cdots$

$= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \left(\frac{x^2}{2} - \frac{x^4}{6} + \cdots\right) + \left(\frac{x^3}{6} - \frac{x^5}{12} + \cdots\right) + \left(\frac{x^4}{24} - \frac{x^6}{36} + \cdots\right) + \cdots = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \cdots$

47. $(\sin x)(\exp x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots$

$(\sin x)(\exp x) = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$

$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) + \left(x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \cdots\right) + \left(\frac{x^3}{2} - \frac{x^5}{12} + \cdots\right) + \left(\frac{x^4}{6} - \frac{x^6}{36} + \cdots\right) + \left(\frac{x^5}{24} - \frac{x^7}{144} + \cdots\right) + \cdots$

$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots$

48. $\frac{\sin x}{\exp x} = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$

$\frac{\sin x}{\exp x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots} = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$

9.9 Concepts Review

1. $f(1)$; $f'(1)$; $f''(1)$

2. $\dfrac{f^{(6)}(0)}{6!}$

3. error of the method; error of calculation

4. increase; decrease

Problem Set 9.9

1. $f(x) = e^{2x}$ $\quad f(0) = 1$
$f'(x) = 2e^{2x}$ $\quad f'(0) = 2$
$f''(x) = 4e^{2x}$ $\quad f''(0) = 4$
$f^{(3)}(x) = 8e^{2x}$ $\quad f^{(3)}(0) = 8$
$f^{(4)}(x) = 16e^{2x}$ $\quad f^{(4)}(0) = 16$
$$f(x) \approx 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$
$$f(0.12) \approx 1 + 2(0.12) + 2(0.12)^2 + \frac{4}{3}(0.12)^3 + \frac{2}{3}(0.12)^4 \approx 1.2712$$

2. $f(x) = e^{-3x}$ $\quad f(0) = 1$
$f'(x) = -3e^{-3x}$ $\quad f'(0) = -3$
$f''(x) = 9e^{-3x}$ $\quad f''(0) = 9$
$f^{(3)}(x) = -27e^{-3x}$ $\quad f^{(3)}(0) = -27$
$f^{(4)}(x) = 81e^{-3x}$ $\quad f^{(4)}(0) = 81$
$$f(x) \approx 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4$$
$$f(0.12) \approx 1 - 3(0.12) + \frac{9}{2}(0.12)^2 - \frac{9}{2}(0.12)^3 + \frac{27}{8}(0.12)^4 \approx 0.6977$$

3. $f(x) = \sin 2x$ $\quad f(0) = 0$
$f'(x) = 2\cos 2x$ $\quad f'(0) = 2$
$f''(x) = -4\sin 2x$ $\quad f''(0) = 0$
$f^{(3)}(x) = -8\cos 2x$ $\quad f^{(3)}(0) = -8$
$f^{(4)}(x) = 16\sin 2x$ $\quad f^{(4)}(0) = 0$
$$f(x) \approx 2x - \frac{8}{3!}x^3 = 2x - \frac{4}{3}x^3$$
$$f(0.12) \approx 2(0.12) - \frac{4}{3}(0.12)^3 \approx 0.2377$$

4. $f(x) = \tan x$ $\quad f(0) = 0$
$f'(x) = \sec^2 x$ $\quad f'(0) = 1$
$f''(x) = 2\sec^2 x \tan x$ $\quad f''(0) = 0$
$f^{(3)}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$
$f^{(3)}(0) = 2$
$f^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$
$f^{(4)}(0) = 0$
$$f(x) \approx x + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$
$$f(0.12) \approx 0.12 + \frac{1}{3}(0.12)^3 \approx 0.1206$$

5. $f(x) = \ln(1 + x)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\quad f''(0) = -1$

$f^{(3)}(x) = \frac{2}{(1+x)^3}$ $\quad f^{(3)}(0) = 2$

$f^{(4)}(x) = -\frac{6}{(1+x)^4}$ $\quad f^{(4)}(0) = -6$

$f(x) \approx x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

$f(0.12) \approx 0.12 - \frac{1}{2}(0.12)^2 + \frac{1}{3}(0.12)^3 - \frac{1}{4}(0.12)^4$

≈ 0.1133

6. $f(x) = \sqrt{1+x}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad f''(0) = -\frac{1}{4}$

$f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2}$ $\quad f^{(3)}(0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2}$ $\quad f^{(4)}(0) = -\frac{15}{16}$

$f(x) \approx 1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3 - \frac{\frac{15}{16}}{4!}x^4$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$

$f(0.12) \approx 1 + \frac{1}{2}(0.12) - \frac{1}{8}(0.12)^2$

$+ \frac{1}{16}(0.12)^3 - \frac{5}{128}(0.12)^4 \approx 1.0583$

7. $f(x) = \tan^{-1} x$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $\quad f'(0) = 1$

$f''(x) = -\frac{2x}{(1+x^2)^2}$ $\quad f''(0) = 0$

$f'''(x) = \frac{6x^2 - 2}{(1+x^2)^3}$ $\quad f'''(0) = -2$

$f^{(4)}(x) = \frac{-24x^3 + 24x}{(1+x^2)^4}$ $\quad f^{(4)}(0) = 0$

$f(x) \approx x - \frac{2}{3!}x^3 = x - \frac{1}{3}x^3$

$f(0.12) \approx 0.12 - \frac{1}{3}(0.12)^3 \approx 0.1194$

8. $f(x) = \sinh x \quad f(0) = 0$

$f'(x) = \cosh x \quad f'(0) = 1$

$f''(x) = \sinh x \quad f''(0) = 0$

$f'''(x) = \cosh x \quad f'''(0) = 1$

$f^{(4)}(x) = \sinh x \quad f^{(4)}(0) = 0$

$f(x) \approx x + \frac{1}{3!}x^3 = x + \frac{1}{6}x^3$

$f(0.12) \approx 0.12 + \frac{1}{6}(0.12)^3 \approx 0.1203$

9. $f(x) = e^x$ $\quad f(1) = e$

$f'(x) = e^x \quad f'(1) = e$

$f''(x) = e^x \quad f''(1) = e$

$f'''(x) = e^x \quad f'''(1) = e$

$P_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$

10. $f(x) = \sin x \; f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x \quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x \quad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$$

11. $f(x) = \tan x; \quad f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$f'(x) = \sec^2 x; \quad f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$

$f''(x) = 2\sec^2 x \tan x; \quad f''\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{9}$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x; \quad f'''\left(\frac{\pi}{6}\right) = \frac{16}{3}$

$$P_3(x) = \frac{\sqrt{3}}{3} + \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{4\sqrt{3}}{9}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3$$

12. $f(x) = \sec x$; $f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x$; $f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x$; $f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$f'''(x) = 5\sec^3 x \tan x + \sec x \tan^3 x$;

$f'''\left(\frac{\pi}{4}\right) = 11\sqrt{2}$

$$P_3(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3$$

13. $f(x) = \cot^{-1} x$; $f(1) = \frac{\pi}{4}$

$f'(x) = -\frac{1}{1+x^2}$; $f'(1) = -\frac{1}{2}$

$f''(x) = \frac{2x}{(1+x^2)^2}$; $f''(1) = \frac{1}{2}$

$f'''(x) = \frac{-6x^2+2}{(1+x^2)^3}$; $f'''(1) = -\frac{1}{2}$

$$P_3(x) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{12}(x-1)^3$$

14. $f(x) = \sqrt{x}$; $f(2) = \sqrt{2}$

$f'(x) = \frac{1}{2}x^{-1/2}$; $f'(2) = \frac{\sqrt{2}}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$; $f''(2) = -\frac{\sqrt{2}}{16}$

$f'''(x) = \frac{3}{8}x^{-5/2}$; $f'''(2) = \frac{3\sqrt{2}}{64}$

$$P_3(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + \frac{\sqrt{2}}{128}(x-2)^3$$

15. $f(x) = x^3 - 2x^2 + 3x + 5$; $f(1) = 7$

$f'(x) = 3x^2 - 4x + 3$; $f'(1) = 2$

$f''(x) = 6x - 4$; $f''(1) = 2$

$f^{(3)}(x) = 6$; $f^{(3)}(1) = 6$

$P_3(x) = 7 + 2(x-1) + (x-1)^2 + (x-1)^3$

$= 5 + 3x - 2x^2 + x^3 = f(x)$

16. $f(x) = x^4$; $f(2) = 16$

$f'(x) = 4x^3$; $f'(2) = 32$

$f''(x) = 12x^2$; $f''(2) = 48$

$f^{(3)}(x) = 24x$; $f^{(3)}(2) = 48$

$f^{(4)}(x) = 24$; $f^{(4)}(2) = 24$

$$P_4(x) = 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4$$

$= x^4 = f(x)$

17. $f(x) = \frac{1}{1-x}$; $f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2}$; $f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3}$; $f''(0) = 2$

$f^{(3)}(x) = \frac{6}{(1-x)^4}$; $f^{(3)}(0) = 6$

$f^{(4)}(x) = \frac{24}{(1-x)^5}$; $f^{(4)}(0) = 24$

$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$; $f^{(n)}(0) = n!$

$f(x) \approx 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \ldots + \frac{n!}{n!}x^n$

$= 1 + x + x^2 + x^3 + \ldots + x^n$

Using $n = 4$, $f(x) \approx 1 + x + x^2 + x^3 + x^4$

a. $f(0.1) \approx 1.1111$

b. $f(0.5) \approx 1.9375$

c. $f(0.9) \approx 4.0951$

d. $f(2) \approx 31$

18. $f(x) = \sin x$; $f(0) = 0$

$f'(x) = \cos x$; $f'(0) = 1$

$f''(x) = -\sin x$; $f''(0) = 0$

$f^{(3)}(x) = -\cos x$; $f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$; $f^{(4)}(0) = 0$

When n is odd,

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{(n-1)/2}x^n}{n!}$$

Using $n = 5$, $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

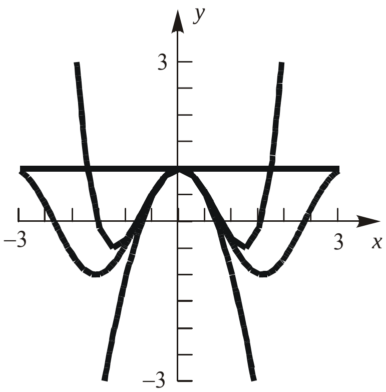
a. $\sin(0.1) \approx 0.0998$

b. $\sin(0.5) \approx 0.4794$

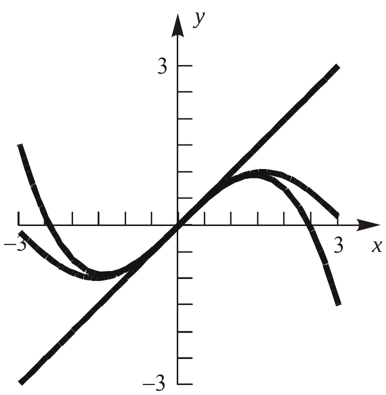
c. $\sin(1) \approx 0.8417$

d. $\sin(10) \approx 676.67$

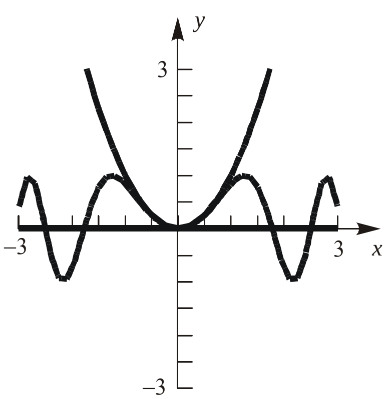
19.



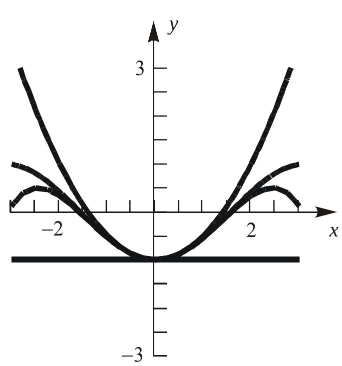
20.



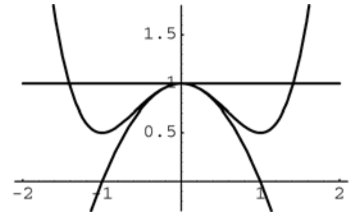
21.



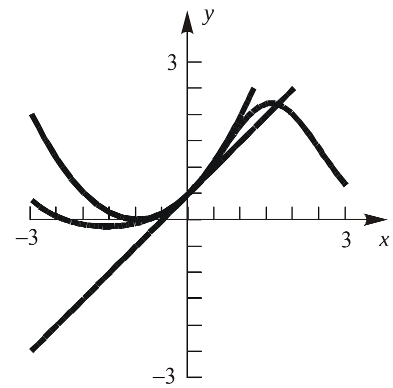
22.



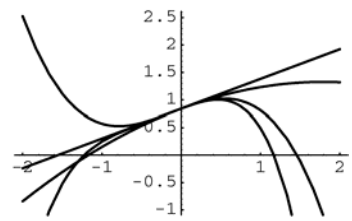
23.



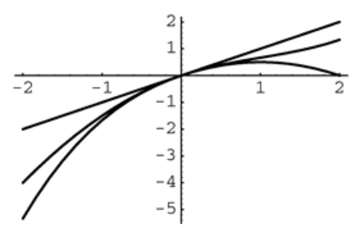
24.



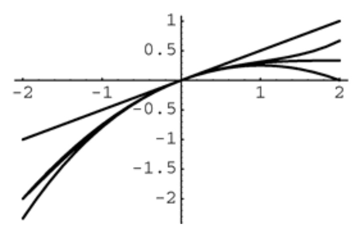
25.



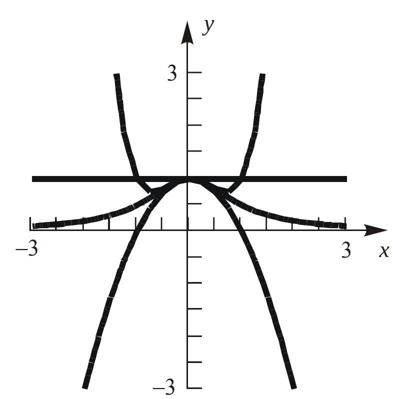
26.



27.



28.



29. $\left|e^{2c}+e^{-2c}\right| \le \left|e^{2c}\right|+\left|\frac{1}{e^{2c}}\right| \le e^6+1$

30. $\left|\tan c+\sec c\right| \le \left|\tan c\right|+\left|\sec c\right| \le 1+\sqrt{2}$

31. $\left|\frac{4c}{\sin c}\right| = \frac{|4c|}{|\sin c|} \le \frac{2\pi}{\frac{1}{\sqrt{2}}} = 2\sqrt{2}\pi$

32. $\left|\frac{4c}{c+4}\right| = \frac{|4c|}{|c+4|} \le \frac{4}{4} = 1$

33. $\left|\frac{e^c}{c+5}\right| = \frac{\left|e^c\right|}{|c+5|} \le \frac{e^4}{3}$

34. $\left|\frac{\cos c}{c+2}\right| = \frac{|\cos c|}{|c+2|} \le \frac{1}{2}$

35. $\left|\frac{c^2+\sin c}{10\ln c}\right| = \frac{\left|c^2+\sin c\right|}{|10\ln c|} \le \frac{\left|c^2\right|+|\sin c|}{|10\ln c|}$
$\le \frac{16+1}{10\ln 2} = \frac{17}{10\ln 2}$

36. $\left|\frac{c^2-c}{\cos c}\right| = \frac{\left|c^2-c\right|}{|\cos c|} \le \sqrt{2}\left|c^2-c\right| \le \sqrt{2}\left|\left(\frac{1}{2}\right)^2-\frac{1}{2}\right|$
$= \frac{\sqrt{2}}{4}$ (Note that $\left|x^2-x\right|$ is maximum at $\frac{1}{2}$ in $[0,\pi/4]$.)

37. $f(x)=\ln(2+x);\ f'(x)=\frac{1}{2+x};$
$f''(x)=-\frac{1}{(2+x)^2};\ f^{(3)}(x)=\frac{2}{(2+x)^3};$
$f^{(4)}(x)=-\frac{6}{(2+x)^4};\ f^{(5)}(x)=\frac{24}{(2+x)^5};$
$f^{(6)}(x)=-\frac{120}{(2+x)^6};\ f^{(7)}(x)=\frac{720}{(2+x)^7}$
$R_6(x)=\frac{1}{7!}\cdot\frac{720}{(2+c)^7}x^7 = \frac{x^7}{7(2+c)^7}$
$\left|R_6(0.5)\right| \le \left|\frac{0.5^7}{7\cdot 2^7}\right| \approx 8.719\times 10^{-6}$

38. $f(x)=e^{-x};\ f'(x)=-e^{-x};$
$f^{(n)}(x)=\begin{cases} e^{-x} & \text{if } n \text{ is even} \\ -e^{-x} & \text{if } n \text{ is odd}\end{cases}$
$R_6(x)=\frac{-e^{-c}}{7!}(x-1)^7 = -\frac{(x-1)^7}{5040e^c}$
$\left|R_6(0.5)\right| \le \left|\frac{(-0.5)^7}{5040e^{0.5}}\right| \approx 9.402\times 10^{-7}$

39. $f(x)=\sin x;\ f^{(7)}(x)=-\cos x$
$R_6(x)=\frac{-\cos c}{7!}\left(x-\frac{\pi}{4}\right)^7 = \frac{-\cos c\left(x-\frac{\pi}{4}\right)^7}{5040}$
$\left|R_6(0.5)\right| \le \left|\frac{\cos 0.5\left(0.5-\frac{\pi}{4}\right)^7}{5040}\right| \approx 2.685\times 10^{-8}$

40. $f(x)=\frac{1}{x-3};\ f'(x)=-\frac{1}{(x-3)^2};\ f''(x)=\frac{2}{(x-3)^3};\ f^{(3)}(x)=-\frac{6}{(x-3)^4};\ f^{(4)}(x)=\frac{24}{(x-3)^5};$
$f^{(5)}(x)=-\frac{120}{(x-3)^6};\ f^{(6)}(x)=\frac{720}{(x-3)^7};\ f^{(7)}(x)=-\frac{5040}{(x-3)^8}$
$R_6(x)=\frac{1}{7!}\cdot-\frac{5040}{(c-3)^8}(x-1)^7 = -\frac{(x-1)^7}{(c-3)^8}$
$\left|R_6(0.5)\right| \le \left|\frac{(0.5-1)^7}{(1-3)^8}\right| = \left|\frac{0.5^7}{2^8}\right| \approx 3.052\times 10^{-5}$

41. If $f(x)=\dfrac{1}{x}$, it is easily verified that

$f^{(n)}(x)=\dfrac{(-1)^n n!}{x^{(n+1)}}$. Thus for $a=1$

$R_6(x)=\dfrac{-(x-1)^7}{c^8}$, where c is between x and 1.

Thus, $|R_6(0.5)|=\dfrac{(0.5)^7}{c^8}$, where $c\in(0.5,1)$.

Therefore, $|R_6(0.5)|\le\dfrac{(0.5)^7}{(0.5)^8}=2$

42. If $f(x)=\dfrac{1}{x^2}$, it is easily verified that

$f^{(n)}(x)=\dfrac{(-1)^n (n+1)!}{x^{(n+2)}}$. Thus for $a=1$

$R_6(x)=\dfrac{(-1)^7 8!(x-1)^7}{(7!)c^9}=\dfrac{-8(x-1)^7}{c^9}$, where c is between x and 1. Thus,

$|R_6(0.5)|=\dfrac{8(0.5)^7}{c^9}$, where $c\in(0.5,1)$.

Therefore, $|R_6(0.5)|\le\dfrac{8(0.5)^7}{(0.5)^9}=32$

43. $R_n(x)=\dfrac{e^c}{(n+1)!}x^{n+1}$

Note that $e^1<3$.

$|R_n(1)|<\dfrac{3}{(n+1)!}$

$\dfrac{3}{(n+1)!}<0.000005$ or $600000<(n+1)!$ when $n\ge 9$.

44. To find a formula for $f^{(n)}(x)$ (and thus for $R_n(x)$), is difficult, but we can use another approach: From section 9.8 we know that

$4(\arctan x)=4\sum_{k=1}^{\infty}\dfrac{(-1)^{k+1}x^{2k-1}}{(2k-1)}$, which is an alternating series.(because of the $(-1)^{k+1}$ and the fact that all powers are odd) for all $x\in[-1,1]$. Thus, by the Alternating Series Test,

$|R_n(x)|\le\dfrac{4|x|^{2(n+1)-1}}{2(n+1)-1}$ and so $|R_n(1)|\le\dfrac{4}{2n+1}$. Since we want $|R_n(1)|\le 0.000005$, we set

$\dfrac{4}{2n+1}\le 0.000005$, which yields $n>399{,}999$.

45. This is a Binomial Series ($p=\frac{1}{2}$), so the third-order Maclaurin polynomial is (see section 9.8, Thm D and example 6) $P_3(x)=1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}$; further,

$R_3(x)=\dfrac{-5x^4}{128(1+c)^{7/2}}$. Now if $x\in[-0.5,0.5]$ and c is between 0 and x, then

$1+c>0.5$ and $x^4\le\left(\dfrac{1}{2}\right)^4=\dfrac{1}{16}$ so that, for all x,

$|R_3(x)|\le\dfrac{5\left(\frac{1}{16}\right)}{128(0.5)^{7/2}}=\dfrac{5\sqrt{2}}{256}\approx 0.0276$

46.

$f(x)=(1+x)^{3/2}$ $\quad f(0)=1$

$f'(x)=\dfrac{3}{2}(1+x)^{1/2}$ $\quad f'(0)=\dfrac{3}{2}$

$f''(x)=\dfrac{3}{4}(1+x)^{-1/2}$ $\quad f''(0)=\dfrac{3}{4}$

$f^{(3)}(x)=-\dfrac{3}{8}(1+x)^{-3/2}$ $\quad f'''(0)=-\dfrac{3}{8}$

$f^{(4)}(x)=\dfrac{9}{16}(1+x)^{-5/2}$ $\quad f^{(4)}(c)=\dfrac{9}{16}(1+c)^{-5/2}$

$(1+x)^{3/2}\approx 1+\dfrac{3}{2}x+\dfrac{3}{8}x^2-\dfrac{1}{16}x^3$

$R_3(x)=\dfrac{3}{128}(1+c)^{-5/2}x^4$

$|R_3(x)|\le\left|\dfrac{3}{128}(0.9)^{-5/2}(-0.1)^4\right|\approx 3.05\times10^{-6}$

47.

$f(x)=(1+x)^{-1/2}$ $\quad f(0)=1$

$f'(x)=-\dfrac{1}{2}(1+x)^{-3/2}$ $\quad f'(0)=-\dfrac{1}{2}$

$f''(x)=\dfrac{3}{4}(1+x)^{-5/2}$ $\quad f''(0)=\dfrac{3}{4}$

$f^{(3)}(x)=-\dfrac{15}{8}(1+x)^{-7/2}$ $\quad f^{(3)}(0)=-\dfrac{15}{8}$

$f^{(4)}(x)=\dfrac{105}{16}(1+x)^{-9/2}$ $\quad f^{(4)}(c)=\dfrac{105}{16}(1+c)^{-9/2}$

$(1+x)^{-1/2}\approx 1-\dfrac{1}{2}x+\dfrac{3}{8}x^2-\dfrac{5}{16}x^3$

$R_3(x)=\dfrac{35}{128}(1+c)^{-9/2}x^4$

$|R_3(x)|\le\left|\dfrac{35}{128}(0.95)^{-9/2}(0.05)^4\right|\approx 2.15\times10^{-6}$

48. $f(x)=\ln\left(\frac{1+x}{1-x}\right)\quad f(0)=0$

$f'(x)=\frac{2}{1-x^2}\quad f'(0)=2$

$f''(x)=\frac{4x}{(1-x^2)^2}\quad f''(0)=0$

$f^{(3)}(x)=\frac{4(1+3x^2)}{(1-x^2)^3}\quad f^{(3)}(0)=4$

$f^{(4)}(x)=\frac{48x(1+x^2)}{(1-x^2)^4}\quad f^{(4)}(0)=0$

$f^{(5)}(x)=\frac{48(1+10x^2+5x^4)}{(1-x^2)^5}$

$f^{(5)}(c)=\frac{48(1+10c^2+5c^4)}{(1-c^2)^5}$

$\ln\left(\frac{1+x}{1-x}\right)\approx 2x+\frac{2}{3}x^3$

$R_4(x)=\frac{2}{5}\left[\frac{1+10c^2+5c^4}{(1-c^2)^5}\right]x^5$

$|R_4(x)|<\frac{2}{5}\left[\frac{1+10(0.5)^2+5(0.5)^4}{(1-(0.5)^2)^5}\right](0.5)^5$

≈ 0.201

49. $R_4(x)=\frac{\cos c}{5!}x^5$

$|R_4(x)|\le\frac{(0.5)^5}{5!}\approx 0.00026042\le 0.0002605$

$\int_0^{0.5}\sin x\,dx\approx\int_0^{0.5}\left(x-\frac{1}{6}x^3\right)dx$

$=\left[\frac{1}{2}x^2-\frac{1}{24}x^4\right]_0^{0.5}\approx 0.1224$

Error $\le 0.0002605(0.5-0)=0.00013025$

50. $R_5(x)=-\frac{\cos c}{6!}x^6$

$|R_5(x)|\le\frac{1}{6!}\approx 0.001389$

$\int_0^1\cos x\,dx\approx\int_0^1\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)dx$

$=\left[x-\frac{x^3}{6}+\frac{x^5}{120}\right]_0^1\approx 0.8417$

Error $\le 0.001389(1-0)=0.001389$

51. Assume n is odd; that is $n=2m+1$ for $m\ge 0$.

Then, $R_{n+1}(x)=R_{2m+2}(x)=\frac{f^{(2m+3)}(c)}{(2m+3)!}x^{2m+3}$.

Note that, for all m,

$f^{(4m)}(x)=\sin x,\ f^{(4m+1)}(x)=\cos x$

$f^{(4m+2)}(x)=-\sin x,\ f^{(4m+3)}(x)=-\cos x$;

therefore, $|R_{n+1}(x)|=\frac{\cos c}{(2m+3)!}x^{2m+3}$ where c is between 0 and x. For $x\in[0,\pi/2]$, $c\in(0,x)$ so that $\cos c<1$ and $x\le\frac{\pi}{2}$; hence

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+3}}{(2m+3)!}$. Now, for

$k=2,3,\ldots,2m+3,\quad\frac{\frac{\pi}{2}}{k}\le\frac{\pi}{4}$ so that

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+3}}{(2m+3)!}\le\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+2}$ for all $x\in[0,\frac{\pi}{2}]$. Now

$\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+2}\le 0.00005\Rightarrow$

$(2m+2)\ln\left(\frac{\pi}{4}\right)\le\ln\left(\frac{2(0.00005)}{\pi}\right)\Rightarrow$

$2m+2\ge 42.8666\Rightarrow n=2m+1>42$

52. Assume n is even; that is $n=2m$ for $m\ge 0$.

Then, $R_{n+1}(x)=R_{2m+1}(x)=\frac{f^{(2m+2)}(c)}{(2m+2)!}x^{2m+2}$.

Note that, for all m,

$f^{(4m)}(x)=\cos x,\ f^{(4m+1)}(x)=-\sin x$

$f^{(4m+2)}(x)=-\cos x,\ f^{(4m+3)}(x)=\sin x$;

therefore, $|R_{n+1}(x)|=\frac{\cos c}{(2m+2)!}x^{2m+2}$ where c is between 0 and x. For $x\in[0,\pi/2]$, $c\in(0,x)$ so that $\cos c<1$ and $x\le\frac{\pi}{2}$; hence

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+2}}{(2m+2)!}$. Now, for

$k=2,3,\ldots,2m+2,\quad\frac{\frac{\pi}{2}}{k}\le\frac{\pi}{4}$ so that

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+2}}{(2m+2)!}\le\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+1}$ for all $x\in[0,\frac{\pi}{2}]$. Now

$$\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+1} \le 0.00005 \Rightarrow$$

$$(2m+1)\ln\left(\frac{\pi}{4}\right) \le \ln\left(\frac{2(0.00005)}{\pi}\right) \Rightarrow$$

$$2m+1 \ge 42.8666 \Rightarrow n = 2m > 42$$

53. The area of the sector with angle t is $\frac{1}{2}tr^2$. The area of the triangle is

$$\frac{1}{2}\left(r\sin\frac{t}{2}\right)\left(2r\cos\frac{t}{2}\right) = r^2\sin\frac{t}{2}\cos\frac{t}{2} = \frac{1}{2}r^2\sin t$$

$$A = \frac{1}{2}tr^2 - \frac{1}{2}r^2\sin t$$

Using $n = 3$, $\sin t \approx t - \frac{1}{6}t^3$.

$$A \approx \frac{1}{2}tr^2 - \frac{1}{2}r^2\left(t - \frac{1}{6}t^3\right) = \frac{1}{12}r^2t^3$$

54. $m(v) = \dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$; $m(0) = m_0$

$$m'(v) = \frac{m_0 v}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}};\quad m'(0) = 0$$

$$m''(v) = \frac{2m_0v^2 + m_0c^2}{c^4\left(1-\frac{v^2}{c^2}\right)^{5/2}};\quad m''(0) = \frac{m_0}{c^2}$$

The Maclaurin polynomial of order 2 is:

$$m(v) \approx m_0 + \frac{1}{2}\frac{m_0}{c^2}v^2 = m_0 + \frac{m_0}{2}\left(\frac{v}{c}\right)^2.$$

55. a. $\ln\left(1+\frac{r}{12}\right)^{12n} = \ln 2$

$$12n\ln\left(1+\frac{r}{12}\right) = \ln 2$$

$$n = \frac{\ln 2}{12\ln\left(1+\frac{r}{12}\right)}$$

b. $f(x) = \ln(1+x)$; $f(0) = 0$

$$f'(x) = \frac{1}{1+x};\quad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2};\quad f''(0) = -1$$

$$\ln(1+x) \approx x - \frac{x^2}{2}$$

$$n \approx \frac{\ln 2}{r - \frac{r^2}{24}} = \left[\frac{24}{r(24-r)}\right]\ln 2$$

$$= \frac{\ln 2}{r} + \frac{\ln 2}{24-r}$$

$$\approx \frac{\ln 2}{r} + \frac{\ln 2}{24} \approx \frac{0.693}{r} + 0.029$$

We let $24 - r \approx 24$ since the interest rate r is going to be close to 0.

c.

r	n (exact)	n (approx.)	n (rule 72)
0.05	13.8918	13.889	14.4
0.10	6.9603	6.959	7.2
0.15	4.6498	4.649	4.8
0.20	3.4945	3.494	3.6

56. $f(x) = 1 - e^{-(1+k)x}$; $f(0) = 0$

$$f'(x) = (1+k)e^{-(1+k)x};\quad f'(0) = (1+k)$$

$$f''(x) = -(1+k)^2e^{-(1+k)x};\quad f''(0) = -(1+k)^2$$

$$1 - e^{-(1+k)x} \approx (1+k)x - \frac{(1+k)^2}{2}x^2$$

For $x = 2k$, the polynomial is

$2k - 4k^3 - 2k^4 \approx 2k$ when k is very small.

$$1 - e^{-(1+0.01)(0.02)} \approx 0.019997 \approx 0.02$$

57. $f(x) = x^4 - 3x^3 + 2x^2 + x - 2$; $f(1) = -1$

$$f'(x) = 4x^3 - 9x^2 + 4x + 1;\quad f'(1) = 0$$

$$f''(x) = 12x^2 - 18x + 4;\quad f''(1) = -2$$

$$f^{(3)}(x) = 24x - 18;\quad f^{(3)}(1) = 6$$

$$f^{(4)}(x) = 24;\quad f^{(4)}(1) = 24$$

$$f^{(5)}(x) = 0$$

Since $f^{(5)}(x) = 0$, $R_5(x) = 0$.

$$x^4 - 3x^3 + 2x^2 + x - 2$$

$$= -1 - (x-1)^2 + (x-1)^3 + (x-1)^4$$

58. $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$P_n'(x) = f'(a) + \frac{f''(a)}{2!}2(x-a) + \frac{f'''(a)}{3!}3(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}n(x-a)^{n-1}$

$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$

$P_n'(a) = f'(a) + 0 + 0 + \cdots + 0 = f'(a)$

$P_n'' = 0 + f''(a) + \frac{f'''(a)}{2!}2(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(n-1)(x-a)^{n-2}$

$= f''(a) + f'''(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}$

$P_n'' = f''(a) + 0 + 0 + \cdots + 0 = f''(a)$

$\vdots \quad \vdots \quad \vdots$

$P_n^{(n)}(x) = \frac{f^{(n)}(a)}{0!}(x-a)^0 = f^{(n)}(a)$

$P_n^{(n)}(a) = f^{(n)}(a)$

59. $f(x) = \sin x$; $\quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$; $\quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$; $\quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(3)}(x) = -\cos x$; $\quad f^{(3)}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$; $\quad f^{(4)}(c) = \sin c$

$43° = \frac{\pi}{4} - \frac{\pi}{90}$ radians

$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$

$-\frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + R_3(x)$

$\sin\left(\frac{\pi}{4} - \frac{\pi}{90}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{\pi}{90}\right) - \frac{\sqrt{2}}{4}\left(-\frac{\pi}{90}\right)^2$

$-\frac{\sqrt{2}}{12}\left(-\frac{\pi}{90}\right)^3 + R_3\left(\frac{\pi}{4} - \frac{\pi}{90}\right)$

$\approx 0.681998 + R_3$

$|R_3| = \left|\frac{\sin c}{4!}\left(-\frac{\pi}{90}\right)^4\right| < \frac{1}{24}\left(\frac{\pi}{90}\right)^4 \approx 6.19 \times 10^{-8}$

60. $63° = \frac{\pi}{3} + \frac{\pi}{60}$ radians

Since $f^{(n)}(x)$ is $\pm\sin x$ or $\pm\cos x$,

$|R_n(x)| \le \frac{1}{(n+1)!}\left(x - \frac{\pi}{3}\right)^{n+1}$

$\left|R_n\left(\frac{\pi}{3} + \frac{\pi}{60}\right)\right| \le \frac{1}{(n+1)!}\left(\frac{\pi}{60}\right)^{n+1}$

$\frac{1}{(n+1)!}\left(\frac{\pi}{60}\right)^{n+1} \le 0.0005$ when $n \ge 2$.

$f(x) = \cos x$; $\quad f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x) = -\sin x$; $\quad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$; $\quad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + R_3(x)$

$\cos 63° \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{60}\right) - \frac{1}{4}\left(\frac{\pi}{60}\right)^2 \approx 0.45397$

61. $|R_9(x)| \le \frac{1}{10!}x^{10} \le \frac{1}{10!}\left(\frac{\pi}{2}\right)^{10} \approx 2.5202 \times 10^{-5}$

62. a. $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{\sin c}{720}x^6$

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0}\left(\frac{1}{120} - \frac{\sin c}{720}x\right) = \frac{1}{120}$$

b. $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{\sin c}{5040}x^7$

$$\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} =$$

$$\lim_{x\to 0}\left(-\frac{1}{720} + \frac{\sin c}{5040}x\right) = -\frac{1}{720}$$

63. The kth derivative of $h(x)f(x)$ is $\sum_{i=0}^{k}\binom{k}{i}h^{(i)}(x)f^{(k-i)}(x)$. If $h(x) = x^{n+1}$,

$h^{(i)}(x) = \frac{(n+1)!}{(n+1-i)!}x^{n+1-i}$.

Thus for $i \le n + 1$, $h^{(i)}(0) = 0$. Let $q(x) = x^{n+1}f(x)$. Then

$q^{(k)}(0) = \sum_{i=0}^{k}\binom{k}{i}h^{(i)}(0)f^{(k-i)}(0) = 0$

for $k \le n + 1$.

$g^{(k)}(x) = p^{(k)}(x) + q^{(k)}(x)$, so

$g^{(k)}(0) = p^{(k)}(0) + q^{(k)}(0) = p^{(k)}(0)$

for $k \le n + 1$.

The Maclaurin polynomial of order n for g is

$p(0) + p'(0)x + \frac{p''(0)}{2!}x^2 + \ldots + \frac{p^{(n)}(0)}{n!}x^n$ which is the Maclaurin polynomial of order n for $p(x)$. Since $p(x)$ is a polynomial of degree at most n, the remainder $R_n(x)$ of Maclaurin's Formula for $p(x)$ is 0, so the Maclaurin polynomial of order n for $g(x)$ is $p(x)$.

64. Using Taylor's formula,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

Since $f'(c) = f''(c) = f'''(c) = \ldots = f^{(n)}(c) = 0$,

$f(x) = f(c) + R_n(x)$

$R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(a)(x-c)^{n+1}$ where a is between x and c.

(i) Since $f^{(n+1)}(x)$ is continuous near c, then $f^{(n+1)}(a) < 0$ when a is near c.
Thus $R_n(x) < 0$ when x is near c, so $f(x) < f(c)$ when x is near c. $f(c)$ is a local maximum.

(ii) Since $f^{(n+1)}(x)$ is continuous near c, then $f^{(n+1)}(a) > 0$ when a is near c.
Thus $R_n(x) > 0$ when x is near c, so $f(x) > f(c)$ when x is near c.
$f(c)$ is a local minimum.

Suppose $f(x) = x^4$. $f(x) > 0$ when $x > 0$ and $f(x) < 0$ when $x > 0$. Thus $x = 0$ is a local minimum.

$f'(0) = f''(0) = f'''(0) = 0$, $f^{(4)}(0) = 24 > 0$

9.10 Chapter Review

Concepts Test

1. False: If $b_n = 100$ and $a_n = 50 + (-1)^n$ then since $a_n = \begin{cases} 51 & \text{if } n \text{ is even} \\ 49 & \text{if } n \text{ is odd} \end{cases}$,

$0 \le a_n \le b_n$ for all n and $\lim_{n\to\infty} b_n = 100$ while $\lim_{n\to\infty} a_n$ does not exist.

2. True: It is clear that $n! \le n^n$. The inequality $n! \le n^n \le (2n-1)!$ is equivalent to

$1 \le \frac{n^n}{n!} \le \frac{(2n-1)!}{n!}$.

Expanding the terms gives

$\frac{n^n}{n!} = \frac{n}{1}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot\cdots\cdot\frac{n}{n-1}\cdot\frac{n}{n}$

$\le (n+1)(n+2)\cdot\cdots\cdot(n+n-1)$ or

$n\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot\cdots\cdot\frac{n}{n-1} \le (n+1)(n+2)\cdot$
$\cdots(n+n-1)$

The left-hand side consists of $n - 1$ terms, each of which is less than or equal to n, while the right-hand side consists of $n - 1$ terms, each of which is greater than n. Thus, the inequality is true so $n! \le n^n \le (2n-1)!$

3. True: If $\lim_{n\to\infty} a_n = L$ then for any $\varepsilon > 0$ there is a number $M > 0$ such that $|a_n - L| < \varepsilon$ for all $n \ge M$. Thus, for the same ε, $|a_{3n+4} - L| < \varepsilon$ for $3n + 4 \ge M$ or $n \ge \frac{M-4}{3}$. Since ε was arbitrary, $\lim_{n\to\infty} a_{3n+4} = L$.

4. False: Suppose $a_n = 1$ if $n = 2k$ or $n = 3k$ where k is any positive integer and $a_n = 0$ if n is not a multiple of 2 or 3. Then $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{3n} = 1$ but $\lim_{n\to\infty} a_n$ does not exist.

5. False: Let a_n be given by

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{if } n \text{ is composite} \end{cases}$$

Then $a_{mn} = 0$ for all mn since $m \ge 2$, hence $\lim_{n\to\infty} a_{mn} = 0$ for $m \ge 2$.

$\lim_{n\to\infty} a_n$ does not exist since for any $M > 0$ there will be a_n's with $a_n = 1$ since there are infinitely many prime numbers.

6. True: Given $\varepsilon > 0$ there are numbers M_1 and M_2 such that $|a_{2n} - L| < \varepsilon$ when $n \ge M_1$ and $|a_{2n+1} - L| < \varepsilon$ when $n \ge M_2$. Let $M = \max\{M_1, M_2\}$, then when $n \ge M$ we have $|a_{2n} - L| < \varepsilon$ and $|a_{2n+1} - L| < \varepsilon$ so $|a_n - L| < \varepsilon$ for all $n \ge 2M + 1$ since every $k \ge 2M + 1$ is either even ($k = 2n$) or odd ($k = 2n + 1$).

7. False: Let $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then

$$a_n - a_{n+1} = -\frac{1}{n+1} \text{ so}$$

$\lim_{n\to\infty}(a_n - a_{n+1}) = 0$ but $\lim_{n\to\infty} a_n$ is not finite since $\lim_{n\to\infty} a_n = \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges.

8. False: $\{(-1)^n\}$ and $\{(-1)^{n+1}\}$ both diverge but

$$\{(-1)^n + (-1)^{n+1}\} = \{(-1)^n(1-1)\} = \{0\}$$

converges.

9. True: If $\{a_n\}$ converges, then for some N, there are numbers m and M with $m \le a_n \le M$ for all $n \ge N$. Thus $\frac{m}{n} \le \frac{a_n}{n} \le \frac{M}{n}$ for all $n \ge N$. Since $\left\{\frac{m}{n}\right\}$ and $\left\{\frac{M}{n}\right\}$ both converge to 0, $\left\{\frac{a_n}{n}\right\}$ must also converge to 0.

10. False: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges.

$a_n = (-1)^n \frac{1}{\sqrt{n}}$ so $a_n{}^2 = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

11. True: The series converges by the Alternating Series Test.

$S_1 = a_1, S_2 = a_1 - a_2, S_3 = a_1 - a_2 + a_3$,
$S_4 = a_1 - a_2 + a_3 - a_4$, etc.
$0 < a_2 < a_1 \Rightarrow 0 < a_1 - a_2 = S_2 < a_1$;
$0 < a_3 < a_2 \Rightarrow -a_2 < -a_2 + a_3 < 0$ so
$0 < a_1 - a_2 < a_1 - a_2 + a_3 = S_3 < a_1$;
$0 < a_4 < a_3 \Rightarrow 0 < a_3 - a_4 < a_3$, so
$-a_2 < -a_2 + a_3 - a_4 < -a_2 + a_3 < 0$,
hence
$0 < a_1 - a_2 < a_1 - a_2 + a_3 - a_4 = S_4$
$< a_1 - a_2 + a_3 < a_1$; etc.
For each even n, $0 < S_{n-1} - a_n$ while for each odd n, $n > 1$, $S_{n-1} + a_n < a_1$.

12. True: For $n \ge 2$, $\frac{1}{n} \le \frac{1}{2}$ so

$$\sum_{n=2}^{\infty} \left(\frac{1}{n}\right)^n \le \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$$

which converges since it is a geometric series with $r = \frac{1}{2}$.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n = 1 + \frac{1}{4} + \frac{1}{27} + \ldots > 1$$

since all terms are positive.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n = 1+\sum_{n=2}^{\infty}\left(\frac{1}{n}\right)^n \le 1+\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n$$

$$= 1+\frac{1}{4}+\frac{1}{8}+\dots$$

$$= -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} = -\frac{1}{2}+\frac{1}{1-\frac{1}{2}}$$

$$= -\frac{1}{2}+2 = \frac{3}{2}$$

Thus, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n = S$ with

$1 < S \le \frac{3}{2} < 2.$

13. False: $\sum_{n=1}^{\infty}(-1)^n$ diverges but the partial sums are bounded ($S_n = -1$ for odd n and $S_n = 0$ for even n.)

14. False: $0 < \frac{1}{n^2} \le \frac{1}{n}$ for all n in N but $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges while $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

15. True: $\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1,$ Ratio Test is inconclusive. (See the discussion before Example 5 in Section 9.4.)

16. False: $\frac{1}{n^2} > 0$ for all n in N and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, but

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1.$$

17. False: $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e} \ne 0$ so the series cannot converge.

18. False: Since $\lim_{n\to\infty}\frac{n^4+1}{e^n} = 0,$ there is some number M such that $e^n > n^4+1$ for all $n \ge M$, thus $n > \ln(n^4+1)$ and $\frac{1}{n} < \frac{1}{\ln(n^4+1)}$ for $n \ge M$. Hence, $\sum_{n=M}^{\infty}\frac{1}{n} < \sum_{n=M}^{\infty}\frac{1}{\ln(n^4+1)}$ and so $\sum_{n=1}^{\infty}\frac{1}{\ln(n^4+1)}$ diverges by the Comparison Test.

19. True: $\sum_{n=2}^{\infty}\frac{n+1}{(n\ln n)^2}$

$$= \sum_{n=2}^{\infty}\left[\frac{n}{(n\ln n)^2}+\frac{1}{(n\ln n)^2}\right]$$

$$= \sum_{n=2}^{\infty}\left[\frac{1}{n(\ln n)^2}+\frac{1}{n^2(\ln n)^2}\right]$$

$\frac{1}{x(\ln x)^2}$ is continuous, positive, and nonincreasing on $[2, \infty)$. Using $u = \ln x,\ du = \frac{1}{x}dx,$

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx = \int_{\ln 2}^{\infty}\frac{1}{u^2}du$$

$= \left[-\frac{1}{u}\right]_{\ln 2}^{\infty} = 0+\frac{1}{\ln 2} < \infty$ so $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges.

For $n \ge 3$, $\ln n > 1$, so $(\ln n)^2 > 1$ and $\frac{1}{n^2(\ln n)^2} < \frac{1}{n^2}$. Thus $\sum_{n=3}^{\infty}\frac{1}{n^2(\ln n)^2} < \sum_{n=3}^{\infty}\frac{1}{n^2}$ so $\sum_{n=3}^{\infty}\frac{1}{n^2(\ln n)^2}$ converges by the Comparison Test. Since both series converge, so does their sum.

20. False: This series is $\sum_{n=1}^{\infty}\frac{1}{2n-1} = 1+\frac{1}{3}+\frac{1}{5}+\cdots$ which diverges.

21. True: If $0 \le a_{n+100} \le b_n$ for all n in N, then $\sum_{n=101}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$ so $\sum_{n=1}^{\infty} a_n$ also converges, since adding a finite number of terms does not affect the convergence or divergence of a series.

22. True: If $ca_n \ge \frac{1}{n}$ for all n in N with $c > 0$, then $a_n \ge \frac{1}{cn}$ for all n in N so $\sum_{n=1}^{\infty} a_n \ge \sum_{n=1}^{\infty} \frac{1}{cn} = \frac{1}{c}\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges. Thus, $\sum_{n=1}^{\infty} a_n$ diverges by the Comparison Test.

23. True: $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$
$= \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$, so the sum of the first thousand terms is less than $\frac{1}{2}$.

24. False: Consider the series with $a_n = \frac{(-1)^{n+1}}{n}$. Then $(-1)^n a_n = \frac{(-1)^{2n+1}}{n} = \frac{-1}{n}$ so $\sum_{n=1}^{\infty}(-1)^n a_n = -1 - \frac{1}{2} - \frac{1}{3} - \cdots$ which diverges.

25. True: If $b_n \le a_n \le 0$ for all n in N then $0 \le -a_n \le -b_n$ for all n in N.
$\sum_{n=1}^{\infty} -b_n = (-1)\sum_{n=1}^{\infty} b_n$ which converges since $\sum_{n=1}^{\infty} b_n$ converges.
Thus, by the Comparison Test, $\sum_{n=1}^{\infty} -a_n$ converges, hence $\sum_{n=1}^{\infty} a_n = (-1)\sum_{n=1}^{\infty}(-a_n)$ also converges.

26. True Since $a_n \ge 0$ for all n, $\sum_{n=1}^{\infty}\left|(-1)^n a_n\right| = \sum_{n=1}^{\infty} a_n$ so the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely.

27. True: $\left|\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} - \sum_{n=1}^{99}(-1)^{n+1}\frac{1}{n}\right|$
$= \left|-\frac{1}{100} + \frac{1}{101} - \frac{1}{102} - \cdots\right| < \frac{1}{100} = 0.01$

28. True: Suppose $\sum|a_n|$ converges. Thus, $\sum 2|a_n|$ converges, so $\sum\left(|a_n| + a_n\right)$ converges since $0 \le |a_n| + a_n \le 2|a_n|$. But by the linearity of convergent series $\sum a_n = \sum\left(|a_n| + a_n\right) - \sum|a_n|$ converges, which is a contradiction.

29. True: $|3-(-1.1)| = 4.1$, so the radius of convergence of the series is at least 4.1.
$|3-7| = 4 < 4.1$ so $x = 7$ is within the interval of convergence.

30. False: If the radius of convergence is 2, then the convergence at $x = 2$ is independent of the convergence at $x = -2$.

31. True: The radius of convergence is at least 1.5, so 1 is within the interval of convergence.
Thus $\int_0^1 f(x)dx = \left[\sum_{n=0}^{\infty}\frac{a_n x^{n+1}}{n+1}\right]_0^1$
$= \sum_{n=0}^{\infty}\frac{a_n}{n+1}$.

32. False: The convergence set of a power series may consist of a single point.

33. False: Consider the function
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \ne 0 \\ 0 & x = 0 \end{cases}.$$
The Maclaurin series for this function represents the function only at $x = 0$.

34. True: On $(-1, 1)$, $f(x) = \frac{1}{1-x}$.
$f'(x) = \frac{1}{(1-x)^2} = [f(x)]^2$.

35. True: $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}=e^{-x}$, $\frac{d}{dx}e^{-x}+e^{-x}=0$

36. True: $P(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2$

37. True: If $p(x)$ and $q(x)$ are polynomials of degree less than or equal to n, satisfying $p(a)=q(a)=f(a)$ and $p^{(k)}(a)=q^{(k)}(a)=f^{(k)}(a)$ for $k \le n$, then $p(x)=q(x)$.

38. True: $f(0)=f'(0)=f''(0)=0$, its second order Maclaurin polynomial is 0.

39. True: After simplifying, $P_3(x)=f(x)$.

40. True: Any Maclaurin polynomial for $\cos x$ involves only even powers of x.

41. True: The Maclaurin polynomial of an even function involves only even powers of x, so $f'(0)=0$ if $f(x)$ is an even function.

42. True: Taylor's Formula with Remainder for $n=0$ is $f(x)=f(a)+f'(c)(x-a)$ which is equivalent to the Mean Value Theorem.

Sample Test Problems

1. $\lim_{n\to\infty}\frac{9n}{\sqrt{9n^2+1}}=\lim_{n\to\infty}\frac{9}{\sqrt{9+\frac{1}{n^2}}}=3$

The sequence converges to 3.

2. Using l'Hôpital's Rule,

$$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}=\lim_{n\to\infty}\frac{2\sqrt{n}}{n}$$

$$=\lim_{n\to\infty}\frac{2}{\sqrt{n}}=0.$$

3. $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n=\lim_{n\to\infty}\left(\left(1+\frac{4}{n}\right)^{n/4}\right)^4=e^4$

The sequence converges to e^4.

4. $a_{n+1}=\frac{n+1}{3}a_n$ thus for $n>3$, since $\frac{n+1}{3}>1$, $a_{n+1}>a_n$ and the sequence diverges.

5. Let $y=\sqrt[n]{n}=n^{1/n}$ then $\ln y=\frac{1}{n}\ln n$.

$\lim_{n\to\infty}\frac{1}{n}\ln n=\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=\lim_{n\to\infty}\frac{1}{n}=0$ by using l'Hôpital's Rule. Thus, $\lim_{n\to\infty}\sqrt[n]{n}=\lim_{n\to\infty}e^{\ln y}=1$. The sequence converges to 1.

6. $\lim_{n\to\infty}\frac{1}{\sqrt[3]{n}}=0$ while $\frac{1}{\sqrt[n]{3}}=\left(\frac{1}{3}\right)^{1/n}$. As $n\to\infty$, $\frac{1}{n}\to 0$ so

$$\lim_{n\to\infty}\left(\frac{1}{3}\right)^{1/n}=\lim_{\frac{1}{n}\to 0}\left(\frac{1}{3}\right)^{1/n}=\left(\frac{1}{3}\right)^0=1.$$

The sequence converges to 1.

7. $a_n\ge 0$; $\lim_{n\to\infty}\frac{\sin^2 n}{\sqrt{n}}\le\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$

The sequence converges to 0.

8. The sequence does not converge, since whenever n is an even multiple of 6, $a_n=1$, while whenever n is an odd multiple of 6, $a_n=-1$.

9. $S_n = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \ldots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$, so

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$. The series converges to 1.

10. $S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$, so

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}$. The series converges to $\frac{3}{2}$.

11. $\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots = \sum_{n=1}^{\infty} \ln\frac{n}{n+1} = \sum_{n=1}^{\infty}[\ln n - \ln(n+1)]$

$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \ldots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) = \ln 1 - \ln(n+1) = \ln\frac{1}{n+1}$

As $n \to \infty, \frac{1}{n+1} \to 0$ so $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\frac{1}{n+1} = -\infty$.
The series diverges.

12. $\cos k\pi = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$ so

$\sum_{k=0}^{\infty} \cos k\pi = \sum_{k=0}^{\infty} (-1)^k$ which diverges since

$S_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ so $\{S_n\}$ does not converge.

13. $\sum_{k=0}^{\infty} e^{-2k} = \sum_{k=0}^{\infty}\left(\frac{1}{e^2}\right)^k = \frac{1}{1-\frac{1}{e^2}} = \frac{e^2}{e^2-1} \approx 1.1565$

since $\frac{1}{e^2} < 1$.

14. $\sum_{k=0}^{\infty} \frac{3}{2^k} = 3\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{3}{1-\frac{1}{2}} = 6$

$\sum_{k=0}^{\infty} \frac{4}{3^k} = 4\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k = \frac{4}{1-\frac{1}{3}} = 6$

Since both series converge, their sum converges to 6 + 6 = 12.

15. $\sum_{k=1}^{\infty} 91\left(\frac{1}{100}\right)^k = \frac{91}{1-\frac{1}{100}} - 91 = \frac{9100}{99} - 91 = \frac{91}{99}$

The series converges since $\left|\frac{1}{100}\right| < 1$.

16. $\sum_{k=1}^{\infty} \left(\frac{1}{\ln 2}\right)^k$ diverges since $\left|\frac{1}{\ln 2}\right| > 1$.

17. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, so

$1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \cdots$ converges to $\cos 2 \approx -0.41615$.

18. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, so

$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = e^{-1} \approx 0.3679$.

19. Let $a_n = \frac{n}{1+n^2}$ and $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2}+1} = 1$;

$0 < 1 < \infty$.
By the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{1+n^2}$ also diverges.

20. Let $a_n = \frac{n+5}{1+n^3}$ and $b_n = \frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+5n^2}{1+n^3} = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{\frac{1}{n^3}+1} = 1$;

$0 < 1 < \infty$.
By the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+5}{1+n^2}$ also converges.

21. Since the series alternates, $\frac{1}{\sqrt[3]{n}} > \frac{1}{\sqrt[3]{n+1}} > 0$, and

$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, the series converges by the Alternating Series Test.

22. The series diverges since

$\lim_{n\to\infty} \frac{1}{\sqrt[n]{3}} = \lim_{n\to\infty} 3^{-1/n} = 1$.

23. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right)$

$= \left(\frac{1}{1-\frac{1}{2}} - 1\right) + \left(\frac{1}{1-\frac{3}{4}} - 1\right) = 1 + 3 = 4$

The series converges to 4. The 1's must be subtracted since the index starts with $n = 1$.

24. $\rho = \lim_{n\to\infty} \left(\frac{n+1}{e^{(n+1)^2}} \div \frac{n}{e^{n^2}}\right) = \lim_{n\to\infty} \left(\frac{n+1}{ne^{2n+1}}\right)$

$= \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{e^{2n+1}}\right) = 0 < 1,$ so the series converges.

25. $\lim_{n\to\infty} \frac{n+1}{10n+12} = \frac{1}{10} \neq 0$, so the series diverges.

26. Let $a_n = \frac{\sqrt{n}}{n^2+7}$ and $b_n = \frac{1}{n^{3/2}}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+7} = \lim_{n\to\infty} \frac{1}{1+\frac{7}{n^2}} = 1;$

$0 < 1 < \infty$.

By the Limit Comparison Test, since

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $\left(\frac{3}{2} > 1\right)$,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7}$ also converges.

27. $\rho = \lim_{n\to\infty} \left|\frac{(n+1)^2}{(n+1)!} \div \frac{n^2}{n!}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n^2}\right| = 0 < 1$, so the series converges.

28. $\rho = \lim_{n\to\infty} \left|\frac{(n+1)^3 3^{n+1}}{(n+2)!} \div \frac{n^3 3^n}{(n+1)!}\right|$

$= \lim_{n\to\infty} \left|\frac{3(n+1)^3}{n^3(n+2)}\right| = \lim_{n\to\infty} \left|\frac{\frac{3}{n}\left(1+\frac{1}{n}\right)^3}{1+\frac{2}{n^4}}\right| = 0 < 1$

The series converges.

29. $\rho = \lim_{n\to\infty} \left|\frac{2^{n+1}(n+1)!}{(n+3)!} \div \frac{2^n n!}{(n+2)!}\right|$

$= \lim_{n\to\infty} \left|\frac{2(n+1)}{n+3}\right| = 2 > 1$

The series diverges.

30. $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \neq 0$, so the series does not converge.

31. $\rho = \lim_{n\to\infty} \left|\frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n}\right| = \lim_{n\to\infty} \left|\frac{2}{3}\right| \left|\frac{(n+1)^2}{n^2}\right|$

$= \frac{2}{3} < 1$, so the series converges.

32. Since the series alternates, $\frac{1}{1+\ln n} > \frac{1}{1+\ln(n+1)}$,

and $\lim_{n\to\infty} \frac{1}{1+\ln n} = 0,$ the series converges by the Alternating Series Test.

33. $a_n = \frac{1}{3n-1}; \frac{1}{3n-1} > \frac{1}{3n+2}$ so $a_n > a_{n+1};$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n-1} = 0,$ so the series

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{3n-1}$ converges by the Alternating Series Test.

Let $b_n = \frac{1}{n},$ then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3};$

$0 < \frac{1}{3} < \infty.$ By the Limit Comparison Test, since

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3n-1}$ also diverges.

The series is conditionally convergent.

34. $\rho = \lim_{n\to\infty} \left|\frac{(n+1)^3}{2^{n+1}} \div \frac{n^3}{2^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{2n^3}\right|$

$= \lim_{n\to\infty} \left|\frac{\left(1+\frac{1}{n}\right)^3}{2}\right| = \frac{1}{2} < 1$

The series is absolutely convergent.

35. $\dfrac{3^n}{2^{n+8}} = \dfrac{1}{2^8}\left(\dfrac{3}{2}\right)^n$;

$\lim\limits_{n\to\infty} \dfrac{1}{2^8}\left(\dfrac{3}{2}\right)^n = \dfrac{1}{2^8}\lim\limits_{n\to\infty}\left(\dfrac{3}{2}\right)^n = \infty$ since $\dfrac{3}{2} > 1$.

The series is divergent.

36. Let $f(x) = \dfrac{\sqrt[x]{x}}{\ln x}$, then

$f'(x) = \dfrac{1}{(\ln x)^2}\left[\dfrac{x^{1/x}}{x^2}(1-\ln x)\ln x - \dfrac{x^{1/x}}{x}\right]$

$= \dfrac{x^{1/x}}{(x\ln x)^2}[\ln x - (\ln x)^2 - x]$, for $x \ge 3$, $\ln x > 1$

so $(\ln x)^2 > \ln x$ hence $f(x)$ is decreasing on

$[3, \infty)$. Thus, if $a_n = \dfrac{\sqrt[n]{n}}{\ln n}, a_n > a_{n+1}$.

Let $y = \sqrt[n]{n} = n^{1/n}$, so $\ln y = \dfrac{1}{n}\ln n$.

Using l'Hôpital's Rule,

$\lim\limits_{n\to\infty}\dfrac{\ln n}{n} = \lim\limits_{n\to\infty}\dfrac{\frac{1}{n}}{1} = \lim\limits_{n\to\infty}\dfrac{1}{n} = 0$, thus

$\lim\limits_{n\to\infty}\sqrt[n]{n} = \lim\limits_{n\to\infty} e^{\ln y} = e^0 = 1$. Hence, $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n}}{\ln n}$

is of the form $\dfrac{1}{\infty}$ so $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n}}{\ln n} = 0$.

Thus, by the Alternating Series Test,

$\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n\sqrt[n]{n}}{\ln n}$ converges.

$\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$ hence $\sum\limits_{n=2}^{\infty}\dfrac{\sqrt[n]{n}}{n} < \sum\limits_{n=2}^{\infty}\dfrac{\sqrt[n]{n}}{\ln n}$.

Thus if $\sum\limits_{n=2}^{\infty}\dfrac{\sqrt[n]{n}}{n}$ diverges, $\sum\limits_{n=2}^{\infty}\dfrac{\sqrt[n]{n}}{\ln n}$ also diverges.

Let $a_n = \dfrac{\sqrt[n]{n}}{n}$ and $b_n = \dfrac{1}{n}$. Then

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\sqrt[n]{n} = 1$ as shown above;

$0 < 1 < \infty$. Since $\sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ diverges,

$\sum\limits_{n=2}^{\infty}\dfrac{\sqrt[n]{n}}{n}$ also diverges by the Limit Comparison

Test, hence $\sum\limits_{n=2}^{\infty}\dfrac{\sqrt[n]{n}}{\ln n}$ also diverges.

The series is conditionally convergent.

37. $\rho = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^3+1} \div \dfrac{x^n}{n^3+1}\right|$

$= \lim\limits_{n\to\infty}|x|\left|\dfrac{n^3+1}{(n+1)^3+1}\right| = |x|$

When $x = 1$, the series is

$\sum\limits_{n=0}^{\infty}\dfrac{1}{n^3+1} = 1 + \sum\limits_{n=1}^{\infty}\dfrac{1}{n^3+1} \le 1 + \sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$, which converges.

When $x = -1$, the series is $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{n^3+1}$ which

converges absolutely since $\sum\limits_{n=0}^{\infty}\dfrac{1}{n^3+1}$ converges.

The series converges on $-1 \le x \le 1$.

38. $\rho = \lim\limits_{n\to\infty}\left|\dfrac{(-2)^{n+2}x^{n+1}}{2n+5} \div \dfrac{(-2)^{n+1}x^n}{2n+3}\right|$

$= \lim\limits_{n\to\infty}|2x|\left|\dfrac{2n+3}{2n+5}\right| = |2x|$; $|2x| < 1$ when

$-\dfrac{1}{2} < x < \dfrac{1}{2}$.

When $x = \dfrac{1}{2}$, the series is

$\sum\limits_{n=0}^{\infty}\dfrac{(-2)^{n+1}\left(\frac{1}{2}\right)^n}{2n+3} = \sum\limits_{n=0}^{\infty}\dfrac{(-2)\left(\frac{-2}{2}\right)^n}{2n+3}$

$= \sum\limits_{n=0}^{\infty}\dfrac{2(-1)^{n+1}}{2n+3}$. $a_n = \dfrac{2}{2n+3}$; $\dfrac{2}{2n+3} > \dfrac{2}{2n+5}$, so

$a_n > a_{n+1}$; $\lim\limits_{n\to\infty}\dfrac{2}{2n+3} = 0$ so $\sum\limits_{n=0}^{\infty}\dfrac{2(-1)^{n+1}}{2n+3}$

converges by the Alternating Series Test.

When $x = -\dfrac{1}{2}$, the series is

$\sum\limits_{n=0}^{\infty}\dfrac{(-2)^{n+1}\left(-\frac{1}{2}\right)^n}{2n+3} = \sum\limits_{n=0}^{\infty}\dfrac{(-2)\left(\frac{-2}{-2}\right)^n}{2n+3} = -\sum\limits_{n=0}^{\infty}\dfrac{2}{2n+3}$.

$a_n = \dfrac{2}{2n+3}$, let $b_n = \dfrac{1}{n}$ then

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{2n}{2n+3} = \lim\limits_{n\to\infty}\dfrac{2}{2+\frac{3}{n}} = 1$;

$0 < 1 < \infty$ hence since $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ diverges,

$\sum\limits_{n=0}^{\infty}\dfrac{2}{2n+3}$ and also $-\sum\limits_{n=0}^{\infty}\dfrac{2}{2n+3}$ diverges.

The series converges on $-\dfrac{1}{2} < x \le \dfrac{1}{2}$.

39. $\rho = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{n+2} \div \frac{(x-4)^n}{n+1}\right|$

$= \lim_{n\to\infty}|x-4|\left|\frac{n+1}{n+2}\right| = |x-4|$; $|x-4|<1$ when $3 < x < 5$.

When $x = 5$, the series is

$\sum_{n=0}^{\infty}\frac{(-1)^n(1)^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$.

$a_n = \frac{1}{n+1}; \frac{1}{n+1} > \frac{1}{n+2}$, so $a_n > a_{n+1}$;

$\lim_{n\to\infty}\frac{1}{n+1} = 0$ so $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ converges by the Alternating Series Test.

When $x = 3$, the series is

$\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1}$. $a_n = \frac{1}{n+1}$, let

$b_n = \frac{1}{n}$ then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{n+1} = 1;\ \ 0 < 1 < \infty$

hence since $\sum_{n=1}^{\infty}b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=0}^{\infty}\frac{1}{n+1}$ also diverges.

The series converges on $3 < x \le 5$.

40. $\rho = \lim_{n\to\infty}\left|\frac{3^{n+1}x^{3n+3}}{(3n+3)!} \div \frac{3^n x^{3n}}{(3n)!}\right|$

$= \lim_{n\to\infty}\left|3x^3\right|\left|\frac{1}{(3n+3)(3n+2)(3n+1)}\right| = 0$

The series converges for all x.

41. $\rho = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}+1} \div \frac{(x-3)^n}{2^n+1}\right|$

$= \lim_{n\to\infty}|x-3|\left|\frac{1+\frac{1}{2^n}}{2+\frac{1}{2^n}}\right| = \frac{|x-3|}{2}$; $\frac{|x-3|}{2} < 1$

when $1 < x < 5$.

When $x = 5$, the series is

$\sum_{n=0}^{\infty}\frac{2^n}{2^n+1} = \sum_{n=0}^{\infty}\frac{1}{1+\left(\frac{1}{2}\right)^n}$; $\lim_{n\to\infty}\frac{1}{1+\left(\frac{1}{2}\right)^n} = 1 \ne 0$

so the series diverges.

When $x = 1$, the series is

$\sum_{n=0}^{\infty}\frac{(-2)^n}{2^n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{1+\left(\frac{1}{2}\right)^n}$; $\lim_{n\to\infty}\frac{1}{1+\left(\frac{1}{2}\right)^n} = 1 \ne 0$

so the series diverges.

The series converges on $1 < x < 5$.

42. $\rho = \lim_{n\to\infty}\left|\frac{(n+1)!(x+1)^{n+1}}{3^{n+1}} \div \frac{n!(x+1)^n}{3^n}\right|$

$= \lim_{n\to\infty}\left|\frac{x+1}{3}\right|\left|n+1\right| = \infty$ unless $x = -1$.

The series converges only for $x = -1$.

43. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ for $-1 < x < 1$.

If $f(x) = \frac{1}{1+x}$, then $f'(x) = -\frac{1}{(1+x)^2}$. Thus, differentiating the series for $\frac{1}{1+x}$ and multiplying by -1 yields

$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$. The series converges on $-1 < x < 1$.

44. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $-1 < x < 1$. If

$f(x) = \frac{1}{1+x}$, then $f''(x) = \frac{2}{(1+x)^3}$.

Differentiating the series for $\frac{1}{1+x}$ twice and dividing by 2 gives

$\frac{1}{(1+x)^3} = 1 - 3x + \frac{1}{2}(4\cdot 3)x^2 - \frac{1}{2}(5\cdot 4)x^3 + \cdots$

$= 1 - 3x + 6x^2 - 10x^3 + \cdots$.

The series converges on $-1 < x < 1$.

45. $\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)^2$

$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots$

Since the series for $\sin x$ converges for all x, so does the series for $\sin^2 x$.

46. If $f(x) = e^x$, then $f^{(n)}(x) = e^x$. Thus, $e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^3}{3!}(x-2)^3 + \frac{e^4}{4!}(x-2)^4 + \cdots$.

47. $\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$

Since the series for $\sin x$ and $\cos x$ converge for all x, so does the series for $\sin x + \cos x$.

48. Let $a_k = \dfrac{1}{9+k^2}$ and define $f(x) = \dfrac{1}{9+x^2}$; then $f(k) = a_k$ and f is positive, continuous and non-increasing (since $f'(x) = \dfrac{-2x}{(9+x^2)^2} < 0$) on $[1,\infty)$. Thus, by the Integral Test, $E_n < \int_n^\infty \dfrac{1}{9+x^2}dx = \lim_{A\to\infty}\left[\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}\right]_n^A =$ $\dfrac{1}{3}\left[\lim_{A\to\infty}\tan^{-1}\dfrac{A}{3} - \tan^{-1}\dfrac{n}{3}\right] = \dfrac{\pi}{6} - \dfrac{1}{3}\tan^{-1}\dfrac{n}{3}$. Now $\dfrac{\pi}{6} - \dfrac{1}{3}\tan^{-1}\dfrac{n}{3} \le 0.00005 \Rightarrow n \ge 3\left[\tan\left(3\left(\frac{\pi}{6} - 0.00005\right)\right)\right] \approx 20,000$.

49. Let $a_k = \dfrac{k}{e^{k^2}}$ and define $f(x) = \dfrac{x}{e^{x^2}}$; then $f(k) = a_k$ and f is positive, continuous and non-increasing (since $f'(x) = \dfrac{1-2x^2}{e^{x^2}} < 0$) on $[1,\infty)$. Thus, by the Integral Test, $E_n < \int_n^\infty \dfrac{x}{e^{x^2}}dx = \lim_{A\to\infty}\left[-\dfrac{1}{2e^{x^2}}\right]_n^A =$ $\dfrac{1}{2}\left[-\lim_{A\to\infty}\dfrac{1}{2e^{A^2}} + \dfrac{1}{2e^{n^2}}\right] = \dfrac{1}{4e^{n^2}}$. Now $\dfrac{1}{4e^{n^2}} \le 0.000005 \Rightarrow e^{n^2} \ge 50,000 \Rightarrow n^2 \ge \ln(50,000) \approx 10.82 \Rightarrow \ n > 3$.

50. One million terms are needed to approximate the sum to within 0.001 since $\dfrac{1}{\sqrt{n+1}} < 0.001$ is equivalent to $999,999 < n$.

51. **a.** From the Maclaurin series for $\dfrac{1}{1-x}$, we have $\dfrac{1}{1-x^3} = 1 + x^3 + x^6 + \cdots$.

b. In Example 6 of Section 9.8 it is shown that $\sqrt{1+x} = 1 + \dfrac{1}{2}x - \dfrac{1}{8}x^2 + \dfrac{1}{16}x^3 - \dfrac{5}{128}x^4 + \cdots$ so $\sqrt{1+x^2} = 1 + \dfrac{1}{2}x^2 - \dfrac{1}{8}x^4 + \cdots$.

c. $e^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \dfrac{x^5}{5!} + \cdots$, so $e^{-x} - 1 + x = \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots$.

d. Using division with the Maclaurin series for cos x, we get $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + \cdots$.

Thus, $x\sec x = x + \dfrac{x^3}{2} + \dfrac{5x^5}{4!} + \cdots$.

e. $e^{-x}\sin x = \left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots\right)\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots\right) = x - x^2 + \dfrac{x^3}{3} - \cdots$

f. $1 + \sin x = 1 + x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$; Using division, we get $\dfrac{1}{1+\sin x} = 1 - x + x^2 - \cdots$.

52.
$$f(x) = \cos x \qquad f(0) = 1$$
$$f^{(1)}(x) = -\sin x \qquad f^{(1)}(0) = 0$$
$$f^{(2)}(x) = -\cos x \qquad f^{(2)}(0) = -1$$
$$\therefore\ P_2(x) = 1 - \frac{x^2}{2}$$
Thus, $\cos(0.1) \approx 1 - \dfrac{(0.1)^2}{2} = 1 - 0.005 = 0.995$

53. $f(0) = 0$

$f'(x) = \cos^2 x - 2x^2\sin^2 x$

$f'(0) = 1$

$p(x) = x$; $p(0.2) = 0.2$; $f(0.2) = 0.1998$

54. a. $f(x) = xe^x \qquad f(0) = 0$

$f'(x) = e^x + xe^x \quad f'(0) = 1$

$f''(x) = 2e^x + xe^x \quad f''(0) = 2$

$f^{(3)}(x) = 3e^x + xe^x \qquad f^{(3)}(0) = 3$

$f^{(4)}(x) = 4e^x + xe^x \qquad f^{(4)}(0) = 4$

$f(x) \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

$f(0.1) \approx 0.11052$

b. $f(x) = \cosh x \qquad f(0) = 1$

$f'(x) = \sinh x \qquad f'(0) = 0$

$f''(x) = \cosh x \qquad f''(0) = 1$

$f^{(3)}(x) = \sinh x \quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cosh x \quad f^{(4)}(0) = 1$

$f(x) \approx 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$

$f(0.1) \approx 1.0050042$

55. $g(x) = x^3 - 2x^2 + 5x - 7 \qquad g(2) = 3$

$g'(x) = 3x^2 - 4x + 5 \quad g'(2) = 9$

$g''(x) = 6x - 4 \qquad g''(2) = 8$

$g^{(3)}(x) = 6 \qquad g^{(3)}(2) = 6$

Since $g^{(4)}(x) = 0, R_3(x) = 0,$ so the Taylor polynomial of order 3 based at 2 is an exact representation.

$g(x) = P_4(x) = 3 + 9(x-2) + 4(x-2)^2 + (x-2)^3$

56. $g(2.1) = 3 + 9(0.1) + 4(0.1)^2 + (0.1)^3 = 3.941$

57. $f(x) = \frac{1}{x+1} \qquad f(1) = \frac{1}{2}$

$f'(x) = -\frac{1}{(x+1)^2} \qquad f'(1) = -\frac{1}{4}$

$f''(x) = \frac{2}{(x+1)^3} \qquad f''(1) = \frac{1}{4}$

$f^{(3)}(x) = -\frac{6}{(x+1)^4} \quad f^{(3)}(1) = -\frac{3}{8}$

$f^{(4)}(x) = \frac{24}{(x+1)^5} \qquad f^{(4)}(1) = \frac{3}{4}$

$$f(x) \approx \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4$$

58. $f^{(5)}(x) = -\frac{120}{(x+1)^6}$, so $R_4(x) = -\frac{(x-1)^5}{(c+1)^6}$.

$$|R_4(1.2)| = \frac{(0.2)^5}{(c+1)^6} \le \frac{(0.2)^5}{(2)^6} = 0.000005$$

59. $f(x) = \frac{1}{2}(1 - \cos 2x) \qquad f(0) = 0$

$f'(x) = \sin 2x \qquad f'(0) = 0$

$f''(x) = 2\cos 2x \qquad f''(0) = 2$

$f^{(3)}(x) = -4\sin 2x \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = -8\cos 2x \qquad f^{(4)}(0) = -8$

$f^{(5)}(x) = 16\sin 2x \qquad f^{(5)}(0) = 0$

$f^{(6)}(x) = 32\cos 2x \qquad f^{(6)}(c) = 32\cos 2c$

$$\sin^2 x \approx \frac{2}{2!}x^2 - \frac{8}{4!}x^4 = x^2 - \frac{1}{3}x^4$$

$$|R_4(x)| = |R_5(x)| = \left|\frac{32}{6!}(\cos 2c)x^6\right| \le \frac{2}{45}(0.2)^6 < 2.85 \times 10^{-6}$$

60. $f^{(n+1)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

$$|R_n(x)| = \left|\frac{(-1)^n}{(n+1)c^{n+1}}(x-1)^{n+1}\right| \le \frac{0.2^{n+1}}{(n+1)0.8^{n+1}} = \frac{(0.25)^{n+1}}{(n+1)}$$

$\frac{(0.25)^{n+1}}{(n+1)} < 0.00005$ when $n \ge 5$.

61. From Problem 60,

$$\ln x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5.$$

$$|R_5(x)| = \left|\frac{1}{6c^6}(x-1)^6\right| \le \frac{0.2^6}{6 \cdot 0.8^6} < 4.07 \times 10^{-5}$$

$$\int_{0.8}^{1.2} \ln x\, dx \approx \int_{0.8}^{1.2} \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5\right] dx$$

$$= \left[\frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{30}(x-1)^6\right]_{0.8}^{1.2}$$

≈ -0.00269867

Error $\le (1.2 - 0.8)4.07 \times 10^{-5} < 1.63 \times 10^{-5}$

Review and Preview Problems

1. $f(x)=\frac{x^2}{4}$ so that $f'(x)=\frac{x}{2}$ and $f'(2)=1$

a. The tangent line will be the line through the point $(2,1)$ having slope $m=1$. Using the point slope formula: $(y-1)=1(x-2)$ or $y=x-1$.

b. The normal line will be the line through the point $(2,1)$ having slope $m=-\frac{1}{1}=-1$. Using the point slope formula: $(y-1)=-1(x-2)$ or $y=-x+3$ or $x+y=3$.

2. $f(x)=y=\frac{x^2}{4},\ f'(x)=\frac{x}{2}$

a. The line $y=x$ has slope = 1, so we seek x such that $f'(x)=1$ or $x=2$. The point is $(2,1)$.

b. Since $f'(x)=\frac{x}{2}$ is the slope of the tangent line at the point $(x, x^2/4)$, $-\frac{2}{x}$ will be the slope of the normal line at the same point. Since $y=x$ has slope 1, we seek x such that $-\frac{2}{x}=1$ or $x=-2$. The point is $(-2,1)$.

3. Solving equation 1 for y^2: $y^2=9-\frac{9}{16}x^2$ and putting this result into equation 2:

$$\frac{x^2}{9}+\frac{1}{16}\left(9-\frac{9}{16}x^2\right)=1$$

$175x^2=1008 \quad x^2=5.76 \quad x=\pm 2.4$

Putting these values into equation 1 we get

$y^2=9-\frac{9}{16}(5.76)=9-3.24=5.76$ so $y=\pm 2.4$ also. Thus the points of intersection are $(2.4,2.4),\ (-2.4,2.4),\ (2.4,-2.4),\ (-2.4,-2.4)$

4. Solving equation 2 for y^2: $y^2=9-x^2$ and putting this result into equation 1: $\frac{x^2}{16}-\frac{x^2}{9}=0$

Thus $x=0$ and $y^2=9,\ y=\pm 3$. Thus the points of intersection are $(0,3),\quad (0,-3)$

5. Since we are given a point, all we need is the slope to determine the equation of our tangent line.

$$\frac{d}{dx}\left(x^2+\frac{y^2}{4}\right)=\frac{d}{dx}(1)$$

$$2x+\frac{y}{2}\frac{dy}{dx}=0$$

$$\frac{y}{2}\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-4x}{y}$$

At the point $\left(-\frac{\sqrt{3}}{2},1\right)$, we get

$$\frac{dy}{dx}=\frac{-4\left(-\sqrt{3}/2\right)}{1}=2\sqrt{3}=m_{\tan}$$

Therefore, the equation of the tangent line to the curve at $\left(-\frac{\sqrt{3}}{2},1\right)$ is given by

$$y-1=2\sqrt{3}\left(x-\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$y-1=2\sqrt{3}\,x+3$$

$$y=2\sqrt{3}\,x+4$$

6. Since we are given a point, all we need is the slope to determine the equation of our tangent line.

$$\frac{d}{dx}\left(\frac{x^2}{9}-\frac{y^2}{16}\right)=\frac{d}{dx}(1)$$

$$\frac{2x}{9}-\frac{y}{8}\frac{dy}{dx}=0$$

$$-\frac{y}{8}\frac{dy}{dx}=-\frac{2x}{9}$$

$$\frac{dy}{dx}=\frac{16x}{9y}$$

At the point $\left(9,8\sqrt{2}\right)$, we get

$$\frac{dy}{dx}=\frac{16(9)}{9\left(8\sqrt{2}\right)}=\frac{2}{\sqrt{2}}=\sqrt{2}=m_{\tan}$$

Therefore, the equation of the tangent line to the curve at $\left(9,8\sqrt{2}\right)$ is given by

$$y-8\sqrt{2}=\sqrt{2}\left(x-9\right)$$

$$y-8\sqrt{2}=\sqrt{2}\,x-9\sqrt{2}$$

$$y=\sqrt{2}\,x-\sqrt{2}$$

7. Denote the curves as

$$C_1 : \frac{x^2}{100}+\frac{y^2}{64}=1 \quad \text{and} \quad C_2 : \frac{x^2}{9}-\frac{y^2}{27}=1$$

a. From C_2, $3x^2-27=y^2$ and so, from C_1,

$$16x^2+25\left(3x^2-27\right)=1600$$

$$91x^2=2275,\ x^2=25,\ x=\pm 5$$

$$y^2=3(25)-27,\ y^2=48,\ y=\pm 4\sqrt{3}$$

Thus the point of intersection in the first quadrant is $P=(5,4\sqrt{3})$.

b. Slope m_1 of the line tangent to C_1 at P:

$$C_1' : \frac{x}{50}+\frac{y}{32}y'=0,\ \ y'=-\frac{16x}{25y} \text{ so } m_1=-\frac{4\sqrt{3}}{15}$$

The line T_1 tangent to C_1 at P is

$$T_1 : (y-4\sqrt{3})=-\frac{4\sqrt{3}}{15}(x-5) \text{ or}$$

$$4\sqrt{3}\,x+15y=80\sqrt{3}$$

Slope m_2 of the line tangent to C_2 at P:

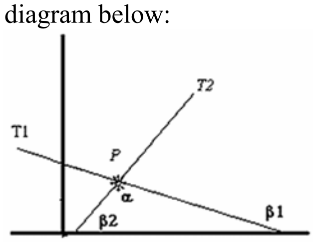
$$C_2' : \frac{2}{9}x-\frac{2}{27}yy'=0, \qquad y'=\frac{3x}{y} \text{ so}$$

$$m_2=\frac{15}{4\sqrt{3}}=\frac{5\sqrt{3}}{4}.$$

The line T_2 tangent to C_2 at P is

$$T_2 : (y-4\sqrt{3})=\frac{5\sqrt{3}}{4}(x-5) \text{ or } 5\sqrt{3}\,x-4y=9\sqrt{3}$$

c. To find the angles between the tangent lines, you can use problem 40 of section 0.7 or consider the diagram below:



Note that

$\alpha+\beta 2+(180-\beta 1)=180$ or $\alpha=\beta 1-\beta 2$; further

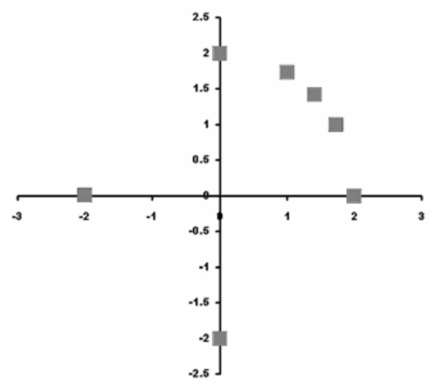
$$\beta 1=180-\tan^{-1}|m_1|=180-\tan^{-1}\left(\frac{4\sqrt{3}}{15}\right)=155.2^\circ$$

$$\beta 2=\tan^{-1}m_2=\tan^{-1}\left(\frac{5\sqrt{3}}{4}\right)=65.2$$

Thus $\alpha=155.2-65.2=90^\circ$ so the tangent lines are perpendicular. This can be verified by noting that $-\frac{1}{m_1}=\frac{15}{4\sqrt{3}}=\frac{15\sqrt{3}}{12}=\frac{5\sqrt{3}}{4}=m_2$

8.

t	$x=2\cos t$	$y=2\sin t$
0	2	0
$\pi/6$	$\sqrt{3}$	1
$\pi/4$	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	1	$\sqrt{3}$
$\pi/2$	0	2
π	-2	0
$3\pi/2$	0	-2
2π	2	0



Note that

$$x^2+y^2=(2\cos t)^2+(2\sin t)^2=$$

$$4(\cos^2 t+\sin^2 t)=4$$

so all the points will lie on the circle of radius 2 that is centered at the origin.

9. By the Pythagorean Theorem, $r^2=3^2+4^2$ or $r=\sqrt{9+16}=5$. Since

$$\sin\theta=\frac{3}{r}=\frac{3}{5},\quad \theta=\sin^{-1}(0.6)=36.9^\circ$$

10. By the Pythagorean Theorem, $r^2=2^2+5^2$ or $r=\sqrt{4+25}=\sqrt{29}$. Since

$$\sin\theta=\frac{2}{r}=\frac{2}{\sqrt{29}},\quad \theta=\sin^{-1}\left(\frac{2\sqrt{29}}{29}\right)=21.8^\circ$$

11. Since the triangle is an isosceles right triangle, $x=y$ and $x^2+y^2=8^2$. Thus $2x^2=64$ and $x=y=\sqrt{32}=4\sqrt{2}$

12. Since $\sin\frac{\pi}{6}=\frac{1}{2},\ \frac{y}{12}=\frac{1}{2}$ or $y=6$. Further $x^2+y^2=12^2$ or $x^2=144-36=108$. Hence $x=\sqrt{108}=6\sqrt{3}$.

CHAPTER 10 Conics and Polar Coordinates

10.1 Concepts Review

1. $e < 1$; $e = 1$; $e > 1$

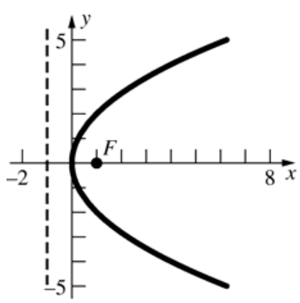
2. $y^2 = 4px$

3. (0, 1); $y = -1$

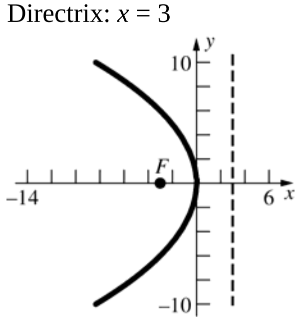
4. parallel to the axis

Problem Set 10.1

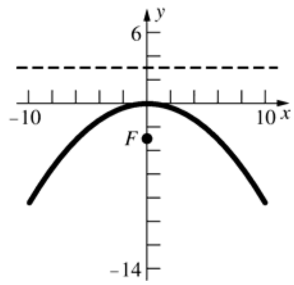
1. $y^2 = 4(1)x$
   Focus at (1, 0)
   Directrix: $x = -1$



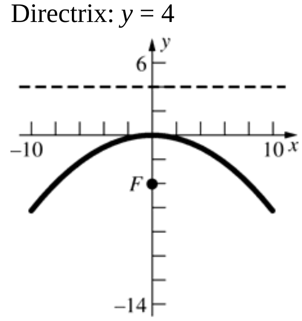
2. $y^2 = -4(3)x$
   Focus at (–3, 0)
   Directrix: $x = 3$



3. $x^2 = -4(3)y$
   Focus at (0, –3)
   Directrix: $y = 3$



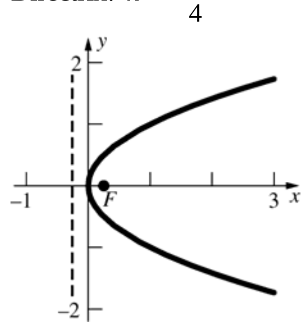
4. $x^2 = -4(4)y$
   Focus at (0, –4)
   Directrix: $y = 4$



5. $y^2 = 4\left(\frac{1}{4}\right)x$

   Focus at $\left(\frac{1}{4}, 0\right)$

   Directrix: $x = -\frac{1}{4}$

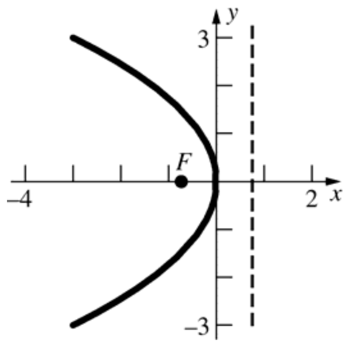


6. $y^2 = -3x$

$y^2 = -4\left(\frac{3}{4}\right)x$

Focus at $\left(-\frac{3}{4}, 0\right)$

Directrix: $x = \frac{3}{4}$

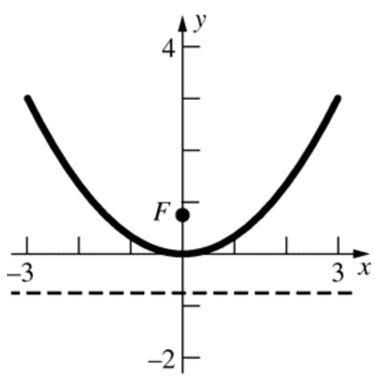


7. $2x^2 = 6y$

$x^2 = 4\left(\frac{3}{4}\right)y$

Focus at $\left(0, \frac{3}{4}\right)$

Directrix: $y = -\frac{3}{4}$

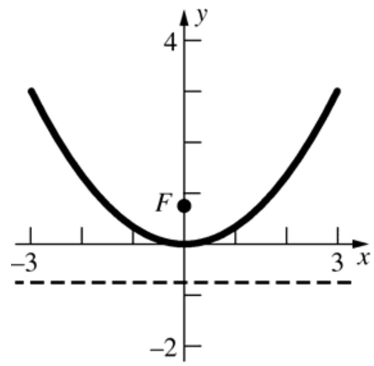


8. $3x^2 = 9y$

$x^2 = 4\left(\frac{3}{4}\right)y$

Focus at $\left(0, \frac{3}{4}\right)$

Directrix: $y = -\frac{3}{4}$



9. The parabola opens to the right, and $p = 2$.
$y^2 = 8x$

10. The parabola opens to the left, and $p = 3$.
$y^2 = -12x$

11. The parabola opens downward, and $p = 2$.
$x^2 = -8y$

12. The parabola opens downward, and $p = \frac{1}{9}$.

$x^2 = -\frac{4}{9}y$

13. The parabola opens to the left, and $p = 4$.
$y^2 = -16x$

14. The parabola opens downward, and $p = \frac{7}{2}$.

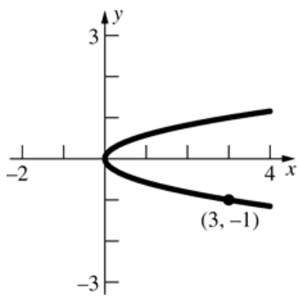
$x^2 = -14y$

15. The equation has the form $y^2 = cx$, so

$(-1)^2 = 3c.$

$c = \frac{1}{3}$

$y^2 = \frac{1}{3}x$

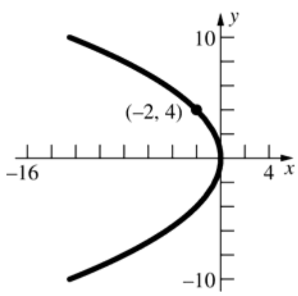


16. The equation has the form $y^2 = cx$, so

$(4)^2 = -2c.$

$c = -8$

$y^2 = -8x$

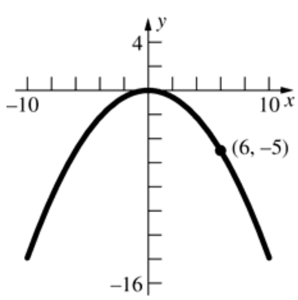


17. The equation has the form $x^2 = cy$, so

$(6)^2 = -5c.$

$c = -\frac{36}{5}$

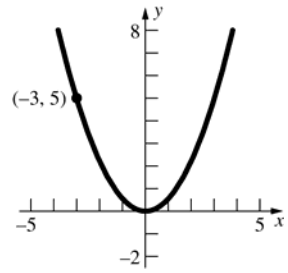
$x^2 = -\frac{36}{5}y$



18. The equation has the form $x^2 = cy$, so

$(-3)^2 = 5c$.

$c = \frac{9}{5} \quad \Rightarrow \quad x^2 = \frac{9}{5}y$



19. $y^2 = 16x$

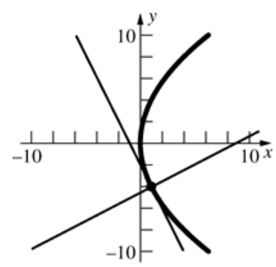
$2yy' = 16$

$y' = \frac{16}{2y}$

At (1, –4), $y' = -2$.

Tangent: $y + 4 = -2(x - 1)$ or $y = -2x - 2$

Normal: $y + 4 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{9}{2}$



20. $x^2 = -10y$

$2x = -10y'$

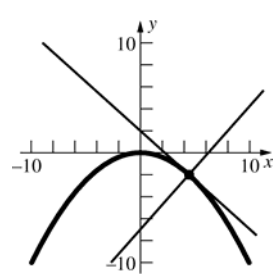
$y' = -\frac{x}{5}$

At $\left(2\sqrt{5}, -2\right)$, $y' = -\frac{2\sqrt{5}}{5}$.

Tangent: $y + 2 = -\frac{2\sqrt{5}}{5}\left(x - 2\sqrt{5}\right)$ or

$y = -\frac{2\sqrt{5}}{5}x + 2$

Normal: $y + 2 = \frac{\sqrt{5}}{2}\left(x - 2\sqrt{5}\right)$ or $y = \frac{\sqrt{5}}{2}x - 7$



21. $x^2 = 2y$

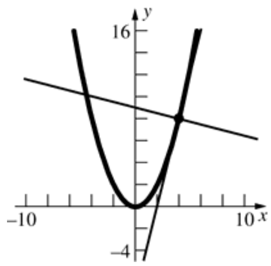
$2x = 2y'$

$y' = x$

At (4, 8), $y' = 4$.

Tangent: $y - 8 = 4(x - 4)$ or $y = 4x - 8$

Normal: $y - 8 = -\frac{1}{4}(x - 4)$ or $y = -\frac{1}{4}x + 9$



22. $y^2 = -9x$

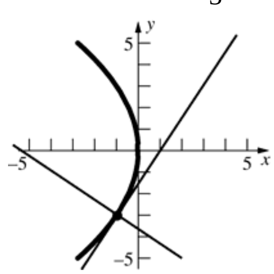
$2yy' = -9$

$y' = -\frac{9}{2y}$

At (–1, –3), $y' = \frac{3}{2}$

Tangent: $y + 3 = \frac{3}{2}(x + 1)$ or $y = \frac{3}{2}x - \frac{3}{2}$

Normal: $y + 3 = -\frac{2}{3}(x + 1)$ or $y = -\frac{2}{3}x - \frac{11}{3}$



23. $y^2 = -15x$

$2yy' = -15$

$y' = -\frac{15}{2y}$

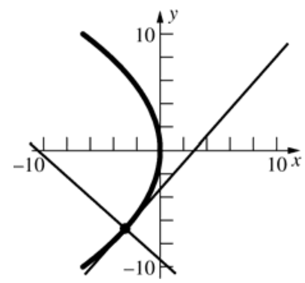
At $\left(-3, -3\sqrt{5}\right)$, $y' = \frac{\sqrt{5}}{2}$.

Tangent: $y + 3\sqrt{5} = \frac{\sqrt{5}}{2}(x + 3)$ or

$y = \frac{\sqrt{5}}{2}x - \frac{3\sqrt{5}}{2}$

Normal: $y + 3\sqrt{5} = -\frac{2\sqrt{5}}{5}(x + 3)$ or

$y = -\frac{2\sqrt{5}}{5}x - \frac{21\sqrt{5}}{5}$



24. $x^2 = 4y$

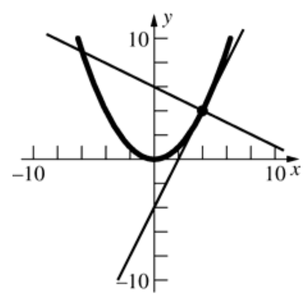
$2x = 4y'$

$y' = \frac{x}{2}$

At (4, 4), $y' = 2$.

Tangent: $y - 4 = 2(x - 4)$ or $y = 2x - 4$

Normal: $y - 4 = -\frac{1}{2}(x - 4)$ or $y = -\frac{1}{2}x + 6$



25. $x^2 = -6y$

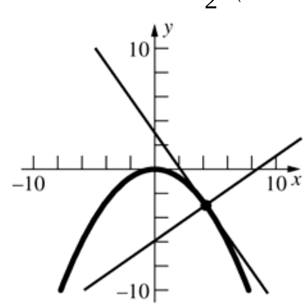
$2x = -6y'$

$y' = -\frac{x}{3}$

At $\left(3\sqrt{2}, -3\right)$, $y' = -\sqrt{2}$.

Tangent: $y + 3 = -\sqrt{2}\left(x - 3\sqrt{2}\right)$ or $y = -\sqrt{2}x + 3$

Normal: $y + 3 = \frac{\sqrt{2}}{2}\left(x - 3\sqrt{2}\right)$ or $y = \frac{\sqrt{2}}{2}x - 6$



26. $y^2 = 20x$

$2yy' = 20$

$y' = \frac{10}{y}$

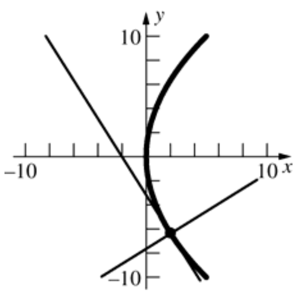
At $\left(2, -2\sqrt{10}\right)$, $y' = -\frac{\sqrt{10}}{2}$.

Tangent: $y + 2\sqrt{10} = -\frac{\sqrt{10}}{2}(x-2)$ or

$y = -\frac{\sqrt{10}}{2}x - \sqrt{10}$

Normal: $y + 2\sqrt{10} = \frac{\sqrt{10}}{5}(x-2)$ or

$y = \frac{\sqrt{10}}{5}x - \frac{12\sqrt{10}}{5}$



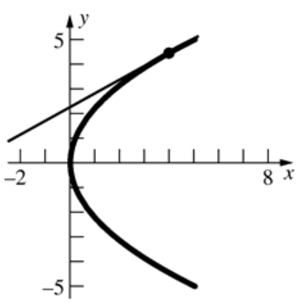
27. $y^2 = 5x$

$2yy' = 5$

$y' = \frac{5}{2y}$

$y' = \frac{\sqrt{5}}{4}$ when $y = 2\sqrt{5}$, so $x = 4$.

The point is $\left(4, 2\sqrt{5}\right)$.



28. $x^2 = -14y$

$2x = -14y'$

$y' = -\frac{x}{7}$

$y' = -\frac{2\sqrt{7}}{7}$ when $x = 2\sqrt{7}$, so $y = -2$.

The point is $\left(2\sqrt{7}, -2\right)$.

29. The slope of the line is $\frac{3}{2}$.

$y^2 = -18x; 2yy' = -18$

$2y\left(\frac{3}{2}\right) = -18; y = -6$

$(-6)^2 = -18x; x = -2$

The equation of the tangent line is

$y + 6 = \frac{3}{2}(x+2)$ or $y = \frac{3}{2}x - 3$.

30. Place the x-axis along the axis of the parabola such that the equation $y^2 = 4px$ describes the parabola. Let $\left(\frac{y_0^2}{4p}, y_0\right)$ be one of the extremities and $\left(\frac{y_1^2}{4p}, y_1\right)$ be the other.

First solve for y_1 in terms of y_0 and p. Since the focal chord passes through the focus $(p, 0)$, we have the following relation.

$$\frac{y_1}{\frac{y_1^2}{4p} - p} = \frac{y_0}{\frac{y_0^2}{4p} - p}$$

$$y_1(y_0^2 - 4p^2) = y_0(y_1^2 - 4p^2)$$

$$y_0 y_1^2 - (y_0^2 - 4p^2)y_1 - 4p^2 y_0 = 0$$

$$(y_1 - y_0)(y_0 y_1 + 4p^2) = 0$$

$$y_1 = y_0 \text{ or } y_1 = -\frac{4p^2}{y_0}$$

Thus, the other extremity is $\left(\frac{4p^3}{y_0^2}, -\frac{4p^2}{y_0}\right)$.

Implicitly differentiate $y^2 = 4px$ to get $2yy' = 4p$, so $y' = \frac{2p}{y}$.

At $\left(\frac{y_0^2}{4p}, y_0\right)$, $y' = \frac{2p}{y_0}$. The equation of the tangent line is $y - y_0 = \frac{2p}{y_0}\left(x - \frac{y_0^2}{4p}\right)$. When $x = -p$, $y = -\frac{2p^2}{y_0} + \frac{y_0}{2}$.

At $\left(\frac{4p^3}{y_0^2}, -\frac{4p^2}{y_0}\right)$, $y' = -\frac{y_0}{2p}$. The equation of the tangent line is $y + \frac{4p^2}{y_0} = -\frac{y_0}{2p}\left(x - \frac{4p^3}{y_0^2}\right)$.

When $x = -p$, $y = \frac{y_0}{2} - \frac{2p^2}{y_0}$.

Thus, the two tangent lines intersect on the directrix at $\left(-p, \frac{y_0}{2} - \frac{2p^2}{y_0}\right)$.

31. From Problem 30, if the parabola is described by the equation $y^2 = 4px$, the slopes of the tangent lines are $\frac{2p}{y_0}$ and $-\frac{y_0}{2p}$. Since they are negative reciprocals, the tangent lines are perpendicular.

32. Place the x-axis along the axis of the parabola such that the equation $y^2 = 4px$ describes the parabola. The endpoints of the chord are $\left(1, \frac{1}{2}\right)$ and $\left(1, -\frac{1}{2}\right)$, so $\left(\frac{1}{2}\right)^2 = 4(1)p$ or $p = \frac{1}{16}$. The distance from the vertex to the focus is $\frac{1}{16}$.

33. Assume that the x- and y-axes are positioned such that the axis of the parabola is the y-axis with the vertex at the origin and opening upward. Then the equation of the parabola is $x^2 = 4py$ and $(0, p)$ is the focus. Let D be the distance from a point on the parabola to the focus.

$$D = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + \left(\frac{x^2}{4p} - p\right)^2}$$

$$= \sqrt{\frac{x^4}{16p^2} + \frac{x^2}{2} + p^2} = \frac{x^2}{4p} + p$$

$$D' = \frac{x}{2p}; \frac{x}{2p} = 0, x = 0$$

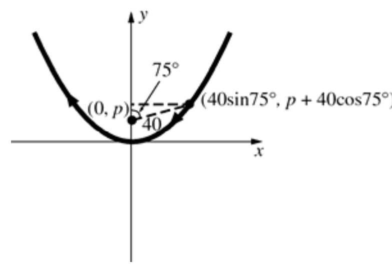
$D'' > 0$ so at $x = 0$, D is minimum. $y = 0$
Therefore, the vertex $(0, 0)$ is the point closest to the focus.

34. Let the y-axis be the axis of the parabola, so Earth's coordinates are $(0, p)$ and the equation of the path is $x^2 = 4py$, where the coordinates are in millions of miles. When the line from Earth to the asteroid makes an angle of 90° with the axis of the parabola, the asteroid is at $(40, p)$.

$(40)^2 = 4p(p), p = 20$

The closest point to Earth is $(0, 0)$, so the asteroid will come within 20 million miles of Earth.

35. Let the y-axis be the axis of the parabola, so the Earth's coordinates are $(0, p)$ and the equation of the path is $x^2 = 4py$, where the coordinates are in millions of miles. When the line from Earth to the asteroid makes an angle of 75° with the axis of the parabola, the asteroid is at $(40\sin 75^\circ, p + 40\cos 75^\circ)$. (See figure.)



$(40\sin 75^\circ)^2 = 4p(p + 40\cos 75^\circ)$

$p^2 + 40p\cos 75^\circ - 400\sin^2 75^\circ = 0$

$$p = \frac{-40\cos 75^\circ \pm \sqrt{1600\cos^2 75^\circ + 1600\sin^2 75^\circ}}{2}$$

$= -20\cos 75^\circ \pm 20$

$p = 20 - 20\cos 75^\circ \approx 14.8\ (p > 0)$

The closest point to Earth is $(0, 0)$, so the asteroid will come within 14.8 million miles of Earth.

36. Let the equation $x^2 = 4py$ describe the cables. The cables are attached to the towers at $(\pm 400, 400)$.

$(400)^2 = 4p(400), p = 100$

The vertical struts are at $x = \pm 300$.

$(300)^2 = 4(100)y, y = 225$

The struts must be 225 m long.

37. Let $|RL|$ be the distance from R to the directrix. Observe that the distance from the latus rectum to the directrix is $2p$ so $|RG| = 2p - |RL|$. From the definition of a parabola, $|RL| = |FR|$. Thus, $|FR| + |RG| = |RL| + 2p - |RL| = 2p$.

38. Let the coordinates of P be (x_0, y_0).

$2yy' = 4p$, so $y' = \frac{2p}{y}$. Thus the slope of the normal line at P is $-\frac{y_0}{2p}$.

The equation of the normal line is

$y - y_0 = -\frac{y_0}{2p}(x - x_0)$. When $y = 0$,

$x = 2p + x_0$, so B is at $(2p + x_0, 0)$. A is at $(x_0, 0)$. Thus, $|AB| = 2p + x_0 - x_0 = 2p$.

39. Let P_1 and P_2 denote (x_1, y_1) and (x_2, y_2), respectively. $|P_1P_2| = |P_1F| + |P_2F|$ since the focal chord passes through the focus. By definition of a parabola, $|P_1F| = p + x_1$ and $|P_2F| = p + x_2$.

Thus, the length of the chord is

$|P_1P_2| = x_1 + x_2 + 2p$.

$L = p + p + 2p = 4p$

40. Let C denote the center of the circle and r the radius. Observe that the distance from a point P to the circle is $|PC| - r$. Let l be the line and $|PL|$ the distance from the point to the line. Thus, $|PC| - r = |PL|$. Let l' be the line parallel to l, r units away and on the side opposite from the circle. Then $|PL'|$, the distance from P to l', is $|PL| + r$; so $|PL| = |PL'| - r$. Therefore, $|PC| - r = |PL'| - r$ or $|PC| = |PL'|$. The set of points is a parabola by definition.

41. $\frac{dy}{dx} = \frac{\delta x}{H}$

$y = \frac{\delta x^2}{2H} + C$

$y(0) = 0$ implies that $C = 0$. $y = \frac{\delta x^2}{2H}$

This is an equation for a parabola.

42. a. $A(T_1)$ is the area of the trapezoid formed by $(a, 0)$, P, Q, $(b, 0)$ minus the area the two trapezoids formed by $(a, 0)$, P, (c, c^2), $(c, 0)$ and by $(c, 0)$, (c, c^2), Q, $(b, 0)$. Observe that since $c = \frac{a+b}{2}, \frac{b-c}{2} = \frac{c-a}{2} = \frac{b-a}{4}$.

$$A(T_1) = \frac{b-a}{2}[a^2 + b^2] - \frac{c-a}{2}[a^2 + c^2] - \frac{b-c}{2}[c^2 + b^2] = \frac{b-a}{2}[a^2 + b^2] - \frac{b-a}{4}[a^2 + 2c^2 + b^2]$$

$$= \frac{b-a}{2}[a^2 + b^2] - \frac{b-a}{4}\left[a^2 + 2\left(\frac{a+b}{2}\right)^2 + b^2\right] = \frac{b-a}{4}\left[a^2 + b^2 - \frac{a^2}{2} - ab - \frac{b^2}{2}\right]$$

$$= \frac{b-a}{4}\left(\frac{a^2}{2} - ab + \frac{b^2}{2}\right) = \frac{(b-a)^3}{8}$$

b. $$A(T_2) = \frac{(c-a)^3}{8} + \frac{(b-c)^3}{8} = \frac{\left(\frac{b-a}{2}\right)^3}{8} + \frac{\left(\frac{b-a}{2}\right)^3}{8} = \frac{(b-a)^3}{32} = \frac{A(T_1)}{4}$$

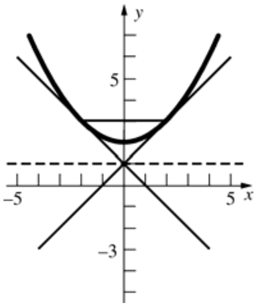
c. Using reasoning similar to part b, $A(T_n) = \frac{A(T_{n-1})}{4}$, so $A(T_n) = \frac{A(T_1)}{4^{n-1}}$.

$$A(S) = A(T_1) + A(T_2) + A(T_3) + \ldots = \sum_{n=1}^{\infty} \frac{A(T_1)}{4^{n-1}} = A(T_1)\left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{4}{3}A(T_1)$$

d. $$\text{Area} = \frac{b-a}{2}[a^2 + b^2] - A(S) = \frac{(b-a)(a^2 + b^2)}{2} - \frac{(b-a)^3}{6}$$

$$= \frac{b-a}{6}[3a^2 + 3b^2 - (b^2 - 2ab + a^2)] = \frac{(b-a)}{6}[2a^2 + 2ab + 2b^2] = \frac{1}{3}(b^3 - a^3) = \frac{b^3}{3} - \frac{a^3}{3}$$

43.



44. a. Since the vertex is on the positive y-axis and the parabola crosses the x-axis, its equation is of the form: $x^2 = 4p(y-k)$, where k is the y-coordinate of the vertex; that is $x^2 = 4p(y-630)$. Since the point $(315, 0)$ is on the parabola, we have

$$(315)^2 = 4p(0-630) \text{ or } 4p = \frac{-315}{2} = -157.5$$

Thus the parabola has the equation

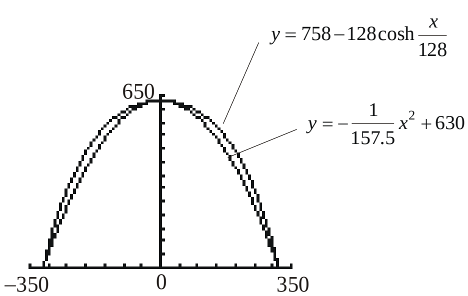
$$x^2 = -157.5(y-630)$$

b. Solving for y, we get

$$y = -\frac{1}{157.5}x^2 + 630$$

The catenary for the Gateway Arch is

$$y = 758 - 128\cosh\frac{x}{128}.$$

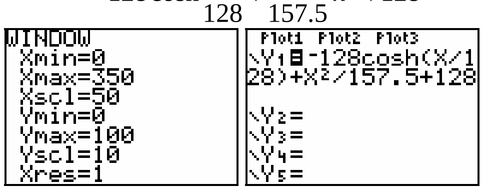


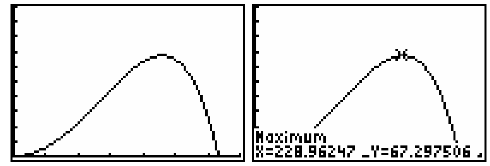
c. Because of symmetry, we can focus on the largest vertical distance between the graphs for positive x.

Let $f(x) = y_{\text{Arch}} - y_{\text{parabola}}$. That is,

$$f(x) = \left(758 - 128\cosh\frac{x}{128}\right) - \left(-\frac{1}{157.5}x^2 + 630\right)$$

$$= -128\cosh\frac{x}{128} + \frac{1}{157.5}x^2 + 128$$





Using a CAS, we find that the largest vertical distance between the catenary and the parabola is roughly 67 feet.

10.2 Concepts Review

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

2. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

3. foci

4. to the other focus; directly away from the other focus

Problem Set 10.2

1. Horizontal ellipse

2. Horizontal hyperbola

3. Vertical hyperbola

4. Horizontal hyperbola

5. Vertical parabola

6. Vertical parabola

7. Vertical ellipse

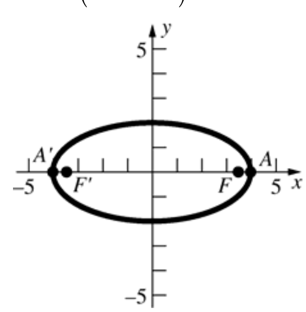
8. Horizontal hyperbola

9. $\frac{x^2}{16} + \frac{y^2}{4} = 1$; horizontal ellipse

   $a = 4$, $b = 2$, $c = 2\sqrt{3}$

   Vertices: $(\pm 4, 0)$

   Foci: $\left(\pm 2\sqrt{3}, 0\right)$



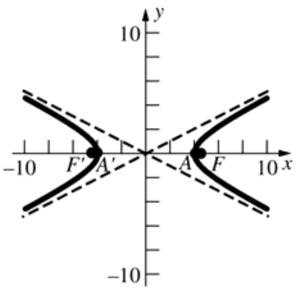
10. $\dfrac{x^2}{16}-\dfrac{y^2}{4}=1$; horizontal hyperbola

$a = 4$, $b = 2$, $c = 2\sqrt{5}$

Vertices: $(\pm 4, 0)$

Foci: $\left(\pm 2\sqrt{5}, 0\right)$

Asymptotes: $y=\pm\dfrac{1}{2}x$



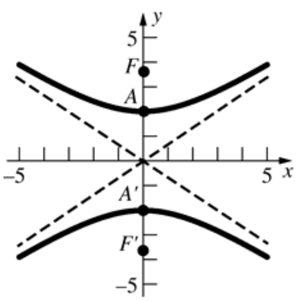
11. $\dfrac{y^2}{4}-\dfrac{x^2}{9}=1$; vertical hyperbola

$a = 2$, $b = 3$, $c = \sqrt{13}$

Vertices: $(0, \pm 2)$

Foci: $\left(0, \pm\sqrt{13}\right)$

Asymptotes: $y=\pm\dfrac{2}{3}x$

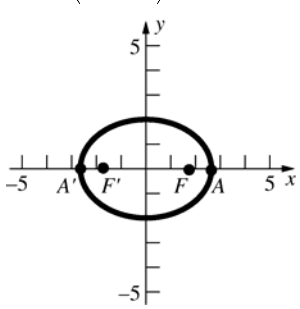


12. $\dfrac{x^2}{7}+\dfrac{y^2}{4}=1$; horizontal ellipse

$a=\sqrt{7}, b=2, c=\sqrt{3}$

Vertices: $\left(\pm\sqrt{7}, 0\right)$

Foci: $\left(\pm\sqrt{3}, 0\right)$

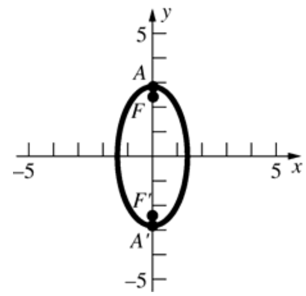


13. $\dfrac{x^2}{2}+\dfrac{y^2}{8}=1$; vertical ellipse

$a=2\sqrt{2}, b=\sqrt{2}, c=\sqrt{6}$

Vertices: $(0, \pm 2\sqrt{2})$

Foci: $\left(0, \pm\sqrt{6}\right)$

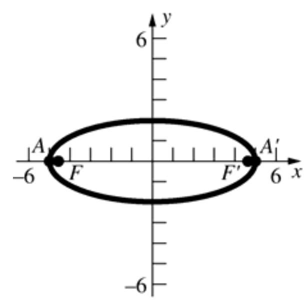


14. $\dfrac{x^2}{25}+\dfrac{y^2}{4}=1$; horizontal ellipse

$a = 5$, $b = 2$, $c = \sqrt{21}$

Vertices: $(\pm 5, 0)$

Foci: $\left(\pm\sqrt{21}, 0\right)$



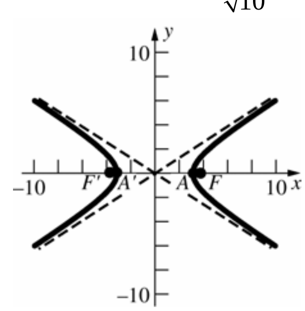
15. $\dfrac{x^2}{10}-\dfrac{y^2}{4}=1$; horizontal hyperbola

$a=\sqrt{10}, b=2, c=\sqrt{14}$

Vertices: $\left(\pm\sqrt{10}, 0\right)$

Foci: $\left(\pm\sqrt{14}, 0\right)$

Asymptotes: $y=\pm\dfrac{2}{\sqrt{10}}x$



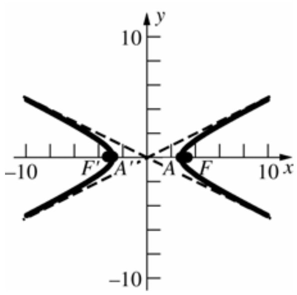
16. $\frac{x^2}{8}-\frac{y^2}{2}=1$; horizontal hyperbola

$a=2\sqrt{2}, b=\sqrt{2}, c=\sqrt{10}$

Vertices: $\left(\pm 2\sqrt{2}, 0\right)$

Foci: $\left(\pm\sqrt{10}, 0\right)$

Asymptotes: $y=\pm\frac{1}{2}x$



17. This is a horizontal ellipse with $a = 6$ and $c = 3$.

$b=\sqrt{36-9}=\sqrt{27}$

$\frac{x^2}{36}+\frac{y^2}{27}=1$

18. This is a horizontal ellipse with $c = 6$.

$a=\frac{c}{e}=\frac{6}{\frac{2}{3}}=9, b=\sqrt{81-36}=\sqrt{45}$

$\frac{x^2}{81}+\frac{y^2}{45}=1$

19. This is a vertical ellipse with $c = 5$.

$a=\frac{c}{e}=\frac{5}{\frac{1}{3}}=15, b=\sqrt{225-25}=\sqrt{200}$

$\frac{x^2}{200}+\frac{y^2}{225}=1$

20. This is a vertical ellipse with $b = 4$ and $c = 3$.

$a=\sqrt{16+9}=5$

$\frac{x^2}{16}+\frac{y^2}{25}=1$

21. This is a horizontal ellipse with $a = 5$.

$\frac{x^2}{25}+\frac{y^2}{b^2}=1$

$\frac{4}{25}+\frac{9}{b^2}=1$

$b^2=\frac{225}{21}$

$\frac{x^2}{25}+\frac{y^2}{\frac{225}{21}}=1$

22. This is a horizontal hyperbola with $a = 4$ and $c = 5$.

$b=\sqrt{25-16}=3$

$\frac{x^2}{16}-\frac{y^2}{9}=1$

23. This is a vertical hyperbola with $a = 4$ and $c = 5$.

$b=\sqrt{25-16}=3$

$\frac{y^2}{16}-\frac{x^2}{9}=1$

24. This is a vertical hyperbola with $a = 3$.

$c=ae=3\left(\frac{3}{2}\right)=\frac{9}{2}, b=\sqrt{\frac{81}{4}-9}=\frac{\sqrt{45}}{2}$

$\frac{y^2}{9}-\frac{x^2}{\frac{45}{4}}=1$

25. This is a horizontal hyperbola with $a = 8$.

The asymptotes are $y=\pm\frac{1}{2}x$, so $\frac{b}{8}=\frac{1}{2}$ or $b = 4$.

$\frac{x^2}{64}-\frac{y^2}{16}=1$

26. $c=ae=\frac{\sqrt{6}}{2}a, b^2=c^2-a^2=\frac{3}{2}a^2-a^2=\frac{1}{2}a^2$

$\frac{y^2}{a^2}-\frac{x^2}{\frac{1}{2}a^2}=1$

$\frac{16}{a^2}-\frac{4}{\frac{1}{2}a^2}=1$

$a^2=8$

$\frac{y^2}{8}-\frac{x^2}{4}=1$

27. This is a horizontal ellipse with $c = 2$.

$8=\frac{a}{e}, 8=\frac{a}{\frac{c}{a}}$, so $a^2=8c=16$.

$b=\sqrt{16-4}=\sqrt{12}$

$\frac{x^2}{16}+\frac{y^2}{12}=1$

28. This is a horizontal hyperbola with $c = 4$.

$1=\frac{a}{e}, 1=\frac{a}{\frac{c}{a}}$, so $a^2=c=4$.

$b=\sqrt{16-4}=\sqrt{12}$

$\frac{x^2}{4}-\frac{y^2}{12}=1$

29. The asymptotes are $y = \pm\frac{1}{2}x$. If the hyperbola is horizontal $\frac{b}{a} = \frac{1}{2}$ or $a = 2b$. If the hyperbola is vertical, $\frac{a}{b} = \frac{1}{2}$ or $b = 2a$.

Suppose the hyperbola is horizontal.

$$\frac{x^2}{4b^2} - \frac{y^2}{b^2} = 1$$
$$\frac{16}{4b^2} - \frac{9}{b^2} = 1$$
$$b^2 = -5$$

This is not possible.

Suppose the hyperbola is vertical.

$$\frac{y^2}{a^2} - \frac{x^2}{4a^2} = 1$$
$$\frac{9}{a^2} - \frac{16}{4a^2} = 1$$
$$a^2 = 5$$
$$\frac{y^2}{5} - \frac{x^2}{20} = 1$$

30. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\left.\begin{aligned}\frac{25}{a^2} + \frac{1}{b^2} = 1\\ \frac{16}{a^2} + \frac{4}{b^2} = 1\end{aligned}\right\} -\frac{84}{a^2} = -3$$
$$a^2 = 28$$
$$b^2 = \frac{28}{3}$$
$$\frac{x^2}{28} + \frac{y^2}{\frac{28}{3}} = 1$$

31. This is an ellipse whose foci are (0, 9) and (0, - 9) and whose major diameter has length $2a = 26$. Since the foci are on the y-axis, it is the major axis of the ellipse so the equation has the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. Since $2a = 26,\ a^2 = 169$ and since $a^2 = b^2 + c^2,\ b^2 = 169 - (9)^2 = 88$. Thus the equation is $\frac{y^2}{169} + \frac{x^2}{88} = 1$

32. This is an ellipse whose foci are (4, 0) and (- 4, 0) and whose major diameter has length $2a = 14$. Since the foci are on the x-axis, it is the major axis of the ellipse so the equation has the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since $2a = 14,\ a^2 = 49$ and since $a^2 = b^2 + c^2,\ b^2 = 49 - (4)^2 = 33$. Thus the equation is $\frac{x^2}{49} + \frac{y^2}{33} = 1$

33. This is an hyperbola whose foci are (7, 0) and (- 7, 0) and whose axis is the x-axis. So the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since $2a = 12,\ a^2 = 36$ and $b^2 = c^2 - a^2 = (7^2) - 36 = 13$ Thus the equation is $\frac{x^2}{36} - \frac{y^2}{13} = 1$

34. This is an hyperbola whose foci are (0, 6) and (0, - 6) and whose axis is the y-axis. So the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Since $2a = 10,\ a^2 = 25$ and $b^2 = c^2 - a^2 = (6^2) - 25 = 11$ Thus the equation is $\frac{y^2}{25} - \frac{x^2}{11} = 1$

35. Use implicit differentiation to find the slope: $\frac{2}{27}x + \frac{2}{9}y\,y' = 0$. At the point $(3, \sqrt{6}),\ \frac{2}{9} + \frac{2\sqrt{6}}{9}y' = 0$, or $y' = -\frac{\sqrt{6}}{6}$ so the equation of the tangent line is $(y - \sqrt{6}) = -\frac{\sqrt{6}}{6}(x - 3)$ or $x + \sqrt{6}y = 9$.

36. Use implicit differentiation to find the slope: $\frac{1}{12}x + \frac{y}{8}y' = 0$. At the point $(3\sqrt{2}, -2),\ \frac{\sqrt{2}}{4} - \frac{1}{4}y' = 0$, or $y' = \sqrt{2}$ so the equation of the tangent line is $(y + 2) = \sqrt{2}(x - 3\sqrt{2})$ or $\sqrt{2}x - y = 8$.

37. Use implicit differentiation to find the slope: $\frac{2}{27}x + \frac{2}{9}y\,y' = 0$. At the point $(3, -\sqrt{6})$, $\frac{2}{9} - \frac{2\sqrt{6}}{9}y' = 0$, or $y' = \frac{\sqrt{6}}{6}$ so the equation of the tangent line is $(y + \sqrt{6}) = -\frac{\sqrt{6}}{6}(x - 3)$ or $x - \sqrt{6}y = 9$.

38. Use implicit differentiation to find the slope: $x - \frac{1}{2}y\,y' = 0$. At the point $(\sqrt{3}, \sqrt{2})$ $\sqrt{3} - \frac{\sqrt{2}}{2}y' = 0$, or $y' = \sqrt{6}$ so the equation of the tangent line is $(y - \sqrt{2}) = \sqrt{6}(x - \sqrt{3})$ or $6x - \sqrt{6}y = 4\sqrt{3}$.

39. Use implicit differentiation to find the slope: $2x + 2y\,y' = 0$. At the point $(5,12)$ $10 + 24y' = 0$, or $y' = -\frac{5}{12}$ so the equation of the tangent line is $(y - 12) = -\frac{5}{12}(x - 5)$ or $5x + 12y = 169$.

40. Use implicit differentiation to find the slope: $2x - 2y\,y' = 0$. At the point $(\sqrt{2}, \sqrt{3})$ $2\sqrt{2} - 2\sqrt{3}y' = 0$, or $y' = \frac{\sqrt{6}}{3}$ so the equation of the tangent line is $(y - \sqrt{3}) = \frac{\sqrt{6}}{3}(x - \sqrt{2})$ or $3y - \sqrt{6}\,x = \sqrt{3}$.

41. Use implicit differentiation to find the slope: $\frac{1}{44}x + \frac{2}{169}y\,y' = 0$. At the point $(0,13)$, $\frac{2}{13}y' = 0$, or $y' = 0$. The tangent line is horizontal and thus has equation $y = 13$.

42. Use implicit differentiation to find the slope: $\frac{2}{49}x + \frac{2}{33}y\,y' = 0$. At the point $(7,0)$ $\frac{2}{7} + 0y' = 0$, or y' is undefined.. The tangent line is vertical and thus has equation $x = 7$.

43. Let the y-axis run through the center of the arch and the x-axis lie on the floor. Thus a = 5 and b = 4 and the equation of the arch is $\frac{x^2}{25} + \frac{y^2}{16} = 1$.
When y = 2, $\frac{x^2}{25} + \frac{(2)^2}{16} = 1$, so $x = \pm\frac{5\sqrt{3}}{2}$.
The width of the box can at most be $5\sqrt{3} \approx 8.66$ ft.

44. Let the y-axis run through the center of the arch and the x-axis lie on the floor.
The equation of the arch is $\frac{x^2}{25} + \frac{y^2}{16} = 1$.
When x = 2, $\frac{(2)^2}{25} + \frac{y^2}{16} = 1$, so $y = \pm\frac{4\sqrt{21}}{5}$.
The height at a distance of 2 feet to the right of the center is $\frac{4\sqrt{21}}{5} \approx 3.67$ ft.

45. The foci are at $(\pm c, 0)$.
$$c = \sqrt{a^2 - b^2}$$
$$\frac{a^2 - b^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$y^2 = \frac{b^4}{a^2},\ y = \pm\frac{b^2}{a}$$
Thus, the length of the latus rectum is $\frac{2b^2}{a}$.

46. The foci are at $(\pm c, 0)$
$$c = \sqrt{a^2 + b^2}$$
$$\frac{a^2 + b^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$y^2 = \frac{b^4}{a^2},\ y = \pm\frac{b^2}{a}$$
Thus, the length of the latus rectum is $\frac{2b^2}{a}$.

47. a = 18.09, b = 4.56,
$$c = \sqrt{(18.09)^2 - (4.56)^2} \approx 17.51$$
The comet's minimum distance from the sun is 18.09 – 17.51 $\approx$ 0.58 AU.

48. $a - c = 0.13, c = ae, a(1-e) = 0.13,$
$$a = \frac{0.13}{1 - 0.999925} \approx 1733$$
$$a + c = a(1+e) \approx 1733(1 + 0.999925) \approx 3466 \text{ AU}$$

49. $a - c = 4132$; $a + c = 4583$
$2a = 8715$; $a = 4357.5$
$c = 4357.5 - 4132 = 225.5$
$$e = \frac{c}{a} = \frac{225.5}{4357.5} \approx 0.05175$$

50. (See Example 5) Since $a + c = 49.31$ and $a - c = 29.65$, we conclude that $2a = 78.96,\ 2c = 19.66$ and so $a = 39.48,\ c = 9.83$. Thus $b = \sqrt{a^2 - c^2} = \sqrt{1462.0415} \approx 38.24$. So the major diameter $= 2a = 78.96$ and the minor diameter $= 2b = 76.48$.

51. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

Equation of tangent line at $(x_0,\ y_0)$:
$$\frac{xx_0}{4} + \frac{yy_0}{9} = 1$$
At (0, 6), $y_0 = \dfrac{3}{2}$.

When $y = \dfrac{3}{2}, \dfrac{x^2}{4} + \dfrac{1}{4} = 1,\quad x = \pm\sqrt{3}.$

The points of tangency are $\left(\sqrt{3}, \dfrac{3}{2}\right)$ and $\left(-\sqrt{3}, \dfrac{3}{2}\right)$

52. $\dfrac{x^2}{4} - \dfrac{y^2}{36} = 1$

Equation of tangent line at $(x_0,\ y_0)$:
$$\frac{xx_0}{4} - \frac{yy_0}{36} = 1$$
At (0, 6), $y_0 = -6$.

When $y = -6,\ \dfrac{x^2}{4} - \dfrac{36}{36} = 1, x = \pm 2\sqrt{2}.$

The points of tangency are $\left(2\sqrt{2}, -6\right)$ and $\left(-2\sqrt{2}, -6\right)$.

53. $2x^2 - 7y^2 - 35 = 0; 4x - 14yy' = 0$
$$y' = \frac{2x}{7y}; -\frac{2}{3} = \frac{2x}{7y}; x = -\frac{7y}{3}$$
Substitute $x = -\dfrac{7y}{3}$ into the equation of the hyperbola.
$$\frac{98}{9}y^2 - 7y^2 - 35 = 0,\ y = \pm 3$$
The coordinates of the points of tangency are $(-7, 3)$ and $(7, -3)$.

54. The slope of the line is $\dfrac{1}{\sqrt{2}}$.
$$x^2 + 2y^2 - 2 = 0; 2x + 4yy' = 0$$
$$y' = -\frac{x}{2y}; \frac{1}{\sqrt{2}} = -\frac{x}{2y}; x = -\sqrt{2}y$$
Substitute $x = -\sqrt{2}y$ into the equation of the ellipse.
$$2y^2 + 2y^2 - 2 = 0;\ y = \pm\frac{1}{\sqrt{2}}$$
The tangent lines are tangent at $\left(-1, \dfrac{1}{\sqrt{2}}\right)$ and $\left(1, -\dfrac{1}{\sqrt{2}}\right)$. The equations of the tangent lines are
$y - \dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}(x+1)$ and $y + \dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}(x-1)$ or $x - \sqrt{2}y + 2 = 0$ and $x - \sqrt{2}\,y - 2 = 0$.

55. $y = \pm b\sqrt{1 - \dfrac{x^2}{a^2}}$
$$A = 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx$$
Let $x = a \sin t$ then $dx = a \cos t\ d\ t$. Then the limits are 0 and $\dfrac{\pi}{2}$.
$$A = 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx = 4ab\int_0^{\pi/2} \cos^2 t\,dt$$
$$= 2ab\int_0^{\pi/2} (1 + \cos 2t)dt = 2ab\left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2}$$
$$= \pi ab$$

56. $x = \pm a\sqrt{1 - \dfrac{y^2}{b^2}}$
$$V = 2 \cdot \pi\int_0^b a^2\left(1 - \frac{y^2}{b^2}\right)dy = 2\pi a^2\left[y - \frac{y^3}{3b^2}\right]_0^b$$
$$= \frac{4\pi a^2 b}{3}$$

57. $y = \pm b\sqrt{\frac{x^2}{a^2} - 1}$

The vertical line at one focus is $x = \sqrt{a^2 + b^2}$.

$$V = \pi\int_a^{\sqrt{a^2+b^2}} \left(b\sqrt{\frac{x^2}{a^2} - 1}\right)^2 dx$$

$$= b^2\pi\int_a^{\sqrt{a^2+b^2}} \left(\frac{x^2}{a^2} - 1\right) dx = b^2\pi\left[\frac{x^3}{3a^2} - x\right]_a^{\sqrt{a^2+b^2}}$$

$$= b^2\pi\left[\frac{(a^2+b^2)^{3/2}}{3a^2} - \sqrt{a^2+b^2} + \frac{2}{3}a\right]$$

$$= \frac{\pi b^2}{3a^2}\left[(a^2+b^2)^{3/2} - 3a^2\sqrt{a^2+b^2} + 2a^3\right]$$

58. $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

$$V = 2\cdot\pi\int_0^a \left(b\sqrt{1 - \frac{x^2}{a^2}}\right)^2 dx$$

$$= 2\pi b^2\int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi b^2\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{4}{3}\pi ab^2$$

59. If one corner of the rectangle is at (x, y) the sides have length $2x$ and $2y$.

$$x = \pm a\sqrt{1 - \frac{y^2}{b^2}}$$

$$A = 4xy = 4ya\sqrt{1 - \frac{y^2}{b^2}} = 4a\sqrt{y^2 - \frac{y^4}{b^2}}$$

$$\frac{dA}{dy} = \frac{2a\left(2y - \frac{4y^3}{b^2}\right)}{\sqrt{y^2 - \frac{y^4}{b^2}}};\ \frac{dA}{dy} = 0 \text{ when}$$

$$y - \frac{2y^3}{b^2} = 0$$

$$y\left(1 - \frac{2y^2}{b^2}\right) = 0$$

$y = 0$ or $y = \pm\frac{b}{\sqrt{2}}$

The Second Derivative Test shows that $y = \frac{b}{\sqrt{2}}$ is a maximum.

$$x = a\sqrt{1 - \frac{\left(\frac{b}{\sqrt{2}}\right)^2}{b^2}} = \frac{a}{\sqrt{2}}$$

Therefore, the rectangle is $a\sqrt{2}$ by $b\sqrt{2}$.

60. Position the x-axis on the axis of the hyperbola such that the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ describes the hyperbola. The equation of the tangent line at (x_0, y_0) is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$. The equations of the asymptotes are $y = \pm\frac{b}{a}x$.

Substitute $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ into the equation of the tangent line.

$$\frac{x_0 x}{a^2} - \frac{y_0 x}{ab} = 1 \qquad \frac{x_0 x}{a^2} + \frac{y_0 x}{ab} = 1$$

$$x\left(\frac{bx_0 - ay_0}{a^2 b}\right) = 1 \qquad x\left(\frac{bx_0 + ay_0}{a^2 b}\right) = 1$$

$$x = \frac{a^2 b}{bx_0 - ay_0} \qquad x = \frac{a^2 b}{bx_0 + ay_0}$$

Thus the tangent line intersects the asymptotes at $\left(\frac{a^2 b}{bx_0 - ay_0}, \frac{ab^2}{bx_0 - ay_0}\right)$ and $\left(\frac{a^2 b}{bx_0 + ay_0}, -\frac{ab^2}{bx_0 + ay_0}\right)$.

Observe that $b^2 x_0 - a^2 y_0 = a^2 b^2$.

$$\frac{1}{2}\left(\frac{a^2 b}{bx_0 - ay_0} + \frac{a^2 b}{bx_0 + ay_0}\right)$$

$$= \frac{a^2 b^2 x_0}{b^2 x_0{}^2 - a^2 y_0{}^2} = x_0$$

$$\frac{1}{2}\left(\frac{ab^2}{bx_0 - ay_0} - \frac{ab^2}{bx_0 + ay_0}\right) = \frac{a^2 b^2 y_0}{b^2 x_0{}^2 - a^2 y_0{}^2} = y_0$$

Thus, the point of contact is midway between the two points of intersection.

61. Add the two equations to get $9y^2 = 675$.

$y = \pm 5\sqrt{3}$

Substitute $y = 5\sqrt{3}$ into either of the two equations and solve for $x \Rightarrow \quad x = \pm 6$

The point in the first quadrant is $\left(6, 5\sqrt{3}\right)$.

62. Substitute $x = 6 - 2y$ into $x^2 + 4y^2 = 20$.

$(6-2y)^2 + 4y^2 = 20$

$8y^2 - 24y + 16 = 0$

$y^2 - 3y + 2 = 0$

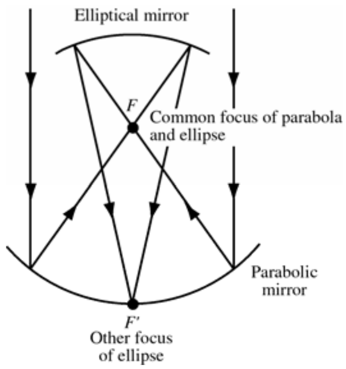
$(y-1)(y-2) = 0$

$y = 1$ or $y = 2$

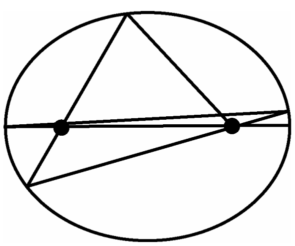
$x = 4$ or $x = 2$

The points of intersection are (4, 1) and (2, 2).

63.

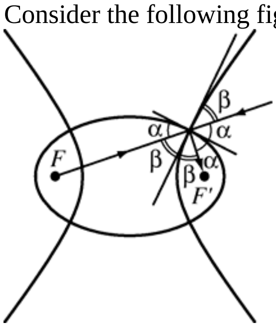


64. If the original path is not along the major axis, the ultimate path will approach the major axis.



65. Written response. Possible answer: the ball will follow a path that does not go between the foci.

66. Consider the following figure.



Observe that $2(\alpha + \beta) = 180°$, so $\alpha + \beta = 90°$. The ellipse and hyperbola meet at right angles.

67. Possible answer: Attach one end of a string to F and attach one end of another string to F'. Place a spool at a vertex. Tightly wrap both strings in the same direction around the spool. Insert a pencil through the spool. Then trace out a branch of the hyperbola by unspooling the strings while keeping both strings taut.

68. $\frac{|AP|}{u} = \frac{|AB|}{v} + \frac{|BP|}{u}$

$|AP| - |BP| = \frac{2uc}{v}$

Thus the curve is the right branch of the horizontal hyperbola with $a = \frac{uc}{v}$, so $b = \sqrt{\left(1 - \frac{u^2}{v^2}\right)}c.$

The equation of the curve is

$\frac{x^2}{\frac{u^2c^2}{v^2}} - \frac{y^2}{\left(1-\frac{u^2}{v^2}\right)c^2} = 1\left(x \ge \frac{uc}{v}\right).$

69. Let $P(x, y)$ be the location of the explosion.

$3|AP| = 3|BP| + 12$

$|AP| - |BP| = 4$

Thus, P lies on the right branch of the horizontal hyperbola with $a = 2$ and $c = 8$, so $b = 2\sqrt{15}.$

$\frac{x^2}{4} - \frac{y^2}{60} = 1$

Since $|BP| = |CP|$, the y-coordinate of P is 5.

$\frac{x^2}{4} - \frac{25}{60} = 1, x = \pm\sqrt{\frac{17}{3}}$

P is at $\left(\sqrt{\frac{17}{3}}, 5\right).$

70. $\lim_{x\to\infty}\left(\sqrt{x^2 - a^2} - x\right)$

$= \lim_{x\to\infty}\left[\frac{\left(\sqrt{x^2-a^2} - x\right)}{1} \cdot \frac{\left(\sqrt{x^2-a^2} + x\right)}{\left(\sqrt{x^2-a^2} + x\right)}\right]$

$= \lim_{x\to\infty} \frac{-a^2}{\sqrt{x^2-a^2} + x} = 0$

71. $2a = p + q,\ 2c = |p - q|$

$b^2 = a^2 - c^2 = \frac{(p+q)^2}{4} - \frac{(p-q)^2}{4} = pq$

$b = \sqrt{pq}$

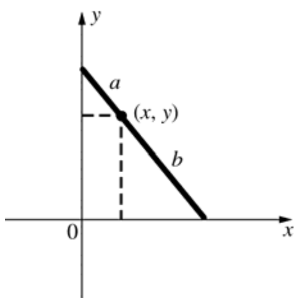
72. $x = a\cos t,\ y = a\sin t - b\sin t = (a-b)\sin t$

$\cos t = \frac{x}{a}, \sin t = \frac{y}{a-b}$

$\frac{x^2}{a^2} + \frac{y^2}{(a-b)^2} = 1$

Thus the coordinates of R at time t lie on an ellipse.

73. Let (x, y) be the coordinates of P as the ladder slides. Using a property of similar triangles, $\frac{x}{a} = \frac{\sqrt{b^2 - y^2}}{b}$.



Square both sides to get $\frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2}$ or $b^2x^2 + a^2y^2 = a^2b^2$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

74. Place the x-axis on the axis of the hyperbola such that the equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. One focus is at $(c, 0)$ and the asymptotes are $y = \pm\frac{b}{a}x$. The equations of the lines through the focus, perpendicular to the asymptotes, are $y = \pm\frac{a}{b}(x - c)$. Then solve for x in

$$\frac{b}{a}x = -\frac{a}{b}(x - c).$$

$$\frac{a^2 + b^2}{ab}x = \frac{ac}{b}$$

$$x = \frac{a^2c}{a^2 + b^2}$$

Since $c^2 = a^2 + b^2$, $x = \frac{a^2}{c}$. The equation of the directrix nearest the focus is $x = \frac{a^2}{c}$, so the line through a focus and perpendicular to an asymptote intersects that asymptote on the directrix nearest the focus.

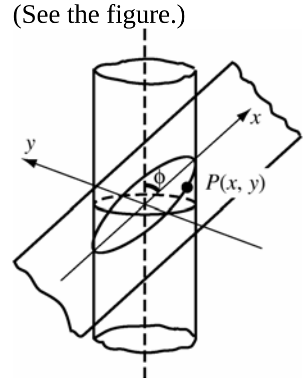
75. The equations of the hyperbolas are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$$

$$E = \frac{c}{b} = \frac{\sqrt{a^2 + b^2}}{b}$$

$$e^{-2} + E^{-2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

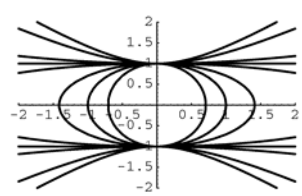
76. Position the x-axis on the plane so that it makes the angle ϕ with the axis of the cylinder and the y-axis is perpendicular to the axis of the cylinder. (See the figure.)



If $P(x, y)$ is a point on C, $(x\sin\phi)^2 + y^2 = r^2$ where r is the radius of the cylinder. Then

$$\frac{x^2}{\frac{r^2}{\sin^2\phi}} + \frac{y^2}{r^2} = 1.$$

77.



When $a < 0$, the conic is an ellipse. When $a > 0$, the conic is a hyperbola. When $a = 0$, the graph is two parallel lines.

10.3 Concepts Review

1. $\frac{a^2}{4}$

2. 14; ellipse

3. $\frac{A-C}{B}$

4. rotation, translation

Problem Set 10.3

1. $x^2 + y^2 - 2x + 2y + 1 = 0$
$(x^2 - 2x + 1) + (y^2 + 2y + 1) = -1 + 1 + 1$
$(x-1)^2 + (y+1)^2 = 1$
This is a circle.

2. $x^2 + y^2 + 6x - 2y + 6 = 0$
$(x^2 + 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$
$(x+3)^2 + (y-1)^2 = 4$
This is a circle.

3. $9x^2 + 4y^2 + 72x - 16y + 124 = 0$
$9(x^2 + 8x + 16) + 4(y^2 - 4y + 4) = -124 + 144 + 16$
$9(x+4)^2 + 4(y-2)^2 = 36$
This is an ellipse.

4. $16x^2 - 9y^2 + 192x + 90y - 495 = 0$
$16(x^2 + 12x + 36) - 9(y^2 - 10y + 25)$
$= 495 + 576 - 225$
$16(x+6)^2 - 9(y-5)^2 = 846$
This is a hyperbola.

5. $9x^2 + 4y^2 + 72x - 16y + 160 = 0$
$9(x^2 + 8x + 16) + 4(y^2 - 4y + 4) = -160 + 144 + 16$
$9(x+4)^2 + 4(y-2)^2 = 0$
This is a point.

6. $16x^2 + 9y^2 + 192x + 90y + 1000 = 0$
$16(x^2 + 12x + 36) + 9(y^2 + 10y + 25)$
$= -1000 + 576 + 225$
$16(x+6)^2 + 9(y+5)^2 = -199$
This is the empty set.

7. $y^2 - 5x - 4y - 6 = 0$
$(y^2 - 4y + 4) = 5x + 6 + 4$
$(y-2)^2 = 5(x+2)$
This is a parabola.

8. $4x^2 + 4y^2 + 8x - 28y - 11 = 0$
$4(x^2 + 2x + 1) + 4\left(y^2 - 7y + \frac{49}{4}\right) = 11 + 4 + 49$
$4(x+1)^2 + 4\left(y - \frac{7}{2}\right)^2 = 64$
This is a circle.

9. $3x^2 + 3y^2 - 6x + 12y + 60 = 0$
$3(x^2 - 2x + 1) + 3(y^2 + 4y + 4) = -60 + 3 + 12$
$3(x-1)^2 + 3(y+2)^2 = -45$
This is the empty set.

10. $4x^2 - 4y^2 - 2x + 2y + 1 = 0$
$4\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - 4\left(y^2 - \frac{1}{2}y + \frac{1}{16}\right) = -1 + \frac{1}{4} - \frac{1}{4}$
$4\left(x - \frac{1}{4}\right)^2 - 4\left(y - \frac{1}{4}\right)^2 = -1$
$4\left(y - \frac{1}{4}\right)^2 - 4\left(x - \frac{1}{4}\right)^2 = 1$
This is a hyperbola.

11. $4x^2 - 4y^2 + 8x + 12y - 5 = 0$
$4(x^2 + 2x + 1) - 4\left(y^2 - 3y + \frac{9}{4}\right) = 5 + 4 - 9$
$4(x+1)^2 - 4\left(y - \frac{3}{2}\right)^2 = 0$
This is two intersecting lines.

12. $4x^2 - 4y^2 + 8x + 12y - 6 = 0$
$4(x^2 + 2x + 1) - 4\left(y^2 - 3y + \frac{9}{4}\right) = 6 + 4 - 9$
$4(x+1)^2 - 4\left(y - \frac{3}{2}\right)^2 = 1$
This is a hyperbola.

13. $4x^2 - 24x + 36 = 0$
$4(x^2 - 6x + 9) = -36 + 36$
$4(x-3)^2 = 0$
This is a line.

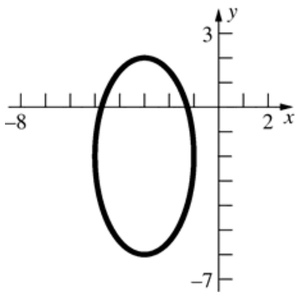
14. $4x^2 - 24x + 35 = 0$

$4(x^2 - 6x + 9) = -35 + 36$

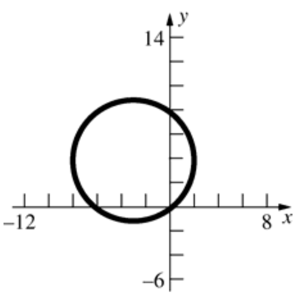
$4(x-3)^2 = 1$

This is two parallel lines.

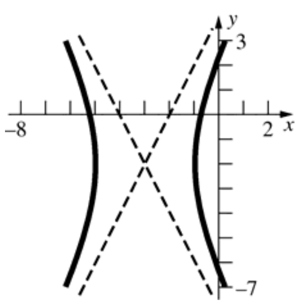
15. $\dfrac{(x+3)^2}{4} + \dfrac{(y+2)^2}{16} = 1$



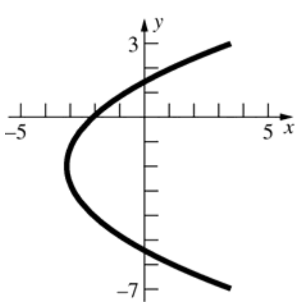
16. $(x+3)^2 + (y-4)^2 = 25$



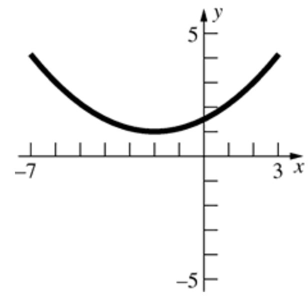
17. $\dfrac{(x+3)^2}{4} - \dfrac{(y+2)^2}{16} = 1$



18. $4(x+3) = (y+2)^2$



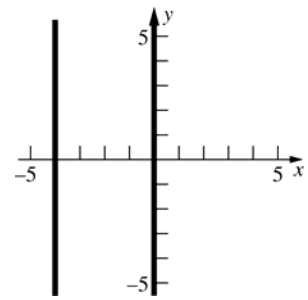
19. $(x+2)^2 = 8(y-1)$



20. $(x+2)^2 = 4$

$x + 2 = \pm 2$

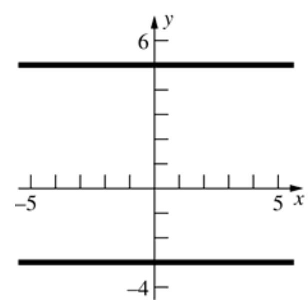
$x = -4, x = 0$



21. $(y-1)^2 = 16$

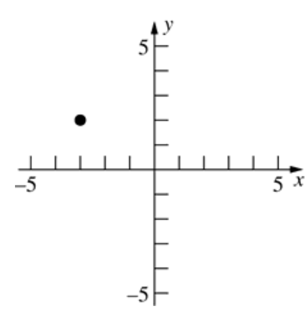
$y - 1 = \pm 4$

$y = 5, y = -3$



22. $\dfrac{(x+3)^2}{4} + \dfrac{(y-2)^2}{8} = 0$

$(-3, 2)$

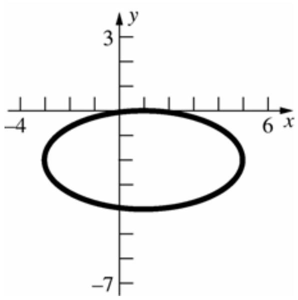


23. $x^2+4y^2-2x+16y+1=0$

$(x^2-2x+1)+4(y^2+4y+4)=-1+1+16$

$(x-1)^2+4(y+2)^2=16$

$$\frac{(x-1)^2}{16}+\frac{(y+2)^2}{4}=1$$

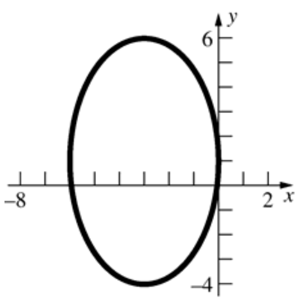


24. $25x^2+9y^2+150x-18y+9=0$

$25(x^2+6x+9)+9(y^2-2y+1)=-9+225+9$

$25(x+3)^2+9(y-1)^2=225$

$$\frac{(x+3)^2}{9}+\frac{(y-1)^2}{25}=1$$

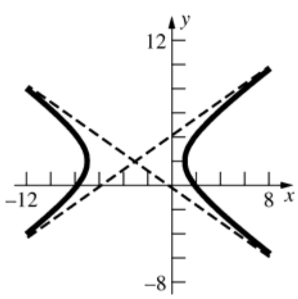


25. $9x^2-16y^2+54x+64y-127=0$

$9(x^2+6x+9)-16(y^2-4y+4)=127+81-64$

$9(x+3)^2-16(y-2)^2=144$

$$\frac{(x+3)^2}{16}-\frac{(y-2)^2}{9}=1$$

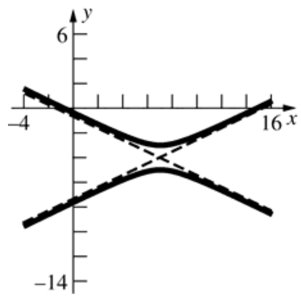


26. $x^2-4y^2-14x-32y-11=0$

$(x^2-14x+49)-4(y^2+8y+16)=11+49-64$

$4(y+4)^2-(x-7)^2=4$

$$(y+4)^2-\frac{(x-7)^2}{4}=1$$

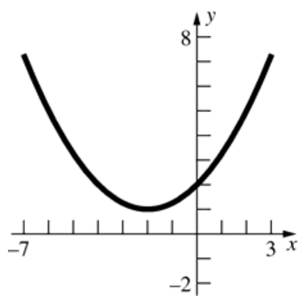


27. $4x^2+16x-16y+32=0$

$4(x^2+4x+4)=16y-32+16$

$4(x+2)^2=16(y-1)$

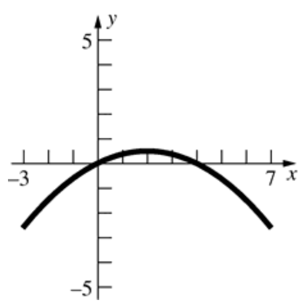
$(x+2)^2=4(y-1)$



28. $x^2-4x+8y=0$

$x^2-4x+4=-8y+4$

$$(x-2)^2=-8\left(y-\frac{1}{2}\right)$$



29. $2y^2 - 4y - 10x = 0$

$2(y^2 - 2y + 1) = 10x + 2$

$(y-1)^2 = 5\left(x + \frac{1}{5}\right)$

$(y-1)^2 = 4\left(\frac{5}{4}\right)\left(x + \frac{1}{5}\right)$

Horizontal parabola, $p = \frac{5}{4}$

Vertex $\left(-\frac{1}{5}, 1\right)$; Focus is at $\left(\frac{21}{20}, 1\right)$ and directrix is at $x = -\frac{29}{20}$.

30. $-9x^2 + 18x + 4y^2 + 24y = 9$

$-9(x^2 - 2x + 1) + 4(y^2 + 6y + 9) = 9 - 9 + 36$

$4(y+3)^2 - 9(x-1)^2 = 36$

$\frac{(y+3)^2}{9} - \frac{(x-1)^2}{4} = 1$

$a^2 = 9, a = 3$

The distance between the vertices is $2a = 6$.

31. $16(x-1)^2 + 25(y+2)^2 = 400$

$\frac{(x-1)^2}{25} + \frac{(y+2)^2}{16} = 1$

Horizontal ellipse, center (1, –2), $a = 5$, $b = 4$,

$c = \sqrt{25 - 16} = 3$

Foci are at (–2, –2) and (4, –2).

32. $x^2 - 6x + 4y + 3 = 0$

$x^2 - 6x + 9 = -4y - 3 + 9$

$(x-3)^2 = -4\left(y - \frac{3}{2}\right)$

Vertical parabola, opens downward, vertex $\left(3, \frac{3}{2}\right)$, $p = 1$

Focus is at $\left(3, \frac{1}{2}\right)$ and directrix is $y = \frac{5}{2}$.

33. $a = 5, b = 4$

$\frac{(x-5)^2}{25} + \frac{(y-1)^2}{16} = 1$

34. Horizontal hyperbola, $a = 2$, $c = 3$,

$b = \sqrt{9-4} = \sqrt{5}$

$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{5} = 1$

35. Vertical parabola, opens upward, $p = 5 - 3 = 2$

$(x-2)^2 = 4(2)(y-3)$

$(x-2)^2 = 8(y-3)$

36. An equation for the ellipse can be written in the form $\frac{(x-2)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$.

Substitute the points into the equation.

$\frac{16}{a^2} = 1, \frac{4}{b^2} = 1$

Therefore, $a = 4$ and $b = 2$.

$\frac{(x-2)^2}{16} + \frac{(y-3)^2}{4} = 1$

37. Vertical hyperbola, center (0, 3), $2a = 6$, $a = 3$,

$c = 5,\ b = \sqrt{25-9} = 4$

$\frac{(y-3)^2}{9} - \frac{x^2}{16} = 1$

38. Vertical ellipse; center (2, 6), $a = 8$, $c = 6$,

$b = \sqrt{64-36} = \sqrt{28}$

$\frac{(x-2)^2}{28} + \frac{(y-6)^2}{64} = 1$

39. Horizontal parabola, opens to the left

Vertex (6, 5), $p = \frac{10-2}{2} = 4$

$(y-5)^2 = -4(4)(x-6)$

$(y-5)^2 = -16(x-6)$

40. Vertical parabola, opens downward, $p = 1$

$(x-2)^2 = -4(y-6)$

41. Horizontal ellipse, center (0, 2), $c = 2$

Since it passes through the origin and center is at (0, 2), $b = 2$.

$a = \sqrt{4+4} = \sqrt{8}$

$\frac{x^2}{8} + \frac{(y-2)^2}{4} = 1$

42. Vertical hyperbola, center (0, 2), $c = 2$,

$b^2 = 4 - a^2$

An equation for the hyperbola can be written in the form $\frac{(y-2)^2}{a^2} - \frac{x^2}{4-a^2} = 1$.

Substitute (12, 9) into the equation.

$\frac{49}{a^2} - \frac{144}{4-a^2} = 1$

$49(4-a^2) - 144a^2 = a^2(4-a^2)$

$a^4 - 197a^2 + 196 = 0$

$(a^2 - 196)(a^2 - 1) = 0$

$a^2 = 196, a^2 = 1$

Since $a < c$, $a = 1$, $b = \sqrt{4-1} = \sqrt{3}$

$(y-2)^2 - \frac{x^2}{3} = 1$

43. $x^2 + xy + y^2 = 6$

$\cot 2\theta = 0$

$2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

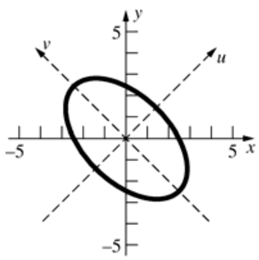
$x = u\frac{\sqrt{2}}{2} - v\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(u - v)$

$y = u\frac{\sqrt{2}}{2} + v\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(u + v)$

$\frac{1}{2}(u - v)^2 + \frac{1}{2}(u - v)(u + v) + \frac{1}{2}(u + v)^2 = 6$

$\frac{3}{2}u^2 + \frac{1}{2}v^2 = 6$

$\frac{u^2}{4} + \frac{v^2}{12} = 1$



44. $3x^2 + 10xy + 3y^2 + 10 = 0$

$\cot 2\theta = 0,\ 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

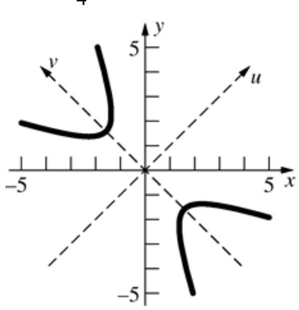
$x = \frac{\sqrt{2}}{2}(u - v)$

$y = \frac{\sqrt{2}}{2}(u + v)$

$\frac{3}{2}(u - v)^2 + 5(u - v)(u + v) + \frac{3}{2}(u + v)^2 + 10 = 0$

$8u^2 - 2v^2 = -10$

$\frac{v^2}{5} - \frac{u^2}{\frac{5}{4}} = 1$



45. $4x^2 + xy + 4y^2 = 56$

$\cot 2\theta = 0,\ 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

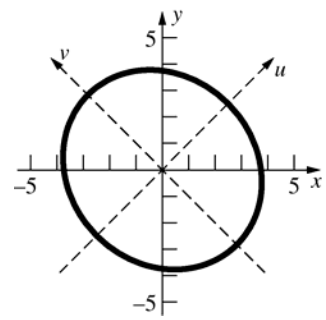
$x = \frac{\sqrt{2}}{2}(u - v)$

$y = \frac{\sqrt{2}}{2}(u + v)$

$2(u - v)^2 + \frac{1}{2}(u - v)(u + v) + 2(u + v)^2 = 56$

$\frac{9}{2}u^2 + \frac{7}{2}v^2 = 56$

$\frac{u^2}{\frac{112}{9}} + \frac{v^2}{16} = 1$



46. $4xy - 3y^2 = 64$

$\cot 2\theta = \frac{3}{4},\ r = 5$

$\cos 2\theta = \frac{3}{5}$

$\cos\theta = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$

$\sin\theta = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$

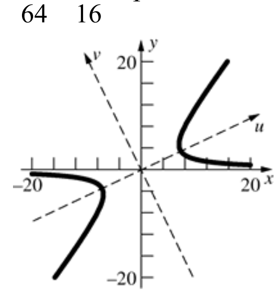
$x = \frac{1}{\sqrt{5}}(2u - v)$

$y = \frac{1}{\sqrt{5}}(u + 2v)$

$\frac{4}{5}(2u - v)(u + 2v) - \frac{3}{5}(u + 2v)^2 = 64$

$u^2 - 4v^2 = 64$

$\frac{u^2}{64} - \frac{v^2}{16} = 1$



47. $-\frac{1}{2}x^2 + 7xy - \frac{1}{2}y^2 - 6\sqrt{2}x - 6\sqrt{2}y = 0$

$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

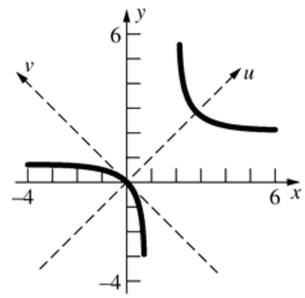
$x = \frac{\sqrt{2}}{2}(u-v)$; $\quad y = \frac{\sqrt{2}}{2}(u+v)$

$-\frac{1}{4}(u-v)^2 + \frac{7}{2}(u-v)(u+v) - \frac{1}{4}(u+v)^2 - 6(u-v) - 6(u+v) = 0$

$3u^2 - 4v^2 - 12u = 0$

$3(u^2 - 4u + 4) - 4v^2 = 12$

$\frac{(u-2)^2}{4} - \frac{v^2}{3} = 1$



48. $\frac{3}{2}x^2 + xy + \frac{3}{2}y^2 + \sqrt{2}x + \sqrt{2}y = 13$

$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

$x = \frac{\sqrt{2}}{2}(u-v)$; $\quad y = \frac{\sqrt{2}}{2}(u+v)$

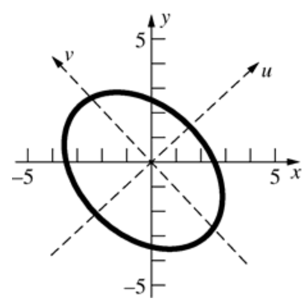
$\frac{3}{4}(u-v)^2 + \frac{1}{2}(u-v)(u+v) + \frac{3}{4}(u+v)^2 + (u-v) + (u+v) = 13$

$2u^2 + v^2 + 2u = 13$

$2\left(u^2 + u + \frac{1}{4}\right) + v^2 = 13 + \frac{1}{2}$

$2\left(u + \frac{1}{2}\right)^2 + v^2 = \frac{27}{2}$

$\frac{\left(u+\frac{1}{2}\right)^2}{\frac{27}{4}} + \frac{v^2}{\frac{27}{2}} = 1$



49. $A = 4,\ B = -3,\ C = D = E = 0,\ F = -18$

$\cot 2\theta = \frac{4-0}{-3} = -\frac{4}{3}$

Since $0 \le 2\theta \le \pi$, $\sin 2\theta$ is positive, so $\cos 2\theta$ is negative; using a 3-4-5 right triangle, we conclude $\cos 2\theta = -\frac{4}{5}$. Thus

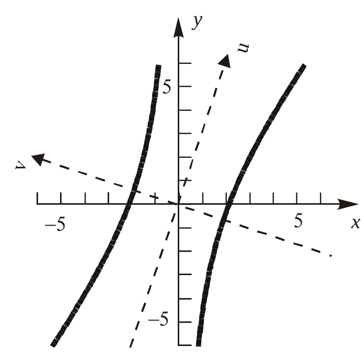
$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-(-4/5)}{2}} = \frac{3\sqrt{10}}{10}$ and

$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+(-4/5)}{2}} = \frac{\sqrt{10}}{10}$. Rotating through the angle

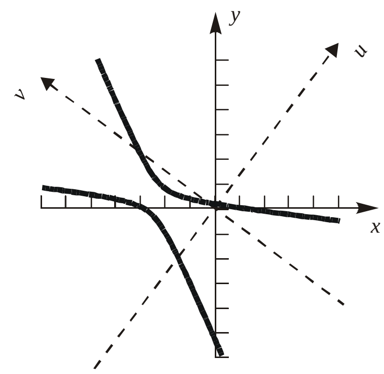
$\theta = \frac{1}{2}\cos^{-1}(-0.8) = 71.6^\circ$, we have

$4\left(\frac{\sqrt{10}}{10}u - \frac{3\sqrt{10}}{10}v\right)^2 - 3\left(\frac{\sqrt{10}}{10}u - \frac{3\sqrt{10}}{10}v\right)\left(\frac{3\sqrt{10}}{10}u + \frac{\sqrt{10}}{10}v\right) = 18$ or

$45v^2 - 5u^2 = 180$ or $\frac{v^2}{4} - \frac{u^2}{36} = 1$. This is a hyperbola in standard position in the uv-system; its axis is the v-axis, and $a = 2, b = 6$.



50. $A = 11,\ B = 96,\ C = 39,\ D = 240,\ E = 570,\ F = 875$



$\cot 2\theta = \frac{11-39}{96} = -\frac{7}{24}$

Since $0 \le 2\theta \le \pi$, $\cos 2\theta$ is negative; using a 7-24-25 right triangle, we conclude $\cos 2\theta = -\frac{7}{25}$.

Thus $\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-(-7/25)}{2}} = \frac{4}{5}$ and

$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+(-7/25)}{2}} = \frac{3}{5}$.

Rotating through the angle $\theta = \frac{1}{2}\cos^{-1}(-0.28) = 53.13^\circ$, we have

$11\left(\frac{3}{5}u - \frac{4}{5}v\right)^2 + 96\left(\frac{3}{5}u - \frac{4}{5}v\right)\left(\frac{4}{5}u + \frac{3}{5}v\right) +$

$39\left(\frac{4}{5}u + \frac{3}{5}v\right)^2 + 240\left(\frac{3}{5}u - \frac{4}{5}v\right) + 570\left(\frac{4}{5}u + \frac{3}{5}v\right) = -875$

or

$3u^2 - v^2 + 24u + 6v = -35$

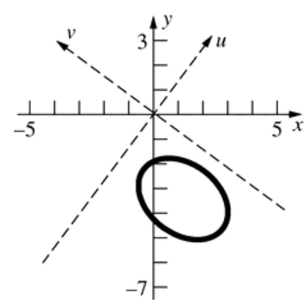
$3(u^2 + 8u + 16) - (v^2 - 6v + 9) = -35 + 48 - 9$

$3(u+4)^2 - (v-3)^2 = 4$

$\frac{(u+4)^2}{4/3} - \frac{(v-3)^2}{4} = 1$

This is a hyperbola in standard position in the *uv*-system; its axis is the *u*-axis, its center is $(u,v) = (-4,3)$ and $a = \frac{2\sqrt{3}}{3}$, $b = 2$.

51. $34x^2 + 24xy + 41y^2 + 250y = -325$



$\cot 2\theta = -\frac{7}{24}, r = 25$

$\cos 2\theta = -\frac{7}{25}$

$\cos\theta = \sqrt{\frac{1-\frac{7}{25}}{2}} = \frac{3}{5}$; $\quad \sin\theta = \sqrt{\frac{1+\frac{7}{25}}{2}} = \frac{4}{5}$

$x = \frac{1}{5}(3u - 4v)$; $\quad y = \frac{1}{5}(4u + 3v)$

$\frac{34}{25}(3u-4v)^2 + \frac{24}{25}(3u-4v)(4u+3v) + \frac{41}{25}(4u+3v)^2 + 50(4u+3v) = -325$

$50u^2 + 25v^2 + 200u + 150v = -325$

$2u^2 + v^2 + 8u + 6v = -13$

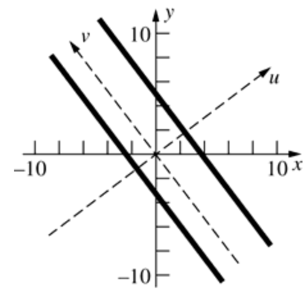
$2(u^2 + 4u + 4) + (v^2 + 6v + 9) = -13 + 8 + 9$

$2(u+2)^2 + (v+3)^2 = 4$

$\frac{(u+2)^2}{2} + \frac{(v+3)^2}{4} = 1$

This is an ellipse in standard position in the *uv*-system, with major axis parallel to the *v*-axis. Its center is $(u,v) = (-2,-3)$ and $a = 2$, $b = \sqrt{2}$.

52. $16x^2+24xy+9y^2-20x-15y-150=0$



$\cot 2\theta=\frac{7}{24},\ r=25$

$\cos 2\theta=\frac{7}{25}$

$\cos\theta=\sqrt{\frac{1+\frac{7}{25}}{2}}=\frac{4}{5};\quad \sin\theta=\sqrt{\frac{1-\frac{7}{25}}{2}}=\frac{3}{5}$

$x=\frac{1}{5}(4u-3v);\quad y=\frac{1}{5}(3u+4v)$

$\frac{16}{25}(4u-3v)^2+\frac{24}{25}(4u-3v)(3u+4v)+\frac{9}{25}(3u+4v)^2\ 25u^2-25u=150$

$-4(4u-3v)-3(3u+4v)=150$

$u^2-u=6$

$u^2-u+\frac{1}{4}=6+\frac{1}{4}$

$\left(u-\frac{1}{2}\right)^2=\frac{25}{4}$

The graph consists of the two parallel lines $u=-2$ and $u=3$.

53. a. If C is a vertical parabola, the equation for C can be written in the form $y=ax^2+bx+c$. Substitute the three points into the equation.

$2=a-b+c$
$0=c$
$6=9a+3b+c$

Solve the system to get $a=1$, $b=-1$, $c=0$.

$y=x^2-x$

b. If C is a horizontal parabola, an equation for C can be written in the form $x=ay^2+by+c$. Substitute the three points into the equation.

$-1=4a+2b+c$
$0=c$
$3=36a+6b+c$

Solve the system to get $a=\frac{1}{4}$, $b=-1$, $c=0$.

$x=\frac{1}{4}y^2-y$

c. If C is a circle, an equation for C can be written in the form $(x-h)^2+(y-k)^2=r^2$. Substitute the three points into the equation.

$(-1-h)^2+(2-k)^2=r^2$

$h^2+k^2=r^2$

$(3-h)^2+(6-k)^2=r^2$

Solve the system to get $h=\frac{5}{2}, k=\frac{5}{2}$, and $r^2=\frac{25}{2}$.

$\left(x-\frac{5}{2}\right)^2+\left(y-\frac{5}{2}\right)^2=\frac{25}{2}$

54. Let (p, q) be the coordinates of P. By properties of similar triangles and since $\alpha=\frac{|KP|}{|AP|}, \alpha=\frac{x-p}{a-p}$ and $\alpha=\frac{y-q}{b-q}$. Solve for p and q to get $p=\frac{x-\alpha a}{1-\alpha}$ and $q=\frac{y-\alpha b}{1-\alpha}$. Since $P(p, q)$ is a point on a circle of radius r centered at (0, 0), $p^2+q^2=r^2$

Therefore, $\left(\frac{x-\alpha a}{1-\alpha}\right)^2+\left(\frac{y-\alpha b}{1-\alpha}\right)^2=r^2$ or $(x-\alpha a)^2+(y-\alpha b)^2=(1-\alpha)^2r^2$ is the equation for C.

55. $y^2 = Lx + Kx^2$

$$K\left(x^2 + \frac{L}{K}x + \frac{L^2}{4K^2}\right) - y^2 = \frac{L^2}{4K}$$

$$K\left(x + \frac{L}{2K}\right)^2 - y^2 = \frac{L^2}{4K}$$

$$\frac{(x+\frac{L}{2K})^2}{\frac{L^2}{4K^2}} - \frac{y^2}{\frac{L^2}{4K}} = 1$$

If $K < -1$, the conic is a vertical ellipse. If $K = -1$, the conic is a circle. If $-1 < K < 0$, the conic is a horizontal ellipse. If $K = 0$, the original equation is $y^2 = Lx$, so the conic is a horizontal parabola. If $K > 0$, the conic is a horizontal hyperbola.

If $-1 < K < 0$ (a horizontal ellipse) the length of the latus rectum is (see problem 45, Section 10.2)

$$\frac{2b^2}{a} = 2\frac{L^2}{4|K|}\frac{1}{\frac{|L|}{2|K|}} = |L|$$

From general considerations, the result for a vertical ellipse is the same as the one just obtained.
For $K = -1$ (a circle) we have

$$\left(x - \frac{L}{2}\right)^2 + y^2 = \frac{L^2}{4} \Rightarrow 2\frac{|L|}{2} = |L|$$

If $K = 0$ (a horizontal parabola) we have $y^2 = Lx; y^2 = 4\frac{L}{4}x; p = \frac{L}{4}$, and the latus rectum is

$$2\sqrt{Lp} = 2\sqrt{L\frac{L}{4}} = |L|.$$

If $K > 0$ (a horizontal hyperbola) we can use the result of Problem 46, Section 10.2. The length of the latus rectum is $\frac{2b^2}{a}$, which is equal to $|L|$.

56. Parabola: horizontal parabola, opens to the right, $p = c - a,\ y^2 = 4(c-a)(x-a)$

Hyperbola: horizontal hyperbola, $b^2 = c^2 - a^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ Now show that y^2 (hyperbola) is greater than y^2 (parabola).

$$\frac{b^2}{a^2}(x^2 - a^2) = \frac{c^2 - a^2}{a^2}(x^2 - a^2)$$

$$= \frac{(c+a)(c-a)}{a^2}(x+a)(x-a)$$

$$\frac{(c+a)(x+a)}{a^2}(c-a)(x-a) > \frac{(2a)(2a)}{a^2}(c-a)(x-a)$$

$$\frac{(2a)(2a)}{a^2}(c-a)(x-a) = 4(c-a)(x-a)$$

$c + a > 2a$ and $x + a > 2a$ since $c > a$ and $x > a$ except at the vertex.

57. $x = u\cos\alpha - v\sin\alpha$
$y = u\sin\alpha + v\cos\alpha$
$(u\cos\alpha - v\sin\alpha)\cos\alpha + (u\sin\alpha + v\cos\alpha)\sin\alpha = d$
$u(\cos^2\alpha + \sin^2\alpha) = d$
$u = d$
Thus, the perpendicular distance from the origin is d.

58. $x=\frac{\sqrt{2}}{2}(u-v)$; $y=\frac{\sqrt{2}}{2}(u+v)$

$$\left[\frac{\sqrt{2}}{2}(u-v)\right]^{1/2}+\left[\frac{\sqrt{2}}{2}(u+v)\right]^{1/2}=a^{1/2}$$

$$\frac{\sqrt{2}}{2}(u-v)+2\left[\frac{1}{2}(u-v)(u+v)\right]^{1/2}+\frac{\sqrt{2}}{2}(u+v)=a$$

$$\sqrt{2}u+\sqrt{2}(u^2-v^2)^{1/2}=a$$

$$\sqrt{2}(u^2-v^2)^{1/2}=a-\sqrt{2}u$$

$$2(u^2-v^2)=a^2-2\sqrt{2}au+2u^2$$

$$v^2=\sqrt{2}au-\frac{1}{2}a^2$$

The corresponding curve is a parabola with $x > 0$ and $y > 0$.

59. $x = u\cos\theta - v\sin\theta$; $y = u\sin\theta + v\cos\theta$

$$x(\cos\theta)+y(\sin\theta)=(u\cos^2\theta-v\cos\theta\sin\theta)+(u\sin^2\theta+v\cos\theta\sin\theta)=u$$

$$x(-\sin\theta)+y(\cos\theta)=(-u\cos\theta\sin\theta+v\sin^2\theta)+(u\cos\theta\sin\theta+v\cos^2\theta)=v$$

Thus, $u = x\cos\theta + y\sin\theta$ and $v = -x\sin\theta + y\cos\theta$.

60. $u=5\cos 60°-3\sin 60°=\frac{5}{2}-\frac{3\sqrt{3}}{2}$; $v=-5\sin 60°-3\cos 60°=-\frac{5\sqrt{3}}{2}-\frac{3}{2}$

$$(u,\ v)=\left(\frac{5}{2}-\frac{3\sqrt{3}}{2},-\frac{5\sqrt{3}}{2}-\frac{3}{2}\right)$$

61. Rotate to eliminate the *xy*-term.

$$x^2+14xy+49y^2=100$$

$$\cot 2\theta=-\frac{24}{7}$$

$$\cos 2\theta=-\frac{24}{25}$$

$$\cos\theta=\sqrt{\frac{1-\frac{24}{25}}{2}}=\frac{1}{5\sqrt{2}};\quad \sin\theta=\sqrt{\frac{1+\frac{24}{25}}{2}}=\frac{7}{5\sqrt{2}}$$

$$x=\frac{1}{5\sqrt{2}}(u-7v);\quad y=\frac{1}{5\sqrt{2}}(7u+v)$$

$$\frac{1}{50}(u-7v)^2+\frac{14}{50}(u-7v)(7u+v)+\frac{49}{50}(7u+v)^2=100$$

$$50u^2=100$$

$$u^2=2$$

$$u=\pm\sqrt{2}$$

Thus the points closest to the origin in *uv*-coordinates are $\left(\sqrt{2},0\right)$ and $\left(-\sqrt{2},0\right)$.

$$x=\frac{1}{5\sqrt{2}}\left(\sqrt{2}\right)=\frac{1}{5}\ \text{or}\ x=\frac{1}{5\sqrt{2}}\left(-\sqrt{2}\right)=-\frac{1}{5}$$

$$y=\frac{1}{5\sqrt{2}}\left(7\sqrt{2}\right)=\frac{7}{5}\ \text{or}\ y=\frac{1}{5\sqrt{2}}\left(-7\sqrt{2}\right)=-\frac{7}{5}$$

The points closest to the origin in *xy*-coordinates are $\left(\frac{1}{5},\frac{7}{5}\right)$ and $\left(-\frac{1}{5},-\frac{7}{5}\right)$.

62. $x = u \cos \theta - v \sin \theta$
$y = u \sin \theta + v \cos \theta$

$$Ax^2 = A(u\cos\theta - v\sin\theta)^2 = A(u^2\cos^2\theta - 2uv\cos\theta\sin\theta + v^2\sin^2\theta)$$

$$Bxy = B(u\cos\theta - v\sin\theta)(u\sin\theta + v\cos\theta) = B(u^2\cos\theta\sin\theta + uv(\cos^2\theta - \sin^2\theta) - v^2\cos\theta\sin\theta)$$

$$Cy^2 = C(u\sin\theta + v\cos\theta)^2 = C(u^2\sin^2\theta + 2uv\cos\theta\sin\theta + v^2\cos^2\theta)$$

$$Ax^2 + Bxy + Cy^2 = (A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta)u^2 + (-2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\cos\theta\sin\theta)uv$$
$$+(A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta)v^2$$

Thus, $a = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$ and $c = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta$.

$$a + c = A(\cos^2\theta + \sin^2\theta) + B(\cos\theta\sin\theta - \cos\theta\sin\theta) + C(\sin^2\theta + \cos^2\theta) = A + C$$

63. From Problem 62, $a = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$,

$b = -2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\cos\theta\sin\theta$, and

$c = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta$.

$$b^2 = B^2\cos^4\theta + 4(-AB + BC)\cos^3\theta\sin\theta + 2(2A^2 - B^2 - 4AC + 2C^2)\cos^2\theta\sin^2\theta$$
$$+4(AB - BC)\cos\theta\sin^3\theta + B^2\sin^4\theta$$

$$4ac = 4AC\cos^4\theta + 4(-AB + BC)\cos^3\theta\sin\theta + 4(A^2 - B^2 + C^2)\cos^2\theta\sin^2\theta + 4(AB - BC)\cos\theta\sin^3\theta + 4AC\sin^4\theta$$

$$b^2 - 4ac = (B^2 - 4AC)\cos^4\theta + 2(B^2 - 4AC)\cos^2\theta\sin^2\theta + (B^2 - 4AC)\sin^4\theta$$
$$= (B^2 - 4AC)(\cos^2\theta)(\cos^2\theta + \sin^2\theta) + (B^2 - 4AC)(\sin^2\theta)(\cos^2\theta + \sin^2\theta)$$
$$= (B^2 - 4AC)(\cos^2\theta + \sin^2\theta) = B^2 - 4AC$$

64. By choosing an appropriate angle of rotation, the second-degree equation can be written in the form $au^2 + cv^2 + du + ev + f = 0.$ From Problem 63, $-4ac = B^2 - 4AC$.

a. If $B^2 - 4AC = 0,$ then $4ac = 0$, so the graph is a parabola or limiting form.

b. If $B^2 - 4AC < 0,$ then $4ac > 0$, so the graph is an ellipse or limiting form.

c. If $B^2 - 4AC > 0,$ then $4ac < 0$, so the graph is a hyperbola or limiting form.

65. a. From Problem 63, $-4ac = B^2 - 4AC = -\Delta$ or $\frac{1}{ac} = \frac{4}{\Delta}.$

b. From Problem 62, $a + c = A + C$.
$\frac{1}{a} + \frac{1}{c} = \frac{a+c}{ac} = \frac{4(A+C)}{\Delta}$

c. $\frac{2}{\Delta}\left(A + C \pm \sqrt{(A-C)^2 + B^2}\right)$

$$= \frac{2}{\Delta}\left[\frac{\Delta}{4}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \sqrt{A^2 + 2AC + C^2 + B^2 - 4AC}\right]$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{2}{\Delta}\sqrt{(A+C)^2 - \Delta}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{2}{\Delta}\sqrt{\frac{\Delta^2}{16}\left(\frac{1}{a} + \frac{1}{c}\right)^2 - \Delta}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{a} + \frac{1}{c}\right)^2 - 4\left(\frac{4}{\Delta}\right)}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \sqrt{\frac{1}{a^2} + \frac{2}{ac} + \frac{1}{c^2} - 4\left(\frac{1}{ac}\right)}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{a} - \frac{1}{c}\right)^2}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c} \pm \left|\frac{1}{a} - \frac{1}{c}\right|\right)$$

The two values are $\frac{1}{a}$ and $\frac{1}{c}$.

66. $Ax^2 + Bxy + Cy^2 = 1$ can be transformed to $au^2 + cv^2 = 1$. Since $4ac = \Delta > 0$, the graph is an ellipse or a limiting form. $\frac{1}{a} + \frac{1}{c} = \frac{4}{\Delta}(A + C) > 0$, so $a > 0$ and $c > 0$. Thus, the graph is an ellipse (or circle).

The area of $au^2 + cv^2 = 1$ is

$$\pi\frac{1}{\sqrt{ac}} = \pi\sqrt{\frac{4}{\Delta}} = \frac{2\pi}{\sqrt{\Delta}}.$$

67. $\cot 2\theta = 0,\ \theta = \frac{\pi}{4}$

$$x = \frac{\sqrt{2}}{2}(u - v)$$

$$y = \frac{\sqrt{2}}{2}(u + v)$$

$$\frac{1}{2}(u - v)^2 + \frac{B}{2}(u - v)(u + v) + \frac{1}{2}(u + v)^2 = 1$$

$$\frac{2 + B}{2}u^2 + \frac{2 - B}{2}v^2 = 1$$

a. The graph is an ellipse if $\frac{2 + B}{2} > 0$ and $\frac{2 - B}{2} > 0$, so $-2 < B < 2$.

b. The graph is a circle if $\frac{2 + B}{2} = \frac{2 - B}{2}$, so $B = 0$.

c. The graph is a hyperbola if $\frac{2 + B}{2} > 0$ and $\frac{2 - B}{2} < 0$ or if $\frac{2 + B}{2} < 0$ and $\frac{2 - B}{2} > 0$, so $B < -2$ or $B > 2$.

d. The graph is two parallel lines if $\frac{2 + B}{2} = 0$ or $\frac{2 - B}{2} = 0$, so $B = \pm 2$.

68. $\Delta = 4(25)(1) - 8^2 = 36$

Since $c < a$, $\frac{1}{c} = \frac{2}{\Delta}\left(A + C + \sqrt{(A - C)^2 + B^2}\right)$

$$\frac{1}{c} = \frac{1}{18}\left(25 + 1 + \sqrt{24^2 + 8^2}\right) = \frac{1}{9}\left(13 + 4\sqrt{10}\right)$$

$$c = \left(\frac{9}{13 + 4\sqrt{10}}\right)\left(\frac{13 - 4\sqrt{10}}{13 - 4\sqrt{10}}\right) = 13 - 4\sqrt{10}$$

$$\frac{2\pi}{\sqrt{\Delta}} = \frac{2\pi}{6} = \frac{\pi}{3}$$

Thus, the distance between the foci is $26 - 8\sqrt{10}$ and the area is $\frac{\pi}{3}$.

69. From Figure 6 it is clear that $v = r\sin\phi$ and $u = r\cos\phi$.

Also noting that $y = r\sin(\theta + \phi)$ leads us to

$$y = r\sin(\theta + \phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$$
$$= (r\cos\phi)(\sin\theta) + (r\sin\phi)(\cos\theta)$$
$$= u\sin\theta + v\cos\theta$$

10.4 Concepts Review

1. simple; closed; simple

2. parametric; parameter

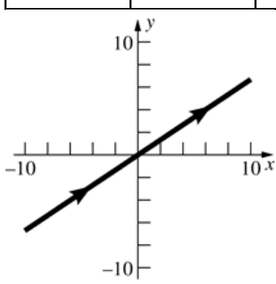
3. cycloid

4. $(dy/dt)/(dx/dt) = g'(t)/f'(t)$

Problem Set 10.4

1. a.

t	x	y
–2	–6	–4
–1	–3	–2
0	0	0
1	3	2
2	6	4

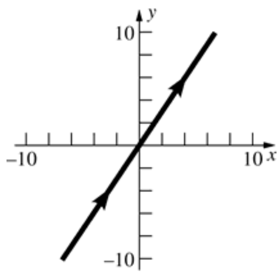


b. Simple; not closed

c. $t=\dfrac{x}{3} \Rightarrow y=\dfrac{2}{3}x$

2. a.

t	x	y
–2	–4	–6
–1	–2	–3
0	0	0
1	2	3
2	4	6

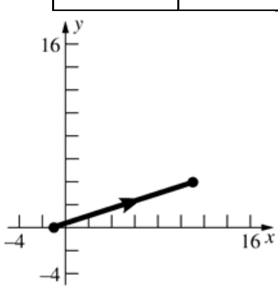


b. Simple; not closed

c. $t=\dfrac{x}{2} \Rightarrow y=\dfrac{3}{2}x$

3. a.

t	x	y
0	–1	0
1	2	1
2	5	2
3	8	3
4	11	4

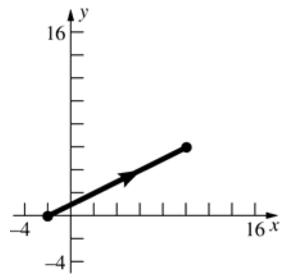


b. Simple; not closed

c. $t=\dfrac{1}{3}(x+1) \Rightarrow y=\dfrac{1}{3}(x+1)$

4. a.

t	x	y
0	–2	0
1	2	2
2	6	4
3	10	6

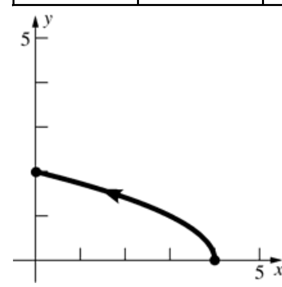


b. Simple; not closed

c. $t=\dfrac{1}{4}(x+2) \Rightarrow y=\dfrac{1}{2}(x+2)$

5. a.

t	x	y
0	4	0
1	3	1
2	2	$\sqrt{2}$
3	1	$\sqrt{3}$
4	0	2



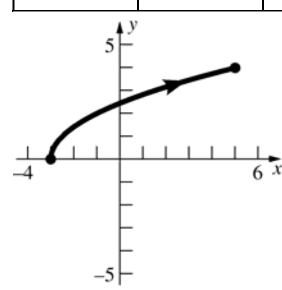
b. Simple; not closed

c. $t=4-x$

$y=\sqrt{4-x}$

6. a.

t	x	y
0	–3	0
2	–1	2
4	1	$2\sqrt{2}$
6	3	$2\sqrt{3}$
8	5	4



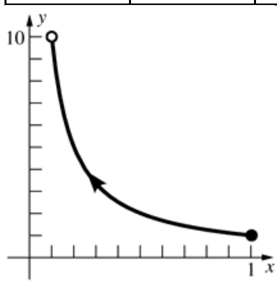
b. Simple; not closed

c. $t=x+3$

$y=\sqrt{2x+6}$

7. a.

s	x	y
1	1	1
3	$\frac{1}{3}$	3
5	$\frac{1}{5}$	5
7	$\frac{1}{7}$	7
9	$\frac{1}{9}$	9



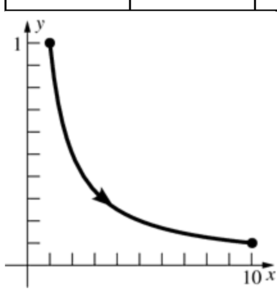
b. Simple; not closed

c. $s = \dfrac{1}{x}$

$y = \dfrac{1}{x}$

8. a.

s	x	y
1	1	1
3	3	$\frac{1}{3}$
5	5	$\frac{1}{5}$
7	7	$\frac{1}{7}$
9	9	$\frac{1}{9}$



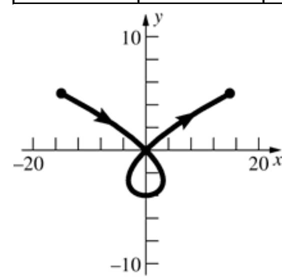
b. Simple; not closed

c. $s = x$

$y = \dfrac{1}{x}$

9. a.

t	x	y
–3	–15	5
–2	0	0
–1	3	–3
0	0	–4
1	–3	–3
2	0	0
3	15	5



b. Not simple; not closed

c. $x^2 = t^6 - 8t^4 + 16t^2$

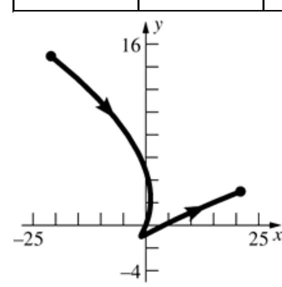
$t^2 = y + 4$

$x^2 = (y+4)^3 - 8(y+4)^2 + 16(y+4)$

$x^2 = y^3 + 4y^2$

10. a.

t	x	y
–3	–21	15
–2	–4	8
–1	1	3
0	0	0
1	–1	–1
2	4	0
3	21	3



b. Simple; not closed

c. $t^2 - 2t - y = 0$

$t = 1 \pm \sqrt{1+y}$

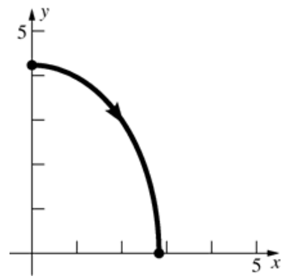
$x = \left(1 \pm \sqrt{1+y}\right)^3 - 2\left(1 \pm \sqrt{1+y}\right)$

$x = 2 + 3y \pm (y+2)\sqrt{1+y}$

$(x - 3y - 2)^2 = (y+1)(y+2)^2$

11. a.

t	x	y
2	0	$3\sqrt{2}$
3	2	3
4	$2\sqrt{2}$	0



b. Simple; not closed

c. $t = \frac{1}{4}x^2 + 2$

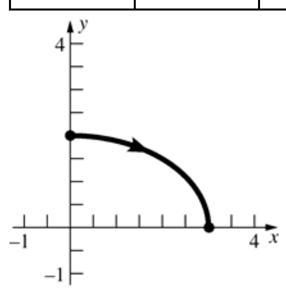
$t = 4 - \frac{1}{9}y^2$

$\frac{1}{4}x^2 + 2 = 4 - \frac{1}{9}y^2$

$\frac{x^2}{8} + \frac{y^2}{18} = 1$

12. a.

t	x	y
3	0	2
$\frac{7}{2}$	$\frac{3}{\sqrt{2}}$	$\sqrt{2}$
4	3	0



b. Simple; not closed

c. $t = \frac{1}{9}x^2 + 3$

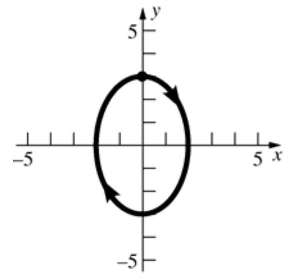
$t = 4 - \frac{1}{4}y^2$

$\frac{1}{9}x^2 + 3 = 4 - \frac{1}{4}y^2$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

13. a.

t	x	y
0	0	3
$\frac{\pi}{2}$	2	0
π	0	–3
$\frac{3\pi}{2}$	–2	0
2π	0	3



b. Simple; closed

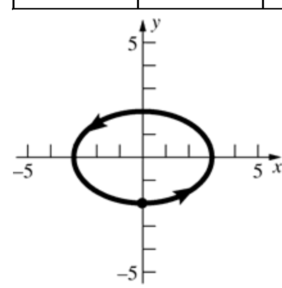
c. $\sin^2 t = \frac{x^2}{4}$

$\cos^2 t = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

14. a.

r	x	y
0	0	–2
$\frac{\pi}{2}$	3	0
π	0	2
$\frac{3\pi}{2}$	–3	0
2π	0	–2



b. Simple; closed

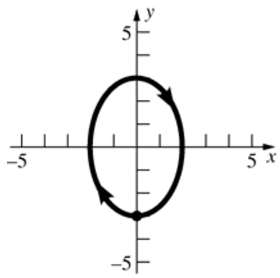
c. $\sin^2 r = \frac{x^2}{9}$

$\cos^2 r = \frac{y^2}{4}$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

15. a.

r	x	y
0	0	–3
$\frac{\pi}{2}$	–2	0
π	0	3
$\frac{3\pi}{2}$	2	0
2π	0	–3
$\frac{5\pi}{2}$	–2	0
3π	0	3
$\frac{7\pi}{2}$	2	0
4π	0	–3



b. Not simple; closed

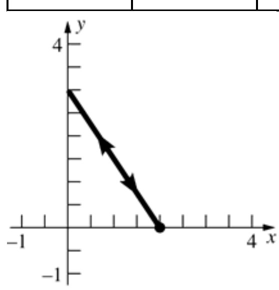
c. $\sin^2 r = \frac{x^2}{4}$

$\cos^2 r = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

16. a.

r	x	y
0	2	0
$\frac{\pi}{2}$	0	3
π	2	0
$\frac{3\pi}{2}$	0	3
2π	2	0



b. Not simple; closed

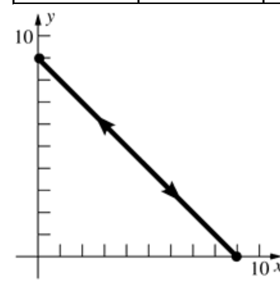
c. $\cos^2 r = \frac{x}{2}$

$\sin^2 r = \frac{y}{3}$

$\frac{x}{2} + \frac{y}{3} = 1$

17. a.

θ	x	y
0	0	9
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	0	9



b. Not simple; closed

c. $\sin^2 \theta = \frac{x}{9}$

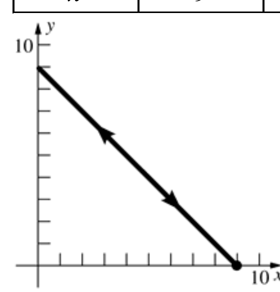
$\cos^2 \theta = \frac{y}{9}$

$\frac{x}{9} + \frac{y}{9} = 1$

$x + y = 9$

18. a.

θ	x	y
0	9	0
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	9	0



b. Not simple; closed

c. $\cos^2\theta = \dfrac{x}{9}$

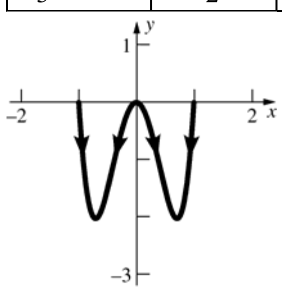
$\sin^2\theta = \dfrac{y}{9}$

$\dfrac{x}{9} + \dfrac{y}{9} = 1$

$x + y = 9$

19. a.

θ	x	y
$0 + 2\pi n$	1	0
$\frac{\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$
$\frac{\pi}{2} + 2\pi n$	0	0
$\frac{2\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\pi + 2\pi n$	-1	0
$\frac{4\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{3\pi}{2} + 2\pi n$	0	0
$\frac{5\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$



b. Not simple; not closed

c. $\cos\,\theta = x$

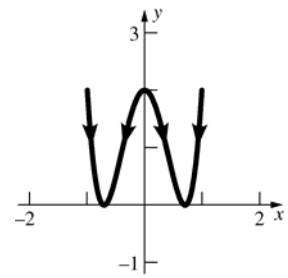
$\sin\theta = \sqrt{1-x^2}$

$y = -8\sin^2\theta\cos^2\theta$

$y = -8x^2(1-x^2)$

20. a.

θ	x	y
$0 + 2\pi n$	0	2
$\frac{\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{2} + 2\pi n$	1	2
$\frac{5\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\pi + 2\pi n$	0	2
$\frac{7\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{3\pi}{2} + 2\pi n$	-1	2
$\frac{11\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$



b. Not simple; not closed

c. $\sin\,\theta = x$

$\cos\theta = \sqrt{1-x^2}$

$y = 2(\cos^2\theta - \sin^2\theta)^2$

$y = 2(2x^2-1)^2$

21. $\dfrac{dx}{d\tau} = 6\tau$

$\dfrac{dy}{d\tau} = 12\tau^2$

$\dfrac{dy}{dx} = 2\tau$

$\dfrac{dy'}{d\tau} = 2$

$\dfrac{d^2y}{dx^2} = \dfrac{1}{3\tau}$

22. $\dfrac{dx}{ds} = 12s$

$\dfrac{dy}{ds} = -6s^2$

$\dfrac{dy}{dx} = -\dfrac{1}{2}s$

$\dfrac{dy'}{ds} = -\dfrac{1}{2}$

$\dfrac{d^2y}{dx^2} = -\dfrac{1}{24s}$

23. $\dfrac{dx}{d\theta} = 4\theta$

$\dfrac{dy}{d\theta} = 3\sqrt{5}\theta^2$

$\dfrac{dy}{dx} = \dfrac{3\sqrt{5}}{4}\theta$

$\dfrac{dy'}{d\theta} = \dfrac{3\sqrt{5}}{4}$

$\dfrac{d^2y}{dx^2} = \dfrac{3\sqrt{5}}{16\theta}$

24. $\frac{dx}{d\theta} = 2\sqrt{3}\theta$

$\frac{dy}{d\theta} = -3\sqrt{3}\theta^2$

$\frac{dy}{dx} = -\frac{3}{2}\theta$

$\frac{dy'}{d\theta} = -\frac{3}{2}$

$\frac{d^2y}{dx^2} = -\frac{\sqrt{3}}{4\theta}$

25. $\frac{dx}{dt} = \sin t$

$\frac{dy}{dt} = \cos t$

$\frac{dy}{dx} = \cot t$

$\frac{dy'}{dt} = -\csc^2 t$

$\frac{d^2y}{dx^2} = -\csc^3 t$

26. $\frac{dx}{dt} = 2\sin t$

$\frac{dy}{dt} = 5\cos t$

$\frac{dy}{dx} = \frac{5}{2}\cot t$

$\frac{dy'}{dt} = -\frac{5}{2}\csc^2 t$

$\frac{d^2y}{dx^2} = -\frac{5}{4}\csc^3 t$

27. $\frac{dx}{dt} = 3\sec^2 t$

$\frac{dy}{dt} = 5\sec t \tan t$

$\frac{dy}{dx} = \frac{5}{3}\sin t$

$\frac{dy'}{dt} = \frac{5}{3}\cos t$

$\frac{d^2y}{dx^2} = \frac{5}{9}\cos^3 t$

28. $\frac{dx}{dt} = -\csc^2 t$

$\frac{dy}{dt} = 2\csc t \cot t$

$\frac{dy}{dx} = -2\cos t$

$\frac{dy'}{dt} = 2\sin t$

$\frac{d^2y}{dx^2} = -2\sin^3 t$

29. $\frac{dx}{dt} = -\frac{2t}{(1+t^2)^2}$

$\frac{dy}{dt} = \frac{2t-1}{t^2(1-t)^2}$

$\frac{dy}{dx} = \frac{(1-2t)(1+t^2)^2}{2t^3(1-t)^2}$

$\frac{dy'}{dt} = -\frac{3t^5+7t^4-6t^3+10t^2-9t+3}{2t^4(1-t)^3}$

$\frac{d^2y}{dx^2} = \frac{(3t^5+7t^4-6t^3+10t^2-9t+3)(1+t^2)^2}{4t^5(1-t)^3}$

30. $\frac{dx}{dt} = -\frac{4t}{(1+t^2)^2}$

$\frac{dy}{dt} = -\frac{2(3t^2+1)}{t^2(1+t^2)^2}$

$\frac{dy}{dx} = \frac{3t^2+1}{2t^3}$

$\frac{dy'}{dt} = -\frac{3(t^2+1)}{2t^4}$

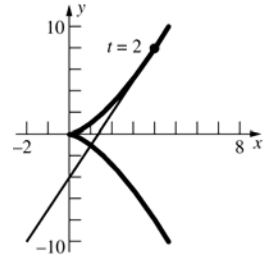
$\frac{d^2y}{dx^2} = \frac{3(t^2+1)^3}{8t^5}$

31. $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{3}{2}t$

At $t = 2$, $x = 4$, $y = 8$, and $\frac{dy}{dx} = 3$.

Tangent line: $y - 8 = 3(x - 4)$ or $3x - y - 4 = 0$

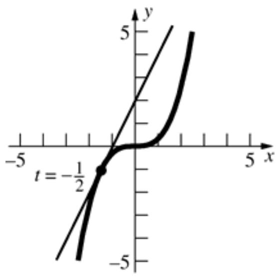


32. $\frac{dx}{dt} = 3, \frac{dy}{dt} = 24t^2$

$\frac{dy}{dx} = 8t^2$

At $t = -\frac{1}{2}, x = -\frac{3}{2}, y = -1$, and $\frac{dy}{dx} = 2$.

Tangent line: $y + 1 = 2\left(x + \frac{3}{2}\right)$ or $2x - y + 2 = 0$



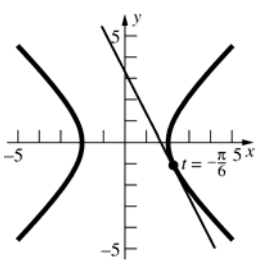
33. $\frac{dx}{dt} = 2\sec t\tan t, \frac{dy}{dt} = 2\sec^2 t$

$\frac{dy}{dx} = \csc t$

At $t = -\frac{\pi}{6}, x = \frac{4}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}$, and $\frac{dy}{dx} = -2$.

Tangent line: $y + \frac{2}{\sqrt{3}} = -2\left(x - \frac{4}{\sqrt{3}}\right)$ or

$2\sqrt{3}x + \sqrt{3}y - 6 = 0$



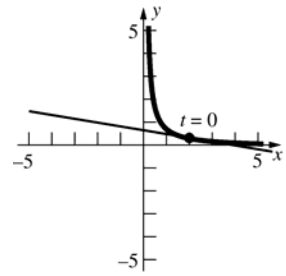
34. $\frac{dx}{dt} = 2e^t, \frac{dy}{dt} = -\frac{1}{3}e^{-t}$

$\frac{dy}{dx} = -\frac{1}{6}e^{-2t}$

At $t = 0$, $x = 2$, $y = \frac{1}{3}$, and $\frac{dy}{dx} = -\frac{1}{6}$.

Tangent line: $y - \frac{1}{3} = -\frac{1}{6}(x - 2)$ or

$x + 6y - 4 = 0$



35. $\frac{dx}{dt} = 2, \frac{dy}{dt} = 3$

$L = \int_0^3 \sqrt{4+9}\,dt = \sqrt{13}\int_0^3 dt = \sqrt{13}[t]_0^3 = 3\sqrt{13}$

36. $\frac{dx}{dt} = -1, \frac{dy}{dt} = 2$

$L = \int_{-3}^3 \sqrt{1+4}\,dt = \sqrt{5}\int_{-3}^3 dt = \sqrt{5}[t]_{-3}^3 = 6\sqrt{5}$

37. $\frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{3}{2}t^{1/2}$

$L = \int_0^3 \sqrt{1 + \frac{9}{4}t}\,dt$

$= \frac{1}{2}\int_0^3 \sqrt{4 + 9t}\,dt$

$= \frac{1}{18}\left[\frac{2}{3}(4 + 9t)^{3/2}\right]_0^3$

$= \frac{1}{27}(31^{3/2} - 8) = \frac{1}{27}(31\sqrt{31} - 8)$

38. $\frac{dx}{dt} = 2\cos t, \frac{dy}{dt} = -2\sin t$

$L = \int_0^\pi \sqrt{4\cos^2 t + 4\sin^2 t}\,dt = 2\int_0^\pi dt = 2[t]_0^\pi = 2\pi$

39. $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 3t^2$

$L = \int_0^2 \sqrt{36t^2 + 9t^4}\,dt = 3\int_0^2 t\sqrt{4 + t^2}\,dt$

$3\left[\frac{1}{3}(4 + t^2)^{3/2}\right]_0^2 = 16\sqrt{2} - 8$

40. $\frac{dx}{dt}=1-\frac{1}{t^2}, \frac{dy}{dt}=\frac{2}{t}$

$$L=\int_1^4\sqrt{\left(1-\frac{2}{t^2}+\frac{1}{t^4}\right)+\frac{4}{t^2}}dt$$

$$=\int_1^4\sqrt{1+\frac{2}{t^2}+\frac{1}{t^4}}dt=\int_1^4\sqrt{\left(1+\frac{1}{t^2}\right)^2}dt$$

$$=\int_1^4\left(1+\frac{1}{t^2}\right)dt=\left[t-\frac{1}{t}\right]_1^4=\frac{15}{4}$$

41. $\frac{dx}{dt}=2e^t, \frac{dy}{dt}=\frac{9}{2}e^{3t/2}$

$$L=\int_{\ln 3}^{2\ln 3}\sqrt{4e^{2t}+\frac{81}{4}e^{3t}}dt=\int_{\ln 3}^{2\ln 3}e^t\sqrt{4+\frac{81}{4}e^t}dt$$

$$=\left[\frac{8}{243}\left(4+\frac{81}{4}e^t\right)^{3/2}\right]_{\ln 3}^{2\ln 3}$$

$$=\frac{745\sqrt{745}-259\sqrt{259}}{243}$$

42. $\frac{dx}{dt}=-\frac{t}{\sqrt{1-t^2}}, \frac{dy}{dt}=-1$

$$L=\int_0^{1/4}\sqrt{\frac{t^2}{1-t^2}+1}dt=\int_0^{1/4}\frac{1}{\sqrt{1-t^2}}dt$$

$$=[\sin^{-1}t]_0^{1/4}=\sin^{-1}\frac{1}{4}$$

43. $\frac{dx}{dt}=\frac{2}{\sqrt{t}}, \frac{dy}{dt}=2t-\frac{1}{2t^2}$

$$L=\int_{1/4}^1\sqrt{\frac{4}{t}+\left(4t^2-\frac{2}{t}+\frac{1}{4t^4}\right)}dt$$

$$=\int_{1/4}^1\sqrt{4t^2+\frac{2}{t}+\frac{1}{4t^4}}dt$$

$$=\int_{1/4}^1\sqrt{\left(2t+\frac{1}{2t^2}\right)^2}dt$$

$$=\int_{1/4}^1\left(2t+\frac{1}{2t^2}\right)dt=\left[t^2-\frac{1}{2t}\right]_{1/4}^1=\frac{39}{16}$$

44. $\frac{dx}{dt}=\operatorname{sech}^2 t, \frac{dy}{dt}=2\tanh t$

$$L=\int_{-3}^3\sqrt{\operatorname{sech}^4 t+4\tanh^2 t}\,dt$$

$$=\int_{-3}^3\sqrt{4-4\operatorname{sech}^2 t+\operatorname{sech}^4 t}\,dt$$

$$=\int_{-3}^3\sqrt{(2-\operatorname{sech}^2 t)^2}dt=\int_{-3}^3(2-\operatorname{sech}^2 t)dt$$

$$=[2t-\tanh t]_{-3}^3=12-2\tanh 3$$

45. $\frac{dx}{dt}=-\sin t,$

$$\frac{dy}{dt}=\frac{\sec t\tan t+\sec^2 t}{\sec t+\tan t}-\cos t=\sec t-\cos t$$

$$L=\int_0^{\pi/4}\sqrt{\sin^2 t+(\sec^2 t-2+\cos^2 t)}dt$$

$$=\int_0^{\pi/4}\tan t\,dt$$

$$=\left[-\ln|\cos t|\right]_0^{\pi/4}=-\ln\frac{1}{\sqrt{2}}=\frac{1}{2}\ln 2$$

46. $\frac{dx}{dt}=t\sin t, \frac{dy}{dt}=t\cos t$

$$L=\int_{\pi/4}^{\pi/2}\sqrt{t^2\sin^2 t+t^2\cos^2 t}\,dt$$

$$=\int_{\pi/4}^{\pi/2}t\,dt=\left[\frac{1}{2}t^2\right]_{\pi/4}^{\pi/2}=\frac{3\pi^2}{32}$$

47. **a.** $\frac{dx}{d\theta}=\cos\theta, \frac{dy}{d\theta}=-\sin\theta$

$$L=\int_0^{2\pi}\sqrt{\cos^2\theta+\sin^2\theta}\,d\theta=\int_0^{2\pi}d\theta$$

$$=[\theta]_0^{2\pi}=2\pi$$

b. $\frac{dx}{d\theta}=3\cos 3\theta, \frac{dy}{d\theta}=-3\sin 3\theta$

$$L=\int_0^{2\pi}\sqrt{9\cos^2 3\theta+9\sin^2 3\theta}\,d\theta$$

$$=3\int_0^{2\pi}d\theta=3[\theta]_0^{2\pi}=6\pi$$

c. The curve in part a goes around the unit circle once, while the curve in part b goes around the unit circle three times.

48. $\Delta S=2\pi x\Delta s$

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

$$S=\int_a^b 2\pi x\,ds=\int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

See Section 5.4 of the text

49. $\frac{dx}{dt}=-\sin t, \frac{dy}{dt}=\cos t$

$$S=\int_0^{2\pi}2\pi(1+\cos t)\sqrt{\sin^2 t+\cos^2 t}\,dt$$

$$=2\pi\int_0^{2\pi}(1+\cos t)dt=2\pi[t+\sin t]_0^{2\pi}=4\pi^2$$

50. $\dfrac{dx}{dt} = -\sin t, \dfrac{dy}{dt} = \cos t$

$$S = \int_0^{2\pi} 2\pi(3+\sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt$$

$$= 2\pi\int_0^{2\pi}(3+\sin t)dt = 2\pi[3t - \cos t]_0^{2\pi} = 12\pi^2$$

51. $\dfrac{dx}{dt} = -\sin t, \dfrac{dy}{dt} = \cos t$

$$S = \int_0^{2\pi} 2\pi(1+\sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt$$

$$= 2\pi\int_0^{2\pi}(1+\sin t)dt = 2\pi[t - \cos t]_0^{2\pi} = 4\pi^2$$

52. $\dfrac{dx}{dt} = \sqrt{t}, \dfrac{dy}{dt} = \dfrac{1}{\sqrt{t}}$

$$S = \int_0^{2\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{t + \frac{1}{t}}dt$$

$$= \frac{4\pi}{3}\int_0^{2\sqrt{3}} t\sqrt{t^2+1}dt$$

$$= \frac{4\pi}{3}\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^{2\sqrt{3}} = \frac{4\pi}{9}(13\sqrt{13}-1)$$

53. $\dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = t + \sqrt{7}$

$$S = \int_{-\sqrt{7}}^{\sqrt{7}} 2\pi\left(t+\sqrt{7}\right)\sqrt{1+\left(t+\sqrt{7}\right)^2}\,dt$$

$$= 2\pi\left[\frac{1}{3}\left(1+\left(t+\sqrt{7}\right)^2\right)^{3/2}\right]_{-\sqrt{7}}^{\sqrt{7}}$$

$$= \frac{2\pi}{3}\left(29\sqrt{29}-1\right)$$

54. $\dfrac{dx}{dt} = t + a, \dfrac{dy}{dt} = 1$

$$S = \int_{-\sqrt{a}}^{\sqrt{a}} 2\pi(t+a)\sqrt{(t+a)^2+1}\,dt$$

$$= 2\pi\left[\frac{1}{3}((t+a)^2+1)^{3/2}\right]_{-\sqrt{a}}^{\sqrt{a}}$$

$$= \frac{2\pi}{3}\left[\left(a^2+2a\sqrt{a}+a+1\right)^{3/2} - \left(a^2-2a\sqrt{a}+a+1\right)^{3/2}\right]$$

55. $dx = dt$; when $x = 0$, $t = -1$; when $x = 1$, $t = 0$.

$$\int_0^1 (x^2-4y)dx = \int_{-1}^0 [(t+1)^2 - 4(t^3+4)]dt$$

$$= \int_{-1}^0 (-4t^3+t^2+2t-15)dt$$

$$= \left[-t^4 + \frac{1}{3}t^3 + t^2 - 15t\right]_{-1}^0 = -\frac{44}{3}$$

56. $dy = \sec^2 t\, dt$; when $y = 1, t = \dfrac{\pi}{4}$;

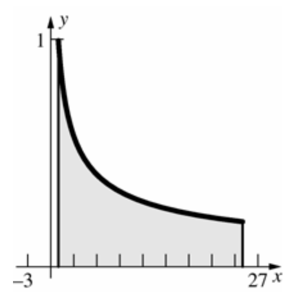
when $y = \sqrt{3}, t = \dfrac{\pi}{3}$.

$$\int_1^{\sqrt{3}} xy\,dy = \int_{\pi/4}^{\pi/3}(\sec t)(\tan t)\sec^2 t\,dt$$

$$= \left[\frac{1}{3}\sec^3 t\right]_{\pi/4}^{\pi/3} = \frac{8}{3} - \frac{2\sqrt{2}}{3}$$

57. $dx = 2e^{2t}dt$

$$A = \int_1^{25} y\,dx = \int_0^{\ln 5} 2e^t dt = [2e^t]_0^{\ln 5} = 8$$



58. a. $t = \dfrac{x}{v_0 \cos\alpha}$

$$y = -16\left(\frac{x}{v_0\cos\alpha}\right)^2 + (v_0\sin\alpha)\left(\frac{x}{v_0\cos\alpha}\right)$$

$$y = -\left(\frac{16}{v_0{}^2\cos^2\alpha}\right)x^2 + (\tan\alpha)x$$

This is an equation for a parabola.

b. Solve for t when $y = 0$.

$$-16t^2 + (v_0\sin\alpha)t = 0$$

$$t(-16t + v_0\sin\alpha) = 0$$

$$t = 0, \frac{v_0\sin\alpha}{16}$$

The time of flight is $\dfrac{v_0\sin\alpha}{16}$ seconds.

c. At $t = \dfrac{v_0\sin\alpha}{16}$, $x = (v_0\cos\alpha)\left(\dfrac{v_0\sin\alpha}{16}\right)$

$$= \frac{v_0^2\sin\alpha\cos\alpha}{16} = \frac{v_0^2\sin 2\alpha}{32}.$$

d. Let R be the range as a function of α.

$$R = \frac{v_0^2 \sin 2\alpha}{32}$$

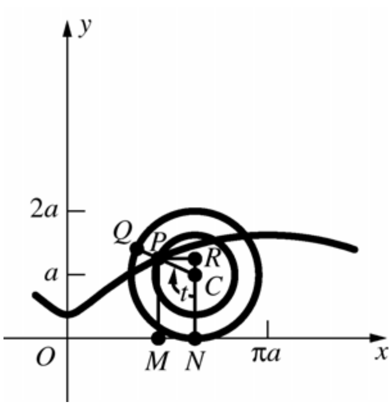
$$\frac{dR}{d\alpha} = \frac{v_0^2 \cos 2\alpha}{16}$$

$$\frac{v_0^2 \cos 2\alpha}{16} = 0, \cos 2\alpha = 0, \alpha = \frac{\pi}{4}$$

$$\frac{d^2R}{d\alpha^2} = -\frac{v_0^2 \sin 2\alpha}{8}; \frac{d^2R}{d\alpha^2} < 0 \text{ at } \alpha = \frac{\pi}{4}.$$

The range is the largest possible when $\alpha = \frac{\pi}{4}$.

59.

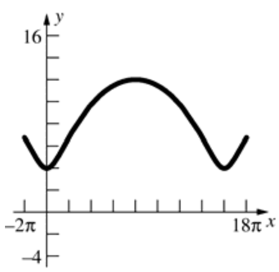


Let the wheel roll along the x-axis with P initially at $(0, a - b)$.

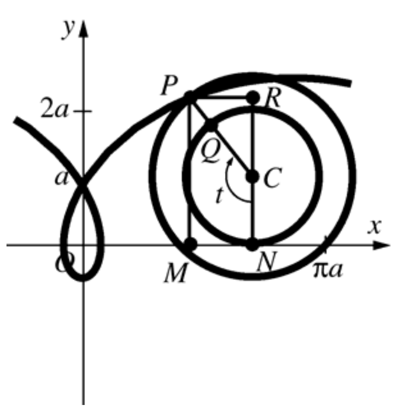
$|ON| = \text{arc } NQ = at$

$x = |OM| = |ON| - |MN| = at - b\sin t$

$y = |MP| = |RN| = |NC| + |CR| = a - b\cos t$



60.

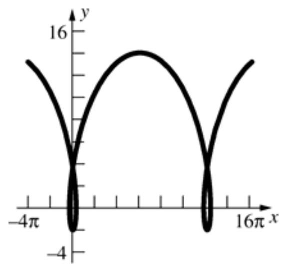


Let the wheel roll along the x-axis with P initially at $(0, a - b)$.

$|ON| = \text{arc } NQ = at$

$x = |OM| = |ON| - |MN| = at - b\sin t$

$y = |MP| = |RN| = |NC| + |CR| = a - b\cos t$



61. The x- and y-coordinates of the center of the circle of radius b are $(a - b)\cos t$ and $(a - b)\sin t$, respectively. The angle measure (in a clockwise direction) of arc BP is $\frac{a}{b}t$. The horizontal change from the center of the circle of radius b to P is $b\cos\left(-\left(\frac{a}{b}t - t\right)\right) = b\cos\left(\frac{a-b}{b}t\right)$ and the vertical change is $b\sin\left(-\left(\frac{a}{b}t - t\right)\right) = -b\sin\left(\frac{a-b}{b}t\right)$.

Therefore, $x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)$ and

$$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right).$$

62. From Problem 61,

$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)$ and

$$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right).$$

Substitute $b = \frac{a}{4}$.

$$x = \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos(3t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos(2t + t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)(\cos 2t\cos t - \sin 2t\sin t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)(\cos^3 t - \sin^2 t\cos t - 2\sin^2 t\cos t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos^3 t - \left(\frac{3a}{4}\right)\cos t\sin^2 t$$

$$= \left(\frac{3a}{4}\right)(\cos t)(1 - \sin^2 t) + \left(\frac{a}{4}\right)\cos^3 t = a\cos^3 t$$

$$y = \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)\sin(3t)$$

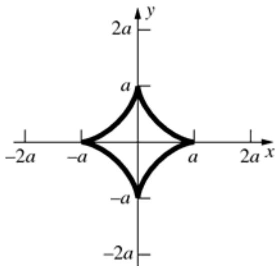
$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)\sin(2t+t)$

$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)(\sin 2t \cos t + \cos 2t \sin t)$

$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)(2\sin t \cos^2 t + \cos^2 t \sin t - \sin^3 t)$

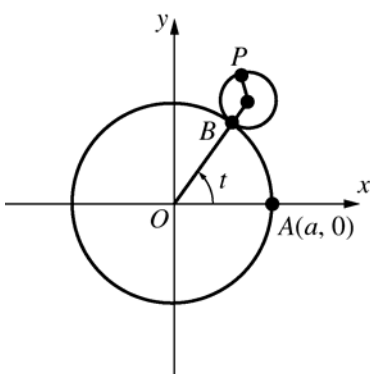
$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{3a}{4}\right)\sin t \cos^2 t + \left(\frac{a}{4}\right)\sin^3 t$

$= \left(\frac{3a}{4}\right)(\sin t)(1-\cos^2 t) + \left(\frac{a}{4}\right)\sin^3 t = a\sin^3 t$



$x^{2/3} + y^{2/3} = a^{2/3}\cos^2 t + a^{2/3}\sin^2 t = a^{2/3}$

63. Consider the following figure similar to the one in the text for Problem 61.



The x- and y-coordinates of the center of the circle of radius b are $(a + b)\cos t$ and $(a + b)\sin t$ respectively. The angle measure (in a counter-clockwise direction) of arc PB is $\frac{a}{b}t$. The horizontal change from the center of the circle of radius b to P is

$b\cos\left(\frac{a}{b}t + t + \pi\right) = -b\cos\left(\frac{a+b}{b}t\right)$ and the vertical change is

$b\sin\left(\frac{a}{b}t + t + \pi\right) = -b\sin\left(\frac{a+b}{b}t\right)$. Therefore,

$x = (a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)$ and

$y = (a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)$.

64. $x = 2a\cos t - a\cos 2t = 2a\cos t - 2a\cos^2 t + a$

$= 2a\cos t(1-\cos t) + a$

$y = 2a\sin t - a\sin 2t = 2a\sin t - 2a\sin t\cos t$

$= 2a\sin t(1-\cos t)$

$x - a = 2a\cos t(1-\cos t)$

$(x-a)^2 + y^2 = 4a^2(1-\cos t)^2$

$(x-a)^2 + y^2 + 2a(x-a) =$

$4a^2(1-\cos t)^2 + 4a^2(1-\cos t)\cos t$

$(x-a)^2 + y^2 + 2a(x-a) = 4a^2(1-\cos t)$

$[(x-a)^2 + y^2 + 2a(x-a)]^2 = 4a^2 \cdot 4a^2(1-\cos t)^2$

$[(x-a)^2 + y^2 + 2a(x-a)]^2 = 4a^2[(x-a)^2 + y^2]$

65. $\frac{dx}{dt} = \left(\frac{a}{3}\right)(-2\sin t - 2\sin 2t)$,

$\frac{dy}{dt} = \left(\frac{a}{3}\right)(2\cos t - 2\cos 2t)$

$\left(\frac{dx}{dt}\right)^2 = \left(\frac{a}{3}\right)^2(4\sin^2 t + 8\sin t\sin 2t + 4\sin^2 2t)$

$\left(\frac{dy}{dt}\right)^2 = \left(\frac{a}{3}\right)^2(4\cos^2 t - 8\cos t\cos 2t + 4\cos^2 2t)$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{a}{3}\right)^2(8 + 8\sin t\sin 2t - 8\cos t\cos 2t)$

$= \left(\frac{a}{3}\right)^2(8 + 16\sin^2 t\cos t - 8\cos^3 t + 8\sin^2 t\cos t)$

$= \left(\frac{a}{3}\right)^2(8 + 24\cos t\sin^2 t - 8\cos^3 t)$

$= \left(\frac{a}{3}\right)^2(8 + 24\cos t - 32\cos^3 t)$

$L = 3\int_0^{2\pi/3}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= a\int_0^{2\pi/3}\sqrt{8 + 24\cos t - 32\cos^3 t}\,dt$

Using a CAS to evaluate the length, $L = \frac{16a}{3}$.

66. a. Let $x = a\cos t$ and $y = b\sin t$.

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\sin^2 t + b^2\cos^2 t$$

$$= a^2 + (b^2 - a^2)\cos^2 t$$

$$= a^2 - c^2\cos^2 t = a^2\left(1 - \frac{c^2}{a^2}\cos^2 t\right)$$

$$= a^2\left(1 - \left(\frac{c}{a}\right)^2\cos^2 t\right) = a^2(1 - e^2\cos^2 t)$$

$$P = 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= 4a\int_0^{\pi/2}\sqrt{1 - e^2\cos^2 t}\,dt$$

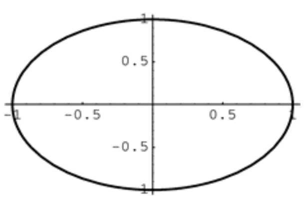
b. $P = 4\int_0^{\pi/2}\sqrt{1 - \frac{\cos^2 t}{16}}\,dt$

$$= \int_0^{\pi/2}\sqrt{16 - \cos^2 t}\,dt \approx 6.1838$$

(The answer is near 2π because it is slightly smaller than a circle of radius 1 whose perimeter is 2π).

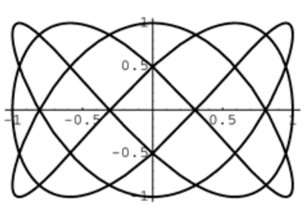
c. $P = \int_0^{\pi/2}\sqrt{16 - \cos^2 t}\,dt \approx 6.1838$

67. a.



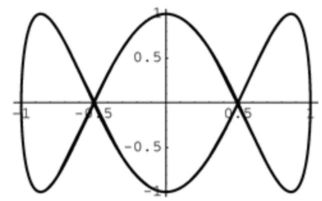
The curve touches a horizontal border twice and touches a vertical border twice.

b.



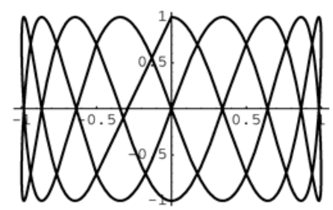
The curve touches a horizontal border five times and touches a vertical border three times.

c.



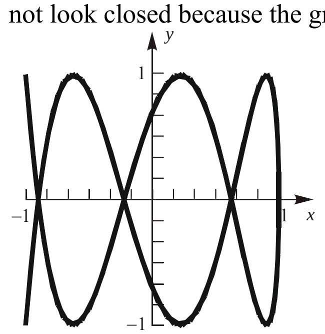
The curve touches a horizontal border six times and touches a vertical border twice. Note that the curve is traced out five times.

d.

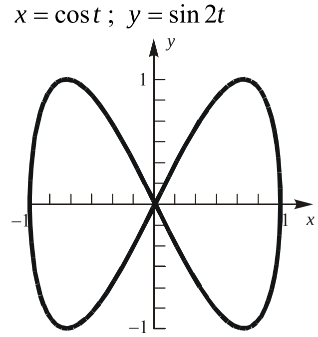


The curve touches a horizontal border 18 times and touches a vertical border four times.

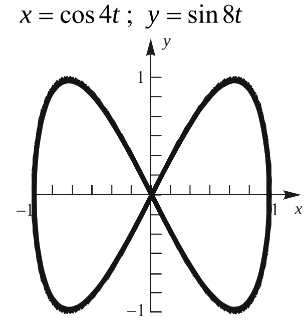
68. This is a closed curve even thought the graph does not look closed because the graph retraces itself.



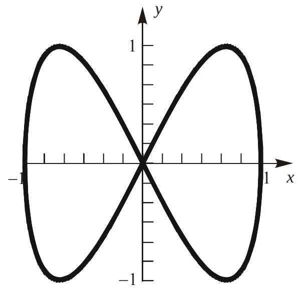
69. a. $x = \cos t$; $y = \sin 2t$



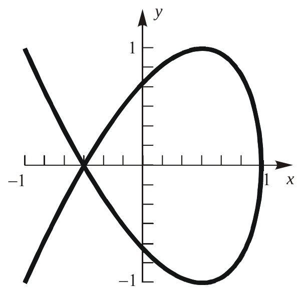
b. $x = \cos 4t$; $y = \sin 8t$



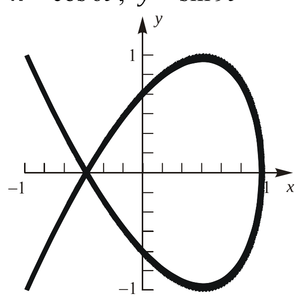
c. $x = \cos 5t$; $y = \sin 10t$



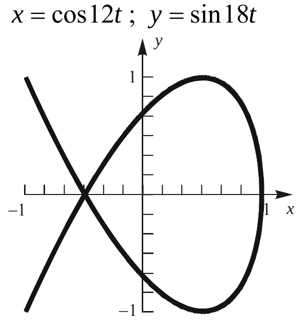
d. $x = \cos 2t$; $y = \sin 3t$



e. $x = \cos 6t$; $y = \sin 9t$



f. $x = \cos 12t$; $y = \sin 18t$



70. Consider the curve defined parametrically by $x = \cos at$, $y = \sin bt$, $t \in [0, 2\pi)$; we assume a and b are integers. This graph will be contained in the box with sides $x = \pm 1$, $y = \pm 1$. Let H be the number of times the graph touches a horizontal side, V the number of times it touches a vertical side, and C the number of times it touches a corner (right now, C is included in H and V). Finally, let w = the *greatest common divisor* of a and b; write $a = u \cdot w$, $b = v \cdot w$. Note: $\frac{a}{b}$ in lowest terms is $\frac{u}{v}$.

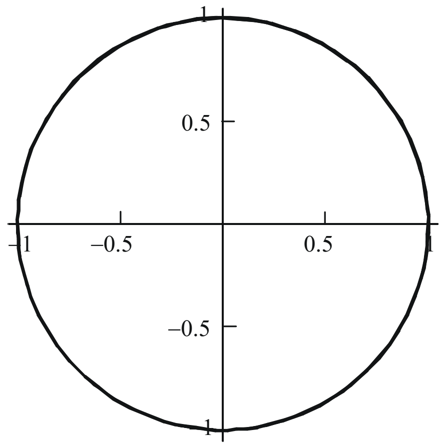
a. The graph touches a horizontal side if $\sin bt = \pm 1$ or $bt = \frac{k}{2}\pi$ (k odd); that is, $t = \frac{k}{2b}\pi$, where $k = 1, 3, \ldots, 4b - 1$.
Hence, H = 2*b*.

b. The graph touches a vertical side if $\cos at = \pm 1$ or $at = n\pi$ (n an integer); that is, $t = \frac{n}{a}\pi$, where $n = 0, 1, \ldots, 2a - 1$.
Hence, V = 2*a*.

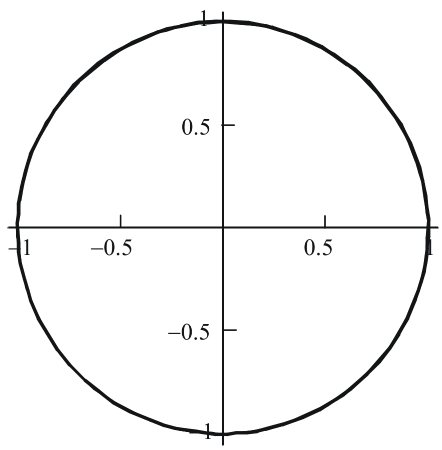
c. If t_0 yields a corner, then (see **a.** and **b.**) then $at_0 = n\pi$, $bt_0 = \frac{k}{2}\pi$ so that $\frac{u}{v} = \frac{a}{b} = \frac{n}{k/2} = \frac{2n}{k}$. Thus corners can only occur if u is even and v is odd. Assume that is the case, write $u = 2r$, and assume we have a corner at t_0; then $t_0 = \frac{k}{2b}\pi$ and $at_0 = \frac{ak}{2b}\pi = n\pi$, ($n$ an integer). Thus $\frac{ak}{2b} = \frac{2rwk}{2vw} = \frac{rk}{v}$ is an integer; hence v is a factor of rk. But v and r have no factors in common, so v must be a factor of k. Conclusion: k is an odd multiple of v. Thus corners occur at $\frac{m}{2b}\pi$ where $m = v, 3v, 5v, \ldots, (4w-1)v$; therefore C = 2*w*.
Now if we count corner contacts as half horizontal and half vertical, the ratio of vertical contacts to horizontal contacts is given by

$$\frac{\text{V-}\frac{1}{2}\text{C}}{\text{H-}\frac{1}{2}\text{C}} = \begin{cases} \frac{2a}{2b} = \frac{a}{b} = \frac{u}{v} & \textit{if } \text{C} = 0 \\ \frac{2a - w}{2b - w} = \frac{2u - 1}{2v - 1} & \textit{if } \text{C} \neq 0 \end{cases}$$

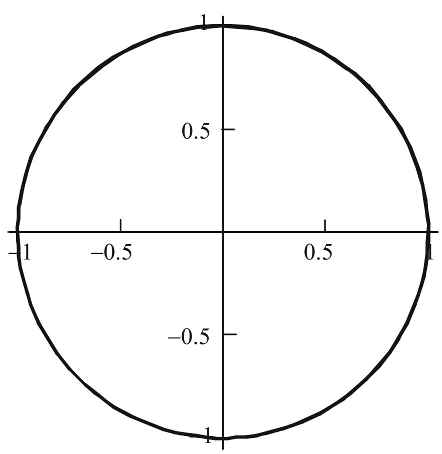
71. a.



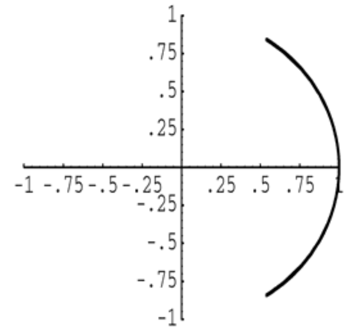
b.



c.

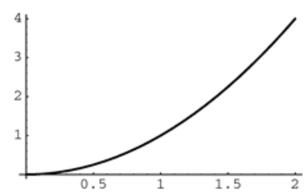


d.



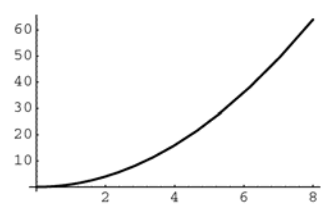
Given a parameterization of the form $x = \cos f(t)$ and $y = \sin f(t)$, the point moves around the curve (which is a circle of radius 1) at a speed of $|f'(t)|$. The point travels clockwise around the circle when $f(t)$ is decreasing and counterclockwise when $f(t)$ is increasing. Note that in part d, only part of the circle will be traced out since the range of $f(t) = \sin t$ is $[-1, 1]$.

72. a.



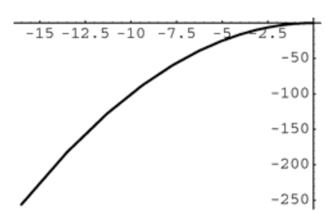
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 2$

b.

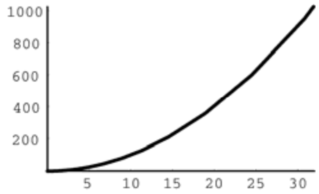


The curve traced out is the graph of $y = x^2$ for $0 \le x \le 8$.

c.



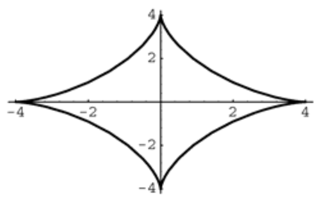
The curve traced out is the graph of $y = -x^2$ for $-16 \le x \le 0$.

d. 

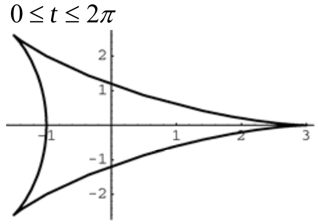
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 32$

All of the curves lie on the graph of $y = \pm x^2$, but trace out different parts because of the parameterization

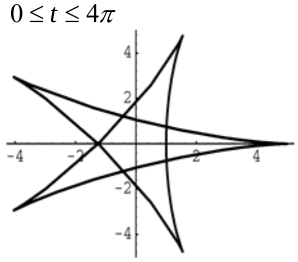
73. a. $0 \le t \le 2\pi$



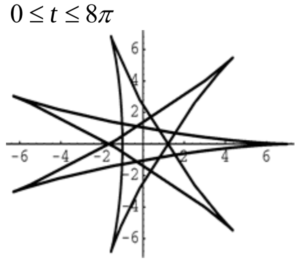
b. $0 \le t \le 2\pi$



c. $0 \le t \le 4\pi$



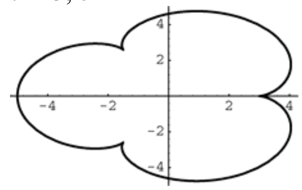
d. $0 \le t \le 8\pi$



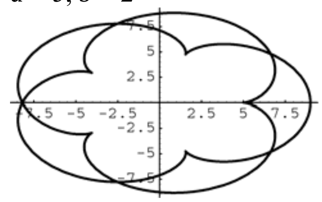
Let $\dfrac{p}{q} = \dfrac{a}{b}$ where $\dfrac{p}{q}$ is the reduced fraction of $\dfrac{a}{b}$. The length of the t-interval is $2q\pi$. The number of times the graph would touch the circle of radius a during the t-interval is p. If $\dfrac{a}{b}$ is irrational, the curve is not periodic.

74. Some possible graphs for different a and b are shown below.

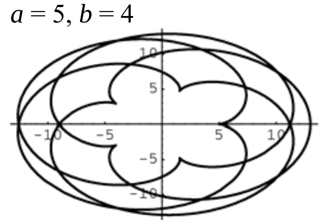
$a = 3$, $b = 1$



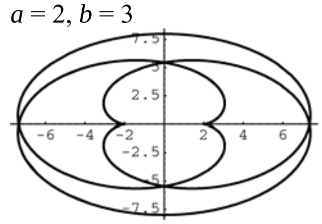
$a = 5$, $b = 2$



$a = 5$, $b = 4$

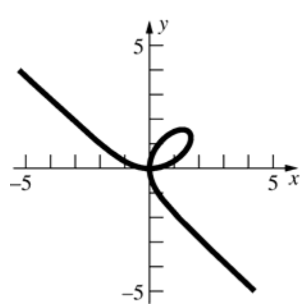


$a = 2$, $b = 3$



Let $\dfrac{p}{q} = \dfrac{a}{b}$ where $\dfrac{p}{q}$ is the reduced fraction of $\dfrac{a}{b}$. The length of the t-interval is $2q\pi$. The number of times the graph would touch the circle of radius a during the t-interval is p. If $\dfrac{a}{b}$ is irrational, the curve is not periodic.

75. $x = \dfrac{3t}{t^3+1}, y = \dfrac{3t^2}{t^3+1}$



When $x > 0$, $t > 0$ or $t < -1$.
When $x < 0$, $-1 < t < 0$.
When $y > 0$, $t > -1$. When $y < 0$, $t < -1$.
Therefore the graph is in quadrant I for $t > 0$, quadrant II for $-1 < t < 0$, quadrant III for no t, and quadrant IV for $t < -1$.

10.5 Concepts Review

1. infinitely many

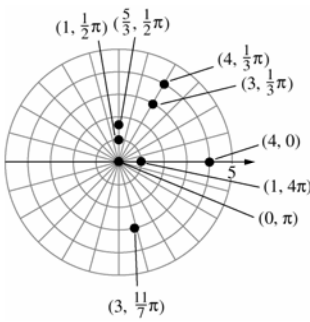
2. $r\cos\theta$; $r\sin\theta$; r^2

3. circle; line

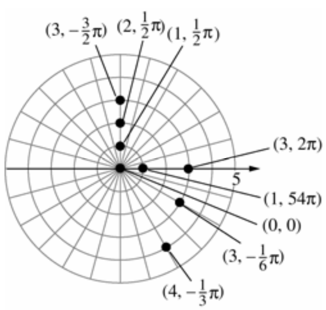
4. conic

Problem Set 10.5

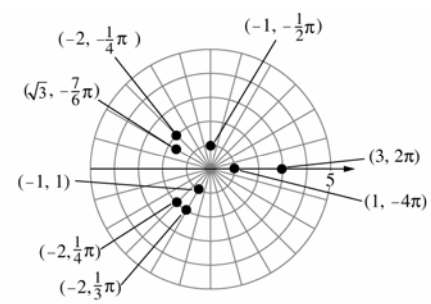
1.



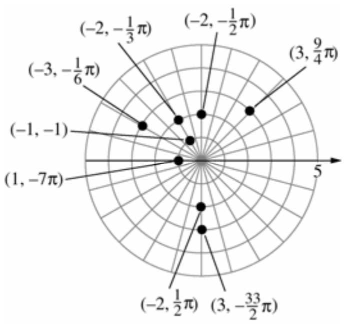
2.



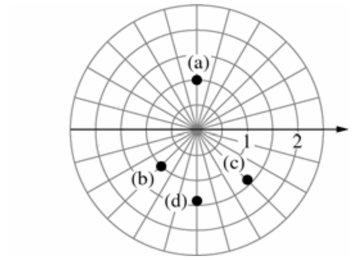
3.



4.



5.



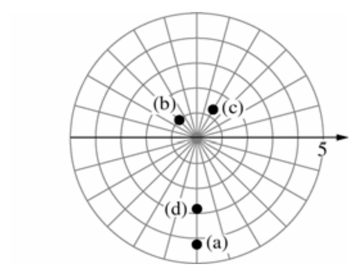
a. $\left(1, -\frac{3}{2}\pi\right), \left(1, \frac{5}{2}\pi\right), \left(-1, -\frac{1}{2}\pi\right), \left(-1, \frac{3}{2}\pi\right)$

b. $\left(1, -\frac{3}{4}\pi\right), \left(1, \frac{5}{4}\pi\right), \left(-1, -\frac{7}{4}\pi\right), \left(-1, \frac{9}{4}\pi\right)$

c. $\left(\sqrt{2}, -\frac{7}{3}\pi\right), \left(\sqrt{2}, \frac{5}{3}\pi\right), \left(-\sqrt{2}, -\frac{4}{3}\pi\right),$
$\left(-\sqrt{2}, \frac{2}{3}\pi\right)$

d. $\left(\sqrt{2}, -\frac{1}{2}\pi\right), \left(\sqrt{2}, \frac{3}{2}\pi\right), \left(-\sqrt{2}, -\frac{3}{2}\pi\right),$
$\left(-\sqrt{2}, \frac{1}{2}\pi\right)$

6.



a. $\left(3\sqrt{2}, -\frac{1}{2}\pi\right), \left(3\sqrt{2}, \frac{3}{2}\pi\right), \left(-3\sqrt{2}, -\frac{3}{2}\pi\right),$
$\left(-3\sqrt{2}, \frac{1}{2}\pi\right)$

b. $\left(1, -\frac{5}{4}\pi\right), \left(1, \frac{3}{4}\pi\right), \left(-1, -\frac{1}{4}\pi\right), \left(-1, \frac{7}{4}\pi\right)$

c. $\left(\sqrt{2}, -\frac{5}{3}\pi\right), \left(\sqrt{2}, \frac{1}{3}\pi\right), \left(-\sqrt{2}, -\frac{8}{3}\pi\right),$
$\left(-\sqrt{2}, \frac{4}{3}\pi\right)$

d. $\left(2\sqrt{2}, -\frac{1}{2}\pi\right), \left(2\sqrt{2}, \frac{3}{2}\pi\right), \left(-2\sqrt{2}, -\frac{3}{2}\pi\right),$
$\left(-2\sqrt{2}, \frac{1}{2}\pi\right)$

7. **a.** $x = 1\cos\frac{1}{2}\pi = 0$

$y = 1\sin\frac{1}{2}\pi = 1$

(0, 1)

b. $x = -1\cos\frac{1}{4}\pi = -\frac{\sqrt{2}}{2}$

$y = -1\sin\frac{1}{4}\pi = -\frac{\sqrt{2}}{2}$

$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

c. $x = \sqrt{2}\cos\left(-\frac{1}{3}\pi\right) = \frac{\sqrt{2}}{2}$

$y = \sqrt{2}\sin\left(-\frac{1}{3}\pi\right) = -\frac{\sqrt{6}}{2}$

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{2}\right)$

d. $x = -\sqrt{2}\cos\frac{5}{2}\pi = 0$

$y = -\sqrt{2}\sin\frac{5}{2}\pi = -\sqrt{2}$

$\left(0, -\sqrt{2}\right)$

8. **a.** $x = 3\sqrt{2}\cos\frac{7}{2}\pi = 0$

$y = 3\sqrt{2}\sin\frac{7}{2}\pi = -3\sqrt{2}$

$\left(0, -3\sqrt{2}\right)$

b. $x = -1\cos\frac{15}{4}\pi = -\frac{\sqrt{2}}{2}$

$y = -1\sin\frac{15}{4}\pi = \frac{\sqrt{2}}{2}$

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

c. $x = -\sqrt{2}\cos\left(-\frac{2}{3}\pi\right) = \frac{\sqrt{2}}{2}$

$y = -\sqrt{2}\sin\left(-\frac{2}{3}\pi\right) = \frac{\sqrt{6}}{2}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$

d. $x = -2\sqrt{2}\cos\frac{29}{2}\pi = 0$

$y = -2\sqrt{2}\sin\frac{29}{2}\pi = -2\sqrt{2}$

$\left(0, -2\sqrt{2}\right)$

9. **a.** $r^2 = \left(3\sqrt{3}\right)^2 + 3^2 = 36,\ r = 6$

$\tan\theta = \frac{1}{\sqrt{3}}, \theta = \frac{\pi}{6}$

$\left(6, \frac{\pi}{6}\right)$

b. $r^2 = \left(-2\sqrt{3}\right)^2 + 2^2 = 16,\ r = 4$

$\tan\theta = \frac{2}{-2\sqrt{3}}, \theta = \frac{5\pi}{6}$

$\left(4, \frac{5\pi}{6}\right)$

c. $r^2 = \left(-\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2 = 4,\ r = 2$

$\tan\theta = \frac{-\sqrt{2}}{-\sqrt{2}}, \theta = \frac{5\pi}{4}$

$\left(2, \frac{5\pi}{4}\right)$

d. $r^2 = 0^2 + 0^2 = 0,\ r = 0$

$\tan\ \theta = 0,\ \theta = 0$

(0, 0)

10. **a.** $r^2 = \left(-\frac{3}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{10}{3}, r = \frac{\sqrt{10}}{\sqrt{3}}$

$\tan\theta = \frac{\frac{1}{\sqrt{3}}}{\frac{-3}{\sqrt{3}}}, \theta = \pi + \tan^{-1}\left(-\frac{1}{3}\right)$

$\left(\frac{\sqrt{10}}{\sqrt{3}}, \pi + \tan^{-1}\left(-\frac{1}{3}\right)\right)$

b. $r^2 = \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{6}{4}, r = \frac{\sqrt{6}}{2}$

$\tan\theta = \frac{\frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2}}, \theta = \frac{3\pi}{4}$

$\left(\frac{\sqrt{6}}{2}, \frac{3\pi}{4}\right)$

c. $r^2 = 0^2 + (-2)^2 = 4,\ r = 2$

$\tan\theta = -\frac{2}{0}, \theta = \frac{3\pi}{2}$

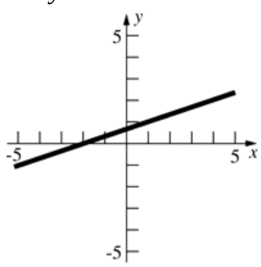
$\left(2, \frac{3\pi}{2}\right)$

d. $r^2 = 3^2 + (-4)^2 = 25,\ r = 5$

$\tan\theta = -\frac{4}{3}, \theta = \tan^{-1}\left(-\frac{4}{3}\right)$

$\left(2, \tan^{-1}\left(-\frac{4}{3}\right)\right)$

11. $x - 3y + 2 = 0$

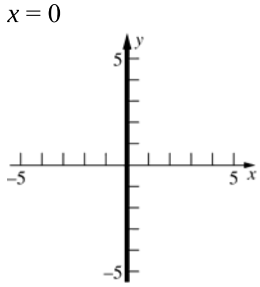


$r\cos\theta - 3r\sin\theta + 2 = 0$

$r = -\frac{2}{\cos\theta - 3\sin\theta}$

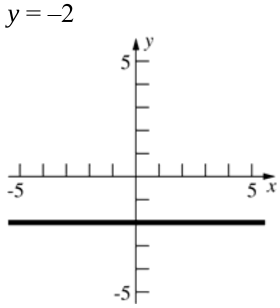
$r = \frac{2}{3\sin\theta - \cos\theta}$

12. $x = 0$



$\theta = \frac{\pi}{2}$

13. $y = -2$

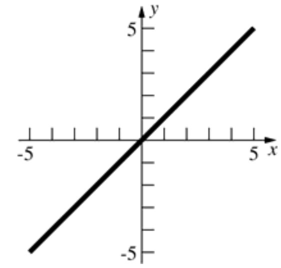


$r\sin\theta = -2$

$r = -\frac{2}{\sin\theta}$

$r = -2\csc\theta$

14. $x - y = 0$

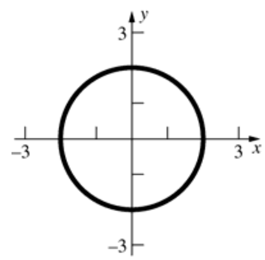


$r\cos\theta - r\sin\theta = 0$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

15. $x^2 + y^2 = 4$

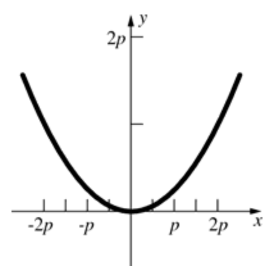


$(r\cos\theta)^2 + (r\sin\theta)^2 = 4$

$r^2 = 4$

$r = 2$

16. $x^2 = 4py$



$(r\cos\theta)^2 = 4p(r\sin\theta)$

$r = \frac{4p\sin\theta}{\cos^2\theta}$

$r = 4p\sec\theta\tan\theta$

17. $\theta = \frac{\pi}{2}$

$\cot\theta = 0$

$\frac{x}{y} = 0$

$x = 0$

18. $r = 3$

$r^2 = 9$

$x^2 + y^2 = 9$

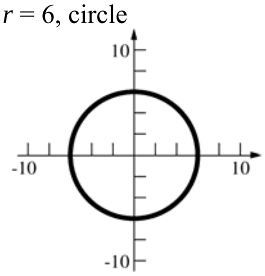
19. $r\cos\theta + 3 = 0$
$x + 3 = 0$
$x = -3$

20. $r - 5\cos\theta = 0$
$r^2 - 5r\cos\theta = 0$
$x^2 + y^2 - 5x = 0$
$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$
$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$

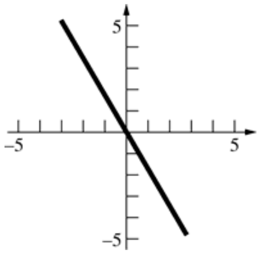
21. $r\sin\theta - 1 = 0$
$y - 1 = 0$
$y = 1$

22. $r^2 - 6r\cos\theta - 4r\sin\theta + 9 = 0$
$x^2 + y^2 - 6x - 4y + 9 = 0$
$(x^2 - 6x + 9) + (y^2 - 4y + 4) = -9 + 9 + 4$
$(x-3)^2 + (y-2)^2 = 4$

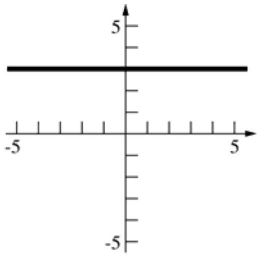
23. $r = 6$, circle



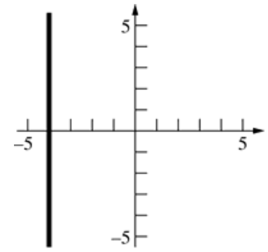
24. $\theta = \frac{2\pi}{3}$, line



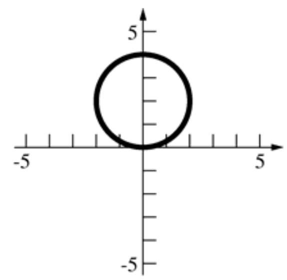
25. $r = \frac{3}{\sin\theta}$
$r = \frac{3}{\cos\left(\theta - \frac{\pi}{2}\right)}$, line



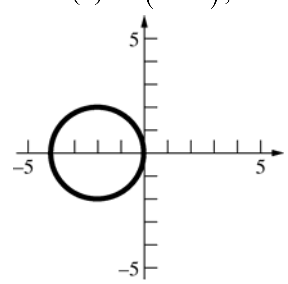
26. $r = -\frac{4}{\cos\theta}$
$r = \frac{4}{\cos(\theta - \pi)}$, line



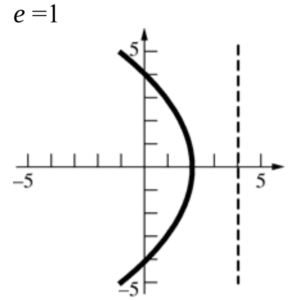
27. $r = 4\sin\theta$
$r = 2(2)\cos\left(\theta - \frac{\pi}{2}\right)$, circle



28. $r = -4\cos\theta$
$r = 2(2)\cos(\theta - \pi)$, circle



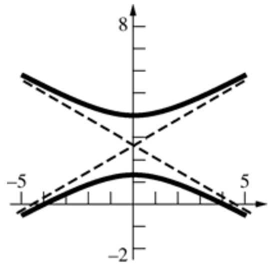
29. $r = \frac{4}{1 + \cos\theta}$
$r = \frac{(1)(4)}{1 + (1)\cos\theta}$, parabola
$e = 1$



30. $r = \dfrac{4}{1 + 2\sin\theta}$

$r = \dfrac{(2)(2)}{1 + 2\cos\left(\theta - \frac{\pi}{2}\right)}$, hyperbola

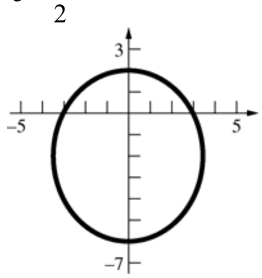
$e = 2$



31. $r = \dfrac{6}{2 + \sin\theta}$

$r = \dfrac{\left(\frac{1}{2}\right)6}{1 + \left(\frac{1}{2}\right)\cos\left(\theta - \frac{\pi}{2}\right)}$, ellipse

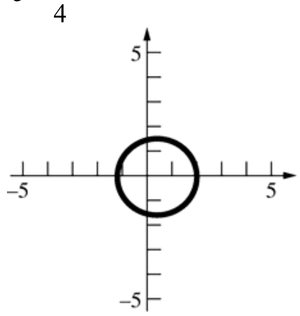
$e = \dfrac{1}{2}$



32. $r = \dfrac{6}{4 - \cos\theta}$

$r = \dfrac{6\left(\frac{1}{4}\right)}{1 + \left(\frac{1}{4}\right)\cos(\theta - \pi)}$, ellipse

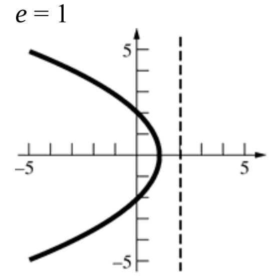
$e = \dfrac{1}{4}$



33. $r = \dfrac{4}{2 + 2\cos\theta}$

$r = \dfrac{(1)(2)}{1 + (1)\cos\theta}$, parabola

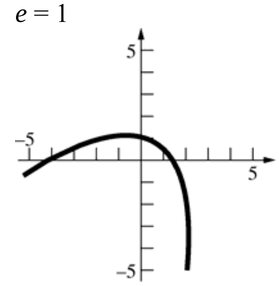
$e = 1$



34. $r = \dfrac{4}{2 + 2\cos\left(\theta - \frac{\pi}{3}\right)}$

$r = \dfrac{2(1)}{1 + (1)\cos\left(\theta - \frac{\pi}{3}\right)}$, parabola

$e = 1$

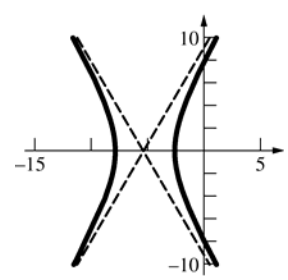


35. $r = \dfrac{4}{\frac{1}{2} + \cos(\theta - \pi)}$

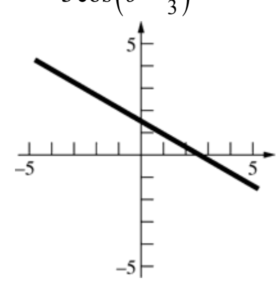
$r = \dfrac{4(2)}{1 + 2\cos(\theta - \pi)}$,

hyperbola

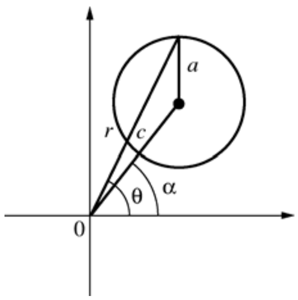
$e = 2$



36. $r = \dfrac{4}{3\cos\left(\theta - \frac{\pi}{3}\right)}$, line



37. By the Law of Cosines,
$a^2 = r^2 + c^2 - 2rc\cos(\theta - \alpha)$ (see figure below).



38. $r = a\sin\theta + b\cos\theta$

$r^2 = ar\sin\theta + br\cos\theta$

$x^2 + y^2 = ay + bx$

$x^2 - bx + y^2 - ay = 0$

$$\left(x^2 - bx + \frac{b^2}{4}\right) + \left(y^2 - ay + \frac{a^2}{4}\right) = \frac{a^2 + b^2}{4}$$

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

This is an equation of a circle with radius $\frac{\sqrt{a^2 + b^2}}{2}$ and center $\left(\frac{b}{2}, \frac{a}{2}\right)$.

39. Recall that the latus rectum is perpendicular to the axis of the conic through a focus.

$$r\left(\theta_0 + \frac{\pi}{2}\right) = \frac{ed}{1 + e\cos\frac{\pi}{2}} = ed$$

Thus the length of the latus rectum is $2ed$.

40. a. The point closest to the pole is at θ_0.

$$r_1 = r_1(\theta_0) = \frac{ed}{1 + e\cos(0)} = \frac{ed}{1+e}$$

The point furthest from the pole is at $\theta_0 + \pi$.

$$r_2 = r(\theta_0 + \pi) = \frac{ed}{1 + e\cos\pi} = \frac{ed}{1-e}$$

b. The length of the major diameter is

$$r_1 + r_2 = \frac{ed}{1+e} + \frac{ed}{1-e} = \frac{ed - e^2d}{1-e^2} + \frac{ed + e^2d}{1-e^2}$$

$$= \frac{2ed}{1-e^2}.$$

$$a = \frac{ed}{1-e^2}$$

$$c = ea = \frac{e^2d}{1-e^2}$$

$$b^2 = a^2 - c^2 = \left(\frac{ed}{1-e^2}\right)^2 - \left(\frac{e^2d}{1-e^2}\right)^2$$

$$= \frac{e^2d^2(1-e^2)}{(1-e^2)^2} = \frac{e^2d^2}{1-e^2}$$

$$b = \frac{ed}{\sqrt{1-e^2}}$$

The length of the minor diameter is $\frac{2ed}{\sqrt{1-e^2}}$.

41. $a + c = 183$, $a - c = 17$
$2a = 200$, $a = 100$
$2c = 166$, $c = 83$

$$e = \frac{c}{a} = 0.83$$

42. $a = \frac{185.8}{2} = 92.9$,

$c = ea = (0.0167)92.9 = 1.55143$

Perihelion $= a - c \approx 91.3$ million miles

43. Let sun lie at the pole and the axis of the parabola lie on the pole so that the parabola opens to the left. Then the path is described by the equation $r = \frac{d}{1 + \cos\theta}$. Substitute $(100, 120^\circ)$ into the equation and solve for d.

$$100 = \frac{d}{1 + \cos 120^\circ}$$

$d = 50$

The closest distance occurs when $\theta = 0^\circ$.

$$r = \frac{50}{1 + \cos 0^\circ} = 25 \text{ million miles}$$

44. a. $4 = \dfrac{d}{1+\cos\left(\frac{\pi}{2}-\theta_0\right)}$ $\qquad$ $3 = \dfrac{d}{1+\cos\left(\frac{\pi}{4}-\theta_0\right)}$

$4+4\left(\cos\dfrac{\pi}{2}\cos\theta_0+\sin\dfrac{\pi}{2}\sin\theta_0\right)=d$ $\qquad$ $3+3\left(\cos\dfrac{\pi}{4}\cos\theta_0+\sin\dfrac{\pi}{4}\sin\theta_0\right)=d$

$d=4+4\sin\theta_0$ $\qquad$ $d=3+\dfrac{3\sqrt{2}}{2}\cos\theta_0+\dfrac{3\sqrt{2}}{2}\sin\theta_0$

$$4+4\sin\theta_0=3+\frac{3\sqrt{2}}{2}\cos\theta_0+\frac{3\sqrt{2}}{2}\sin\theta_0$$

$$\frac{3\sqrt{2}}{2}\cos\theta_0+\left(\frac{3\sqrt{2}}{2}-4\right)\sin\theta_0-1=0$$

$$3\sqrt{2}\cos\theta_0+\left(3\sqrt{2}-8\right)\sin\theta_0-2=0$$

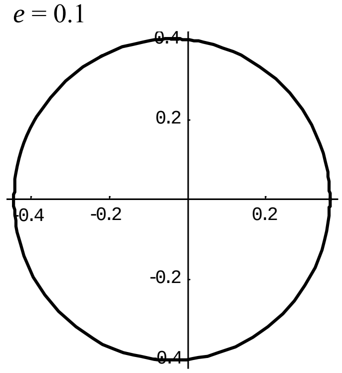
$$4.24\cos\theta_0-3.76\sin\theta_0-2=0$$

b. A graph shows that a root lies near 0.5. Using Newton's Method, $\theta_0 \approx 0.485$.
$d=4+4\sin\theta_0 \approx 5.86$

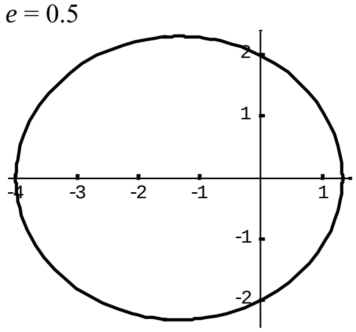
c. The closest the comet gets to the sun is $r=\dfrac{d}{1+\cos(\theta_0-\theta_0)}=\dfrac{d}{2}\approx 2.93$ AU

45. $x=\dfrac{4e}{1+e\cos t}\cos t,\ y=\dfrac{4e}{1+e\cos t}\sin t$

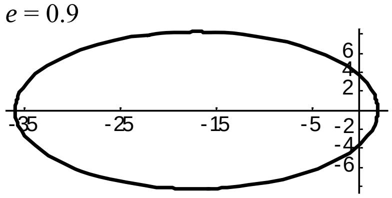
$e = 0.1$



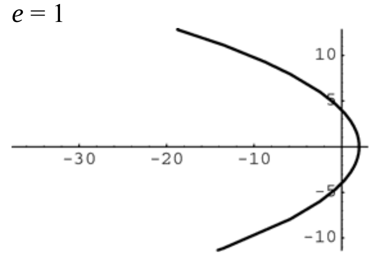
$e = 0.5$



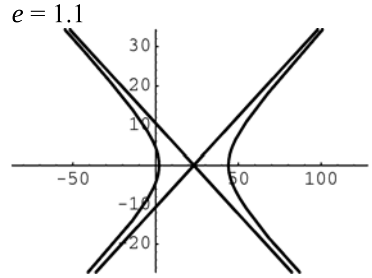
$e = 0.9$



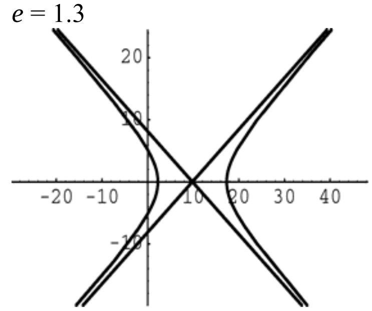
$e = 1$



$e = 1.1$



$e = 1.3$



10.6 Concepts Review

1. limaçon

2. cardioid

3. rose; odd; even

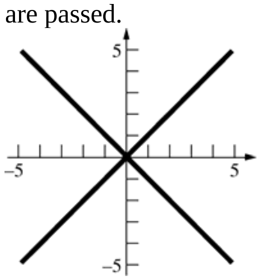
4. spiral

Problem Set 10.6

1. $\theta^2 - \frac{\pi^2}{16} = 0$

$\theta = \pm\frac{\pi}{4}$

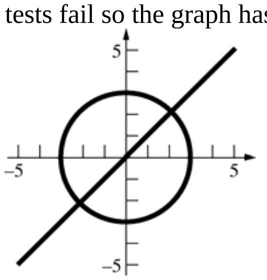
Changing $\theta \to -\theta$ or $r \to -r$ yields an equivalent set of equations. Therefore all 3 tests are passed.



2. $(r-3)\left(\theta - \frac{\pi}{4}\right) = 0$

$r = 3$ or $\theta = \frac{\pi}{4}$

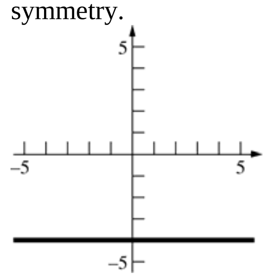
$\theta = \theta_0$ defines a line through the pole. Since a line forms an angle of π radians, changing $\theta \to \pi + \theta$ results in an equivalent set of equations, thus passing test 3. The other two tests fail so the graph has only origin symmetry.



3. $r \sin\theta + 4 = 0$

$r = -\frac{4}{\sin\theta}$

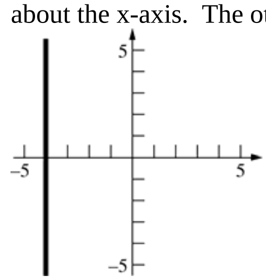
Since $\sin(-\theta) = -\sin\theta$, test 2 is passed. The other two tests fail so the graph has only y-axis symmetry.



4. $r = -4 \sec\theta$

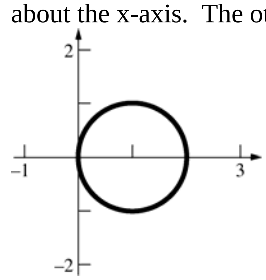
$r = -\frac{4}{\cos\theta}$

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



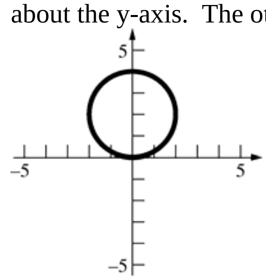
5. $r = 2\cos\theta$

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



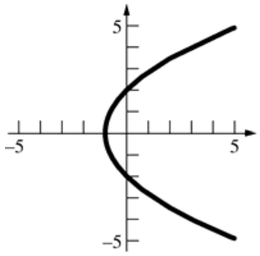
6. $r = 4\sin\theta$

Since $\sin(-\theta) = -\sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



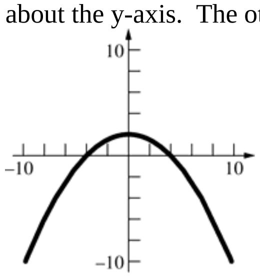
7. $r = \dfrac{2}{1-\cos\theta}$

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



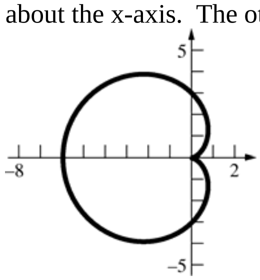
8. $r = \dfrac{4}{1+\sin\theta}$

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



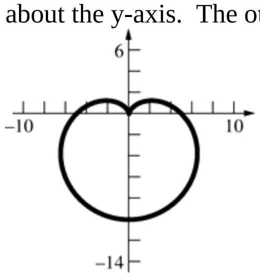
9. $r = 3 - 3\cos\theta$ (cardioid)

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



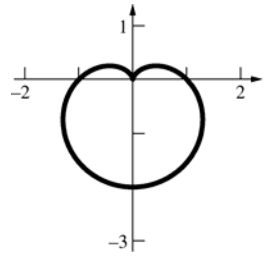
10. $r = 5 - 5\sin\theta$ (cardioid)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



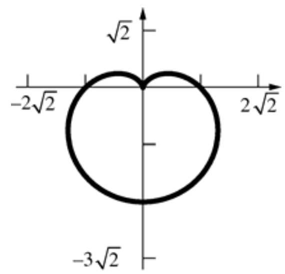
11. $r = 1 - 1\sin\theta$ (cardioid)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



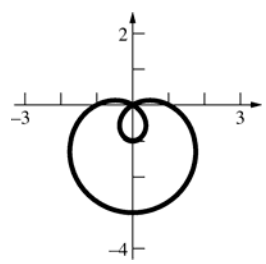
12. $r = \sqrt{2} - \sqrt{2}\sin\theta$ (cardioid)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



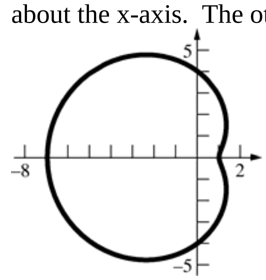
13. $r = 1 - 2\sin\theta$ (limaçon)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



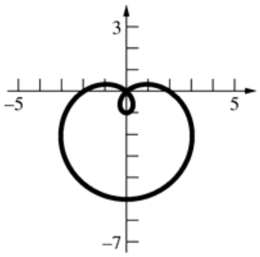
14. $r = 4 - 3\cos\theta$ (limaçon)

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



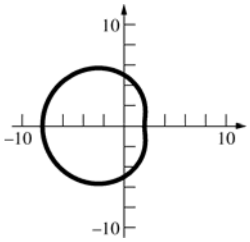
15. $r = 2 - 3\sin\theta$ (limaçon)

Since $\sin(\pi - \theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



16. $r = 5 - 3\cos\theta$ (limaçon)

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



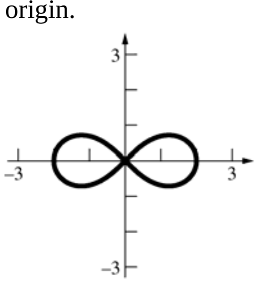
17. $r^2 = 4\cos 2\theta$ (lemniscate)

$r = \pm 2\sqrt{\cos 2\theta}$

Since $\cos(-2\theta) = \cos 2\theta$ and

$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$

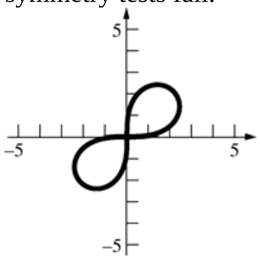
the graph is symmetric about both axes and the origin.



18. $r^2 = 9\sin 2\theta$ (lemniscate)

$r = \pm 3\sqrt{\sin(2\theta)}$

Since $\sin(2(\pi + \theta)) = \sin(2\pi + 2\theta) = \sin 2\theta$, the graph is symmetric about the origin. The other symmetry tests fail.



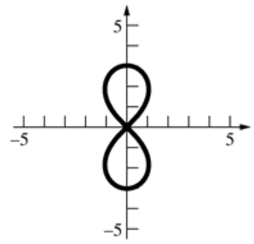
19. $r^2 = -9\cos 2\theta$ (lemniscate)

$r = \pm 3\sqrt{-\cos 2\theta}$

Since $\cos(-2\theta) = \cos 2\theta$ and

$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$

the graph is symmetric about both axes and the origin.



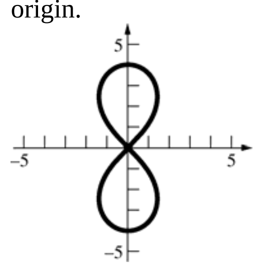
20. $r^2 = -16\cos 2\theta$ (lemniscate)

$r = \pm 4\sqrt{-\cos 2\theta}$

Since $\cos(-2\theta) = \cos 2\theta$ and

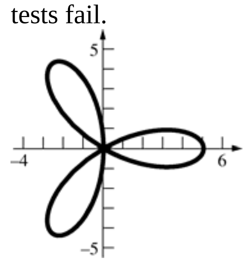
$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$

the graph is symmetric about both axes and the origin.



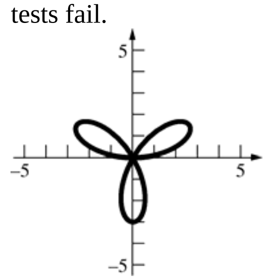
21. $r = 5\cos 3\theta$ (three-leaved rose)

Since $\cos(-3\theta) = \cos(3\theta)$, the graph is symmetric about the x-axis. The other symmetry tests fail.

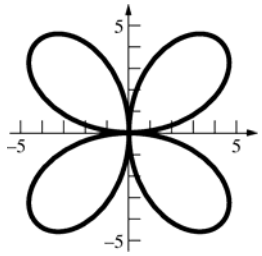


22. $r = 3\sin 3\theta$ (three-leaved rose)

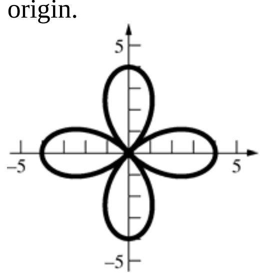
Since $\sin(-3\theta) = -\sin(3\theta)$, the graph is symmetric about the y-axis. The other symmetry tests fail.



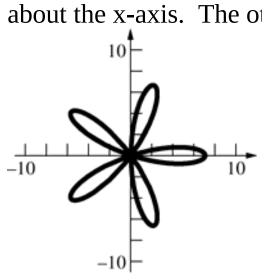
23. $r = 6\sin 2\theta$ (four-leaved rose)
Since
$$\sin(2(\pi-\theta)) = \sin(2\pi - 2\theta) = \sin(-2\theta) = -\sin(2\theta)$$
and $\sin(-2\theta) = -\sin(2\theta)$, the graph is symmetric about both axes and the origin.



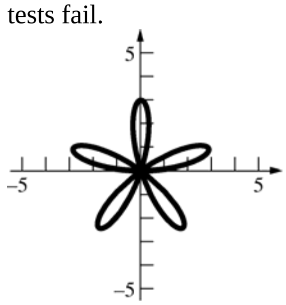
24. $r = 4\cos 2\theta$ (four-leaved rose)
Since $\cos(-2\theta) = \cos 2\theta$ and
$\cos(2(\pi-\theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$
the graph is symmetric about both axes and the origin.



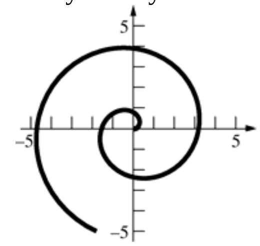
25. $r = 7\cos 5\theta$ (five-leaved rose)
Since $\cos(-5\theta) = \cos 5\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



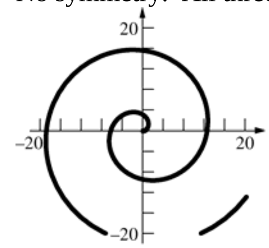
26. $r = 3\sin 5\theta$ (five-leaved rose)
Since $\sin(-5\theta) = -\sin 5\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



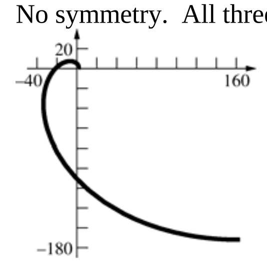
27. $r = \frac{1}{2}\theta, \theta \geq 0$ (spiral of Archimedes)
No symmetry. All three tests fail.



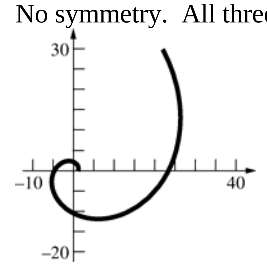
28. $r = 2\theta, \theta \geq 0$ (spiral of Archimedes)
No symmetry. All three tests fail.



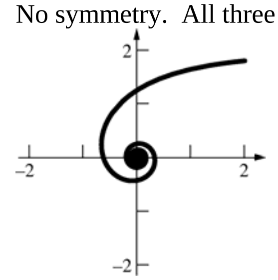
29. $r = e^{\theta}, \theta \geq 0$ (logarithmic spiral)
No symmetry. All three tests fail.



30. $r = e^{\theta/2}, \theta \geq 0$ (logarithmic spiral)
No symmetry. All three tests fail.

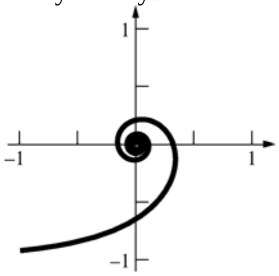


31. $r = \frac{2}{\theta}, \theta > 0$ (reciprocal spiral)
No symmetry. All three tests fail.

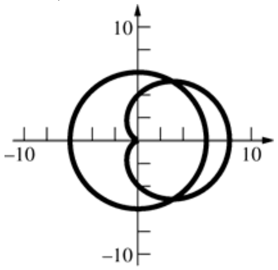


32. $r = -\dfrac{1}{\theta}, \theta > 0$ (reciprocal spiral)

No symmetry. All three tests fail.

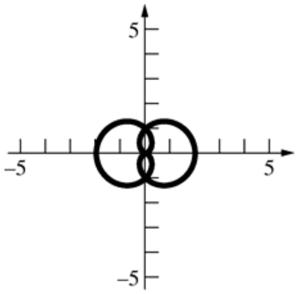


33. $r = 6,\ r = 4 + 4\cos\theta$



$$6 = 4 + 4\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}$$
$$\left(6, \frac{\pi}{3}\right), \left(6, \frac{5\pi}{3}\right)$$

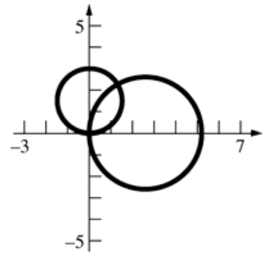
34. $r = 1 - \cos\theta, r = 1 + \cos\theta$



$$1 - \cos\theta = 1 + \cos\theta$$
$$\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}$$
$$\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$$

(0, 0) is also a solution since both graphs include the pole.

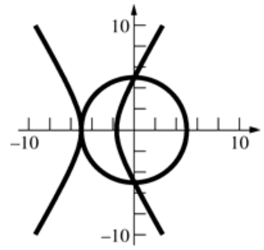
35. $r = 3\sqrt{3}\cos\theta, r = 3\sin\theta$



$$3\sqrt{3}\cos\theta = 3\sin\theta$$
$$\tan\theta = \sqrt{3}$$
$$\theta = \frac{\pi}{3}, \theta = \frac{4\pi}{3}$$
$$\left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right) = \left(-\frac{3\sqrt{3}}{2}, \frac{4\pi}{3}\right)$$

(0, 0) is also a solution since both graphs include the pole.

36. $r = 5, r = \dfrac{5}{1 - 2\cos\theta}$

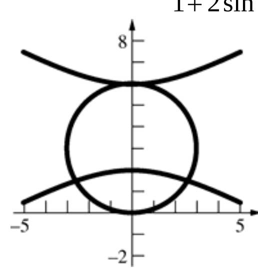


$$5 = \frac{5}{1 - 2\cos\theta}$$
$$\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}\ ;\ \left(5, \frac{\pi}{2}\right), \left(5, \frac{3\pi}{2}\right)$$

Note that r = –5 is equivalent to r = 5.

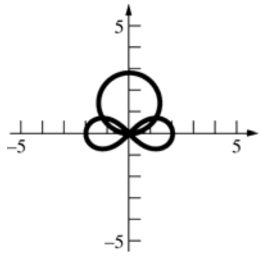
$$-5 = \frac{5}{1 - 2\cos\theta}$$
$$\cos\theta = 1$$
$$\theta = 0\ ;\ (-5, 0)$$

37. $r = 6\sin\theta, r = \dfrac{6}{1 + 2\sin\theta}$



$$6\sin\theta = \frac{6}{1+2\sin\theta}$$
$$12\sin^2\theta + 6\sin\theta - 6 = 0$$
$$6(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2}, \sin\theta = -1$$
$$\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}, \theta = \frac{3\pi}{2}$$
$$\left(3, \frac{\pi}{6}\right), \left(3, \frac{5\pi}{6}\right), \left(-6, \frac{3\pi}{2}\right) \text{ or } \left(6, \frac{\pi}{2}\right)$$

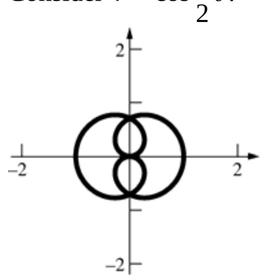
38. $r^2 = 4\cos 2\theta, r = 2\sqrt{2}\sin\theta$



$$4\cos 2\theta = \left(2\sqrt{2}\sin\theta\right)^2$$
$$4 - 8\sin^2\theta = 8\sin^2\theta$$
$$\sin^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \pm\frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}, \theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$$
$$\left(\sqrt{2}, \frac{\pi}{6}\right) = \left(-\sqrt{2}, \frac{7\pi}{6}\right),$$
$$\left(\sqrt{2}, \frac{5\pi}{6}\right) = \left(-\sqrt{2}, \frac{11\pi}{6}\right)$$

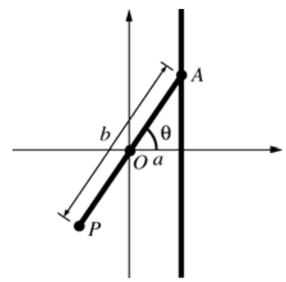
(0, 0) is also a solution since both graphs includes the pole.

39. Consider $r = \cos\frac{1}{2}\theta.$



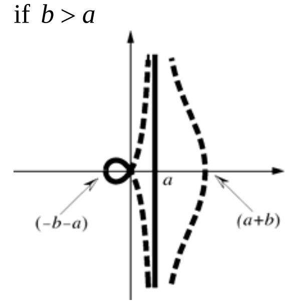
The graph is clearly symmetric with respect to the y-axis.
Substitute (r, θ) by $(-r, -\theta)$.
$$-r = \cos\left(-\frac{1}{2}\theta\right) = \cos\frac{1}{2}\theta$$
$$r = -\cos\frac{1}{2}\theta$$
Substitute (r, θ) by $(r, \pi - \theta)$
$$r = \cos\frac{1}{2}(\pi - \theta) = \cos\frac{1}{2}\pi\cos\frac{1}{2}\theta + \sin\frac{1}{2}\pi\sin\frac{1}{2}\theta$$
$$= \sin\frac{1}{2}\theta$$
$$r = \sin\frac{1}{2}\theta$$

40. Consider the following figure.

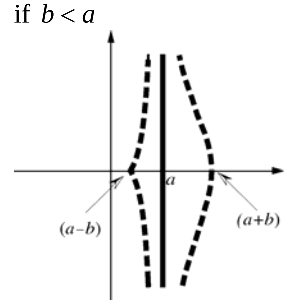


$$r = \frac{a}{\cos\theta} - b$$
$$r\cos\theta = a - b\cos\theta$$
$$x = a - b\cos\theta$$
$$xr = ar - br\cos\theta$$
$$(x-a)r = -bx$$
$$(x-a)^2 r^2 = b^2 x^2$$
$$(x-a)^2(x^2+y^2) = b^2x^2$$
$$y^2 = \frac{b^2x^2}{(x-a)^2} - x^2$$

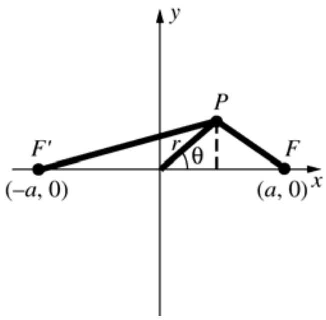
if $b > a$



if $b < a$



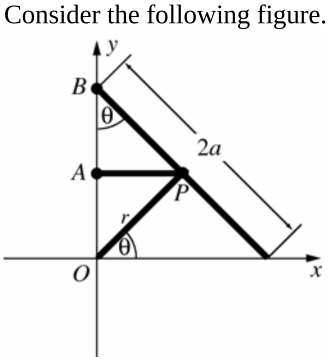
41.



$$|PF| = \sqrt{(a - r\cos\theta)^2 + (r\sin\theta)^2}$$
$$= \sqrt{r^2 + a^2 - 2ar\cos\theta}$$
$$|PF'| = \sqrt{(a + r\cos\theta)^2 + (r\sin\theta)^2}$$
$$= \sqrt{r^2 + a^2 + 2ar\cos\theta}$$
$$|PF||PF'| = \sqrt{(r^2 + a^2)^2 - 4a^2r^2\cos^2\theta} = a^2$$
$$(r^4 + 2a^2r^2 + a^4) - 4a^2r^2\cos^2\theta = a^4$$
$$r^4 - 4a^2r^2\cos^2\theta + 2a^2r^2 = 0$$
$$r^2(r^2 - 2a^2(2\cos^2\theta - 1)) = 0$$
$$r^2 - 2a^2(2\cos^2\theta - 1) = 0$$
$$r^2 = 2a^2(1 + \cos 2\theta - 1)$$
$$r^2 = 2a^2\cos 2\theta$$

This is the equation of a lemniscate.

42. Consider the following figure.



Then $\tan\theta = \frac{\overline{AP}}{\overline{BA}} = \frac{r\cos\theta}{2a\cos\theta - r\sin\theta}$

$$\frac{\sin\theta}{\cos\theta} = \frac{r\cos\theta}{2a\cos\theta - r\sin\theta}$$
$$2a\sin\theta\cos\theta - r\sin^2\theta = r\cos^2\theta$$
$$r\cos^2\theta + r\sin^2\theta = 2a\sin\theta\cos\theta$$
$$r = a\sin 2\theta$$

This is a polar equation for a four-leaved rose.

43. **a.** $y = 45$

$r\sin\theta = 45$

$$r = \frac{45}{\sin\theta}$$

b. $x^2 + y^2 = 36$

$r^2 = 36$

$r = 6$

c. $x^2 - y^2 = 1$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2 = \frac{1}{\cos 2\theta}$$
$$r = \pm\frac{1}{\sqrt{\cos 2\theta}}$$

d. $4xy = 1$

$$4r^2\cos\theta\sin\theta = 1$$
$$r^2 = \frac{1}{2\sin 2\theta}$$
$$r = \pm\frac{1}{\sqrt{2\sin 2\theta}}$$

e. $y = 3x + 2$

$r\sin\theta = 3r\cos\theta + 2$

$r(\sin\theta - 3\cos\theta) = 2$

$$r = \frac{2}{\sin\theta - 3\cos\theta}$$

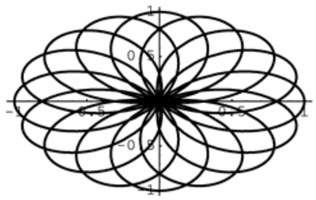
f. $3x^2 + 4y = 2$

$$3r^2\cos^2\theta + 4r\sin\theta = 2$$
$$(3\cos^2\theta)r^2 + (4\sin\theta)r - 2 = 0$$
$$r = \frac{-4\sin\theta \pm \sqrt{16\sin^2\theta + 24\cos^2\theta}}{6\cos^2\theta}$$
$$r = \frac{-2\sin\theta \pm \sqrt{4\sin^2\theta + 6\cos^2\theta}}{3\cos^2\theta}$$

g. $x^2 + 2x + y^2 - 4y - 25 = 0$

$$r^2 + 2r\cos\theta - 4r\sin\theta - 25 = 0$$
$$r^2 + (2\cos\theta - 4\sin\theta)r - 25 = 0$$
$$r = \frac{-2\cos\theta + 4\sin\theta \pm \sqrt{(2\cos\theta - 4\sin\theta)^2 + 100}}{2}$$
$$r = -\cos\theta + 2\sin\theta \pm \sqrt{(\cos\theta - 2\sin\theta)^2 + 25}$$

44.



The curve repeats itself after period p if $f(\theta + p) = f(\theta)$.

$$\cos\left(\frac{8(\theta+p)}{5}\right) = \cos\left(\frac{8\theta}{5} + \frac{p}{5}\right)$$

We need $\frac{p}{5} = 2\pi$.

45. **a.** VII

b. I

c. VIII

d. III

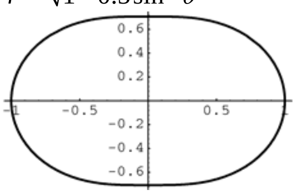
e. V

f. II

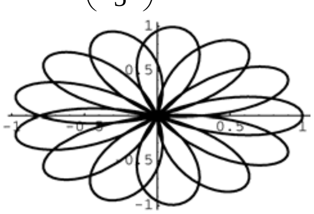
g. VI

h. IV

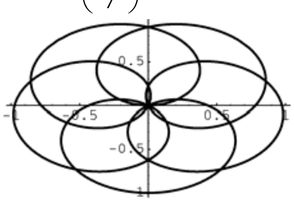
46. $r = \sqrt{1 - 0.5\sin^2\theta}$



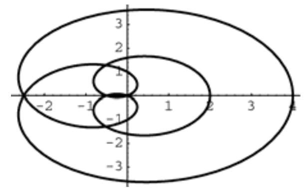
47. $r = \cos\left(\frac{13\theta}{5}\right)$



48. $r = \sin\left(\frac{5\theta}{7}\right)$



49. $r = 1 + 3\cos\left(\frac{\theta}{3}\right)$



50. **a.** The graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is the rotation of the graph of $r = 1 + \sin\theta$ by $\frac{\pi}{3}$ counter-clockwise about the pole. The graph of $r = 1 + \sin\left(\theta + \frac{\pi}{3}\right)$ is the rotation of the graph of $r = 1 + \sin\theta$ by $\frac{\pi}{3}$ clockwise about the pole.

b. $r = 1 - \sin\theta = 1 + \sin(\theta - \pi)$
The graph of $r = 1 + \sin\theta$ is the rotation of the graph of $r = 1 - \sin\theta$ by π about the pole.

c. $r = 1 + \cos\theta = 1 + \sin\left(\theta + \frac{\pi}{2}\right)$
The graph of $r = 1 + \sin\theta$ is the rotation of the graph of $r = 1 + \cos\theta$ by $\frac{\pi}{2}$ counter-clockwise about the pole.

d. The graph of $r = f(\theta)$ is the rotation of the graph of $r = f(\theta - \alpha)$ by a clockwise about the pole.

51. **a.** The graph for $\phi = 0$ is the graph for $\phi \neq 0$ rotated by ϕ counterclockwise about the pole.

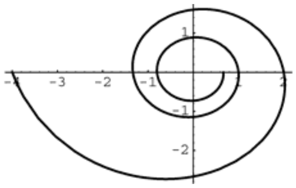
b. As n increases, the number of "leaves" increases.

c. If $|a| > |b|$, the graph will not pass through the pole and will not "loop." If $|b| < |a|$, the graph will pass through the pole and will have $2n$ "loops" (n small "loops" and n large "loops"). If $|a| = |b|$, the graph passes through the pole and will have n "loops." If $ab \neq 0, n > 1,$ and $\phi = 0$, the graph will be symmetric about $\theta = \frac{\pi}{n}k$, where $k = 0, n - 1$.

52. The number of loops is $2n$.

53. The spiral will unwind clockwise for $c < 0$. The spiral will unwind counter-clockwise for $c > 0$.

54. This is for $c = 4\pi$.



The spiral will wind in the counter-clockwise direction.

55. **a.** III

b. IV

c. I

d. II

e. VI

f. V

10.7 Concepts Review

1. $\frac{1}{2}r^2\theta$

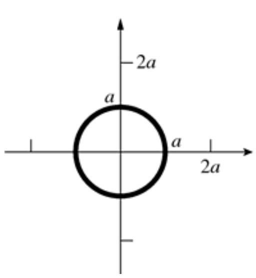
2. $\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta$

3. $\frac{1}{2}\int_{0}^{2\pi}(2+2\cos\theta)^2\,d\theta$

4. $f(\theta) = 0$

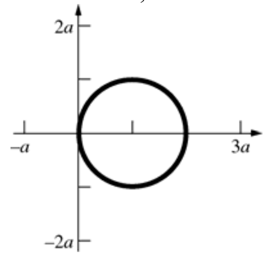
Problem Set 10.7

1. $r = a,\ a > 0$



$$A = \frac{1}{2}\int_0^{2\pi} a^2\,d\theta = \pi a^2$$

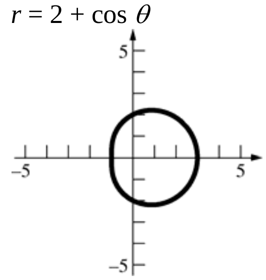
2. $r = 2a\cos\theta,\ a > 0$



$$A = \frac{1}{2}\int_0^{\pi}(2a\cos\theta)^2\,d\theta = 2a^2\int_0^{\pi}\cos^2\theta\,d\theta$$

$$= a^2\int_0^{\pi}(1+\cos 2\theta)d\theta = a^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \pi a^2$$

3. $r = 2 + \cos\theta$



$$A = \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2\,d\theta$$

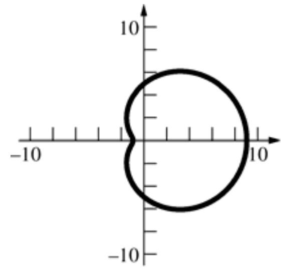
$$= \frac{1}{2}\int_0^{2\pi}(4+4\cos\theta+\cos^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[4+4\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{9}{2}+4\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\left[\frac{9}{2}\theta+4\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{9}{2}\pi$$

4. $r = 5 + 4\cos\theta$



$$A = \frac{1}{2}\int_0^{2\pi}(5+4\cos\theta)^2\,d\theta$$

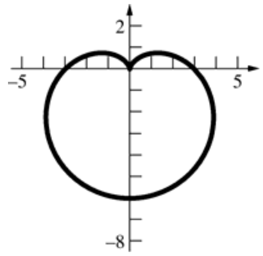
$$= \frac{1}{2}\int_0^{2\pi}(25+40\cos\theta+16\cos^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}[25+40\cos\theta+8(1+\cos 2\theta)]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(33+40\cos\theta+8\cos 2\theta)d\theta$$

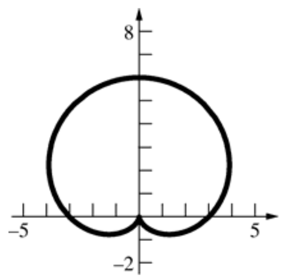
$$= \frac{1}{2}[33\theta+40\sin\theta+4\sin 2\theta]_0^{2\pi} = 33\pi$$

5. $r = 3 - 3\sin\theta$



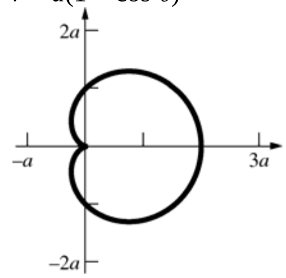
$$A = \frac{1}{2}\int_0^{2\pi}(3-3\sin\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}(9-18\sin\theta+9\sin^2\theta)d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left[9-18\sin\theta+\frac{9}{2}(1-\cos 2\theta)\right]d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{27}{2}-18\sin\theta-\frac{9}{2}\cos 2\theta\right)d\theta$$
$$= \frac{1}{2}\left[\frac{27}{2}\theta+18\cos\theta-\frac{9}{4}\sin 2\theta\right]_0^{2\pi} = \frac{27}{2}\pi$$

6.



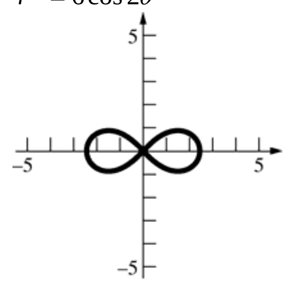
$$A = \frac{1}{2}\int_0^{2\pi}(3+3\sin\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}(9+18\sin\theta+9\sin^2\theta)\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left[9+18\sin\theta+\frac{9}{2}(1-\cos 2\theta)\right]d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{27}{2}+18\sin\theta-\frac{9}{2}\cos 2\theta\right)d\theta$$
$$= \frac{1}{2}\left[\frac{27}{2}\theta-18\cos\theta-\frac{9}{4}\sin 2\theta\right]_0^{2\pi}$$
$$= \frac{27}{2}\pi$$

7. $r = a(1 + \cos\theta)$



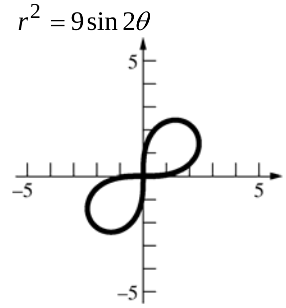
$$A = \frac{1}{2}\int_0^{2\pi}[a(1+\cos\theta)]^2\,d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi}\left[1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$
$$= \frac{a^2}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi a^2}{2}$$

8. $r^2 = 6\cos 2\theta$



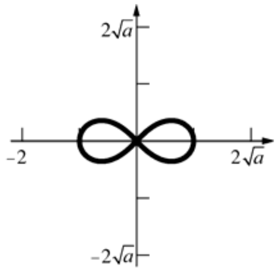
$$A = 2\cdot\frac{1}{2}\int_{-\pi/4}^{\pi/4} 6\cos 2\theta\,d\theta = 6\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta$$
$$= 3[\sin 2\theta]_{-\pi/4}^{\pi/4} = 6$$

9. $r^2 = 9\sin 2\theta$



$$A = 2\cdot\frac{1}{2}\int_0^{\pi/2} 9\sin 2\theta\,d\theta = 9\int_0^{\pi/2}\sin 2\theta\,d\theta$$
$$= \frac{9}{2}[-\cos 2\theta]_0^{\pi/2} = 9$$

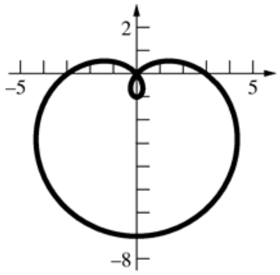
10. $r^2 = a\cos 2\theta$



$$A = 2\cdot\frac{1}{2}\int_{-\pi/4}^{\pi/4} a\cos 2\theta\, d\theta = a\int_{-\pi/4}^{\pi/4} \cos 2\theta\, d\theta$$

$$= \frac{a}{2}[\sin 2\theta]_{-\pi/4}^{\pi/4} = a$$

11. $r = 3 - 4\sin\theta$



$$3 - 4\sin\theta = 0,\ \theta = \sin^{-1}\frac{3}{4}$$

$$A = 2\cdot\frac{1}{2}\int_{\sin^{-1}\frac{3}{4}}^{\pi/2}(3-4\sin\theta)^2\, d\theta$$

$$= \int_{\sin^{-1}\frac{3}{4}}^{\pi/2}(9-24\sin\theta+16\sin^2\theta)d\theta$$

$$= \int_{\sin^{-1}\frac{3}{4}}^{\pi/2}[9-24\sin\theta+8(1-\cos 2\theta)]d\theta$$

$$= \int_{\sin^{-1}\frac{3}{4}}^{\pi/2}(17-24\sin\theta-8\cos 2\theta)d\theta$$

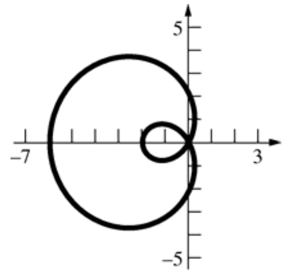
$$= [17\theta+24\cos\theta-4\sin 2\theta]_{\sin^{-1}\frac{3}{4}}^{\pi/2}$$

$$= [17\theta+24\cos\theta-8\sin\theta\cos\theta]_{\sin^{-1}\frac{3}{4}}^{\pi/2}$$

$$= \frac{17\pi}{2} - \left[17\sin^{-1}\frac{3}{4} + 24\left(\frac{\sqrt{7}}{4}\right) - 8\left(\frac{3}{4}\right)\left(\frac{\sqrt{7}}{4}\right)\right]$$

$$= \frac{17\pi}{2} - 17\sin^{-1}\frac{3}{4} - \frac{9\sqrt{7}}{2}$$

12. $r = 2 - 4\cos\theta$



$$2 - 4\cos\theta = 0,\ \theta = \frac{\pi}{3}$$

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/3}(2-4\cos\theta)^2\, d\theta$$

$$= \int_0^{\pi/3}(4-16\cos\theta+16\cos^2\theta)d\theta$$

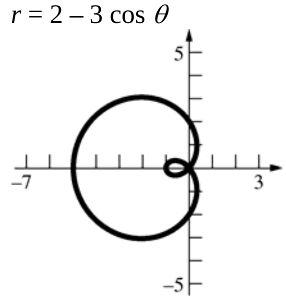
$$= \int_0^{\pi/3}[4-16\cos\theta+8(1+\cos 2\theta)]d\theta$$

$$= \int_0^{\pi/3}(12-16\cos\theta+8\cos 2\theta)d\theta$$

$$= \left[12\theta - 16\sin\theta + 4\sin 2\theta\right]_0^{\pi/3}$$

$$= 4\pi - 6\sqrt{3}$$

13. $r = 2 - 3\cos\theta$



$$2 - 3\cos\theta = 0,\ \theta = \cos^{-1}\frac{2}{3}$$

$$A = 2\cdot\frac{1}{2}\int_{\cos^{-1}\frac{2}{3}}^{\pi}(2-3\cos\theta)^2\, d\theta$$

$$= \int_{\cos^{-1}\frac{2}{3}}^{\pi}(4-12\cos\theta+9\cos^2\theta)d\theta$$

$$= \int_{\cos^{-1}\frac{2}{3}}^{\pi}\left[4-12\cos\theta+\frac{9}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \int_{\cos^{-1}\frac{2}{3}}^{\pi}\left(\frac{17}{2}-12\cos\theta+\frac{9}{2}\cos 2\theta\right)d\theta$$

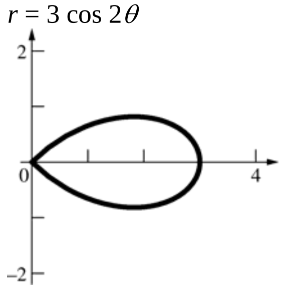
$$= \left[\frac{17}{2}\theta - 12\sin\theta + \frac{9}{4}\sin 2\theta\right]_{\cos^{-1}\frac{2}{3}}^{\pi}$$

$$= \left[\frac{17\theta}{2} - 12\sin\theta + \frac{9}{2}\sin\theta\cos\theta\right]_{\cos^{-1}\frac{2}{3}}^{\pi}$$

$$= \frac{17\pi}{2} - \left[\frac{17}{2}\cos^{-1}\frac{2}{3} - 12\left(\frac{\sqrt{5}}{3}\right) + \frac{9}{2}\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)\right]$$

$$= \frac{17\pi}{2} - \frac{17}{2}\cos^{-1}\frac{2}{3} + 3\sqrt{5}$$

14. $r = 3\cos 2\theta$

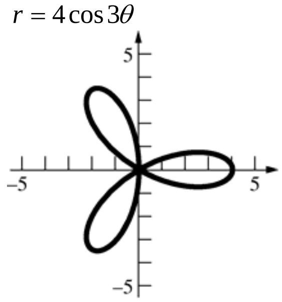


$$A = 2\cdot\frac{1}{2}\int_0^{\pi/4}(3\cos 2\theta)^2\,d\theta = 9\int_0^{\pi/4}\cos^2 2\theta\,d\theta$$

$$= 9\int_0^{\pi/4}\frac{1}{2}(1+\cos 4\theta)d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{9\pi}{8}$$

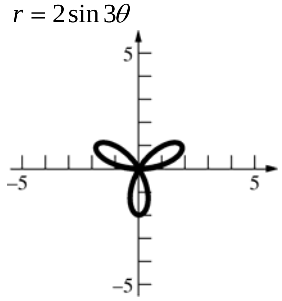
15. $r = 4\cos 3\theta$



$$A = 6\cdot\frac{1}{2}\int_0^{\pi/6}(4\cos 3\theta)^2\,d\theta = 48\int_0^{\pi/6}\cos^2 3\theta\,d\theta$$

$$= 24\int_0^{\pi/6}(1+\cos 6\theta)d\theta = 24\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 4\pi$$

16. $r = 2\sin 3\theta$



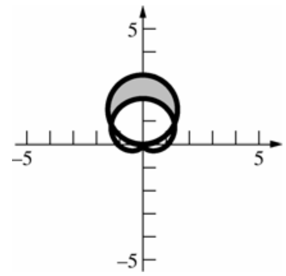
$$A = 3\cdot\frac{1}{2}\int_0^{\pi/3}(2\sin 3\theta)^2\,d\theta = 6\int_0^{\pi/3}\sin^2 3\theta\,d\theta$$

$$= 3\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta$$

$$= 3\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \pi$$

17. $A = \frac{1}{2}\int_0^{2\pi}100\,d\theta - \frac{1}{2}\int_0^{2\pi}49\,d\theta = 51\pi$

18. $r = 3\sin\theta,\ r = 1+\sin\theta$



Solve for the θ-coordinate of the first intersection point.

$3\sin\theta = 1+\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

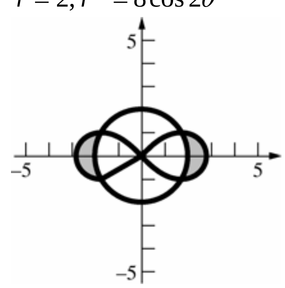
$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}[(3\sin\theta)^2 - (1+\sin\theta)^2]d\theta$$

$$= \int_{\pi/6}^{\pi/2}(8\sin^2\theta - 2\sin\theta - 1)d\theta$$

$$= \int_{\pi/6}^{\pi/2}(3 - 4\cos 2\theta - 2\sin\theta)d\theta$$

$$= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\pi/6}^{\pi/2} = \pi$$

19. $r = 2,\ r^2 = 8\cos 2\theta$



Solve for the θ-coordinate of the first intersection point.

$4 = 8\cos 2\theta$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}$

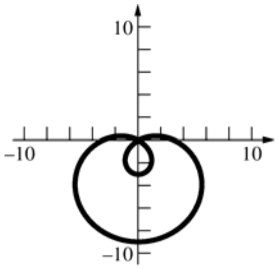
$\theta = \frac{\pi}{6}$

$$A = 4\cdot\frac{1}{2}\int_0^{\pi/6}(8\cos 2\theta - 4)d\theta$$

$$= 2[4\sin 2\theta - 4\theta]_0^{\pi/6}$$

$$= 4\sqrt{3} - \frac{4\pi}{3}$$

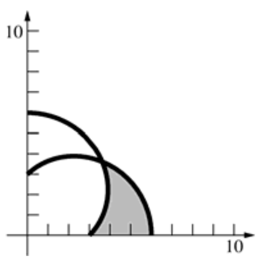
20. $r = 3 - 6\sin\theta$



Let A_1 be the area inside the large loop and let A_2 be the area inside the small loop.

$$A_1 = 2\cdot\frac{1}{2}\int_{-\pi/2}^{\pi/6}(3-6\sin\theta)^2\,d\theta$$
$$= \int_{-\pi/2}^{\pi/6}(9-36\sin\theta+36\sin^2\theta)\,d\theta$$
$$= \int_{-\pi/2}^{\pi/6}(27-36\sin\theta-18\cos 2\theta)\,d\theta$$
$$= \left[27\theta+36\cos\theta-9\sin 2\theta\right]_{-\pi/2}^{\pi/6} = 18\pi+\frac{27\sqrt{3}}{2}$$
$$A_2 = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}(3-6\sin\theta)^2\,d\theta$$
$$= [27\theta+36\cos\theta-9\sin 2\theta]_{\pi/6}^{\pi/2} = 9\pi-\frac{27\sqrt{3}}{2}$$
$$A = A_1 - A_2 = 9\pi + 27\sqrt{3}$$

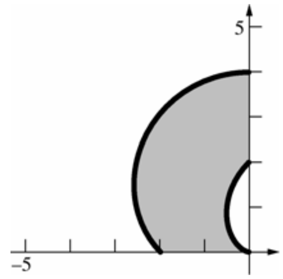
21. $r = 3 + 3\cos\theta,\ r = 3 + 3\sin\theta$



Solve for the θ-coordinate of the intersection point.

$$3+3\cos\theta = 3+3\sin\theta$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$
$$A = \frac{1}{2}\int_0^{\pi/4}[(3+3\cos\theta)^2-(3+3\sin\theta)^2]\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/4}(18\cos\theta+9\cos^2\theta-18\sin\theta-9\sin^2\theta)\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/4}(18\cos\theta-18\sin\theta+9\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\left[18\sin\theta+18\cos\theta+\frac{9}{2}\sin 2\theta\right]_0^{\pi/4}$$
$$= 9\sqrt{2}-\frac{27}{4}$$

22. $r = 2+2\sin\theta,\ r = 2+2\cos\theta$



$$A = \frac{1}{2}\int_{\pi/2}^{\pi}[(2+2\sin\theta)^2-(2+2\cos\theta)^2]\,d\theta$$
$$= \frac{1}{2}\int_{\pi/2}^{\pi}(8\sin\theta+4\sin^2\theta-8\cos\theta-4\cos^2\theta)\,d\theta$$
$$= \frac{1}{2}\int_{\pi/2}^{\pi}(8\sin\theta-8\cos\theta-4\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\left[-8\cos\theta-8\sin\theta-2\sin 2\theta\right]_{\pi/2}^{\pi} = 8$$

23. a. $f(\theta) = 2\cos\theta,\ f'(\theta) = -2\sin\theta$

$$m = \frac{(2\cos\theta)\cos\theta+(-2\sin\theta)\sin\theta}{-(2\cos\theta)\sin\theta+(-2\sin\theta)\cos\theta}$$
$$= \frac{2\cos^2\theta-2\sin^2\theta}{-4\cos\theta\sin\theta} = \frac{\cos 2\theta}{-\sin 2\theta}$$

At $\theta = \frac{\pi}{3}$, $m = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$.

b. $f(\theta) = 1+\sin\theta,\ f'(\theta) = \cos\theta$

$$m = \frac{(1+\sin\theta)\cos\theta+(\cos\theta)\sin\theta}{-(1+\sin\theta)\sin\theta+(\cos\theta)\cos\theta}$$
$$= \frac{\cos\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta-\sin\theta} = \frac{\cos\theta+\sin 2\theta}{\cos 2\theta-\sin\theta}$$

At $\theta = \frac{\pi}{3}$, $m = \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}}{-\frac{1}{2}-\frac{\sqrt{3}}{2}} = -1$.

c. $f(\theta) = \sin 2\theta,\ f'(\theta) = 2\cos 2\theta$

$$m = \frac{(\sin 2\theta)\cos\theta+(2\cos 2\theta)\sin\theta}{-(\sin 2\theta)\sin\theta+(2\cos 2\theta)\cos\theta}$$

At $\theta = \frac{\pi}{3}$, .

$$m = \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)+(-1)\left(\frac{\sqrt{3}}{2}\right)}{-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+(-1)\left(\frac{1}{2}\right)} = \frac{-\frac{\sqrt{3}}{4}}{-\frac{5}{4}} = \frac{\sqrt{3}}{5}$$

d. $f(\theta)=4-3\cos\theta,\ f'(\theta)=3\sin\theta$

$$m=\frac{(4-3\cos\theta)\cos\theta+(3\sin\theta)\sin\theta}{-(4-3\cos\theta)\sin\theta+(3\sin\theta)\cos\theta}$$

$$=\frac{4\cos\theta-3\cos^2\theta+3\sin^2\theta}{-4\sin\theta+6\sin\theta\cos\theta}$$

$$=\frac{4\cos\theta-3\cos 2\theta}{-4\sin\theta+3\sin 2\theta}$$

At $\theta=\frac{\pi}{3}$,

$$m=\frac{4\left(\frac{1}{2}\right)-3\left(-\frac{1}{2}\right)}{-4\left(\frac{\sqrt{3}}{2}\right)+3\left(\frac{\sqrt{3}}{2}\right)}=\frac{\frac{7}{2}}{-\frac{\sqrt{3}}{2}}=-\frac{7}{\sqrt{3}}.$$

24. $f(\theta)=a(1+\cos\theta),\ f'(\theta)=-a\sin\theta$

$$m=\frac{a(1+\cos\theta)\cos\theta+(-a\sin\theta)\sin\theta}{-a(1+\cos\theta)\sin\theta+(-a\sin\theta)\cos\theta}$$

$$=\frac{\cos\theta+\cos^2\theta-\sin^2\theta}{-\sin\theta-2\sin\theta\cos\theta}=\frac{2\cos^2\theta+\cos\theta-1}{-\sin\theta(1+2\cos\theta)}$$

a. $m=0$ when $2\cos^2\theta+\cos\theta-1=0$

$(2\cos\theta-1)(\cos\theta+1)=0.$

$\cos\theta=\frac{1}{2},\cos\theta=-1$

$\theta=\frac{\pi}{3},-\frac{\pi}{3},\theta=\pi$; when

$\theta=\pi,\ f(\theta)=0,$ so $\theta=\pi$ is the tangent line.

$\left(\frac{3a}{2},\frac{\pi}{3}\right),\left(\frac{3a}{2},-\frac{\pi}{3}\right),(0,\pi)$

b. m is undefined when $\sin\theta(1+2\cos\theta)=0$ and $2\cos^2\theta+\cos\theta-1\neq 0$.

$\theta=0,\frac{2\pi}{3},\frac{4\pi}{3}$

$(2a,0),\left(\frac{a}{2},\frac{2\pi}{3}\right),\left(\frac{a}{2},\frac{4\pi}{3}\right)$

There is no vertical tangent at $\theta=\pi$ since $\lim_{\theta\to 0} m(\theta)=0$ (see part (a)).

25. $f(\theta)=1-2\sin\theta,\ f'(\theta)=-2\cos\theta$

$$m=\frac{(1-2\sin\theta)\cos\theta+(-2\cos\theta)\sin\theta}{-(1-2\sin\theta)\sin\theta+(-2\cos\theta)\cos\theta}$$

$$=\frac{\cos\theta-4\sin\theta\cos\theta}{-\sin\theta+2\sin^2\theta-2\cos^2\theta}$$

$$=\frac{\cos\theta(1-4\sin\theta)}{-\sin\theta+2\sin^2\theta-2\cos^2\theta}$$

$m=0$ when $\cos\theta(1-4\sin\theta)=0$

$\cos\theta=0,$ or $1-4\sin\theta=0$

$\theta=\frac{\pi}{2},\theta=\frac{3\pi}{2},\theta=\sin^{-1}\left(\frac{1}{4}\right)\approx 0.25,$

$\theta=\pi-\sin^{-1}\left(\frac{1}{4}\right)\approx 2.89$

$f\left(\frac{\pi}{2}\right)=-1,\ f\left(\frac{3\pi}{2}\right)=3,\ f\left(\sin^{-1}\left(\frac{1}{4}\right)\right)=\frac{1}{2},$

$f\left(\pi-\sin^{-1}\left(\frac{1}{4}\right)\right)=\frac{1}{2}$

$\left(-1,\frac{\pi}{2}\right),\left(3,\frac{3\pi}{2}\right),\left(\frac{1}{2},0.25\right),\left(\frac{1}{2},2.89\right)$

26. Recall from Chapter 5 that $L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ for x and y functions of t and $a\le t\le b$.

$x=r\cos\theta=f(\theta)\cos\theta,\ y=r\sin\theta=f(\theta)\sin\theta$

$$\frac{dx}{d\theta}=f'(\theta)\cos\theta-f(\theta)\sin\theta,\ \frac{dy}{d\theta}=f'(\theta)\sin\theta+f(\theta)\cos\theta$$

$$L=\int_\alpha^\beta\sqrt{(f'(\theta)\cos\theta-f(\theta)\sin\theta)^2+(f'(\theta)\sin\theta+f(\theta)\cos\theta)^2}\,d\theta$$

$$=\int_\alpha^\beta\sqrt{[f(\theta)]^2(\sin^2\theta+\cos^2\theta)+[f'(\theta)]^2(\sin^2\theta+\cos^2\theta)}\,d\theta=\int_\alpha^\beta\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta$$

27. $f(\theta)=a(1+\cos\theta),\ f'(\theta)=-a\sin\theta$

$$L=\int_0^{2\pi}\sqrt{[a(1+\cos\theta)]^2+[-a\sin\theta]^2}\,d\theta=a\int_0^{2\pi}\sqrt{2+2\cos\theta}\,d\theta=2a\int_0^{2\pi}\sqrt{\frac{1+\cos\theta}{2}}\,d\theta=2a\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|d\theta$$

$$=2a\left[\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta-\int_\pi^{2\pi}\cos\frac{\theta}{2}\,d\theta\right]=2a\left(\left[2\sin\frac{\theta}{2}\right]_0^{\pi}-\left[2\sin\frac{\theta}{2}\right]_\pi^{2\pi}\right)=8a$$

28. $f(\theta)=e^{\theta/2},\ f'(\theta)=\frac{1}{2}e^{\theta/2}$

$$L=\int_0^{2\pi}\sqrt{[e^{\theta/2}]^2+\left[\frac{1}{2}e^{\theta/2}\right]^2}\,d\theta=\int_0^{2\pi}\sqrt{\frac{5}{4}e^{\theta}}\,d\theta=\int_0^{2\pi}\frac{\sqrt{5}}{2}e^{\theta/2}\,d\theta=\left[\sqrt{5}e^{\theta/2}\right]_0^{2\pi}=\sqrt{5}(e^{\pi}-1)\approx 49.51$$

29. If n is even, there are $2n$ leaves.

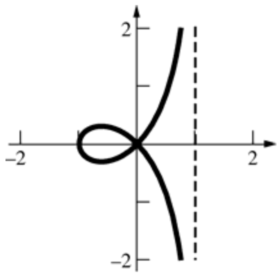
$$A=2n\frac{1}{2}\int_{-\pi/2n}^{\pi/2n}(a\cos n\theta)^2\,d\theta=na^2\int_{-\pi/2n}^{\pi/2n}\cos^2 n\theta\,d\theta=na^2\int_{-\pi/2n}^{\pi/2n}\frac{1+\cos 2n\theta}{2}\,d\theta$$

$$=na^2\left[\frac{1}{2}\theta+\frac{\sin 2n\theta}{4n}\right]_{-\pi/2n}^{\pi/2n}=\frac{1}{2}a^2\pi$$

If n is odd, there are n leaves.

$$A=n\cdot\frac{1}{2}\int_{-\pi/2n}^{\pi/2n}(a\cos n\theta)^2\,d\theta=\frac{na^2}{2}\int_{-\pi/2n}^{\pi/2n}\cos^2 n\theta\,d\theta=\frac{na^2}{2}\left[\frac{1}{2}\theta+\frac{\sin 2n\theta}{4n}\right]_{-\pi/2n}^{\pi/2n}=\frac{1}{4}a^2\pi$$

30. $r=\sec\theta-2\cos\theta$



Solve for the θ-coordinate when $r = 0$.

$\sec\theta-2\cos\theta=0$

$\cos^2\theta=\frac{1}{2}$

$\cos\theta=\pm\frac{1}{\sqrt{2}}$

$\theta=\pm\frac{\pi}{4},\pm\frac{3\pi}{4}$

Notice that the loop is produced for $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$.

$$A=\frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec\theta-2\cos\theta)^2\,d\theta$$

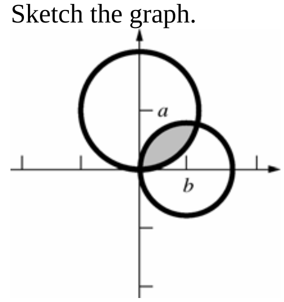
$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec^2\theta-4+4\cos^2\theta)d\theta$$

$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec^2\theta-2+2\cos 2\theta)d\theta$$

$$=\frac{1}{2}\left[\tan\theta-2\theta+\sin 2\theta\right]_{-\pi/4}^{\pi/4}$$

$$=\frac{1}{2}\left[\left(1-\frac{\pi}{2}+1\right)-\left(-1+\frac{\pi}{2}-1\right)\right]=2-\frac{\pi}{2}$$

31. a. Sketch the graph.



Solve for the θ-coordinate of the intersection.

$2a\sin\theta=2b\cos\theta$

$\tan\theta=\frac{b}{a};\quad \theta=\tan^{-1}\left(\frac{b}{a}\right)$

Let $\theta_0=\tan^{-1}\left(\frac{b}{a}\right)$.

$$A=\frac{1}{2}\int_0^{\theta_0}(2a\sin\theta)^2\,d\theta+\frac{1}{2}\int_{\theta_0}^{\pi/2}(2b\cos\theta)^2\,d\theta$$

$$=2a^2\int_0^{\theta_0}\sin^2\theta\,d\theta+2b^2\int_{\theta_0}^{\pi/2}\cos^2\theta\,d\theta$$

$$=a^2\int_0^{\theta_0}(1-\cos 2\theta)d\theta+b^2\int_{\theta_0}^{\pi/2}(1+\cos 2\theta)d\theta$$

$$=a^2\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\theta_0}+b^2\left[\theta+\frac{\sin 2\theta}{2}\right]_{\theta_0}^{\pi/2}$$

$$=a^2\theta_0+b^2\left(\frac{\pi}{2}-\theta_0\right)-\frac{a^2+b^2}{2}\sin 2\theta_0$$

$$=a^2\theta_0+b^2\left(\frac{\pi}{2}-\theta_0\right)-(a^2+b^2)\sin\theta_0\cos\theta_0$$

$$=a^2\tan^{-1}\left(\frac{b}{a}\right)+b^2\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{b}{a}\right)\right)-ab\,.$$

Note that since $\tan\theta=\frac{b}{a},\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$

and $\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$.

b. Let m_1 be the slope of $r = 2a\sin\theta$.

$$m_1 = \frac{2a\sin\theta\cos\theta + 2a\cos\theta\sin\theta}{-2a\sin\theta\sin\theta + 2a\cos\theta\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

At $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, $m_1 = \frac{2ab}{a^2 - b^2}$.

At $\theta = 0$ (the pole), $m_1 = 0$.

Let m_2 be the slope of $r = 2b\cos\theta$.

$$m_2 = \frac{2b\cos\theta\cos\theta - 2b\sin\theta\sin\theta}{-2b\cos\theta\sin\theta - 2b\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{-2\sin\theta\cos\theta}$$

At $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, $m_2 = -\frac{a^2 - b^2}{2ab}$.

At $\theta = \frac{\pi}{2}$ (the pole), m_2 is undefined.

Therefore the two circles intersect at right angles.

32. The area swept from time t_0 to t_1 is

$$A = \int_{\theta(t_0)}^{\theta(t_1)} \frac{1}{2} r^2 d\theta.$$

By the Fundamental Theorem of Calculus,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

So $\frac{dA}{dt} = \frac{k}{2m}$ where k is the constant angular momentum.

Therefore, $\frac{dA}{dt}$ is a constant so $A = \frac{k}{2m}(t_1 - t_0)$.

Equal areas will be swept out in equal time.

33. The edge of the pond is described by the equation $r = 2a\cos\theta$.

Solve for intersection points of the circles $r = ak$ and $r = 2a\cos\theta$.

$ak = 2a\cos\theta$

$\cos\theta = \frac{k}{2}, \theta = \cos^{-1}\left(\frac{k}{2}\right)$

Let A be the grazing area.

$$A = \frac{1}{2}\pi(ka)^2 + 2\cdot\frac{1}{2}\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2} [(ka)^2 - (2a\cos\theta)^2]d\theta = \frac{1}{2}k^2a^2\pi + a^2\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2}(k^2 - 4\cos^2\theta)d\theta$$

$$= \frac{1}{2}k^2a^2\pi + a^2\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2}((k^2 - 2) - 2\cos 2\theta)d\theta = \frac{1}{2}k^2a^2\pi + a^2\left[(k^2 - 2)\theta - \sin 2\theta\right]_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2}$$

$$= \frac{1}{2}k^2a^2\pi + a^2\left[k^2\theta - 2\theta - 2\sin\theta\cos\theta\right]_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2}$$

$$= \frac{1}{2}k^2a^2\pi + a^2\left[\frac{k^2\pi}{2} - \pi - k^2\cos^{-1}\left(\frac{k}{2}\right) + 2\cos^{-1}\left(\frac{k}{2}\right) + \frac{k\sqrt{4-k^2}}{2}\right]$$

$$= a^2\left[(k^2 - 1)\pi + (2 - k^2)\cos^{-1}\left(\frac{k}{2}\right) + \frac{k\sqrt{4-k^2}}{2}\right]$$

34. $|PT| = ka - \phi a$; ϕ goes from 0 to k.

$$A = \frac{1}{2}\int_0^k (ka - \phi a)^2\, d\phi = \frac{1}{2}\left[-\frac{1}{3a}(ka - \phi a)^3\right]_0^k$$

$$= \frac{1}{6}a^2k^3$$

The grazing area is

$$\frac{1}{2}\pi(ka)^2 + 2A = a^2\left(\frac{\pi k^2}{2} + \frac{k^3}{3}\right).$$

35. The untethered goat has a grazing area of πa^2. From Problem 34, the tethered goat has a grazing area of $a^2\left(\frac{\pi k^2}{2} + \frac{k^3}{3}\right)$.

$$\pi a^2 = a^2\left(\frac{\pi k^2}{2} + \frac{k^3}{3}\right)$$

$$\pi = \frac{\pi k^2}{2} + \frac{k^3}{3}$$

$$2k^3 + 3\pi k^2 - 6\pi = 0$$

Using a numerical method or graphing calculator, $k \approx 1.26$. The length of the rope is approximately $1.26a$.

36. $f(\theta) = 2 + \cos\theta,\ f'(\theta) = -\sin\theta$

$$L = 2\int_0^{\pi}\sqrt{[2+\cos\theta]^2 + [-\sin\theta]^2}\, d\theta$$

$$= 2\int_0^{\pi}\sqrt{5 + 4\cos\theta}\, d\theta \approx 13.36$$

$f(\theta) = 2 + 4\cos\theta,\ f'(\theta) = -4\sin\theta$

$$L = 2\int_0^{\pi}\sqrt{[2+4\cos\theta]^2 + [-4\sin\theta]^2}\, d\theta$$

$$= 4\int_0^{\pi}\sqrt{5 + 4\cos\theta}\, d\theta \approx 26.73$$

37. $A = 3\cdot\frac{1}{2}\int_0^{\pi/3}(4\sin 3\theta)^2\, d\theta = 24\int_0^{\pi/3}\sin^2 3\theta\, d\theta$

$$= 12\int_0^{\pi/3}(1 - \cos 6\theta)\, d\theta$$

$$= 12\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = 4\pi$$

$f(\theta) = 4\sin 3\theta,\ f'(\theta) = 12\cos 3\theta$

$$L = 3\int_0^{\pi/3}\sqrt{(4\sin 3\theta)^2 + (12\cos 3\theta)^2}\, d\theta$$

$$= 3\int_0^{\pi/3}\sqrt{16\sin^2 3\theta + 144\cos^2 3\theta}\, d\theta$$

$$= 12\int_0^{\pi/3}\sqrt{1 + 8\cos^2 3\theta}\, d\theta \approx 26.73$$

38. $A = 4\cdot\frac{1}{2}\int_0^{\pi/4} 8\cos 2\theta\, d\theta = 2[4\sin 2\theta]_0^{\pi/4} = 8$

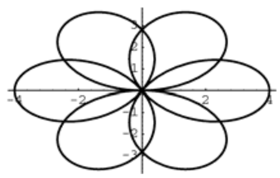
$$f(\theta) = \sqrt{8\cos 2\theta},\ f'(\theta) = \frac{-8\sin 2\theta}{\sqrt{8\cos 2\theta}}$$

$$L = 4\int_0^{\pi/4}\sqrt{8\cos 2\theta + \frac{8\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 4\int_0^{\pi/4}\sqrt{\frac{8}{\cos 2\theta}}\, d\theta$$

$$= 8\sqrt{2}\int_0^{\pi/4}\frac{1}{\sqrt{\cos 2\theta}}\, d\theta \approx 14.83$$

39. $r = 4\sin\left(\frac{3\theta}{2}\right), 0 \le \theta \le 4\pi$



$$f(\theta) = 4\sin\left(\frac{3\theta}{2}\right),\ f'(\theta) = 6\cos\left(\frac{3\theta}{2}\right)$$

$$L = \int_0^{4\pi}\sqrt{\left[4\sin\left(\frac{3\theta}{2}\right)\right]^2 + \left[6\cos\left(\frac{3\theta}{2}\right)\right]^2}\, d\theta$$

$$= \int_0^{4\pi}\sqrt{16\sin^2\left(\frac{3\theta}{2}\right) + 36\cos^2\left(\frac{3\theta}{2}\right)}\, d\theta$$

$$= \int_0^{4\pi}\sqrt{16 + 20\cos^2\left(\frac{3\theta}{2}\right)}\, d\theta \approx 63.46$$

10.8 Chapter Review

Concepts Test

1. False: If $a = 0$, the graph is a line.

2. True: The defining condition of a parabola is $|PF| = |PL|$. Since the axis of a parabola is perpendicular to the directrix and the distance from the vertex to the directrix is equal to the distance to the focus, the vertex is midway between the focus and the directrix.

3. False: The defining condition of an ellipse is $|PF| = e|PL|$ where $0 < e < 1$. Hence the distance from the vertex to a directrix is a greater than the distance to a focus.

4. True: See Problem 33 in Section 10.1.

5. True: The asymptotes for both hyperbolas are $y = \pm \frac{b}{a} x$.

6. True: $C = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt$;

$2\pi b = \int_0^{2\pi} \sqrt{b^2 \sin^2 t + b^2 \cos^2 t}\, dt$

$< C < \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\, dt = 2\pi a$

7. True: As e approaches 0, the ellipse becomes more circular.

8. False: The equation can be rewritten as $\frac{x^2}{4} + \frac{y^2}{6} = 1$ which is a vertical ellipse with foci on the y-axis.

9. False: The equation $x^2 - y^2 = 0$ represents the two lines $y = \pm x$.

10. True: $(y^2 - 4x + 1)^2 = 0$ implies $y^2 - 4x + 1 = 0$ which is an equation for a parabola.

11. True: If $k > 0$, the equation is a horizontal hyperbola; if $k < 0$, the equation is a vertical hyperbola.

12. False: If $k < 0$, there is no graph.

13. False: If $b > a$, the distance is $2\sqrt{b^2 - a^2}$.

14. True: If $y = 0$, $\frac{x^2}{9} = -2$ which is not possible.

15. True: Since light from one focus reflects to the other focus, light emanating from a point between a focus and the nearest vertex will reflect beyond the other focus.

16. True: $a = \frac{8}{2} = 4, c = \frac{2}{2} = 1,$

$b = \sqrt{16-1} = \sqrt{15}$. The length of the minor diameter is $2b = \sqrt{60}$.

17. True: The equation is equivalent to

$\left(x + \frac{C}{2}\right)^2 + \left(y + \frac{D}{2}\right)^2 = -F + \frac{C^2}{4} + \frac{D^2}{4}.$

Thus, the graph is a circle if

$-F + \frac{C^2}{4} + \frac{D^2}{4} > 0$, a point if

$-F + \frac{C^2}{4} + \frac{D^2}{4} = 0$, or the empty set if

$-F^2 + \frac{C^2}{4} + \frac{D^2}{4} < 0.$

18. False: The equation is equivalent to

$2\left(x + \frac{C}{4}\right)^2 + \left(y + \frac{D}{2}\right)^2 = -F + \frac{C^2}{8} + \frac{D^2}{4}.$

Thus, the graph can be a point if

$-F + \frac{C^2}{8} + \frac{D^2}{4} = 0.$

19. False: The limiting forms of two parallel lines and the empty set cannot be formed in such a manner.

20. True: By definition, these curves are conic sections, which can be expressed by an equation of the form

$Ax^2 + Cy^2 + Dx + Ex + F = 0.$

21. False: For example, $xy = 1$ is a hyperbola with coordinates only in the first and third quadrants.

22. False: For example, the graph of $x^2 + 3xy + y^2 = 1$ is a hyperbola that passes through the four points.

23. False: For example, $x = 0$, $y = t$, and $x = 0$, $y = -t$ both represent the line $x = 0$.

24. True: Eliminating the parameter gives $x = 2y$.

25. False: For example, the graph of $x = t^2, y = t$ does not represent y as a function of x. $y = \pm\sqrt{x}$, but $h(x) = \pm\sqrt{x}$ is not a function.

26. True: When $t = 1$, $x = 0$, and $y = 0$.

27. False: For example, if $x = t^3, y = t^3$ then $y = x$ so $\frac{d^2y}{dx^2} = 0$, but $\frac{g''(t)}{f''(t)} = 1$.

28. True: For example, the graph of the four-leaved rose has two tangent lines at the origin.

29. True: The graph of $r = 4\cos\theta$ is a circle of radius 2 centered at (2, 0). The graph $r = 4\cos\left(\theta - \frac{\pi}{3}\right)$ is the graph of $r = 4\cos\theta$ rotated $\frac{\pi}{3}$ counter-clockwise about the pole.

30. True: (r, θ) can be expressed as $(r, \theta + 2\pi n)$ for any integer n.

31. False: For example, if $f(\theta) = \cos\theta$ and $g(\theta) = \sin\theta$, solving the two equations simultaneously does not give the pole (which is $\left(0, \frac{\pi}{2}\right)$ for $f(\theta)$ and (0, 0) for $g(\theta)$).

32. True: Since f is odd $f(-\theta) = -f(\theta)$. Thus, if we replace (r, θ) by $(-r, -\theta)$, the equation $-r = f(-\theta)$ is $-r = -f(\theta)$ or $r = f(\theta)$. Therefore, the graph is symmetric about the y-axis.

33. True: Since f is even $f(-\theta) = f(\theta)$. Thus, if we replace (r, θ) by $(r, -\theta)$, the equation $r = f(-\theta)$ is $r = f(\theta)$. Therefore, the graph is symmetric about the x-axis.

34. True: The graph has 3 leaves and the area is exactly one quarter of the circle $r = 4$. (See Problem 15 of Section 12.8.)

Sample Test Problems

1. **a.** $x^2 - 4y^2 = 0;\ y = \pm\frac{x}{2}$

(5) Two intersecting lines

b. $x^2 - 4y^2 = 0.01;\ \frac{x^2}{0.01} - \frac{y^2}{0.0025} = 1$

(9) A hyperbola

c. $x^2 - 4 = 0;\ x = \pm 2$

(4) Two parallel lines

d. $x^2 - 4x + 4 = 0;\ x = 2$

(3) A single line

e. $x^2 + 4y^2 = 0;\ (0, 0)$

(2) A single point

f. $x^2 + 4y^2 = x;\ x^2 - x + \frac{1}{4} + 4y^2 = \frac{1}{4};$

$$\frac{\left(x - \frac{1}{2}\right)^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{16}} = 1$$

(8) An ellipse

g. $x^2 + 4y^2 = -x;\ x^2 + x + \frac{1}{4} + 4y^2 = \frac{1}{4};$

$$\frac{\left(x + \frac{1}{2}\right)^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{16}} = 1$$

(8) An ellipse

h. $x^2 + 4y^2 = -1$

(1) No graph

i. $(x^2 + 4y - 1)^2 = 0;\ x^2 + 4y - 1 = 0$

(7) A parabola

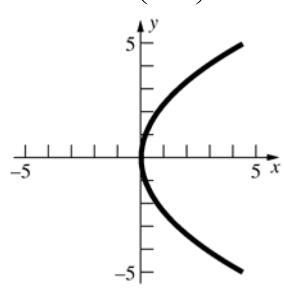
j. $3x^2 + 4y^2 = -x^2 + 1;\ x^2 + y^2 = \frac{1}{4}$

(6) A circle

2. $y^2 - 6x = 0;\ y^2 = 6x;\ y^2 = 4\left(\frac{3}{2}\right)x$

Horizontal parabola; opens to the right; $p = \frac{3}{2}$

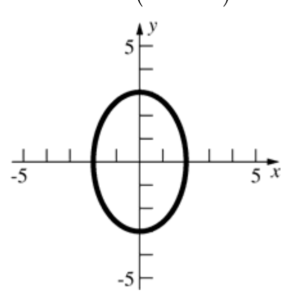
Focus is at $\left(\frac{3}{2}, 0\right)$ and vertex is at (0, 0).



3. $9x^2 + 4y^2 - 36 = 0;\ \frac{x^2}{4} + \frac{y^2}{9} = 1$

Vertical ellipse; $a = 3$, $b = 2$, $c = \sqrt{5}$

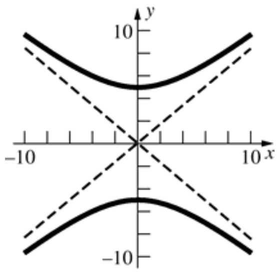
Foci are at $\left(0, \pm\sqrt{5}\right)$ and vertices are at $(0, \pm 3)$.



4. $25x^2-36y^2+900=0; \dfrac{y^2}{25}-\dfrac{x^2}{36}=1$

Vertical hyperbola; $a = 5$, $b = 6$, $c=\sqrt{61}$

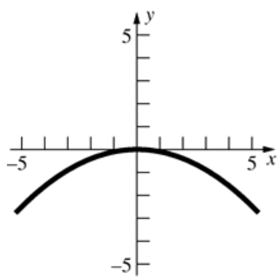
Foci are at $\left(0,\pm\sqrt{61}\right)$ and vertices are at (0, ±5).



5. $x^2+9y=0; x^2=-9y; x^2=-4\left(\dfrac{9}{4}\right)y$

Vertical parabola; opens downward; $p=\dfrac{9}{4}$

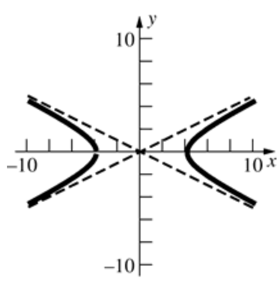
Focus at $\left(0,-\dfrac{9}{4}\right)$ and vertex at (0, 0).



6. $x^2-4y^2-16=0; \dfrac{x^2}{16}-\dfrac{y^2}{4}=1$

Horizontal hyperbola; $a = 4$, $b = 2$, $c=2\sqrt{5}$

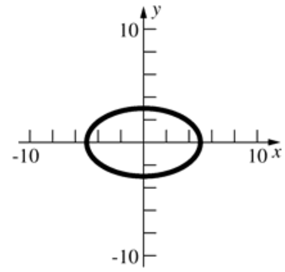
Foci are at $\left(\pm 2\sqrt{5},0\right)$ and vertices are at (±4, 0).



7. $9x^2+25y^2-225=0; \dfrac{x^2}{25}+\dfrac{y^2}{9}=1$

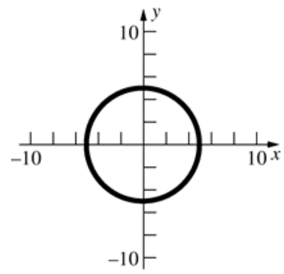
Horizontal ellipse, $a = 5$, $b = 3$, $c = 4$

Foci are at (±4, 0) and vertices are at (±5, 0).



8. $9x^2+9y^2-225=0; x^2+y^2=25$

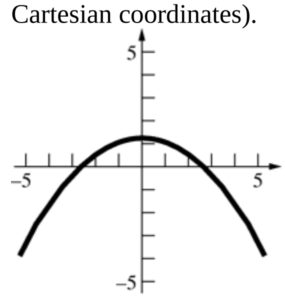
Circle; $r = 5$



9. $r=\dfrac{5}{2+2\sin\theta}=\dfrac{\left(\frac{5}{2}\right)(1)}{1+(1)\cos\left(\theta-\frac{\pi}{2}\right)}$

$e = 1$; parabola

Focus is at (0, 0) and vertex is at $\left(0,\dfrac{5}{4}\right)$ (in Cartesian coordinates).



10. $r(2+\cos\theta)=3; r=\dfrac{3\left(\frac{1}{2}\right)}{1+\frac{1}{2}\cos\theta}$

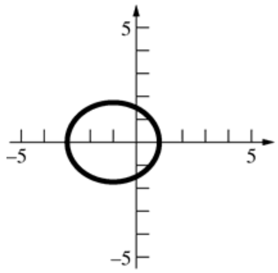
$e=\dfrac{1}{2}$, ellipse

At $\theta=0, r=1.$ At $\theta=\pi, r=3.$.

$a=\dfrac{1+3}{2}=2, c=ea=1$

Center is at (–1, 0) (in Cartesian coordinates).

Foci are (0, 0) and (–2, 0) and vertices are at (1, 0) and (–3, 0) (all in Cartesian coordinates).



11. Horizontal ellipse; center at (0, 0), $a = 4$,

$e = \frac{c}{a} = \frac{1}{2}$, $c = 2$, $b = \sqrt{16-4} = 2\sqrt{3}$

$\frac{x^2}{16} + \frac{y^2}{12} = 1$

12. Vertical parabola; opens downward; $p = 3$

$x^2 = -12y$

13. Horizontal parabola;

$y^2 = ax, (3)^2 = a(-1), a = -9$

$y^2 = -9x$

14. Vertical hyperbola; $a = 3$, $c = ae = 5$,

$b = \sqrt{25-9} = 4$, center at (0, 0)

$\frac{y^2}{9} - \frac{x^2}{16} = 1$

15. Horizontal hyperbola, $a = 2$,

$x = \pm 2y, \frac{a}{b} = 2, b = 1$

$\frac{x^2}{4} - \frac{y^2}{1} = 1$

16. Vertical parabola; opens downward; $p = 1$

$(x-3)^2 = -4(y-3)$

17. Horizontal ellipse; $2a = 10$, $a = 5$, $c = 4 - 1 = 3$,

$b = \sqrt{25-9} = 4$

$\frac{(x-1)^2}{25} + \frac{(y-2)^2}{16} = 1$

18. Vertical hyperbola; $2a = 6$, $a = 3$, $c = ae = 10$,

$b = \sqrt{100-9} = \sqrt{91}$, center at (2, 3)

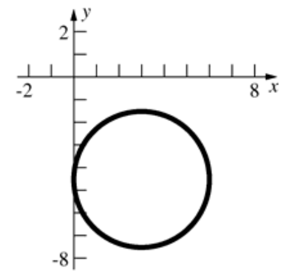
$\frac{(y-3)^2}{9} - \frac{(x-2)^2}{91} = 1$

19. $4x^2 + 4y^2 - 24x + 36y + 81 = 0$

$4(x^2 - 6x + 9) + 4\left(y^2 + 9y + \frac{81}{4}\right) = -81 + 36 + 81$

$4(x-3)^2 + 4\left(y + \frac{9}{2}\right)^2 = 36$

$(x-3)^2 + \left(y + \frac{9}{2}\right)^2 = 9$; circle

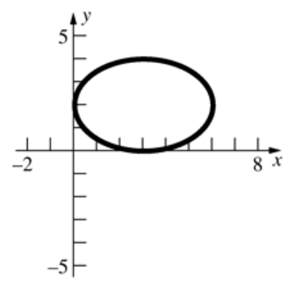


20. $4x^2 + 9y^2 - 24x - 36y + 36 = 0$

$4(x^2 - 6x + 9) + 9(y^2 - 4y + 4) = -36 + 36 + 36$

$4(x-3)^2 + 9(y-2)^2 = 36$

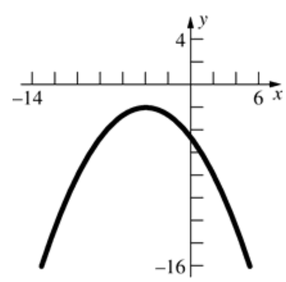
$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$; ellipse



21. $x^2 + 8x + 6y + 28 = 0$

$(x^2 + 8x + 16) = -6y - 28 + 16$

$(x+4)^2 = -6(y+2)$; parabola

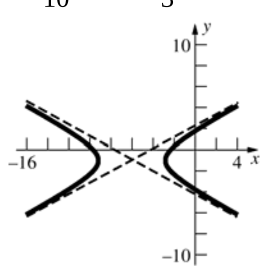


22. $3x^2 - 10y^2 + 36x - 20y + 68 = 0$

$3(x^2 + 12x + 36) - 10(y^2 + 2y + 1) = -68 + 108 - 10$

$3(x+6)^2 - 10(y+1)^2 = 30$

$\frac{(x+6)^2}{10} - \frac{(y+1)^2}{3} = 1$; hyperbola



23. $x = \frac{\sqrt{2}}{2}(u - v)$

$y = \frac{\sqrt{2}}{2}(u + v)$

$\frac{1}{2}(u-v)^2 + \frac{3}{2}(u-v)(u+v) + \frac{1}{2}(u+v)^2 = 10$

$\frac{5}{2}u^2 - \frac{1}{2}v^2 = 10$

$r = \frac{5}{2}, s = -\frac{1}{2}$

$\frac{u^2}{4} - \frac{v^2}{20} = 1$; hyperbola

$a = 2,\ b = 2\sqrt{5}, c = \sqrt{4+20} = 2\sqrt{6}$

The distance between foci is $4\sqrt{6}$.

24. $7x^2 + 8xy + y^2 = 9$

$\cot 2\theta = \frac{3}{4}$

$\cos 2\theta = \frac{3}{5}$

$\cos\theta = \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$

$\sin\theta = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$

$\theta = \sin^{-1}\frac{1}{\sqrt{5}} \approx 0.4636$

$x = \frac{1}{\sqrt{5}}(2u - v)$

$y = \frac{1}{\sqrt{5}}(u + 2v)$

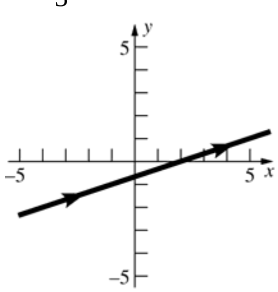
$\frac{7}{5}(2u-v)^2 + \frac{8}{5}(2u-v)(u+2v) + \frac{1}{5}(u+2v)^2 = 9$

$9u^2 - v^2 = 9$

$u^2 - \frac{v^2}{9} = 1$; hyperbola

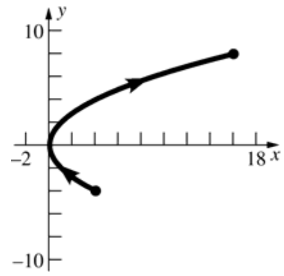
25. $t = \frac{1}{6}(x - 2)$

$y = \frac{1}{3}(x - 2)$



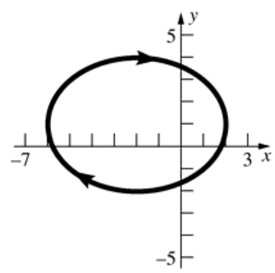
26. $t = \frac{y}{4}$

$x = \frac{y^2}{4}$ or $y^2 = 4x$



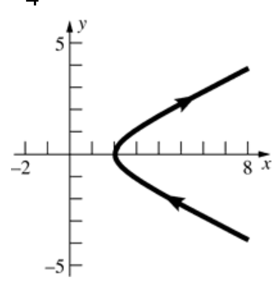
27. $\sin t = \frac{x+2}{4}, \cos t = \frac{y-1}{3}$

$\frac{(x+2)^2}{16} + \frac{(y-1)^2}{9} = 1$



28. $\frac{x^2}{4} = \sec^2 t,\ y^2 = \tan^2 t$

$\frac{x^2}{4} - y^2 = 1$



29. $\frac{dx}{dt} = 6t^2 - 4, \frac{dy}{dt} = 1 + \frac{1}{t+1} = \frac{t+2}{t+1}$

$\frac{dy}{dx} = \frac{\frac{t+2}{t+1}}{6t^2 - 4} = \frac{t+2}{(t+1)(6t^2-4)}$

At $t = 0$, $x = 7$, $y = 0$, and $\frac{dy}{dx} = -\frac{1}{2}$.

Tangent line: $y = -\frac{1}{2}(x - 7)$ or $x + 2y - 7 = 0$

Normal line: $y = 2(x - 7)$ or $2x - y - 14 = 0$.

30. $\frac{dx}{dt} = -3e^{-t}, \frac{dy}{dt} = \frac{1}{2}e^{t}$

$$\frac{dy}{dx} = \frac{\frac{1}{2}e^{t}}{-3e^{-t}} = -\frac{1}{6}e^{2t}$$

At $t = 0$, $x = 3$, $y = \frac{1}{2}$, and $\frac{dy}{dx} = -\frac{1}{6}$.

Tangent line: $y - \frac{1}{2} = -\frac{1}{6}(x-3)$ or $x + 6y - 6 = 0$

Normal line: $y - \frac{1}{2} = 6(x-3)$ or $12x - 2y - 35 = 0$.

31. One approach is to use the arc length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^9 \sqrt{\frac{9t}{4} + \frac{9t}{4}}\, dt =$$

$$\frac{3\sqrt{2}}{2}\int_0^9 \sqrt{t}\, dt = \sqrt{2}\left[t^{3/2}\right]_0^9 = 27\sqrt{2}$$

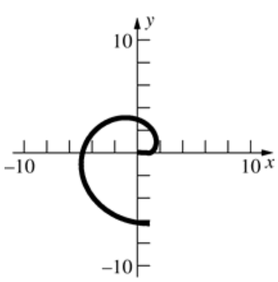
Another way is to note that when $t = 0$, $(x, y) = (1, 2)$, when $t = 9$, $(x, y) = (28, 29)$, and $y = x + 1$, which is a straight line. Thus the curve length is simply the distance between the points $(1, 2)$ and $(28, 29)$ or $\sqrt{(28-1)^2 + (29-2)^2} = 27\sqrt{2}$

32. $\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$

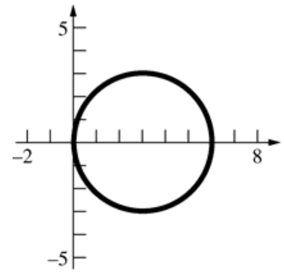
$$\frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t$$

$$L = \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\, dt = \int_0^{2\pi} t\, dt$$

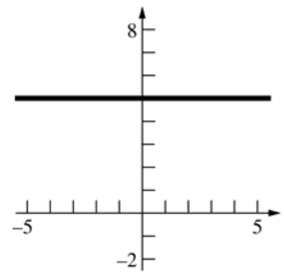
$$= \left[\frac{1}{2}t^2\right]_0^{2\pi} = 2\pi^2$$



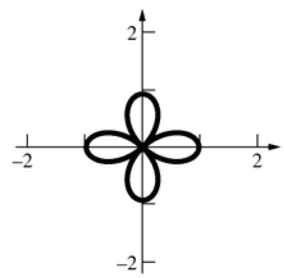
33 $r = 6\cos\theta$



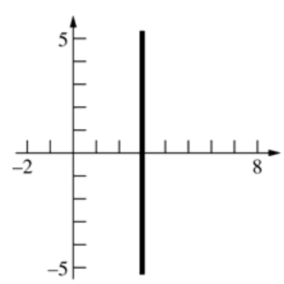
34. $r = \frac{5}{\sin\theta}$



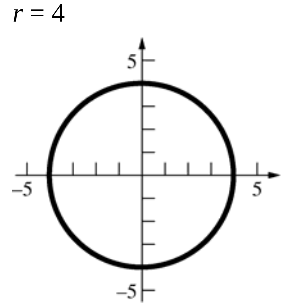
35. $r = \cos 2\theta$



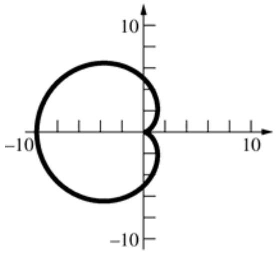
36. $r = \frac{3}{\cos\theta}$



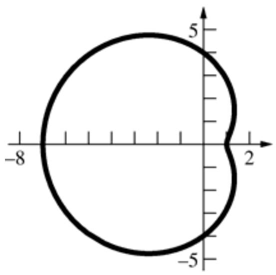
37. $r = 4$



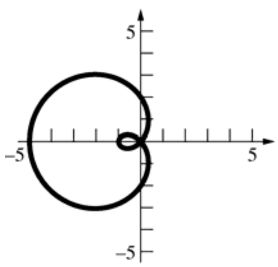
38. $r = 5 - 5\cos\theta$



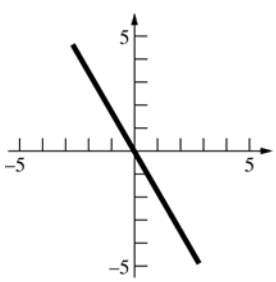
39. $r = 4 - 3\cos\theta$



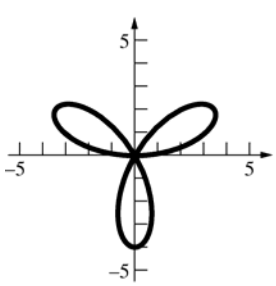
40. $r = 2 - 3\cos\theta$



41. $\theta = \frac{2}{3}\pi$

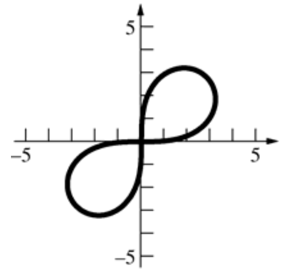


42. $r = 4\sin 3\theta$

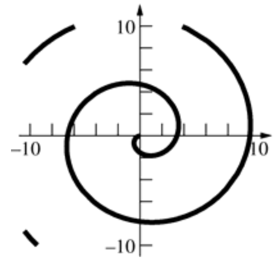


43. $r^2 = 16\sin 2\theta$

$r = \pm 4\sqrt{\sin 2\theta}$



44. $r = -\theta, \theta \geq 0$

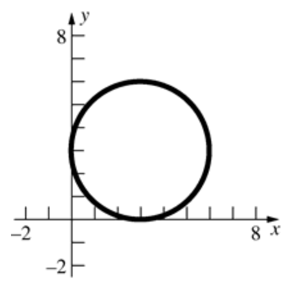


45. $r^2 - 6r(\cos\theta + \sin\theta) + 9 = 0$

$x^2 + y^2 - 6x - 6y + 9 = 0$

$(x^2 - 6x + 9) + (y^2 - 6y + 9) = -9 + 9 + 9$

$(x-3)^2 + (y-3)^2 = 9$

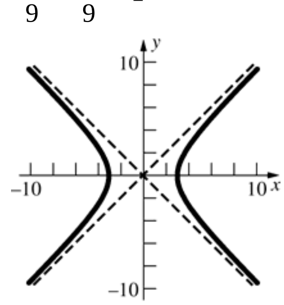


46. $r^2 \cos 2\theta = 9$

$r^2 \cos^2\theta - r^2 \sin^2\theta = 9$

$x^2 - y^2 = 9$

$\frac{x^2}{9} - \frac{y^2}{9} = 1$



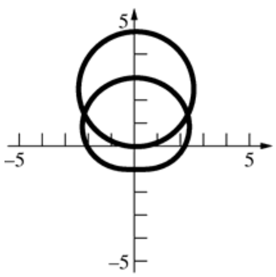
47. $f(\theta)=3+3\cos\theta,\ f'(\theta)=-3\sin\theta$

$$m=\frac{(3+3\cos\theta)\cos\theta+(-3\sin\theta)\sin\theta}{-(3+3\cos\theta)\sin\theta+(-3\sin\theta)\cos\theta}$$

$$=\frac{\cos\theta+\cos^2\theta-\sin^2\theta}{-\sin\theta-2\cos\theta\sin\theta}=\frac{\cos\theta+\cos 2\theta}{-\sin\theta-\sin 2\theta}$$

At $\theta=\frac{\pi}{6}$, $m=\frac{\cos\frac{\pi}{6}+\cos\frac{\pi}{3}}{-\sin\frac{\pi}{6}-\sin\frac{\pi}{3}}=-1$.

48. $r=5\sin\theta, r=2+\sin\theta$



$5\sin\theta=2+\sin\theta$

$\sin\theta=\frac{1}{2}\ \Rightarrow\ \theta=\frac{\pi}{6},\frac{5\pi}{6}$

$\left(\frac{5}{2},\frac{\pi}{6}\right),\left(\frac{5}{2},\frac{5\pi}{6}\right)$

49. $A=2\cdot\frac{1}{2}\int_0^{\pi}(5-5\cos\theta)^2\,d\theta$

$$=25\int_0^{\pi}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$=25\int_0^{\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$=25\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi}=\frac{75\pi}{2}$$

50. $A=2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left[(5\sin\theta)^2-(2+\sin\theta)^2\right]d\theta$

$$=\int_{\pi/6}^{\pi/2}(24\sin^2\theta-4\sin\theta-4)d\theta$$

$$=\int_{\pi/6}^{\pi/2}(8-12\cos 2\theta-4\sin\theta)d\theta$$

$$=\left[8\theta-6\sin 2\theta+4\cos\theta\right]_{\pi/6}^{\pi/2}=\frac{8}{3}\pi+\sqrt{3}$$

51. $\frac{x^2}{400}+\frac{y^2}{100}=1;\ \frac{x}{200}+\frac{yy'}{50}=0$

$y'=-\frac{x}{4y};\ y'=-\frac{2}{3}$ at (16, 6)

Tangent line: $y-6=-\frac{2}{3}(x-16)$

When x = 14, $y=-\frac{2}{3}(14-16)+6=\frac{22}{3}$.

$k=\frac{22}{3}$

52 **a.** III **b.** IV

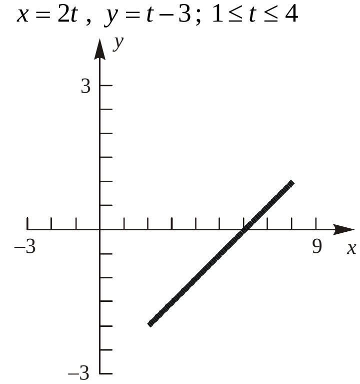
c. I **d.** II

53. **a.** I **b.** IV

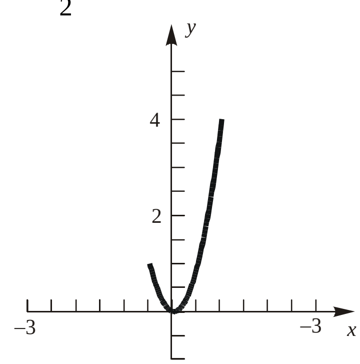
c. III **d.** II

Review and Preview Problems

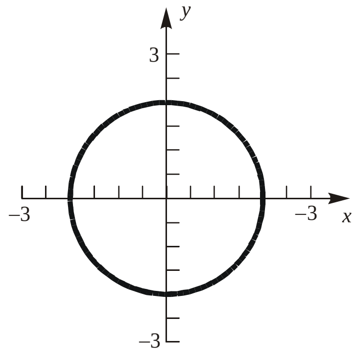
1. $x=2t$, $y=t-3$; $1\le t\le 4$



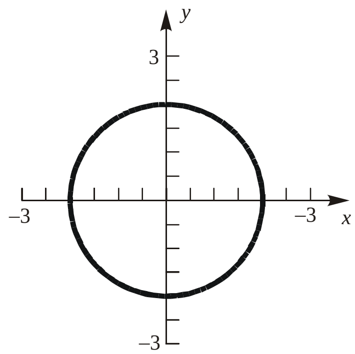
2. $x=\frac{t}{2}$, $y=t^2$; $-1\le t\le 2$



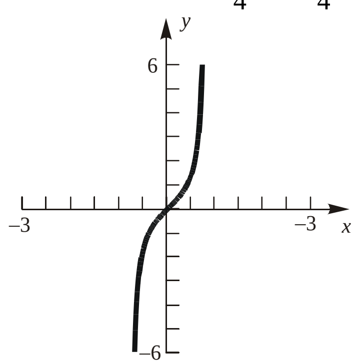
3. $x = 2\cos t$, $y = 2\sin t$; $0 \le t \le 2\pi$



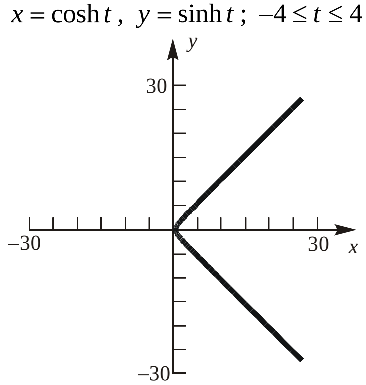
4. $x = 2\sin t$, $y = -2\cos t$; $0 \le t \le 2\pi$



5. $x = t$, $y = \tan 2t$; $-\dfrac{\pi}{4} < t < \dfrac{\pi}{4}$



6. $x = \cosh t$, $y = \sinh t$; $-4 \le t \le 4$



7. $x = h \cdot \cos\theta$

$y = h \cdot \sin\theta$

8. $x = h \cdot \cos\theta$

$y = h \cdot \sin\theta$

For problems 9-12, $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

9. $\dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = \dfrac{9}{2}\sqrt{t}$

$$L = \int_0^4 \sqrt{1 + \tfrac{81}{4}t}\, dt = \int_0^4 \tfrac{1}{2}\sqrt{\underset{\substack{u=4+81t \\ du=81dt}}{4+81t}}\, dt =$$

$$\tfrac{1}{162}\int_4^{328} \sqrt{u}\, du = \frac{2}{3} \cdot \frac{1}{162}\left[u^{3/2}\right]_4^{328} =$$

$$\frac{1}{243}\left[(328)^{3/2} - 8\right] \approx 24.4129$$

10. $\dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = 2$

$$L = \int_1^5 \sqrt{1+4}\, dt = \left[\sqrt{5}\, t\right]_1^5 = 4\sqrt{5} \approx 8.94$$

11. $\dfrac{dx}{dt} = -2a\sin 2t,\ \dfrac{dy}{dt} = 2a\cos 2t$

$$L = \int_0^{\pi/2} \sqrt{4a^2\sin^2 2t + 4a^2\cos^2 2t}\, dt =$$

$$\int_0^{\pi/2} 2|a|\sqrt{1}\ dt = 2|a|\left[\ t\right]_0^{\pi/2} = \pi|a|$$

12. $\dfrac{dx}{dt} = \operatorname{sech}^2 t,\ \dfrac{dy}{dt} = -\operatorname{sech} t \tanh t$

$$L = \int_0^4 \sqrt{\operatorname{sech}^4 t + \operatorname{sech}^2 t \tanh^2 t}\ dt =$$

$$\int_0^4 (\operatorname{sech} t)\sqrt{\operatorname{sech}^2 t + \tanh^2 t}\ dt =$$

$$\int_0^4 (\operatorname{sech}\ t)\sqrt{\frac{1}{\cosh^2 t} + \frac{\sinh^2 t}{\cosh^2 t}}\ dt =$$

$$\int_0^4 (\operatorname{sech}\ t)\sqrt{\frac{\cosh^2 t}{\cosh^2 t}}\ dt = \int_0^4 \operatorname{sech}\ t\ dt =$$

$$\left[2\tan^{-1} e^t\right]_0^4 = 2\tan^{-1} e^4 - 2\tan^{-1} e^0 \approx 1.534$$

13. Let $(x, 2x+1)$ represent any point on the line $y = 2x+1$; then the square of its distance from $(0,3)$ is $d(x) = x^2 + \left[(2x+1) - 3\right]^2 = 5x^2 - 8x + 4$ Now $d'(x) = 10x - 8$ so that $d'(0.8) = 0$; further, $d''(x) = 10 > 0$ so that the absolute minimum of the (square of the) distance occurs at the point $(0.8, 2.6)$ where the distance is

$$\sqrt{d(0.8)} = \sqrt{5(0.8)^2 - 8(0.8) + 4} = \sqrt{0.8} \approx 0.894$$

14. Results will vary. The only limitation on $\{a_1, a_2, b_1, b_2\}$ is that the t value that makes $x = 1$ must also make $y = -1$ *and* the t value that makes $x = 3$ must also make $y = 3$. Let $t = 0$ yield $(1, -1)$ and let $t = 1$ yield $(3, 3)$; then

$1 = a_1(0) + b_1 \quad -1 = a_2(0) + b_2$

$3 = a_1(1) + b_1 \quad\quad 3 = a_2(1) + b_2$

From this we get $b_1 = 1,\ b_2 = -1,\ a_1 = 2,\ a_2 = 4$; thus one parametric representation is $x = 2t + 1,\ \ y = 4t - 1$. Using other t values to yield (1, -1) and (3, 3) will give other representations.

15. $s(t) = t^2 - 6t + 8$

a. $v(t) = s'(t) = 2t - 6$

$a(t) = v'(t) = 2$

b. The object is moving forward (in positive x-direction) when $v(t) > 0$ or $t > 3$.

16. $a(t) = 2$

a. $v(t) = \int a(t)\,dt = 2t + v(0)$; since the object is initially at rest, $v(0) = 0$ so $v(t) = 2t$.

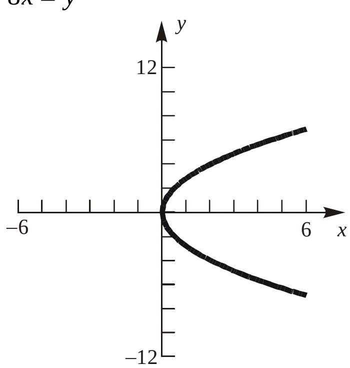
$s(t) = \int v(t)\,dt = t^2 + s(0)$; since $s(0) = 20$,

$s(t) = t^2 + 20$

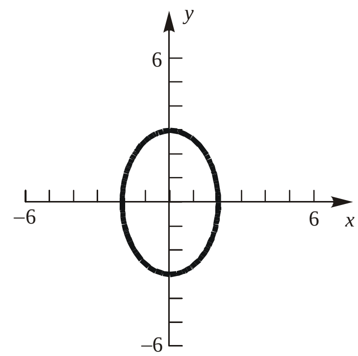
b. $s(t) = 100 \Rightarrow t^2 + 20 = 100 \Rightarrow t = \sqrt{80} \approx 8.944$

The object will reach position 100 after about 8.944 time units.

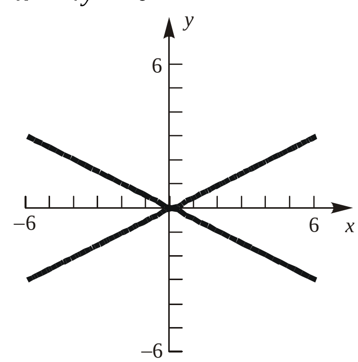
17. $8x = y^2$



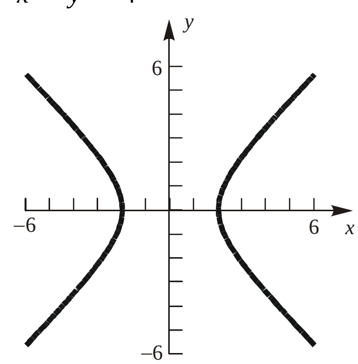
18. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$



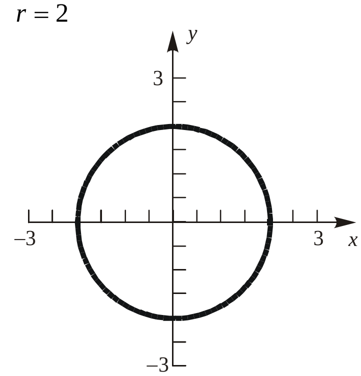
19. $x^2 - 4y^2 = 0$



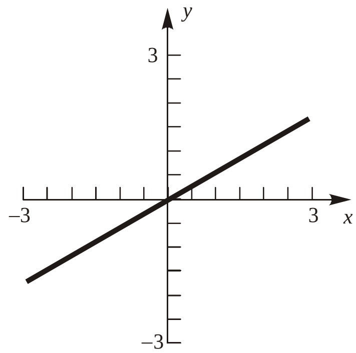
20. $x^2 - y^2 = 4$



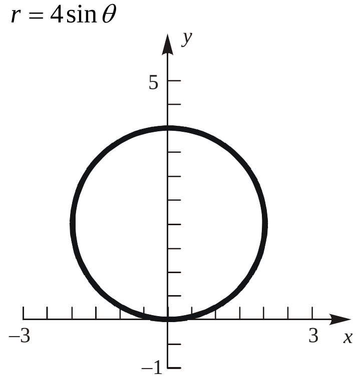
21. $r = 2$



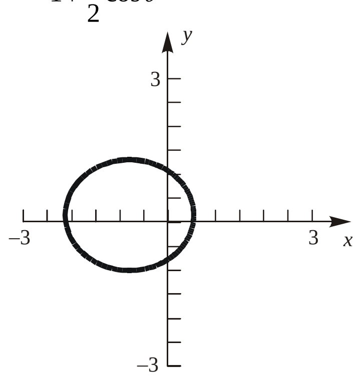
22. $\theta = \frac{\pi}{6}$



23. $r = 4\sin\theta$



24. $r = \dfrac{1}{1+\frac{1}{2}\cos\theta}$



CHAPTER 11 Geometry in Space and Vectors

11.1 Concepts Review

1. coordinates

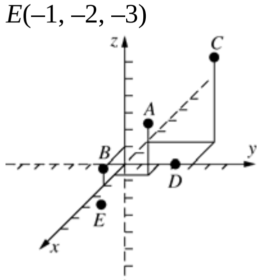
2. $\sqrt{(x+1)^2+(y-3)^2+(z-5)^2}$

3. (–1, 3, 5); 4

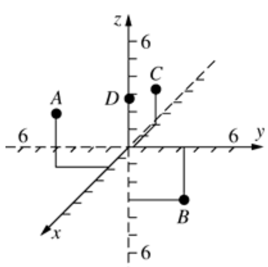
4. plane; 4; –6; 3

Problem Set 11.1

1. $A(1, 2, 3)$, $B(2, 0, 1)$, $C(-2, 4, 5)$, $D(0, 3, 0)$, $E(-1, -2, -3)$



2. $A\left(\sqrt{3}, -3, 3\right)$, $B(0, \pi, -3)$, $C\left(-2, \frac{1}{3}, 2\right)$, $D(0, 0, e)$



3. $x = 0$ in the yz-plane. $x = 0$ and $y = 0$ on the z-axis.

4. $y = 0$ in the xz-plane. $x = 0$ and $z = 0$ on the y-axis.

5. **a.** $\sqrt{(6-1)^2+(-1-2)^2+(0-3)^2} = \sqrt{43}$

   b. $\sqrt{(-2-2)^2+(-2+2)^2+(0+3)^2} = 5$

   c. $\sqrt{(e+\pi)^2+(\pi+4)^2+\left(0-\sqrt{3}\right)^2} \approx 9.399$

6. $P(4, 5, 3)$, $Q(1, 7, 4)$, $R(2, 4, 6)$

$$|PQ| = \sqrt{(4-1)^2+(5-7)^2+(3-4)^2} = \sqrt{14}$$
$$|PR| = \sqrt{(4-2)^2+(5-4)^2+(3-6)^2} = \sqrt{14}$$
$$|QR| = \sqrt{(1-2)^2+(7-4)^2+(4-6)^2} = \sqrt{14}$$

Since the distances are equal, the triangle formed by joining P, Q, and R is equilateral.

7. $P(2, 1, 6)$, $Q(4, 7, 9)$, $R(8, 5, -6)$

$$|PQ| = \sqrt{(2-4)^2+(1-7)^2+(6-9)^2} = 7$$
$$|PR| = \sqrt{(2-8)^2+(1-5)^2+(6+6)^2} = 14$$
$$|QR| = \sqrt{(4-8)^2+(7-5)^2+(9+6)^2} = \sqrt{245}$$

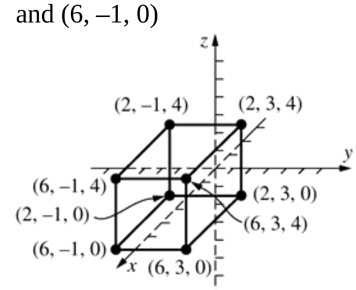
$|PQ|^2+|PR|^2 = 49+196 = 245 = |QR|^2$, so the triangle formed by joining P, Q, and R is a right triangle, since it satisfies the Pythagorean Theorem.

8. **a.** The distance to the xy-plane is 1 since the point is 1 unit below the plane.

   b. The distance is $\sqrt{(2-0)^2+(3-3)^2+(-1-0)^2} = \sqrt{5}$ since the distance from a point to a line is the length of the shortest segment joining the point and the line. Using the point (0, 3, 0) on the y-axis clearly minimizes the length.

   c. $\sqrt{(2-0)^2+(3-0)^2+(-1-0)^2} = \sqrt{14}$

9. Since the faces are parallel to the coordinate planes, the sides of the box are in the planes $x = 2$, $y = 3$, $z = 4$, $x = 6$, $y = -1$, and $z = 0$ and the vertices are at the points where 3 of these planes intersect. Thus, the vertices are (2, 3, 4), (2, 3, 0), (2, –1, 4), (2, –1, 0), (6, 3, 4), (6, 3, 0), (6, –1, 4), and (6, –1, 0)



10. It is parallel to the y-axis; $x = 2$ and $z = 3$. (If it were parallel to the x-axis, the y-coordinate could not change, similarly for the z-axis.)

11. **a.** $(x-1)^2+(y-2)^2+(z-3)^2 = 25$

    b. $(x+2)^2+(y+3)^2+(z+6)^2 = 5$

    c. $(x-\pi)^2+(y-e)^2+\left(z-\sqrt{2}\right)^2 = \pi$

12. Since the sphere is tangent to the *xy*-plane, the point (2, 4, 0) is on the surface of the sphere. Hence, the radius of the sphere is 5 so the equation is $(x-2)^2+(y-4)^2+(z-5)^2=25.$

13. $(x^2-12x+36)+(y^2+14y+49)+(z^2-8z+16)=-1+36+49+16$

$(x-6)^2+(y+7)^2+(z-4)^2=100$

Center: (6, –7, 4); radius 10

14. $(x^2+2x+1)+(y^2-6y+9)+(z^2-10z+25)=-34+1+9+25$

$(x+1)^2+(y-3)^2+(z-5)^2=1$

Center: (–1, 3, 5); radius 1

15. $x^2+y^2+z^2-x+2y+4z=\frac{13}{4}$

$\left(x^2-x+\frac{1}{4}\right)+(y^2+2y+1)+(z^2+4z+4)=\frac{13}{4}+\frac{1}{4}+1+4$

$\left(x-\frac{1}{2}\right)^2+(y+1)^2+(z+2)^2=\frac{17}{2}$

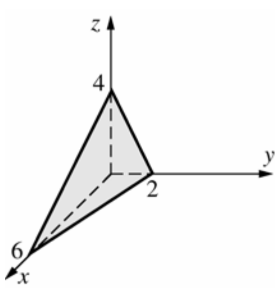
Center: $\left(\frac{1}{2},-1,-2\right)$; radius $\sqrt{\frac{17}{2}}\approx 2.92$

16. $(x^2+8x+16)+(y^2-4y+4)+(z^2-22z+121)=-77+16+4+121$

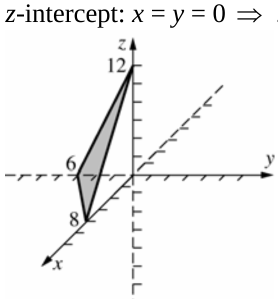
$(x+4)^2+(y-2)^2+(z-11)^2=64$

Center: (–4, 2, 11); radius 8

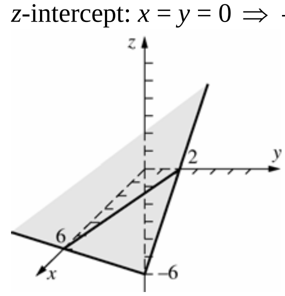
17. *x*-intercept: $y = z = 0 \Rightarrow 2x = 12, x = 6$
y-intercept: $x = z = 0 \Rightarrow 6y = 12, y = 2$
z-intercept: $x = y = 0 \Rightarrow 3z = 12, z = 4$



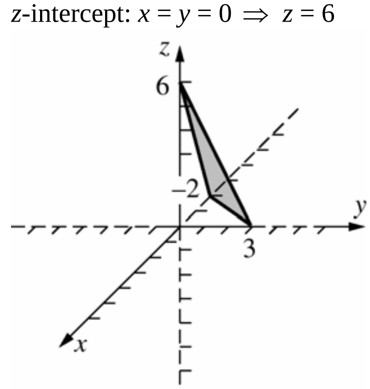
18. *x*-intercept: $y = z = 0 \Rightarrow 3x = 24, x = 8$
y-intercept: $y = z = 0 \Rightarrow -4y = 24, y = -6$
z-intercept: $x = y = 0 \Rightarrow 2z = 24, z = 12$



19. *x*-intercept: $y = z = 0 \Rightarrow x = 6$
y-intercept: $x = z = 0 \Rightarrow 3y = 6, y = 2$
z-intercept: $x = y = 0 \Rightarrow -z = 6, z = -6$

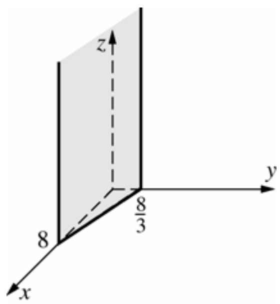


20. *x*-intercept: $y = z = 0 \Rightarrow -3x = 6, x = -2$
y-intercept: $x = z = 0 \Rightarrow 2y = 6, y = 3$
z-intercept: $x = y = 0 \Rightarrow z = 6$

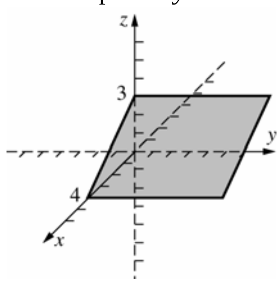


21. x and y cannot both be zero, so the plane is parallel to the z-axis.
x-intercept: $y = z = 0 \Rightarrow x = 8$

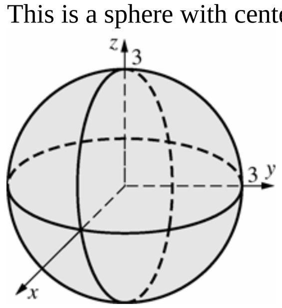
y-intercept: $x = z = 0 \Rightarrow 3y = 8,\ y = \dfrac{8}{3}$



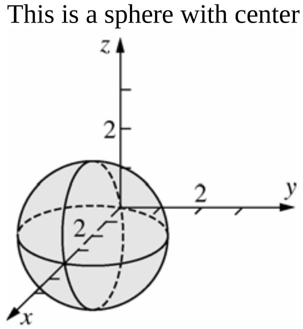
22. x and z cannot both be zero, so the plane is parallel to the y-axis.
x-intercept: $y = z = 0 \Rightarrow 3x = 12,\ x = 4$
z-intercept: $x = y = 0 \Rightarrow 4z = 12,\ z = 3$



23. This is a sphere with center (0, 0, 0) and radius 3.



24. This is a sphere with center (2, 0, 0) and radius 2.



For problems 25-36, $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$

25. $\dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = 1,\ \dfrac{dz}{dt} = 2$

$L = \int_0^2 \sqrt{1^2 + 1^2 + 2^2}\ dt = \int_0^2 \sqrt{6}\, dt =$

$\left[\sqrt{6}t\right]_0^2 = 2\sqrt{6} \approx 4.899$

26. $\dfrac{dx}{dt} = \dfrac{1}{4},\ \dfrac{dy}{dt} = \dfrac{1}{3},\ \dfrac{dz}{dt} = \dfrac{1}{2}$

$L = \int_1^3 \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^2}\ dt = \int_1^3 \sqrt{\frac{61}{144}}\ dt =$

$\left[\sqrt{\frac{61}{144}}\, t\right]_1^3 = 2\sqrt{\frac{61}{144}} \approx 1.302$

27. $\dfrac{dx}{dt} = \dfrac{3}{2}\sqrt{t},\ \dfrac{dy}{dt} = 3,\ \dfrac{dz}{dt} = 4$

$L = \int_1^4 \sqrt{\left(\frac{9}{4}t\right) + 9 + 16}\, dt = \int_1^4 \frac{1}{2}\underset{\substack{u = 9t+100 \\ du = 9\,dt}}{\sqrt{9t+100}}\, dt =$

$\frac{1}{18}\int_{109}^{136} \sqrt{u}\, du = \dfrac{1}{27}\left[u^{3/2}\right]_{109}^{136} \approx 16.59$

28. $\dfrac{dx}{dt} = \dfrac{3}{2}\sqrt{t},\ \dfrac{dy}{dt} = \frac{3}{2}\sqrt{t},\ \dfrac{dz}{dt} = 1$

$L = \int_2^4 \sqrt{\left(\frac{9}{4}t\right) + \left(\frac{9}{4}t\right) + 1}\, dt = \int_2^4 \frac{1}{2}\underset{\substack{u = 18t+4 \\ du = 18\,dt}}{\sqrt{18t+4}}\, dt =$

$\frac{1}{36}\int_{40}^{76} \sqrt{u}\, du = \dfrac{1}{54}\left[u^{3/2}\right]_{40}^{76} \approx 7.585$

29. $\dfrac{dx}{dt} = 2t,\ \dfrac{dy}{dt} = 2\sqrt{t},\ \dfrac{dz}{dt} = 1$

$L = \int_0^8 \sqrt{4t^2 + 4t + 1}\, dt = \int_0^8 \sqrt{(2t+1)^2}\ dt =$

$\int_0^8 (2t+1)\, dt = \left[t^2 + t\right]_0^8 = 72$

30. $\dfrac{dx}{dt} = 2t,\ \dfrac{dy}{dt} = 2\sqrt{3t},\ \dfrac{dz}{dt} = 3$

$L = \int_1^4 \sqrt{4t^2 + 12t + 9}\, dt = \int_1^4 \sqrt{(2t+3)^2}\ dt =$

$\int_1^4 (2t+3)\, dt = \left[t^2 + 3t\right]_1^4 = 28 - 4 = 24$

31. $\dfrac{dx}{dt} = -2\sin t,\ \dfrac{dy}{dt} = 2\cos t,\ \dfrac{dz}{dt} = 3$

$L = \int_{-\pi}^{\pi} \sqrt{4\sin^2 t + 4\cos^2 t + 9}\ dt = \int_{-\pi}^{\pi} \sqrt{13}\, dt =$

$\left[\sqrt{13}\, t\right]_{-\pi}^{\pi} = 2\pi\sqrt{13} \approx 22.654$

32. $\frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = 2\cos t, \frac{dz}{dt} = \frac{1}{20}$

$L = \int_0^{8\pi} \sqrt{4\sin^2 t + 4\cos^2 t + \frac{1}{400}}\ dt =$

$\int_0^{8\pi} \frac{1}{40}\sqrt{1601}\, dt = \left[\frac{1}{40}\sqrt{1601}\, t\right]_0^{8\pi} =$

$\frac{\pi}{5}\sqrt{1601} \approx 25.14$

33. $\frac{dx}{dt} = \frac{1}{2\sqrt{t}},\ \frac{dy}{dt} = 1, \frac{dz}{dt} = 1$

$L = \int_1^6 \sqrt{\left(\frac{1}{4t}\right) + 1 + 1}\, dt = \int_1^6 \sqrt{2 + \left(\frac{1}{4t}\right)}\ dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	1	1.5000	1	1.5000
1	1.5	1.4720	4	5.8878
2	2	1.4577	2	2.9155
3	2.5	1.4491	4	5.7966
4	3	1.4434	2	2.8868
5	3.5	1.4392	4	5.7570
6	4	1.4361	2	2.8723
7	4.5	1.4337	4	5.7349
8	5	1.4318	2	2.8636
9	5.5	1.4302	4	5.7208
10	6	1.4289	1	1.4289
		approximation		**7.2273**

34. $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2t, \frac{dz}{dt} = 3t^2$

$L = \int_1^2 \sqrt{1+4t^2+9t^4}\ dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	1	3.7417	1	3.7417
1	1.1	4.3608	4	17.4433
2	1.2	5.0421	2	10.0841
3	1.3	5.7849	4	23.1395
4	1.4	6.5890	2	13.1779
5	1.5	7.4540	4	29.8161
6	1.6	8.3799	2	16.7598
7	1.7	9.3664	4	37.4655
8	1.8	10.4134	2	20.8268
9	1.9	11.5208	4	46.0832
10	2	12.6886	1	12.6886
		approximation		**7.7075**

35. $\frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = \cos t, \frac{dz}{dt} = 1$

$L = \int_0^{6\pi} \sqrt{4\sin^2 t + \cos^2 t + 1}\ dt =$

$\int_0^{6\pi} \sqrt{3\sin^2 t + 2}\, dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	0	1.4142	1	1.4142
1	1.88	2.1711	4	8.6843
2	3.77	1.7425	2	3.4851
3	5.65	1.7425	4	6.9702
4	7.54	2.1711	2	4.3421
5	9.42	1.4142	4	5.6569
6	11.3	2.1711	2	4.3421
7	13.2	1.7425	4	6.9702
8	15.1	1.7425	2	3.4851
9	17	2.1711	4	8.6843
10	18.8	1.4142	1	1.4142
		approximation		**34.8394**

36. $\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = -\sin t, \frac{dz}{dt} = \cos t$

$L = \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + \cos^2 t}\ dt =$

$\int_0^{2\pi} \sqrt{\cos^2 t + 1}\, dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	0	1.4142	1	1.4142
1	0.63	1.2863	4	5.1451
2	1.26	1.0467	2	2.0933
3	1.88	1.0467	4	4.1866
4	2.51	1.2863	2	2.5726
5	3.14	1.4142	4	5.6569
6	3.77	1.2863	2	2.5726
7	4.4	1.0467	4	4.1866
8	5.03	1.0467	2	2.0933
9	5.65	1.2863	4	5.1451
10	6.28	1.4142	1	1.4142
		approximation		**7.6405**

37. The center of the sphere is the midpoint of the diameter, so it is $\left(\frac{-2+4}{2}, \frac{3-1}{2}, \frac{6+5}{2}\right) = \left(1, 1, \frac{11}{2}\right)$. The radius is $\frac{1}{2}\sqrt{(-2-4)^2 + (3+1)^2 + (6-5)^2} = \frac{\sqrt{53}}{2}$. The equation is $(x-1)^2 + (y-1)^2 + \left(z - \frac{11}{2}\right)^2 = \frac{53}{4}$.

38. Since the spheres are tangent and have equal radii, the radius of each sphere is $\frac{1}{2}$ of the distance between the centers.
$r = \frac{1}{2}\sqrt{(-3-5)^2 + (1+3)^2 + (2-6)^2} = 2\sqrt{6}.$ The spheres are $(x+3)^2 + (y-1)^2 + (z-2)^2 = 24$ and $(x-5)^2 + (y+3)^2 + (z-6)^2 = 24.$

39. The center must be 6 units from each coordinate plane. Since it is in the first octant, the center is (6, 6, 6). The equation is
$(x-6)^2 + (y-6)^2 + (z-6)^2 = 36.$

40. $x + y = 12$ is parallel to the z-axis. The distance from (1, 1, 4) to the plane $x + y = 12$ is the same as the distance in the xy-plane of (1, 1, 0) to the line $x + y - 12 = 0$. That distance is
$\frac{|1+1-12|}{(1+1)^{1/2}} = 5\sqrt{2}.$ The equation of the sphere is
$(x-1)^2 + (y-1)^2 + (z-4)^2 = 50.$

41.
a. Plane parallel to and two units above the xy-plane

b. Plane perpendicular to the xy-plane whose trace in the xy-plane is the line $x = y$.

c. Union of the yz-plane ($x = 0$) and the xz-plane ($y = 0$)

d. Union of the three coordinate planes

e. Cylinder of radius 2, parallel to the z-axis

f. Top half of the sphere with center (0, 0, 0) and radius 3

42. The points of the intersection satisfy both $(x-1)^2 + (y+2)^2 + (z+1)^2 = 10$ and $z = 2$, so $(x-1)^2 + (y+2)^2 + (2+1)^2 = 10$. This simplifies to $(x-1)^2 + (y+2)^2 = 1,$ the equation of a circle of radius 1. The center is (1, –2, 2).

43. If $P(x, y, z)$ denotes the moving point, $\sqrt{(x-1)^2 + (y-2)^2 + (z+3)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2},$ which simplifies to $(x-1)^2 + (y-2)^2 + (z-5)^2 = 16,$ is a sphere with radius 4 and center (1, 2, 5).

44. If $P(x, y, z)$ denotes the moving point, $\sqrt{(x-1)^2 + (y-2)^2 + (z+3)^2} = \sqrt{(x-2)^2 + (y-3)^2 + (z-2)^2},$ which simplifies to $x + y + 5z = 3/2$, a plane.

45. Note that the volume of a segment of height h in a hemisphere of radius r is $\pi h^2\left[r - \left(\frac{h}{3}\right)\right].$

The resulting solid is the union of two segments, one for each sphere. Since the two spheres have the same radius, each segment will have the same value for h. h is the radius minus half the distance between the centers of the two spheres.

$$h = 2 - \frac{1}{2}\sqrt{(2-1)^2 + (4-2)^2 + (3-1)^2} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$V = 2\left[\pi\left(\frac{1}{2}\right)^2\left(2 - \frac{1}{6}\right)\right] = \frac{11\pi}{12}$$

46. As in Problem 45, the resulting solid is the union of two segments. Since the radii are not the same, the segments will have different heights. Let h_1 be the height of the segment from the first sphere and let h_2 be the height from the second sphere. $r_1 = 2$ is the radius of the first sphere and $r_2 = 3$ is the radius of the second sphere .

Solving for the equation of the plane containing the intersection of the spheres $(x-1)^2 + (y-2)^2 + (z-1)^2 - 4 = 0$ and $(x-2)^2 + (y-4)^2 + (z-3)^2 - 9 = 0,$ we get $x + 2y + 2z - 9 = 0$.

The distance from (1, 2, 1) to the plane is $\frac{2}{3},$ and the distance from (2, 4, 3) to the plane is $\frac{7}{3}.$

$$h_1 = 2 - \frac{2}{3} = \frac{4}{3};\ h_2 = 3 - \frac{7}{3} = \frac{2}{3}$$

$$V = \pi\left(\frac{4}{3}\right)^2\left(2 - \frac{4}{9}\right) + \pi\left(\frac{2}{3}\right)^2\left(3 - \frac{2}{9}\right) = 4\pi$$

47. Plots will vary. We first note that the sign of c will influence the vertical direction an object moves (along the helix) with increasing time; if c is negative the object will spiral downward, whereas if c is positive it will spiral upward. The smaller $|c|$ is the "tighter" the spiral will be; that is the space between successive "coils" of the helix decreases as $|c|$ decreases.

48. Plots will vary. We first note that the sign of a will influence the rotational direction that an object moves (along the helix) with increasing time; if a is negative the object will rotate in a clockwise direction, whereas if a is positive rotation will be counterclockwise. The smaller $|a|$ is the narrower the spiral will be; that is the circles traced out will be of smaller radius as $|a|$ decreases

11.2 Concepts Review

1. magnitude; direction

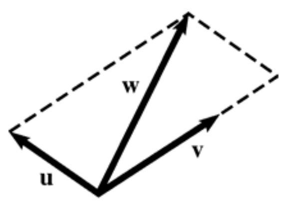
2. they have the same magnitude and direction.

3. the tail of **u**; the head of **v**

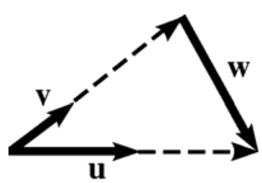
4. 3

Problem Set 11.2

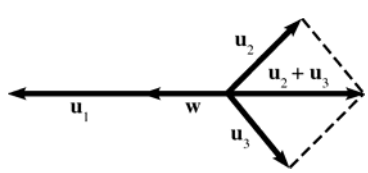
1.



2.



3.



4. **0**

5. $\mathbf{w} = \frac{1}{2}(\mathbf{u}+\mathbf{v}) = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$

6. $\mathbf{n} = \frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$

$\mathbf{m} = \mathbf{v} - \mathbf{n} = \mathbf{v} - \frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{u}$

7. $|\mathbf{w}| = |\mathbf{u}|\cos 60° + |\mathbf{v}|\cos 60° = \frac{1}{2} + \frac{1}{2} = 1$

8. $|\mathbf{w}| = |\mathbf{u}|\cos 45° + |\mathbf{v}|\cos 45° = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

9. $\mathbf{u}+\mathbf{v} = \langle -1+3, 0+4\rangle = \langle 2, 4\rangle$

$\mathbf{u}-\mathbf{v} = \langle -1-3, 0-4\rangle = \langle -4, -4\rangle$

$\|\mathbf{u}\| = \sqrt{(-1)^2+(0)^2} = \sqrt{1} = 1$

$\|\mathbf{v}\| = \sqrt{(3)^2+(4)^2} = \sqrt{25} = 5$

10. $\mathbf{u}+\mathbf{v} = \langle 0+(-3), 0+4\rangle = \langle -3, 4\rangle$

$\mathbf{u}-\mathbf{v} = \langle 0-(-3), 0-4\rangle = \langle 3, -4\rangle$

$\|\mathbf{u}\| = \sqrt{(0)^2+(0)^2} = \sqrt{0} = 0$

$\|\mathbf{v}\| = \sqrt{(-3)^2+(4)^2} = \sqrt{25} = 5$

11. $\mathbf{u}+\mathbf{v} = \langle 12+(-2), 12+2\rangle = \langle 10, 14\rangle$

$\mathbf{u}-\mathbf{v} = \langle 12-(-2), 12-2\rangle = \langle 14, 10\rangle$

$\|\mathbf{u}\| = \sqrt{(12)^2+(12)^2} = \sqrt{288} = 12\sqrt{2}$

$\|\mathbf{v}\| = \sqrt{(-2)^2+(2)^2} = \sqrt{8} = 2\sqrt{2}$

12. $\mathbf{u}+\mathbf{v} = \langle (-0.2)+(-2.1), 0.8+1.3\rangle = \langle -2.3, 2.1\rangle$

$\mathbf{u}-\mathbf{v} = \langle (-0.2)-(-2.1), 0.8-1.3\rangle = \langle 1.9, -0.5\rangle$

$\|\mathbf{u}\| = \sqrt{(-0.2)^2+(0.8)^2} = \sqrt{0.68} \approx 0.825$

$\|\mathbf{v}\| = \sqrt{(-2.1)^2+(1.3)^2} = \sqrt{6.10} \approx 2.47$

13. $\mathbf{u}+\mathbf{v} = \langle -1+3, 0+4, 0+0\rangle = \langle 2, 4, 0\rangle$

$\mathbf{u}-\mathbf{v} = \langle -1-3, 0-4, 0-0\rangle = \langle -4, -4, 0\rangle$

$\|\mathbf{u}\| = \sqrt{(-1)^2+(0)^2+(0)^2} = \sqrt{1} = 1$

$\|\mathbf{v}\| = \sqrt{(3)^2+(4)^2+(0)^2} = \sqrt{25} = 5$

14. $\mathbf{u}+\mathbf{v}=\langle 0+(-3),0+3,0+1\rangle=\langle -3,3,1\rangle$

$\mathbf{u}-\mathbf{v}=\langle 0-(-3),0-3,0-1\rangle=\langle 3,-3,-1\rangle$

$\|\mathbf{u}\|=\sqrt{(0)^2+(0)^2+(0)^2}=\sqrt{0}=0$

$\|\mathbf{v}\|=\sqrt{(-3)^2+(3)^2+(1)^2}=\sqrt{19}\approx 4.359$

15. $\mathbf{u}+\mathbf{v}=\langle 1+(-5),0+0,1+0\rangle=\langle -4,0,1\rangle$

$\mathbf{u}-\mathbf{v}=\langle 1-(-5),0-0,1-0\rangle=\langle 6,0,1\rangle$

$\|\mathbf{u}\|=\sqrt{(1)^2+(0)^2+(1)^2}=\sqrt{2}\approx 1.414$

$\|\mathbf{v}\|=\sqrt{(-5)^2+(0)^2+(0)^2}=\sqrt{25}=5$

16. $\mathbf{u}+\mathbf{v}=\langle 0.3+2.2,0.3+1.3,0.5+(-0.9)\rangle=$
$\langle 2.5,1.6,-0.4\rangle$

$\mathbf{u}-\mathbf{v}=\langle 0.3-2.2,0.3-1.3,0.5-(-0.9)\rangle=$
$\langle -1.9,-1.0,1.4\rangle$

$\|\mathbf{u}\|=\sqrt{(0.3)^2+(0.3)^2+(0.5)^2}=\sqrt{0.43}\approx 0.656$

$\|\mathbf{v}\|=\sqrt{(2.2)^2+(1.3)^2+(-0.9)^2}=\sqrt{7.34}\approx 2.709$

17. Let θ be the angle of **w** measured clockwise from south.

$|\mathbf{w}|\cos\theta=|\mathbf{u}|\cos 30°+|\mathbf{v}|\cos 45°=25\sqrt{3}+25\sqrt{2}$

$=25\left(\sqrt{3}+\sqrt{2}\right)$

$|\mathbf{w}|\sin\theta=|\mathbf{v}|\sin 45°-|\mathbf{u}|\sin 30°=25\sqrt{2}-25$

$=25\left(\sqrt{2}-1\right)$

$|\mathbf{w}|^2=|\mathbf{w}|^2\cos^2\theta+|\mathbf{w}|^2\sin^2\theta$

$=625\left(\sqrt{3}+\sqrt{2}\right)^2+625\left(\sqrt{2}-1\right)^2$

$=625\left(8-2\sqrt{2}+2\sqrt{6}\right)$

$|\mathbf{w}|=\sqrt{625\left(8-2\sqrt{2}+2\sqrt{6}\right)}=25\sqrt{8-2\sqrt{2}+2\sqrt{6}}$

≈ 79.34

$$\tan\theta=\frac{|\mathbf{w}|\sin\theta}{|\mathbf{w}|\cos\theta}=\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}$$

$$\theta=\tan^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right)=7.5°$$

w has magnitude 79.34 lb in the direction S 7.5° W.

18. Let **v** be the resulting force. Let θ be the angle of **v** measured clockwise from south.

$|\mathbf{v}|\cos\theta=60\cos 30°+80\cos 60°=30\sqrt{3}+40$

$=10\left(3\sqrt{3}+4\right)$

$|v|\sin\theta=80\sin 60°-60\sin 30°=40\sqrt{3}-30$

$=10\left(4\sqrt{3}-3\right)$

$|v|^2=|v|^2\cos^2\theta+|v|^2\sin^2\theta$

$=100\left(3\sqrt{3}+4\right)^2+100\left(4\sqrt{3}-3\right)^2$

$= 100(100) = 10,000$

$|\mathbf{v}|=\sqrt{10,000}=100$

$$\tan\theta=\frac{|\mathbf{v}|\sin\theta}{|\mathbf{v}|\cos\theta}=\frac{4\sqrt{3}-3}{3\sqrt{3}+4}$$

$$\theta=\tan^{-1}\left(\frac{4\sqrt{3}-3}{3\sqrt{3}+4}\right)\approx 23.13°$$

The resultant force has magnitude 100 lb in the direction S 23.13° W.

19. The force of 300 N parallel to the plane has magnitude 300 sin 30° = 150 N. Thus, a force of 150 N parallel to the plane will just keep the weight from sliding.

20. Let a be the magnitude of the rope that makes an angle of 27.34°. Let b be the magnitude of the rope that makes an angle of 39.22°.

1. a sin 27.34° = b sin 39.22°
2. a cos 27.34° + b cos 39.22° = 258.5

Solve 1 for b and substitute in 2.

$$a\cos 27.34°+a\frac{\sin 27.34°}{\sin 39.22°}\cos 39.22°=258.5$$

$$a=\frac{258.5}{\cos 27.34°+\sin 27.34°\cot 39.22°}\approx 178.15$$

$$b=\frac{a\sin 27.34°}{\sin 39.22°}\approx 129.40$$

The magnitudes of the forces exerted by the ropes making angles of 27.34° and 39.22° are 178.15 lb and 129.40 lb, respectively.

21. Let θ be the angle the plane makes from north, measured clockwise.

425 sin θ = 45 sin 20°

$$\sin\theta=\frac{9}{85}\sin 20°$$

$$\theta=\sin^{-1}\left(\frac{9}{85}\sin 20°\right)\approx 2.08°$$

Let x be the speed of airplane with respect to the ground.

x = 45 cos 20° + 425 cos $\theta\approx$ 467

The plane flies in the direction N 2.08° E, flying 467 mi/h with respect to the ground.

22. Let **v** be his velocity relative to the surface. Let θ be the angle that his velocity relative to the surface makes with south, measured clockwise.

$|v|\cos\theta = 20, |v|\sin\theta = 3$

$|v|^2 = |v|^2\cos^2\theta + |v|^2\sin^2\theta = 400 + 9 = 409$

$|\mathbf{v}| = \sqrt{409} \approx 20.22$

$\tan\theta = \dfrac{|\mathbf{v}|\sin\theta}{|\mathbf{v}|\cos\theta} = \dfrac{3}{20}$

$\theta = \tan^{-1}\dfrac{3}{20} \approx 8.53°$

His velocity has magnitude 20.22 mi/h in the direction S 8.53° W.

23. Let x be the air speed.

$x \cos 60° = 40$

$x = \dfrac{40}{\cos 60°} = 80$

The air speed of the plane is 80 mi/hr

24. Let x be the air speed. Let θ be the angle that the plane makes with north measured counter-clockwise.

$x \cos\theta = 837 + 63\cos 11.5°$

$x \sin\theta = 63\sin 11.5°$

$x^2 = x^2\cos^2\theta + x^2\sin^2\theta$

$= (837 + 63\cos 11.5°)^2 + (63\sin 11.5°)^2$

$= 704{,}538 + 105{,}462\cos 11.5°$

$x = \sqrt{704{,}538 + 105{,}462\cos 11.5°} \approx 898.82$

$\tan\theta = \dfrac{x\sin\theta}{x\cos\theta} = \dfrac{63\sin 11.5°}{837 + 63\cos 11.5°}$

$\theta = \tan^{-1}\left(\dfrac{63\sin 11.5°}{837 + 63\cos 11.5°}\right) \approx 0.80°$

The plane should fly in the direction N 0.80° W at an air speed of 898.82 mi/h.

25. Let $\mathbf{u} = \langle u_1, u_2\rangle, \mathbf{v} = \langle v_1, v_2\rangle$, and $\mathbf{w} = \langle w_1, w_2\rangle$

a. $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle =$
$\langle v_1 + u_1, v_2 + u_2\rangle = \mathbf{v} + \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle =$
$\langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle =$
$\langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle =$
$\langle u_1, u_2\rangle\langle v_1 + w_1, v_2 + w_2\rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

c. $\mathbf{u} + \mathbf{0} = \langle u_1 + 0, u_2 + 0\rangle = \langle u_1, u_2\rangle = \mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$ by part a.

d. $\mathbf{u} + \mathbf{(-u)} = \langle u_1, u_2\rangle + \langle -u_1, -u_2\rangle =$
$\langle u_1 + (-u_1), u_2 + (-u_2)\rangle = \langle 0, 0\rangle = \mathbf{0}$

e. $a(b\mathbf{u}) = a\left(\langle bu_1, bu_2\rangle\right) = \langle a(bu_1), a(bu_2)\rangle =$
$\langle (ab)u_1, (ab)u_2\rangle = (ab)\mathbf{u}$

f. $a(\mathbf{u} + \mathbf{v}) = a\langle u_1 + v_1, u_2 + v_2\rangle =$
$\langle a(u_1 + v_1), a(u_2 + v_2)\rangle =$
$\langle au_1 + av_1, au_2 + av_2\rangle =$
$\langle au_1, au_2\rangle + \langle av_1, av_2\rangle = a\mathbf{u} + a\mathbf{v}$

g. $(a + b)\mathbf{u} = \langle (a + b)u_1, (a + b)u_2\rangle =$
$\langle au_1 + bu_1, au_2 + bu_2\rangle =$
$\langle au_1, au_2\rangle + \langle bu_1, bu_2\rangle = a\mathbf{u} + b\mathbf{u}$

h. $1\mathbf{u} = \langle 1\cdot u_1, 1\cdot u_2\rangle = \langle u_1, u_2\rangle = \mathbf{u}$

i. $\|a\mathbf{u}\| = \|\langle au_1, au_2\rangle\| = \sqrt{(au_1)^2 + (au_2)^2} =$
$\sqrt{a^2u_1^{\,2} + a^2u_2^{\,2}} = \sqrt{a^2(u_1^{\,2} + u_2^{\,2})} =$
$\sqrt{a^2}\sqrt{u_1^{\,2} + u_2^{\,2}} = |a|\cdot\|\mathbf{u}\|$

26. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle, \mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$

a. $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle =$
$\langle v_1 + u_1, v_2 + u_2, v_3 + u_3\rangle = \mathbf{v} + \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$
$+\langle w_1, w_2, w_3\rangle =$
$\langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3\rangle =$
$\langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3)\rangle =$
$\langle u_1, u_2, u_3\rangle\langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle =$
$\mathbf{u} + (\mathbf{v} + \mathbf{w})$

c. $\mathbf{u} + \mathbf{0} = \langle u_1 + 0, u_2 + 0, u_3 + 0\rangle = \langle u_1, u_2, u_3\rangle = \mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$ by part a.

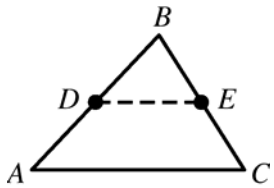
d. $\mathbf{u} + \mathbf{(-u)} = \langle u_1, u_2, u_3\rangle + \langle -u_1, -u_2, -u_3\rangle =$
$\langle u_1 + (-u_1), u_2 + (-u_2), u_3 + (-u_3)\rangle =$
$\langle 0, 0, 0\rangle = \mathbf{0}$

e. $a(b\mathbf{u}) = a\left(\langle bu_1, bu_2, bu_3\rangle\right) =$
$\langle a(bu_1), a(bu_2), a(bu_3)\rangle =$
$\langle (ab)u_1, (ab)u_2, (ab)u_3\rangle = (ab)\mathbf{u}$

g. $(a + b)\mathbf{u} = \langle (a + b)u_1, (a + b)u_2, (a + b)u_3\rangle =$
$\langle au_1 + bu_1, au_2 + bu_2, au_3 + bu_3\rangle =$
$\langle au_1, au_2, au_3\rangle + \langle bu_1, bu_2, bu_3\rangle = a\mathbf{u} + b\mathbf{u}$

h. $1\mathbf{u} = \langle 1\cdot u_1, 1\cdot u_2, 1\cdot u_3\rangle = \langle u_1, u_2, u_3\rangle = \mathbf{u}$

27. Given triangle *ABC*, let *D* be the midpoint of *AB* and *E* be the midpoint of *BC*. $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{AC}$, $\mathbf{z} = \overrightarrow{DE}$

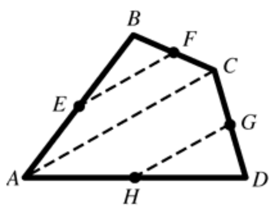


$\mathbf{u} + \mathbf{v} = \mathbf{w}$

$\mathbf{z} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} = \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \frac{1}{2}\mathbf{w}$

Thus, *DE* is parallel to *AC*.

28. Given quadrilateral *ABCD*, let *E* be the midpoint of *AB*, *F* the midpoint of *BC*, *G* the midpoint of *CD*, and *H* the midpoint of *AD*.
ABC and *ACD* are triangles. From Problem 17, *EF* and *HG* are parallel to *AC*. Thus, *EF* is parallel to *HG*. By similar reasoning using triangles *ABD* and *BCD*, *EH* is parallel to *FG*. Therefore, *EFGH* is parallelogram.



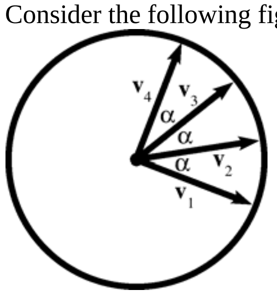
29. Let P_i be the tail of $\mathbf{v}_i$. Then

$\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$

$= \overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \cdots + \overrightarrow{P_nP_1}$

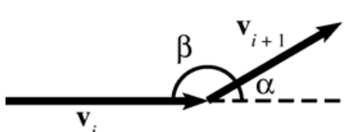
$= \overrightarrow{P_1P_1} = \mathbf{0}.$

30. Consider the following figure of the circle.



$\alpha = \frac{2\pi}{n}$

The vectors have the same length. Consider the following figure for adding vectors $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$.



Then $\beta = \pi - \frac{2\pi}{n}$. Note that the interior angle of a regular *n*-gon is $\pi - \frac{2\pi}{n}$. Thus the vectors (placed head to tail from $\mathbf{v}_1$ to $\mathbf{v}_n$) form a regular *n*-gon. From Problem 19, the sum of the vectors is **0**.

31. The components of the forces along the lines containing *AP*, *BP*, and *CP* are in equilibrium; that is,

$W = W\cos\alpha + W\cos\beta$
$W = W\cos\beta + W\cos\gamma$
$W = W\cos\alpha + W\cos\gamma$

Thus, $\cos\alpha + \cos\beta = 1$, $\cos\beta + \cos\gamma = 1$, and $\cos\alpha + \cos\gamma = 1$. Solving this system of equations results in $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{2}$.

Hence $\alpha = \beta = \gamma = 60°$.
Therefore, $\alpha + \beta = \alpha + \gamma = \beta + \gamma = 120°$.

32. Let A', B', C' be the points where the weights are attached. The center of gravity is located $\frac{|AA'| + |BB'| + |CC'|}{3}$ units below the plane of the triangle. Then, using the hint, the system is in equilibrium when $|AA'| + |BB'| + |CC'|$ is maximum. Hence, it is in equilibrium when $|AP| + |BP| + |CP|$ is minimum, because the total length of the string is $|AP| + |AA'| + |BP| + |BB'| + |CP| + |CC'|$.

33. The components of the forces along the lines containing *AP*, *BP*, and *CP* are in equilibrium; that is,

$5w\cos\alpha + 4w\cos\beta = 3w$
$3w\cos\beta + 5w\cos\gamma = 4w$
$3w\cos\alpha + 4w\cos\gamma = 5w$

Thus,

$5\cos\alpha + 4\cos\beta = 3$
$3\cos\beta + 5\cos\gamma = 4$
$3\cos\alpha + 4\cos\gamma = 5.$

Solving this system of equations results in $\cos\alpha = \frac{3}{5}, \cos\beta = 0, \cos\gamma = \frac{4}{5}$, from which it follows that $\sin\alpha = \frac{4}{5}, \sin\beta = 1, \sin\gamma = \frac{3}{5}$.

Therefore, $\cos(\alpha + \beta) = -\frac{4}{5}, \cos(\alpha + \gamma) = 0$, $\cos(\beta + \gamma) = -\frac{3}{5}$, so

$\alpha + \beta = \cos^{-1}\left(-\frac{4}{5}\right) \approx 143.13°$, $\alpha + \gamma = 90°$,

$\beta + \gamma = \cos^{-1}\left(-\frac{3}{5}\right) \approx 126.87°$.

This problem can be modeled with three strings going through A, four strings through B, and five strings through C, with equal weights attached to the twelve strings. Then the quantity to be minimized is $3|AP| + 4|BP| + 5|CP|$.

34. Written response.

35. By symmetry, the tension on each wire will be the same; denote it by $\|\mathbf{T}\|$ where $\mathbf{T}$ can be the tension vector along any of the wires. The chandelier exerts a force of 100 lbs. vertically downward. Each wire exerts a vertical tension of $\|\mathbf{T}\|\sin 45^\circ$ upward. Since a state of equilibrium exists,

$4 \cdot \|\mathbf{T}\|\sin 45^\circ = 100$ or $\|\mathbf{T}\| = \dfrac{100}{2\sqrt{2}} \approx 35.36$

The tension in each wire is approximately 35.36 lbs.

36. By symmetry, the tension on each wire will be the same; denote it by $\|\mathbf{T}\|$ where $\mathbf{T}$ can be the tension vector along any of the wires. The chandelier exerts a force of 100 lbs. vertically downward. Each wire exerts a vertical tension of $\|\mathbf{T}\|\sin 45^\circ$ upward. Since a state of equilibrium exists, $3 \cdot \|\mathbf{T}\|\sin 45^\circ = 100$ or $\|\mathbf{T}\| = \dfrac{200}{3\sqrt{2}} \approx 47.14$

The tension in each wire is approximately 47.14 lbs.

11.3 Concepts Review

1. $u_1v_1 + u_2v_2 + u_3v_3$; $\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

2. 0

3. $\mathbf{F} \bullet \mathbf{D}$

4. $\langle A, B, C\rangle$

Problem Set 11.3

1. a. $2\mathbf{a} - 4\mathbf{b} = (-4\mathbf{i} + 6\mathbf{j}) + (-8\mathbf{i} + 12\mathbf{j})$
$= -12\mathbf{i} + 18\mathbf{j}$

b. $\mathbf{a} \cdot \mathbf{b} = (-2)(2) + (3)(-3) = -13$

c. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (-2\mathbf{i} + 3\mathbf{j}) \cdot (2\mathbf{i} - 8\mathbf{j})$
$= (-2)(2) + (3)(-8) = -28$

d. $(-2\mathbf{a} + 3\mathbf{b}) \cdot 5\mathbf{c} = 5[(10\mathbf{i} - 15\mathbf{j}) \cdot (-5\mathbf{j})]$
$= 5[(10)(0) + (-15)(-5)] = 375$

e. $\|\mathbf{a}\|\mathbf{c} \cdot \mathbf{a} = \sqrt{4+9}[(0)(-2) + (-5)(3)] = -15\sqrt{13}$

f. $\mathbf{b} \cdot \mathbf{b} - \|\mathbf{b}\| = (2)(2) + (-3)(-3) - \sqrt{4+9}$
$= 13 - \sqrt{13}$

2. a. $-4\mathbf{a} + 3\mathbf{b} = \langle -12, 4\rangle + \langle 3, -3\rangle = \langle -9, 1\rangle$

b. $\mathbf{b} \cdot \mathbf{c} = (1)(0) + (-1)(5) = -5$

c. $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \langle 4, -2\rangle \cdot \langle 0, 5\rangle$
$= (4)(0) + (-2)(5) = -10$

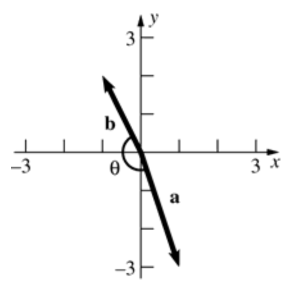
d. $2\mathbf{c} \cdot (3\mathbf{a} + 4\mathbf{b}) = 2\langle 0, 5\rangle \cdot (\langle 9, -3\rangle + \langle 4, -4\rangle)$
$= 2\langle 0, 5\rangle \cdot \langle 13, -7\rangle = 2[(0)(13) + (5)(-7)]$
$= -70$

e. $\|\mathbf{b}\|\mathbf{b} \cdot \mathbf{a} = \sqrt{1+1}[(1)(3) + (-1)(-1)] = 4\sqrt{2}$

f. $\|\mathbf{c}\|^2 - \mathbf{c} \cdot \mathbf{c} = \left(\sqrt{0+25}\right)^2 - [(0)(0) + (5)(5)]$
$= 0$

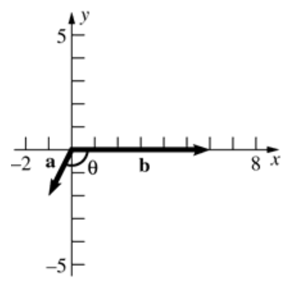
3. a. $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(1)(-1) + (-3)(2)}{\left(\sqrt{10}\right)\left(\sqrt{5}\right)} = -\dfrac{7}{\sqrt{50}}$

$= -\dfrac{7}{5\sqrt{2}} \approx -0.9899$

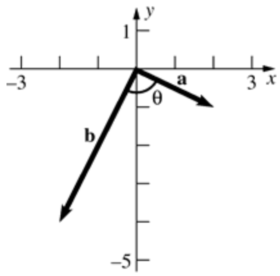


b. $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(-1)(6) + (-2)(0)}{\left(\sqrt{5}\right)(6)} = -\dfrac{6}{6\sqrt{5}}$

$= -\dfrac{1}{\sqrt{5}} \approx -0.4472$

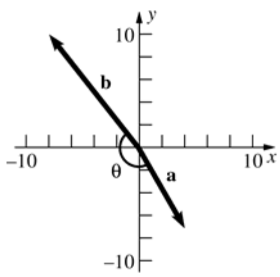


c. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(2)(-2)+(-1)(-4)}{\left(\sqrt{5}\right)\left(2\sqrt{5}\right)} = \dfrac{0}{10} = 0$



d. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(4)(-8)+(-7)(10)}{\left(\sqrt{65}\right)\left(2\sqrt{41}\right)}$

$= \dfrac{-102}{2\sqrt{2665}} = -\dfrac{51}{\sqrt{2665}} \approx -0.9879$

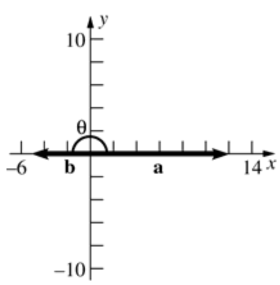


4. a.

$\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(12)(-5)+(0)(0)}{(12)(5)}$

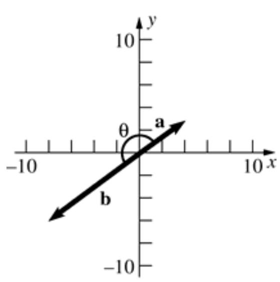
$= \dfrac{-60}{60} = -1$

$\theta = \cos^{-1} -1 = 180^\circ$



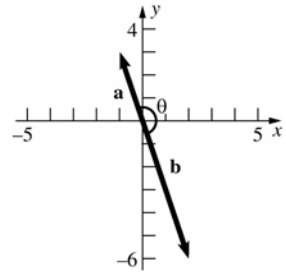
b. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(4)(-8)+(3)(-6)}{(5)(10)} = \dfrac{-50}{50} = -1$

$\theta = \cos^{-1} -1 = 180^\circ$



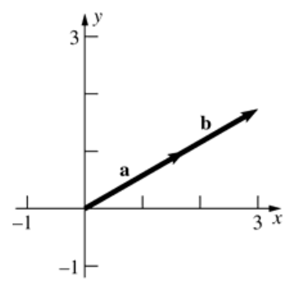
c. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(-1)(2)+(3)(-6)}{\left(\sqrt{10}\right)\left(2\sqrt{10}\right)} = \dfrac{-20}{20} = -1$

$\theta = \cos^{-1} -1 = 180^\circ$



d. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{\left(\sqrt{3}\right)(3)+(1)\left(\sqrt{3}\right)}{(2)\left(2\sqrt{3}\right)} = \dfrac{4\sqrt{3}}{4\sqrt{3}} = 1$

$\theta = \cos^{-1} 1 = 0^\circ$



5. a. $\mathbf{a}\bullet\mathbf{b} = (1)(0)+(2)(1)+(-1)(1) = 1$

b. $(\mathbf{a}+\mathbf{c})\bullet\mathbf{b} = (3\mathbf{j}+\mathbf{k})\bullet(\mathbf{j}+\mathbf{k})$

$= (0)(0)+(3)(1)+(1)(1) = 4$

c. $\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{1^2+2^2+(-1)^2}}(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$= \dfrac{\sqrt{6}}{6}\mathbf{i} + \dfrac{\sqrt{6}}{3}\mathbf{j} - \dfrac{\sqrt{6}}{6}\mathbf{k}$

d. $(\mathbf{b}-\mathbf{c})\bullet\mathbf{a} = (\mathbf{i}-\mathbf{k})\bullet(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$= (1)(1)+(0)(2)+(-1)(-1) = 2$

e. $\dfrac{\mathbf{a}\bullet\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(1)(0)+(2)(1)+(-1)(1)}{\sqrt{1^2+2^2+(-1)^2}\sqrt{0^2+1^2+1^2}}$

$= \dfrac{1}{\sqrt{6}\sqrt{2}} = \dfrac{\sqrt{3}}{6}$

f. By Theorem A (5), $\mathbf{b}\bullet\mathbf{b} - \|\mathbf{b}\|^2 = 0$

6. a. $\mathbf{a}\bullet\mathbf{c} = (\sqrt{2})(-2)+(\sqrt{2})(2)+(0)(1) = 0$

b. $(\mathbf{a}-\mathbf{c})\bullet\mathbf{b} = \left\langle \sqrt{2}-(-2), \sqrt{2}-2, 0-1\right\rangle \bullet \langle 1,-1,1\rangle =$

$(2+\sqrt{2})(1)+(\sqrt{2}-2)(-1)+(-1)(1) = 3$

c. $\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{(\sqrt{2})^2+(\sqrt{2})^2+0^2}}\left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle =$

$\dfrac{1}{2}\left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle = \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, 0 \right\rangle$

d. $(\mathbf{b}-\mathbf{c})\bullet\mathbf{a} = \langle 3, -3, 0\rangle \bullet \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle =$

$(3)(\sqrt{2}) + (-3)(\sqrt{2}) + (0)(0) = 0$

e. $\dfrac{\mathbf{b}\bullet\mathbf{c}}{\|\mathbf{b}\|\|\mathbf{c}\|} = \dfrac{(1)(-2)+(-1)(2)+(1)(1)}{\sqrt{1^2+(-1)^2+1^2}\sqrt{(-2)^2+2^2+1^2}} =$

$\dfrac{-3}{\sqrt{3}\sqrt{9}} = -\dfrac{\sqrt{3}}{3}$

f. By Theorem A (5), $\mathbf{a}\bullet\mathbf{a} - \|\mathbf{a}\|^2 = 0$

7. The basic formula is $\cos\theta = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$

$\theta_{\mathbf{a},\mathbf{b}} = \cos^{-1}\left(\dfrac{\sqrt{2}-\sqrt{2}+0}{\sqrt{4}\cdot\sqrt{3}}\right) =$

$\cos^{-1} 0 = 90^\circ$

$\theta_{\mathbf{a},\mathbf{c}} = \cos^{-1}\left(\dfrac{-2\sqrt{2}+2\sqrt{2}+0}{\sqrt{4}\cdot\sqrt{9}}\right) =$

$\cos^{-1} 0 = 90^\circ$

$\theta_{\mathbf{b},\mathbf{c}} = \cos^{-1}\left(\dfrac{-2-2+1}{\sqrt{3}\cdot\sqrt{9}}\right) =$

$\cos^{-1}\left(-\dfrac{\sqrt{3}}{3}\right) \approx 125.26^\circ$

8. The basic formula is $\cos\theta_{\mathbf{u},\mathbf{v}} = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$

$\theta_{\mathbf{a},\mathbf{b}} = \cos^{-1}\left(\dfrac{\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}+0}{\sqrt{1}\cdot\sqrt{2}}\right) =$

$\cos^{-1} 0 = 90^\circ$

$\theta_{\mathbf{a},\mathbf{c}} = \cos^{-1}\left(\dfrac{\frac{-2\sqrt{3}}{3}+\frac{-2\sqrt{3}}{3}+\frac{\sqrt{3}}{3}}{\sqrt{1}\cdot\sqrt{9}}\right) =$

$\cos^{-1} -\dfrac{\sqrt{3}}{3} \approx 125.26^\circ$

$\theta_{\mathbf{b},\mathbf{c}} = \cos^{-1}\left(\dfrac{-2+2+0}{\sqrt{2}\cdot\sqrt{9}}\right) =$

$\cos^{-1} 0 = 90^\circ$

9. The basic formulae are

$\cos\alpha_{\mathbf{u}} = \dfrac{u_1}{\|\mathbf{u}\|} \quad \cos\beta_{\mathbf{u}} = \dfrac{u_2}{\|\mathbf{u}\|} \quad \cos\gamma_{\mathbf{u}} = \dfrac{u_3}{\|\mathbf{u}\|}$

a. $\mathbf{a} = \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \qquad \|\mathbf{a}\| = 2$

$\cos\alpha_{\mathbf{a}} = \dfrac{a_1}{\|\mathbf{a}\|} = \dfrac{\sqrt{2}}{2}, \quad \alpha_{\mathbf{a}} = 45^\circ$

$\cos\beta_{\mathbf{a}} = \dfrac{a_2}{\|\mathbf{a}\|} = \dfrac{\sqrt{2}}{2}, \quad \beta_{\mathbf{a}} = 45^\circ$

$\cos\gamma_{\mathbf{a}} = \dfrac{a_3}{\|\mathbf{a}\|} = \dfrac{0}{2} = 0, \quad \gamma_{\mathbf{a}} = 90^\circ$

b. $\mathbf{b} = \langle 1, -1, 1\rangle \qquad \|\mathbf{b}\| = \sqrt{3}$

$\cos\alpha_{\mathbf{b}} = \dfrac{b_1}{\|\mathbf{b}\|} = \dfrac{1}{\sqrt{3}} \approx 0.577, \quad \alpha_{\mathbf{b}} \approx 54.74^\circ$

$\cos\beta_{\mathbf{b}} = \dfrac{b_2}{\|\mathbf{b}\|} = \dfrac{-1}{\sqrt{3}} \approx -0.577, \quad \beta_{\mathbf{b}} \approx 125.26^\circ$

$\cos\gamma_{\mathbf{b}} = \dfrac{b_3}{\|\mathbf{b}\|} = \dfrac{1}{\sqrt{3}} \approx 0.577, \quad \gamma_{\mathbf{b}} \approx 54.74^\circ$

c. $\mathbf{c} = \langle -2, 2, 1\rangle \qquad \|\mathbf{c}\| = 3$

$\cos\alpha_{\mathbf{c}} = \dfrac{c_1}{\|\mathbf{c}\|} = -\dfrac{2}{3}, \quad \alpha_{\mathbf{c}} \approx 131.81^\circ$

$\cos\beta_{\mathbf{c}} = \dfrac{c_2}{\|\mathbf{c}\|} = \dfrac{2}{3}, \quad \beta_{\mathbf{c}} \approx 48.19^\circ$

$\cos\gamma_{\mathbf{c}} = \dfrac{c_3}{\|\mathbf{c}\|} = \dfrac{1}{3}, \quad \gamma_{\mathbf{c}} \approx 70.53^\circ$

10. The basic formulae are

$\cos\alpha_{\mathbf{u}} = \dfrac{u_1}{\|\mathbf{u}\|} \quad \cos\beta_{\mathbf{u}} = \dfrac{u_2}{\|\mathbf{u}\|} \quad \cos\gamma_{\mathbf{u}} = \dfrac{u_3}{\|\mathbf{u}\|}$

a. $\mathbf{a} = \left\langle \dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3} \right\rangle \qquad \|\mathbf{a}\| = 1$

$\cos\alpha_{\mathbf{a}} = \dfrac{a_1}{\|\mathbf{a}\|} = \dfrac{\sqrt{3}}{3} \approx 0.577, \quad \alpha_{\mathbf{a}} \approx 54.74^\circ$

$\cos\beta_{\mathbf{a}} = \dfrac{a_2}{\|\mathbf{a}\|} = \dfrac{\sqrt{3}}{3} \approx 0.577, \quad \beta_{\mathbf{a}} \approx 54.74^\circ$

$\cos\gamma_{\mathbf{a}} = \dfrac{a_3}{\|\mathbf{a}\|} = \dfrac{\sqrt{3}}{3} \approx 0.577, \quad \gamma_{\mathbf{a}} \approx 54.74^\circ$

b. $\mathbf{b} = \langle 1, -1, 0\rangle \qquad \|\mathbf{b}\| = \sqrt{2}$

$$\cos\alpha_{\mathbf{b}} = \frac{b_1}{\|\mathbf{b}\|} = \frac{\sqrt{2}}{2}, \quad \alpha_{\mathbf{b}} = 45^\circ$$

$$\cos\beta_{\mathbf{b}} = \frac{b_2}{\|\mathbf{b}\|} = -\frac{\sqrt{2}}{2}, \quad \beta_{\mathbf{b}} \approx 135^\circ$$

$$\cos\gamma_{\mathbf{b}} = \frac{b_3}{\|\mathbf{b}\|} = \frac{0}{\sqrt{2}} = 0, \quad \gamma_{\mathbf{b}} = 90^\circ$$

c. $\mathbf{c} = \langle -2, -2, 1\rangle \qquad \|\mathbf{c}\| = 3$

$$\cos\alpha_{\mathbf{c}} = \frac{c_1}{\|\mathbf{c}\|} = -\frac{2}{3}, \quad \alpha_{\mathbf{c}} \approx 131.81^\circ$$

$$\cos\beta_{\mathbf{c}} = \frac{c_2}{\|\mathbf{c}\|} = -\frac{2}{3}, \quad \beta_{\mathbf{c}} \approx 131.81^\circ$$

$$\cos\gamma_{\mathbf{c}} = \frac{c_3}{\|\mathbf{c}\|} = \frac{1}{3}, \quad \gamma_{\mathbf{c}} \approx 70.53^\circ$$

11. $\langle 6,3\rangle \bullet \langle -1,2\rangle = (6)(-1) + (3)(2) = 0$

Therefore the vectors are orthogonal

12. $\mathbf{a}\bullet\mathbf{b} = (1)(1) + (1)(-1) + (1)(0) = 0$

$\mathbf{a}\bullet\mathbf{c} = (1)(-1) + (1)(-1) + (1)(2) = 0$

$\mathbf{b}\bullet\mathbf{c} = (1)(-1) + (-1)(-1) + (0)(2) = 0$

Therefore the vectors are mutually orthogonal.

13. $\mathbf{a}\bullet\mathbf{b} = (1)(1) + (-1)(1) + (0)(0) = 0$

$\mathbf{a}\bullet\mathbf{c} = (1)(0) + (-1)(0) + (0)(2) = 0$

$\mathbf{b}\bullet\mathbf{c} = (1)(0) + (1)(0) + (0)(2) = 0$

Therefore the vectors are mutually orthogonal.

14. $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}$

$= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$

Thus, $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$ or $\|\mathbf{u}\| = \|\mathbf{v}\|$

15. If $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is perpendicular to $-4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j}$, then $-4x + 5y + z = 0$ and $4x + y = 0$ since the dot product of perpendicular vectors is 0. Solving these equations yields $y = -4x$ and $z = 24x$. Hence, for any x, $x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}$ is perpendicular to the given vectors.

$\|x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}\| = \sqrt{x^2 + 16x^2 + 576x^2}$

$= |x|\sqrt{593}$

This length is 10 when $x = \pm\frac{10}{\sqrt{593}}$. The vectors are $\frac{10}{\sqrt{593}}\mathbf{i} - \frac{40}{\sqrt{593}}\mathbf{j} + \frac{240}{\sqrt{593}}\mathbf{k}$ and

$-\frac{10}{\sqrt{593}}\mathbf{i} + \frac{40}{\sqrt{593}}\mathbf{j} - \frac{240}{\sqrt{593}}\mathbf{k}.$

16. If $\langle x, y, z\rangle$ is perpendicular to both $\langle 1, -2, -3\rangle$ and $\langle -3, 2, 0\rangle$, then $x - 2y - 3z = 0$ and $-3x + 2y = 0$. Solving these equations for y and z in terms of x yields $y = \frac{3}{2}x,\ z = -\frac{2}{3}x.$ Thus, all the vectors have the form $\left\langle x, \frac{3}{2}x, -\frac{2}{3}x\right\rangle$ where x is a real number.

17. A vector equivalent to $\overrightarrow{BA}$ is
$\mathbf{u} = \langle 1+4, 2-5, 3-6\rangle = \langle 5, -3, -3\rangle$.

A vector equivalent to $\overrightarrow{BC}$ is
$\mathbf{v} = \langle 1 + 4, 0 - 5, 1 - 6\rangle = \langle 5, -5, -5\rangle$.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{5\cdot 5 + (-3)(-5) + (-3)(-5)}{\sqrt{25+9+9}\sqrt{25+25+25}}$$

$$= \frac{55}{\sqrt{43}\sqrt{75}} = \frac{11}{\sqrt{129}}, \text{ so } \theta = \cos^{-1}\frac{11}{\sqrt{129}} \approx 14.4^\circ.$$

18. A vector equivalent to $\overrightarrow{BA}$ is
$\mathbf{u} = \langle 6 - 3, 3 - 1, 3 + 1\rangle = \langle 3, 2, 4\rangle$.

A vector equivalent to $\overrightarrow{BC}$ is
$\mathbf{v} = \langle -1 - 3, 10 - 1, -2.5 + 1\rangle = \langle -4, 9, -1.5\rangle$.

$\mathbf{u}\cdot\mathbf{v} = 3(-4) + 2\cdot 9 + 4\cdot(-1.5) = 0$ so $\mathbf{u}$ is perpendicular to $\mathbf{v}$. Thus the angle at B is a right angle.

19. $\langle c, 6\rangle \bullet \langle c, -4\rangle = 0 \Rightarrow c^2 - 24 = 0 \Rightarrow$

$c^2 = 24 \Rightarrow c = \pm 2\sqrt{6}$

20. $(2c\mathbf{i} - 8\mathbf{j})\bullet(3\mathbf{i} + c\mathbf{j}) = 0 \Rightarrow 6c - 8c = 0 \Rightarrow$

$-2c = 0 \Rightarrow c = 0$

21. $(c\mathbf{i} + \mathbf{j} + \mathbf{k})\bullet(0\mathbf{i} + 2\mathbf{j} + d\mathbf{k}) = 0 \Rightarrow 0c + 2 + d = 0 \Rightarrow$

c is any number, $d = -2$

22. $\langle a, 0, 1\rangle \bullet \langle 0, 2, b\rangle = 0 \Rightarrow b = 0$

$\langle a, 0, 1\rangle \bullet \langle 1, c, 1\rangle = 0 \Rightarrow a + 1 = 0$

$\langle 0, 2, b\rangle \bullet \langle 1, c, 1\rangle = 0 \Rightarrow 2c + b = 0$

Thus: $a = -1$ and $c = b = 0$

For problems 23-34, the formula to use is

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{b}\bullet\mathbf{a}}{\|\mathbf{a}\|^2}\right)\mathbf{a}$$

23. $\mathbf{u} = \langle 1, 2\rangle$, $\mathbf{v} = \langle 2, -1\rangle$, $\mathbf{w} = \langle 1, 5\rangle$

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{(1)(2) + (2)(-1)}{2^2 + (-1)^2}\langle 2, -1\rangle = \mathbf{0}$$

24. $\mathbf{u} = \langle 1,2\rangle$, $\mathbf{v} = \langle 2,-1\rangle$, $\mathbf{w} = \langle 1,5\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{(2)(1)+(-1)(2)}{1^2+2^2}\langle 1,2\rangle = \mathbf{0}$$

25. $\mathbf{u} = \langle 1,2\rangle$, $\mathbf{v} = \langle 2,-1\rangle$, $\mathbf{w} = \langle 1,5\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{w} = \frac{(1)(1)+(5)(2)}{1^2+2^2}\langle 1,2\rangle = \left\langle \frac{11}{5}, \frac{22}{5}\right\rangle$$

26. $\mathbf{u} = \langle 1,2\rangle$, $\mathbf{v} = \langle 2,-1\rangle$, $\mathbf{w} = \langle 1,5\rangle$, $\mathbf{w}-\mathbf{v} = \langle -1,6\rangle$

$$\text{proj}_{\mathbf{u}}(\mathbf{w}-\mathbf{v}) = \frac{(-1)(1)+(6)(2)}{1^2+2^2}\langle 1,2\rangle = \left\langle \frac{11}{5}, \frac{22}{5}\right\rangle$$

27. $\mathbf{u} = \langle 1,2\rangle$, $\mathbf{v} = \langle 2,-1\rangle$, $\mathbf{w} = \langle 1,5\rangle$

$$\text{proj}_{\mathbf{j}}\mathbf{u} = \frac{(1)(0)+(2)(1)}{0^2+(1)^2}\langle 0,1\rangle = \langle 0,2\rangle$$

28. $\mathbf{u} = \langle 1,2\rangle$, $\mathbf{v} = \langle 2,-1\rangle$, $\mathbf{w} = \langle 1,5\rangle$

$$\text{proj}_{\mathbf{i}}\mathbf{u} = \frac{(1)(1)+(2)(0)}{1^2+(0)^2}\langle 1,0\rangle = \langle 1,0\rangle$$

29. $\mathbf{u} = \langle 3,2,1\rangle$, $\mathbf{v} = \langle 2,0,-1\rangle$, $\mathbf{w} = \langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{(3)(2)+(2)(0)+(1)(-1)}{2^2+0^2+(-1)^2}\langle 2,0,-1\rangle = \langle 2,0,-1\rangle$$

30. $\mathbf{u} = \langle 3,2,1\rangle$, $\mathbf{v} = \langle 2,0,-1\rangle$, $\mathbf{w} = \langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{(2)(3)+(0)(2)+(-1)(1)}{3^2+2^2+1^2}\langle 3,2,1\rangle = \left\langle \frac{15}{14}, \frac{10}{14}, \frac{5}{14}\right\rangle$$

31. $\mathbf{u} = \langle 3,2,1\rangle$, $\mathbf{v} = \langle 2,0,-1\rangle$, $\mathbf{w} = \langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{w} = \frac{(1)(3)+(5)(2)+(-3)(1)}{3^2+2^2+1^2}\langle 3,2,1\rangle = \left\langle \frac{15}{7}, \frac{10}{7}, \frac{5}{7}\right\rangle$$

32. $\mathbf{u} = \langle 3,2,1\rangle$, $\mathbf{v} = \langle 2,0,-1\rangle$, $\mathbf{w} = \langle 1,5,-3\rangle$

$\mathbf{w}+\mathbf{v} = \langle 3,5,-4\rangle$

$$\text{proj}_{\mathbf{u}}(\mathbf{w}+\mathbf{v}) = \frac{(3)(3)+(5)(2)+(-4)(1)}{3^2+2^2+1^2}\langle 3,2,1\rangle = \left\langle \frac{45}{14}, \frac{30}{14}, \frac{15}{14}\right\rangle$$

33. $\mathbf{u} = \langle 3,2,1\rangle$, $\mathbf{v} = \langle 2,0,-1\rangle$, $\mathbf{w} = \langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{k}}\mathbf{u} = \frac{(3)(0)+(2)(0)+(1)(1)}{0^2+0^2+(1)^2}\langle 0,0,1\rangle = \langle 0,0,1\rangle$$

34. $\mathbf{u} = \langle 3,2,1\rangle$, $\mathbf{v} = \langle 2,0,-1\rangle$, $\mathbf{w} = \langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{i}}\mathbf{u} = \frac{(3)(1)+(2)(0)+(1)(0)}{1^2+0^2+0^2}\langle 1,0,0\rangle = \langle 3,0,0\rangle$$

35. a. $\text{proj}_{\mathbf{u}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right)\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \mathbf{u}$

b. $\text{proj}_{-\mathbf{u}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot(-\mathbf{u})}{\|-\mathbf{u}\|^2}\right)(-\mathbf{u}) = \left(\dfrac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \mathbf{u}$

36. a. $\text{proj}_{\mathbf{u}}(-\mathbf{u}) = \left(\dfrac{(-\mathbf{u})\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right)(-\mathbf{u})$

$= \left(\dfrac{-\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)(-\mathbf{u}) = \mathbf{u}$

b. $\text{proj}_{-\mathbf{u}}(-\mathbf{u}) = \left(\dfrac{(-\mathbf{u})\cdot(-\mathbf{u})}{\|-\mathbf{u}\|^2}\right)(-\mathbf{u})$

$= \left(\dfrac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)(-\mathbf{u}) = -\mathbf{u}$

37. $\mathbf{u}\cdot\mathbf{v} = (-1)(-1)+5\cdot 1+3(-1) = 3$

$\|\mathbf{v}\| = \sqrt{1+1+1} = \sqrt{3}$

$\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{3}{\sqrt{3}} = \sqrt{3}$

38. $\mathbf{u}\cdot\mathbf{v} = 5\left(-\sqrt{5}\right)+5(\sqrt{5})+2(1) = 2$

$\|\mathbf{v}\| = \sqrt{5+5+1} = \sqrt{11}$

$\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{2}{\sqrt{11}}$

39. $\|\mathbf{u}\| = (4+9+z^2)^{1/2} = 5$ and $z > 0$, so $z = 2\sqrt{3} \approx 3.4641$.

40. $\cos^2(46°)+\cos^2(108°)+\cos^2\gamma = 1$

$\Rightarrow \cos\gamma \approx \pm 0.6496$

$\Rightarrow \gamma \approx 49.49°$ or $\gamma \approx 130.51°$

41. There are infinitely many such pairs. Note that $\langle -4, 2, 5\rangle \cdot \langle 1, 2, 0\rangle = -4+4+0 = 0$, so $\mathbf{u} = \langle 1, 2, 0\rangle$ is perpendicular to $\langle -4, 2, 5\rangle$. For any c, $\langle -2, 1, c\rangle \cdot \langle 1, 2, 0\rangle = -2+2+0 = 0$ so $\mathbf{v} = \langle -2, 1, c\rangle$ is a candidate.

$\langle -4, 2, 5\rangle \cdot \langle -2, 1, c\rangle = 8+2+5c$

$8 + 2 + 5c = 0 \Rightarrow c = -2$, so one pair is $\mathbf{u} = \langle 1, 2, 0\rangle$, $\mathbf{v} = \langle -2, 1, -2\rangle$.

42. The midpoint is
$$\left(\frac{3+5}{2},\frac{2-7}{2},\frac{-1+2}{2}\right)=\left(4,-\frac{5}{2},\frac{1}{2}\right),$$ so the vector is $\left\langle 4,-\frac{5}{2},\frac{1}{2}\right\rangle$.

43. The following do not make sense.

a. $\mathbf{v}\cdot\mathbf{w}$ is not a vector.

b. $\mathbf{u}\cdot\mathbf{w}$ is not a vector.

44. The following do not make sense.

c. $\|\mathbf{u}\|$ is not a vector.

d. $\mathbf{u}+\mathbf{v}$ is not a scalar.

45. $a\mathbf{u}+b\mathbf{u}=a\langle u_1, u_2\rangle+b\langle u_1,u_2\rangle$
$=\langle au_1, au_2\rangle+\langle bu_1, bu_2\rangle=\langle au_1+bu_1, au_2+bu_2\rangle$
$\langle (a+b)u_1,(a+b)u_2\rangle=(a+b)\langle u_1, u_2\rangle$
$=(a+b)\mathbf{u}$

46. $\mathbf{u}\bullet\mathbf{v}=u_1v_1+u_2v_2=v_1u_1+v_2u_2=\mathbf{v}\bullet\mathbf{u}$

47. $c(\mathbf{u}\cdot\mathbf{v})=c\left(\langle u_1, u_2\rangle\cdot\langle v_1, v_2\rangle\right)$
$=c(u_1v_1+u_2v_2)=c(u_1v_1)+c(u_2v_2)$
$=(cu_1)v_1+(cu_2)v_2=\langle cu_1, cu_2\rangle\cdot\langle v_1, v_2\rangle$
$=\left(c\langle u_1, u_2\rangle\right)\cdot\langle v_1, v_2\rangle=(c\mathbf{u})\cdot\mathbf{v}$

48. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\langle u_1, u_2\rangle\cdot\left(\langle v_1, v_2\rangle+\langle w_1, w_2\rangle\right)$
$=\langle u_1,u_2\rangle\cdot\langle v_1+w_1, v_2+w_2\rangle$
$=u_1(v_1+w_1)+u_2(v_2+w_2)$
$=(u_1v_1+u_2v_2)+(u_1w_1+u_2w_2)=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

49. $\mathbf{0}\bullet\mathbf{u}=0u_1+0u_2=0$

50. $\mathbf{u}\bullet\mathbf{u}=u_1^{\,2}+u_2^{\,2}=\left(\sqrt{u_1^{\,2}+u_2^{\,2}}\right)^2=\|\mathbf{u}\|^2$

51. $\mathbf{r}=k\mathbf{a}+m\mathbf{b}\Rightarrow 7=k(3)+m(-3)$ and
$-8=k(-2)+m(4)$
$3k-3m=7$
$-2k+4m=-8$
Solve the system of equations to get
$k=\frac{2}{3}, m=-\frac{5}{3}.$

52. $\mathbf{r}=k\mathbf{a}+m\mathbf{b}\Rightarrow 6=k(-4)+m(2)$ and
$-7=k(3)+m(-1)$
$-4k+2m=6$
$3k-m=-7$
Solve the system of equations to get
$k=-4, m=-5.$

53. a and b cannot both be zero. If $a=0$, then the line $ax+by=c$ is horizontal and $\mathbf{n}=b\mathbf{j}$ is vertical, so $\mathbf{n}$ is perpendicular to the line. Use a similar argument if $b=0$. If $a\neq 0$ and $b\neq 0$, then $P_1\left(\frac{c}{a},0\right)$ and $P_2\left(0,\frac{c}{b}\right)$ are points on the line.
$$\mathbf{n}\cdot\overrightarrow{P_1P_2}=(a\mathbf{i}+b\mathbf{j})\cdot\left(-\frac{c}{a}\mathbf{i}+\frac{c}{b}\mathbf{j}\right)=-c+c=0$$

54. $\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$
$+(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=[\mathbf{u}\cdot\mathbf{u}+2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]$
$+[\mathbf{u}\cdot\mathbf{u}-2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]=2(\mathbf{u}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v})$
$=2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$

55. $\|\mathbf{u}+\mathbf{v}\|^2-\|\mathbf{u}-\mathbf{v}\|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$
$-(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=[\mathbf{u}\cdot\mathbf{u}+2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]$
$-[\mathbf{u}\cdot\mathbf{u}-2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}]=4(\mathbf{u}\cdot\mathbf{v})$
so $\mathbf{u}\cdot\mathbf{v}=\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$

56. Place the cube so that its corners are at the points (0, 0, 0), (1, 0, 0), (0,1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0,1, 1), and (1, 1, 1). The main diagonals are (0, 0, 0) to (1, 1, 1), (1, 0, 0) to (0, 1, 1), (0, 1, 0) to (1, 0, 1), and (0, 0, 1) to (1, 1, 0). The corresponding vectors are $\langle 1, 1, 1\rangle$, $\langle -1,1,1\rangle$, $\langle 1,-1,1\rangle$, and $\langle 1,1,-1\rangle$. Because of symmetry, we need only address one situation; let's choose the diagonal from $(0,0,0)$ to $(1,1,1)$ and the face that lies in the $xy-$plane. A vector in the direction of the diagonal is $\mathbf{d}=\langle 1,1,1\rangle$ and a vector normal to the chosen face is $\mathbf{n}=\langle 0,0,1\rangle$. The angle between the diagonal and the face is the *complement* of the angle between $\mathbf{d}$ and $\mathbf{n}$; that is
$$90^\circ-\theta=90^\circ-\cos^{-1}\left(\frac{\mathbf{d}\bullet\mathbf{n}}{\|\mathbf{d}\|\|\mathbf{n}\|}\right)=$$
$$90^\circ-\cos^{-1}\left(\frac{1}{\sqrt{3}\sqrt{1}}\right)=90^\circ-\cos^{-1}\left(\frac{\sqrt{3}}{3}\right)$$
$$\approx 90^\circ-54.74^\circ=35.26^\circ$$

57. Place the box so that its corners are at the points (0, 0, 0), (4, 0, 0), (0, 6, 0), (4, 6, 0), (0, 0, 10), (4, 0, 10), (0, 6, 10), and (4, 6, 10). The main diagonals are (0, 0, 0) to (4, 6, 10), (4, 0, 0) to (0, 6, 10), (0, 6, 0) to (4, 0, 10), and (0, 0, 10) to (4, 6, 0). The corresponding vectors are $\langle 4, 6, 10\rangle$, $\langle -4, 6, 10\rangle$, $\langle 4, -6, 10\rangle$, and $\langle 4, 6, -10\rangle$.

All of these vectors have length $\sqrt{16+36+100} = \sqrt{152}$. Thus, the smallest angle θ between any pair, **u** and **v** of the diagonals is found from the largest value of $\mathbf{u}\cdot\mathbf{v}$, since

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{152}.$$

There are six ways of pairing the four vectors. The largest value of $\mathbf{u}\cdot\mathbf{v}$ is 120 which occurs with

u = $\langle 4, 6, 10\rangle$ and **v** = $\langle -4, 6, 10\rangle$. Thus,

$$\cos\theta = \frac{120}{152} = \frac{15}{19} \text{ so } \theta = \cos^{-1}\frac{15}{19} \approx 37.86°.$$

58. Place the box so that its corners are at the points (0, 0, 0), (1, 0, 0), (0,1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0,1, 1), and (1, 1, 1). The main diagonals are (0, 0, 0) to (1, 1, 1), (1, 0, 0) to (0, 1, 1), (0, 1, 0) to (1, 0, 1), and (0, 0, 1) to (1, 1, 0). The corresponding vectors are $\langle 1, 1, 1\rangle$, $\langle -1, 1, 1\rangle$, $\langle 1, -1, 1\rangle$, and $\langle 1, 1, -1\rangle$.

All of these vectors have length $\sqrt{1+1+1} = \sqrt{3}$. Thus, the angle θ between any pair, **u** and **v** of the diagonals is found from

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{3}.$$

There are six ways of pairing the four vectors, but due to symmetry, there are only two cases we need to consider. In these cases, $\mathbf{u}\cdot\mathbf{v} = 1$ or $\mathbf{u}\cdot\mathbf{v} = -1$. Thus we get that $\cos\theta = \frac{1}{3}$ or $\cos\theta = -\frac{1}{3}$. Solving for θ gives $\theta \approx 70.53^\circ$ or $\theta \approx 109.47^\circ$.

59. Work $= \mathbf{F}\cdot\mathbf{D} = (3\mathbf{i}+10\mathbf{j})\cdot(10\mathbf{j})$
= 0 + 100 = 100 joules

60. **F** = 100 sin 70°**i** – 100 cos 70°**j**
D = 30**i**
Work = $\mathbf{F}\cdot\mathbf{D}$
$= (100\sin 70°)(30) + (-100\cos 70°)(0)$
= 3000 sin 70° ≈ 2819 joules

61. **D** = 5**i** + 8**j**
Work = **F** · **D** = (6)(5) + (8)(8) = 94 ft-lb

62. **D** = 12**j**
Work = **F**· **D** = (–5)(0) + (8)(12) = 96 joules

63. **D** = (4 – 0)**i** + (4 – 0)**j** + (0– 8)**k** = 4 +4**j** – 8 **k**
Thus, $W = \mathbf{F}\cdot\mathbf{D} = 0(4) + 0(4) - 4(-8) = 32$ joules.

64. **D** = (9 – 2)**i** + (4 – 1)**j** + (6 – 3)**k** = 7**i** + 3**j** + 3**k**
Thus, $W = \mathbf{F}\cdot\mathbf{D} = 3(7) - 6(3) + 7(3) = 24$ ft-lbs.

65. $2(x-1) - 4(y-2) + 3(z+3) = 0$
$2x - 4y + 3z = -15$

66. $3(x+2) - 2(y+3) - 1(z-4) = 0$
$3x - 2y - z = -4$

67. $(x-1) + 4(y-2) + 4(z-1) = 0$
$x + 4y + 4z = 13$

68. $z + 3 = 0$
$z = -3$

69. The planes are $2x - 4y + 3z = -15$ and $3x - 2y - z = -4$. The normals to the planes are **u** = $\langle 2, -4, 3\rangle$ and **v** = $\langle 3, -2, -1\rangle$. If θ is the angle between the planes,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{6+8-3}{\sqrt{29}\sqrt{14}} = \frac{11}{\sqrt{406}}, \text{ so}$$

$$\theta = \cos^{-1}\frac{11}{\sqrt{406}} \approx 56.91°.$$

70. An equation of the plane has the form $2x + 4y - z = D$. And this equation must satisfy $2(-1) + 4(2) - (-3) = D$, so $D = 9$. Thus an equation of the plane is $2x + 4y - z = 9$.

71. **a.** Planes parallel to the *xy*-plane may be expressed as $z = D$, so $z = 2$ is an equation of the plane.

b. An equation of the plane is $2(x + 4) - 3(y + 1) - 4(z - 2) = 0$ or $2x - 3y - 4z = -13$.

72. **a.** Planes parallel to the *xy*-plane may be expressed as $z = D$, so $z = 0$ is an equation of the plane.

b. An equation of the plane is $x + y + z = D$; since the origin is in the plane, $0 + 0 + 0 = D$. Thus an equation is $x + y + z = 0$.

73. $$\text{Distance} = \frac{|(1) + 3(-1) + (2) - 7|}{\sqrt{1+9+1}} = \frac{7}{\sqrt{11}} \approx 2.1106$$

74. The distance is 0 since the point is in the plane. $\left(|(-3)(2)+2(6)+(3)-9|=0\right)$

75. (0, 0, 9) is on $-3x + 2y + z = 9$. The distance from (0, 0, 9) to $6x - 4y - 2z = 19$ is
$\frac{|6(0)-4(0)-2(9)-19|}{\sqrt{36+16+4}}=\frac{37}{\sqrt{56}}\approx 4.9443$ is the distance between the planes.

76. (1, 0, 0) is on $5x - 3y - 2z = 5$. The distance from (1, 0, 0) to $-5x + 3y + 2z = 7$ is
$\frac{|-5(1)+3(0)+2(0)-7|}{\sqrt{25+9+4}}=\frac{12}{\sqrt{38}}\approx 1.9467.$

77. The equation of the sphere in standard form is $(x+1)^2+(y+3)^2+(z-4)^2=26,$ so its center is (–1, –3, 4) and radius is $\sqrt{26}$. The distance from the sphere to the plane is the distance from the center to the plane minus the radius of the sphere or
$\frac{|3(-1)+4(-3)+1(4)-15|}{\sqrt{9+16+1}}-\sqrt{26}=\sqrt{26}-\sqrt{26}=0,$
so the sphere is tangent to the plane.

78. The line segment between the points is perpendicular to the plane and its midpoint, (2, 1, 1), is in the plane. Then $\langle 6-(-2),1-1,-2-4\rangle=\langle 8,0,-6\rangle$ is perpendicular to the plane. The equation of the plane is
$8(x-2)+0(y-1)-6(z-1)=0$ or $8x-6z=10$.

79. $|\mathbf{u}\cdot\mathbf{v}|=|\cos\theta|\|\mathbf{u}\|\|\mathbf{v}\|\le\|\mathbf{u}\|\|\mathbf{v}\|$ since $|\cos\theta|\le 1.$

80. $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2$
$\le\|\mathbf{u}\|^2+2|\mathbf{u}\cdot\mathbf{v}|+\|\mathbf{v}\|^2$
$\le\|\mathbf{u}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\|+\|\mathbf{v}\|^2=\left(\|\mathbf{u}\|+\|\mathbf{v}\|\right)^2$
Therefore, $\|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\|.$

81. The 3 wires must offset the weight of the object, thus
$(3\mathbf{i}+4\mathbf{j}+15\mathbf{k})+(-8\mathbf{i}-2\mathbf{j}+10\mathbf{k})+(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$
$=0\mathbf{i}+0\mathbf{j}+30\mathbf{k}$
Thus, $3-8+a=0$, so $a=5$;
$4-2+b=0$, so $b=-2$;
$15+10+c=30$, so $c=5$.

82. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ represent the sides of the polygon connected tail to head in succession around the polygon.
Then $\mathbf{v}_1+\mathbf{v}_2+\ldots+\mathbf{v}_n=\mathbf{0}$ since the polygon is closed, so
$\mathbf{F}\cdot\mathbf{v}_1+\mathbf{F}\cdot\mathbf{v}_2+\ldots+\mathbf{F}\cdot\mathbf{v}_n=\mathbf{F}\cdot(\mathbf{v}_1+\mathbf{v}_2+\ldots+\mathbf{v}_n)$
$=\mathbf{F}\cdot\mathbf{0}=0.$

83. Let $\mathbf{x}=\langle x, y, z\rangle$, so
$(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})=$
$\langle x-a_1, y-a_2, z-a_3\rangle\cdot\langle x-b_1, y-b_2, z-b_3\rangle$
$=x^2-(a_1+b_1)x+a_1b_1+y^2-(a_2+b_2)y+a_2b_2$
$+z^2-(a_3+b_3)z+a_3b_3$
Setting this equal to 0 and completing the squares yields
$$\left(x-\frac{a_1+b_1}{2}\right)^2+\left(y-\frac{a_2+b_2}{2}\right)^2+\left(z-\frac{a_3+b_3}{2}\right)^2.$$
$$=\frac{1}{4}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2\right]$$
A sphere with center $\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},\frac{a_3+b_3}{2}\right)$
and radius
$$\frac{1}{4}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2\right]=\frac{1}{4}|\mathbf{a}-\mathbf{b}|^2$$

84. Let $P(x_0, y_0, z_0)$ be any point on $Ax + By + Cz = D$, so $Ax_0+By_0+Cz_0=D$. The distance between the planes is the distance from $P(x_0, y_0, z_0)$ to $Ax + By + Cz = E$, which is
$$\frac{|Ax_0+By_0+Cz_0-E|}{\sqrt{A^2+B^2+C^2}}=\frac{|D-E|}{\sqrt{A^2+B^2+C^2}}.$$

85. If **a**, **b**, and **c** are the position vectors of the vertices labeled A, B, and C, respectively, then the side BC is represented by the vector **c** – **b**. The position vector of the midpoint of BC is
$\mathbf{b}+\frac{1}{2}(\mathbf{c}-\mathbf{b})=\frac{1}{2}(\mathbf{b}+\mathbf{c}).$ The segment from A to the midpoint of BC is $\frac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{a}.$ Thus, the position vector of P is
$$\mathbf{a}+\frac{2}{3}\left[\frac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{a}\right]=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$$
If the vertices are (2, 6, 5), (4, –1, 2), and (6, 1, 2), the corresponding position vectors are $\langle 2, 6, 5\rangle$, $\langle 4,-1,2\rangle$, and $\langle 6, 1, 2\rangle$. The position vector of P is
$\frac{1}{3}\langle 2+4+6,\ 6-1+1,\ 5+2+2\rangle=\frac{1}{3}\langle 12, 6, 9\rangle=$
$\langle 4, 2, 3\rangle$. Thus P is (4, 2, 3).

86. Let A, B, C, and D be the vertices of the tetrahedron with corresponding position vectors **a**, **b**, **c**, and **d**. The vector representing the segment from A to the centroid of the opposite face, triangle BCD, is $\frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d})-\mathbf{a}$ by Problem 85. Similarly, the segments from B, C, and D to the opposite faces are $\frac{1}{3}(\mathbf{a}+\mathbf{c}+\mathbf{d})-\mathbf{b}, \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{d})-\mathbf{c},$ and $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{d},$ respectively. If these segments meet in one point which has a nice formulation as some fraction of the way from a vertex to the centroid of the opposite face, then there is some number k, such that

$$\mathbf{a}+k\left[\frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d})-\mathbf{a}\right]=\mathbf{b}+k\left[\frac{1}{3}(\mathbf{a}+\mathbf{c}+\mathbf{d})-\mathbf{b}\right]$$
$$=\mathbf{c}+k\left[\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{d})-\mathbf{c}\right]=\mathbf{d}+k\left[\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{d}\right].$$

Thus, $\mathbf{a}(1-k)=\frac{k}{3}\mathbf{a},$ so $k=\frac{3}{4}.$ Hence the segments joining the vertices to the centroids of the opposite faces meet in a common point which is $\frac{3}{4}$ of the way from a vertex to the corresponding centroid. With $k=\frac{3}{4}$, all of the above formulas simplify to $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}),$ the position vector of the point.

87. After reflecting from the xy-plane, the ray has direction $a\mathbf{i}+b\mathbf{j}-c\mathbf{k}$. After reflecting from the xz-plane, the ray now has direction $a\mathbf{i}-b\mathbf{j}-c\mathbf{k}$. After reflecting from the yz-plane, the ray now has direction $-a\mathbf{i}-b\mathbf{j}-c\mathbf{k}$, the opposite of its original direction. ($a<0$, $b<0$, $c<0$)

11.4 Concepts Review

1. $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&2&1\\3&1&-1\end{vmatrix}=\begin{vmatrix}2&1\\1&-1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&1\\3&-1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&2\\3&1\end{vmatrix}\mathbf{k}$

$=(-2-1)\mathbf{i}-(1-3)\mathbf{j}+(-1-6)\mathbf{k}$
$=-3\mathbf{i}+2\mathbf{j}-7\mathbf{k}=\langle -3, 2, -7\rangle$

2. $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$

3. $-(\mathbf{v}\times\mathbf{u})$

4. parallel

Problem Set 11.4

1. a. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&2&-2\\-1&2&-4\end{vmatrix}$

$=\begin{vmatrix}2&-2\\2&-4\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3&-2\\-1&-4\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3&2\\-1&2\end{vmatrix}\mathbf{k}$
$=(-8+4)\mathbf{i}-(12-2)\mathbf{j}+(-6+2)\mathbf{k}$
$=-4\mathbf{i}-10\mathbf{j}-4\mathbf{k}$

b. $\mathbf{b}+\mathbf{c}=6\mathbf{i}+5\mathbf{j}-8\mathbf{k}$, so

$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&2&-2\\6&5&-8\end{vmatrix}$

$=\begin{vmatrix}2&-2\\5&-8\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3&-2\\6&-8\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3&2\\6&5\end{vmatrix}\mathbf{k}$
$=(-16+10)\mathbf{i}-(24+12)\mathbf{j}+(-15-12)\mathbf{k}$
$=-6\mathbf{i}-36\mathbf{j}-27\mathbf{k}$

c. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=-3(6)+2(5)-2(-8)=8$

d. $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&2&-4\\7&3&-4\end{vmatrix}$

$=\begin{vmatrix}2&-4\\3&-4\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&-4\\7&-4\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&2\\7&3\end{vmatrix}\mathbf{k}$
$=(-8+12)\mathbf{i}-(4+28)\mathbf{j}+(-3-14)\mathbf{k}$
$=4\mathbf{i}-32\mathbf{j}-17\mathbf{k}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&2&-2\\4&-32&-17\end{vmatrix}$

$=\begin{vmatrix}2&-2\\-32&-17\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3&-2\\4&-17\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3&2\\4&-32\end{vmatrix}\mathbf{k}$
$=(-34-64)\mathbf{i}-(51+8)\mathbf{j}+(96-8)\mathbf{k}$
$=-98\mathbf{i}-59\mathbf{j}+88\mathbf{k}$

2. a. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&3&1\\-2&-1&0\end{vmatrix}$

$=\begin{vmatrix}3&1\\-1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&1\\-2&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&3\\-2&-1\end{vmatrix}\mathbf{k}$
$=(0+1)\mathbf{i}-(0+2)\mathbf{j}+(-3+6)\mathbf{k}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$
$=\langle 1,-2,3\rangle$

b. $\mathbf{b}+\mathbf{c} = \langle -2 - 2, -1 - 3, 0 - 1\rangle$
$= \langle -4, -4, -1\rangle$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ -4 & -4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ -4 & -4 \end{vmatrix}\mathbf{k}$$

$= (-3 + 4)\mathbf{i} - (-3 + 4)\mathbf{j} + (-12 + 12)\mathbf{k}$
$= \mathbf{i} - \mathbf{j} = \langle 1, -1, 0\rangle$

c. $$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 0 \\ -2 & -3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 0 \\ -2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & -1 \\ -2 & -3 \end{vmatrix}\mathbf{k}$$

$= (1 - 0)\mathbf{i} - (2 - 0)\mathbf{j} + (6 - 2)\mathbf{k} = \langle 1, -2, 4\rangle$

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 3(1) + 3(-2) + 1(4) = 1$

d. $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$= (12 + 2)\mathbf{i} - (12 - 1)\mathbf{j} + (-6 - 3)\mathbf{k}$
$= \langle 14, -11, -9\rangle$

3. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 2 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$= (-8 - 6)\mathbf{i} - (-4 + 6)\mathbf{j} + (2 + 4)\mathbf{k} = -14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
is perpendicular to both **a** and **b**. All perpendicular vectors will have the form $c(-14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ where c is a real number.

4. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & -2 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -2 \\ -2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -2 \\ 3 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 5 \\ 3 & -2 \end{vmatrix}\mathbf{k}$$

$= (20 - 4)\mathbf{i} - (-8 + 6)\mathbf{j} + (4 - 15)\mathbf{k}$
$= 16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$
All vectors perpendicular to both **a** and **b** will have the form $c(16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k})$ where c is a real number.

5. $\mathbf{u} = \langle 3-1, -1-3, 2-5\rangle = \langle 2, -4, -3\rangle$ and
$\mathbf{v} = \langle 4-1, 0-3, 1-5\rangle = \langle 3, -3, -4\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -3 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & -3 \\ -3 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -4 \\ 3 & -3 \end{vmatrix}\mathbf{k}$$

$= (16 - 9)\mathbf{i} - (-8 + 9)\mathbf{j} + (-6 + 12)\mathbf{k}$
$= \langle 7, -1, 6\rangle$ is perpendicular to the plane.

$$\pm\frac{1}{\sqrt{49+1+36}}\langle 7, -1, 6\rangle = \pm\left\langle \frac{7}{\sqrt{86}}, -\frac{1}{\sqrt{86}}, \frac{6}{\sqrt{86}}\right\rangle$$

are the unit vectors perpendicular to the plane.

6. $\mathbf{u} = \langle 5+1, 1-3, 2-0\rangle = \langle 6, -2, 2\rangle$ and
$\mathbf{v} = \langle 4+1, -3-3, -1-0\rangle = \langle 5, -6, -1\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 2 \\ 5 & -6 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 2 \\ -6 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 6 & 2 \\ 5 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 6 & -2 \\ 5 & -6 \end{vmatrix}\mathbf{k}$$

$= (2 + 12)\mathbf{i} - (-6 - 10)\mathbf{j} + (-36 + 10)\mathbf{k}$
$= \langle 14, 16, -26\rangle$
is perpendicular to the plane.

$$\pm\frac{1}{\sqrt{196+256+676}}\langle 14, 16, -26\rangle$$
$$= \pm\left\langle \frac{7}{\sqrt{282}}, \frac{8}{\sqrt{282}}, -\frac{13}{\sqrt{282}}\right\rangle$$

are the unit vectors perpendicular to the plane.

7. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -3 \\ 4 & 2 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -3 \\ 4 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ 4 & 2 \end{vmatrix}\mathbf{k}$$

$= (-4 + 6)\mathbf{i} - (4 + 12)\mathbf{j} + (-2 - 4)\mathbf{k}$
$= 2\mathbf{i} - 16\mathbf{j} - 6\mathbf{k}$
Area of parallelogram = $\|\mathbf{a}\times\mathbf{b}\|$
$= \sqrt{4 + 256 + 36} = 2\sqrt{74}$

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 1 & -4 \end{vmatrix}$

$= \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{k}$

$= (-8 + 1)\mathbf{i} - (-8 - 1)\mathbf{j} + (2 + 2)\mathbf{k}$

$= -7\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$

Area of parallelogram $= \|\mathbf{a}\times\mathbf{b}\| = \sqrt{49+81+16}$

$= \sqrt{146}$

9. $\mathbf{a} = \langle 2-3,\ 4-2,\ 6-1\rangle = \langle -1,\ 2,\ 5\rangle$ and $\mathbf{b} = \langle -1-3, 2-2, 5-1\rangle = \langle -4, 0, 4\rangle$ are adjacent sides of the triangle. The area of the triangle is half the area of the parallelogram with **a** and **b** as adjacent sides.

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 5 \\ -4 & 0 & 4 \end{vmatrix}$

$= \begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 5 \\ -4 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ -4 & 0 \end{vmatrix}\mathbf{k}$

$= (8 - 0)\mathbf{i} - (-4 + 20)\mathbf{j} + (0 + 8)\mathbf{k} = \langle 8, -16, 8\rangle$

Area of triangle $= \frac{1}{2}\sqrt{64+256+64} = \frac{1}{2}\left(8\sqrt{6}\right)$

$= 4\sqrt{6}$

10. $\mathbf{a} = \langle 3-1, 1-2, 5-3\rangle = \langle 2, -1, 2\rangle$ and $\mathbf{b} = \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$ are adjacent sides of the triangle.

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 3 & 3 & 3 \end{vmatrix}$

$= \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}\mathbf{k}$

$= (-3 - 6)\mathbf{i} - (6 - 6)\mathbf{j} + (6 + 3)\mathbf{k} = \langle -9, 0, 9\rangle$

Area of triangle $= \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\sqrt{81+81} = \frac{9\sqrt{2}}{2}$

11. $\mathbf{u} = \langle 0-1, 3-3, 0-2\rangle = \langle -1, 0, -2\rangle$ and $\mathbf{v} = \langle 2-1, 4-3, 3-2\rangle = \langle 1, 1, 1\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{k}$

$= (0 + 2)\mathbf{i} - (-1 + 2)\mathbf{j} + (-1 - 0)\mathbf{k} = \langle 2, -1, -1\rangle$

The plane through (1, 3, 2) with normal $\langle 2, -1, -1\rangle$ has equation

$2(x - 1) - 1(y - 3) - 1(z - 2) = 0$ or $2x - y - z = -3$.

12. $\mathbf{u} = \langle 0-1, 0-1, 1-2\rangle = \langle -1, -1, -1\rangle$ and $\mathbf{v} = \langle -2-1, -3-1, 0-2\rangle = \langle -3, -4, -2\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ -3 & -4 & -2 \end{vmatrix}$

$= \begin{vmatrix} -1 & -1 \\ -4 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -1 \\ -3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & -1 \\ -3 & -4 \end{vmatrix}\mathbf{k}$

$= (2 - 4)\mathbf{i} - (2 - 3)\mathbf{j} + (4 - 3)\mathbf{k} = \langle -2, 1, 1\rangle$

The plane through (0, 0, 1) with normal $\langle -2, 1, 1\rangle$ has equation $-2(x - 0) + 1(y - 0) + 1(z - 1) = 0$ or $-2x + y + z = 1$.

13. $\mathbf{u} = \langle 0-7, 3-0, 0-0\rangle = \langle -7, 3, 0\rangle$ and $\mathbf{v} = \langle 0-7, 0-0, 5-0\rangle = \langle -7, 0, 5\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & 3 & 0 \\ -7 & 0 & 5 \end{vmatrix} =$

$\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -7 & 0 \\ -7 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -7 & 3 \\ -7 & 0 \end{vmatrix}\mathbf{k}$

$= (15 - 0)\mathbf{i} - (-35 - 0)\mathbf{j} + (0 + 21)\mathbf{k} = \langle 15, 35, 21\rangle$

The plane through (7, 0, 0) with normal $\langle 15, 35, 21\rangle$ has equation $15(x - 7) + 35(y - 0) + 21(z - 0) = 0$ or $15x + 35y + 21z = 105$.

14. $\mathbf{u} = \langle 0-a, b-0, 0-0\rangle = \langle -a, b, 0\rangle$ and $\mathbf{v} = \langle 0-a, 0-0, c-0\rangle = \langle -a, 0, c\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} =$$

$$\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix}\mathbf{i} - \begin{vmatrix} -a & 0 \\ -a & c \end{vmatrix}\mathbf{j} + \begin{vmatrix} -a & b \\ -a & 0 \end{vmatrix}\mathbf{k}$$

$= bc\mathbf{i} - (-ac)\mathbf{j} + ab\mathbf{k} = \langle bc, ac, ab\rangle$

The plane through $(a, 0, 0)$ with normal $\langle bc, ac, ab\rangle$ has equation $bc(x - a) + ac(y - 0) + ab(z - 1) = 0$ or $bcx + ac\,y + abz = abc$.

15. An equation of the plane is $1(x - 2) - 1(y - 5) + 2(z - 1) = 0$ or $x - y + 2z = -1$.

16. An equation of the plane is $1(x - 0) + 1(y - 0) + 1(z - 2) = 0$ or $x + y + z = 2$.

17. The plane's normals will be perpendicular to the normals of the other two planes. Then a normal is $\langle 1, -3, 2\rangle \times \langle 2, -2, -1\rangle = \langle 7, 5, 4\rangle$. An equation of the plane is $7(x + 1) + 5(y + 2) + 4(z - 3) = 0$ or $7x + 5y + 4z = -5$.

18. $\mathbf{u} = \langle 1, 1, 1\rangle$ is normal to the plane $x + y + z = 2$ and $\mathbf{v} = \langle 1, -1, -1\rangle$ is normal to the plane $x - y - z = 4$. A normal vector to the required plane must be perpendicular to both the other normals; thus one possibility is

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \langle 0, 2, -2\rangle$$ The plane has an equation of the form: $2y - 2z = D$. Since the point $(2, -1, 4)$ is in the plane, $2(-1) - 2(4) = D$; thus $D = -10$ and an equation for the plane is $2y - 2z = -10$ or $y - z = -5$.

19. $(4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}) = 13\mathbf{i} - 26\mathbf{j} - 26\mathbf{k}$
$= 13(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$
is normal to the plane. An equation of the plane is $1(x - 2) - 2(y + 3) - 2(z - 2) = 0$ or $x - 2y - 2z = 4$.

20. $\mathbf{k} = \langle 0, 0, 1\rangle$ is normal to the xy-plane and $\mathbf{v} = \langle 3, -2, 1\rangle$ is normal to the plane $3x - 2y + z = 4$. A normal vector to the required plane must be perpendicular to both the other normals; thus one possibility is:

$$\mathbf{k}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix} = \langle 2, 3, 0\rangle$$ The plane has an equation of the form: $2x + 3y = D$. Since the point $(0, 0, 0)$ is in the plane, $2(0) + 3(0) = D$; thus $D = 0$ and an equation for the plane is $2x + 3y = 0$.

21. Each vector normal to the plane we seek is parallel to the line of intersection of the given planes. Also, the cross product of vectors normal to the given planes will be parallel to both planes, hence parallel to the line of intersection. Thus, $\langle 4, -3, 2\rangle \times \langle 3, 2, -1\rangle = \langle -1, 10, 17\rangle$ is normal to the plane we seek. An equation of the plane is $-1(x - 6) + 10(y - 2) + 17(z + 1) = 0$ or $-x + 10y + 17z = -3$.

22. $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, hence $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$ hence it is parallel to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

23. $\text{Volume} = |\langle 2, 3, 4\rangle \cdot (\langle 0, 4, -1\rangle \times \langle 5, 1, 3\rangle)| = |\langle 2, 3, 4\rangle \cdot \langle 13, -5, -20\rangle| = |-69| = 69$

24. $\text{Volume} = |(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot [(-\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})| = |(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (12\mathbf{i} + 8\mathbf{j} - 4\mathbf{k})| = |-4| = 4$

25. **a.** $\text{Volume} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\langle 3, 2, 1\rangle \cdot \langle -3, -1, 2\rangle| = |-9| = 9$

b. $\text{Area} = \|\mathbf{u} \times \mathbf{v}\| = |\langle 3, -5, 1\rangle| = \sqrt{9 + 25 + 1} = \sqrt{35}$

c. Let θ be the angle. Then θ is the complement of the smaller angle between **u** and **v** × **w**.

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=\frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{u}\|\|\mathbf{v}\times\mathbf{w}\|}=\frac{9}{\sqrt{9+4+1}\sqrt{9+1+4}}=\frac{9}{14},\ \theta\approx 40.01°$$

26. From Theorem C, $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$, which leads to $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$. Again from Theorem C, $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|=|(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}|=|-(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b}|$, which leads to $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|=|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|$. Therefore, we have that $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|=|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$.

27. Choice (c) does not make sense because $(\mathbf{a}\cdot\mathbf{b})$ is a scalar and can't be crossed with a vector. Choice (d) does not make sense because $(\mathbf{a}\times\mathbf{b})$ is a vector and can't be added to a constant.

28. **a** × **b** and **c** × **d** will both be perpendicular to the common plane. Hence **a** × **b** and **c** × **d** are parallel so (**a** × **b**) × (**c** × **d**) = **0**.

29. Let **b** and **c** determine the (triangular) base of the tetrahedron. Then the area of the base is $\frac{1}{2}\|\mathbf{b}\times\mathbf{c}\|$ which is half of the area of the parallelogram determined by **b** and **c**. Thus,

$$\frac{1}{3}(\text{area of base})(\text{height})=\frac{1}{3}\left[\frac{1}{2}(\text{area of corresponding parallelogram})(\text{height})\right]$$

$$=\frac{1}{6}(\text{area of corresponding parallelpiped})=\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

30. $\mathbf{a}=\langle 4+1,-1-2,2-3\rangle=\langle 5,-3,-1\rangle$,

$\mathbf{b}=\langle 5+1,6-2,3-3\rangle=\langle 6,4,0\rangle$,

$\mathbf{c}=\langle 1+1,1-2,-2-3\rangle=\langle 2,-1,-5\rangle$

$$\text{Volume}=\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=\frac{1}{6}|\langle 5,-3,-1\rangle\cdot\langle -20,30,-14\rangle|=\frac{1}{6}|-176|=\frac{88}{3}$$

31. Let $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ then

$$\mathbf{u}\times\mathbf{v}=\langle u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1\rangle$$

$$\|\mathbf{u}\times\mathbf{v}\|^2=u_2^2v_3^2-2u_2u_3v_2v_3+u_3^2v_2^2+u_3^2v_1^2-2u_1u_3v_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_1u_2v_1v_2+u_2^2v_1^2$$

$$=u_1^2(v_1^2+v_2^2+v_3^2)-u_1^2v_1^2+u_2^2(v_1^2+v_2^2+v_3^2)-u_2^2v_2^2+u_3^2(v_1^2+v_2^2+v_3^2)$$
$$-u_3^2v_3^2-2u_2u_3v_2v_3-2u_1u_3v_1v_3-2u_1u_2v_1v_2$$

$$=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2+2u_2u_3v_2v_3+2u_1u_3v_1v_3+2u_1u_2v_1v_2)$$

$$=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2=\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2$$

32. $\mathbf{u}=\langle u_1,u_2,u_3\rangle,\ \mathbf{v}=\langle v_1,v_2,v_3\rangle,\ \mathbf{w}=\langle w_1,w_2,w_3\rangle$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=\langle (u_2v_3-u_3v_2)+(u_2w_3-u_3w_2),\ (u_3v_1-u_1v_3)+(u_3w_1-u_1w_3),\ (u_1v_2-u_2v_1)+(u_1w_2-u_2w_1)\rangle$$

= (**u** × **v**) + (**u** × **w**)

33. (**v** + **w**) × **u** = –[**u** × (**v** + **w**)]
= –[(**u** × **v**) + (**u** × **w**)] = –(**u** × **v**) – (**u** × **w**)
= (**v** × **u**) + (**w** × **u**)

34. **u** × **v** = **0** ⇒ **u** and **v** are parallel. $\mathbf{u}\cdot\mathbf{v}=0\Rightarrow$ **u** and **v** are perpendicular. Thus, either **u** or **v** is **0**.

35. $\overrightarrow{PQ} = \langle -a, b, 0\rangle$, $\overrightarrow{PR} = \langle -a, 0, c\rangle$,
The area of the triangle is half the area of the parallelogram with $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as adjacent sides, so area

$$(\Delta PQR) = \frac{1}{2}\left\|\langle -a, b, 0\rangle \times \langle -a, 0, c\rangle\right\|$$

$$= \frac{1}{2}\left\|\langle bc, ac, ab\rangle\right\| = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}.$$

36. The area of the triangle is

$$\frac{1}{2}\left\|\langle x_2 - x_1, y_2 - y_1, 0\rangle \times \langle x_3 - x_1, y_3 - y_1, 0\rangle\right\| =$$

$$\frac{1}{2}\left\|\langle 0, 0, (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\rangle\right\|$$

$$= \frac{1}{2}\left|(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right|$$

which is half of the absolute value of the determinant given. (Expand the determinant along the third column to see the equality.)

37. From Problem 35, $D^2 = \frac{1}{4}(b^2c^2 + a^2c^2 + a^2b^2)$

$$= \left(\frac{1}{2}bc\right)^2 + \left(\frac{1}{2}ac\right)^2 + \left(\frac{1}{2}ab\right)^2 = A^2 + B^2 + C^2.$$

38. Note that the area of the face determined by $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$.

Label the tetrahedron so that $\mathbf{m} = \frac{1}{2}(\mathbf{a}\times\mathbf{b})$,

$\mathbf{n} = \frac{1}{2}(\mathbf{b}\times\mathbf{c})$, and $\mathbf{p} = \frac{1}{2}(\mathbf{c}\times\mathbf{a})$ point outward.
The fourth face is determined by $\mathbf{a} - \mathbf{c}$ and $\mathbf{b} - \mathbf{c}$, so

$$\mathbf{q} = \frac{1}{2}[(\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{c})]$$

$$= \frac{1}{2}[(\mathbf{b}\times\mathbf{a}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a}) + (\mathbf{c}\times\mathbf{c})]$$

$$= \frac{1}{2}[-(\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})].$$

$$\mathbf{m}+\mathbf{n}+\mathbf{p}+\mathbf{q} = \frac{1}{2}[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a})$$
$$-(\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})] = \mathbf{0}$$

39. The area of the triangle is $A = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$. Thus,

$$A^2 = \frac{1}{4}\|\mathbf{a}\times\mathbf{b}\|^2 = \frac{1}{4}\left(\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2\right)$$

$$= \frac{1}{4}\left[\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \frac{1}{4}\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right)^2\right]$$

$$= \frac{1}{16}\left[4a^2b^2 - (a^2 + b^2 - c^2)^2\right]$$

$$= \frac{1}{16}(2a^2b^2 - a^4 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4).$$

Note that $s - a = \frac{1}{2}(b + c - a)$,

$s - b = \frac{1}{2}(a + c - b)$, and $s - c = \frac{1}{2}(a + b - c)$.

$$s(s-a)(s-b)(s-c)$$

$$= \frac{1}{16}(a+b+c)(b+c-a)(a+c-b)(a+b-c)$$

$$= \frac{1}{16}(2a^2b^2 - a^4 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4)$$

which is the same as was obtained above.

40. $\mathbf{u}\times\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\times(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$

$$= (u_1v_1)(\mathbf{i}\times\mathbf{i}) + (u_1v_2)(\mathbf{i}\times\mathbf{j}) + (u_1v_3)(\mathbf{i}\times\mathbf{k}) +$$
$$(u_2v_1)(\mathbf{j}\times\mathbf{i}) + (u_2v_2)(\mathbf{j}\times\mathbf{j}) + (u_2v_3)(\mathbf{j}\times\mathbf{k}) +$$
$$(u_3v_1)(\mathbf{k}\times\mathbf{i}) + (u_3v_2)(\mathbf{k}\times\mathbf{j}) + (u_3v_3)(\mathbf{k}\times\mathbf{k})$$
$$= (u_1v_1)(0) + (u_1v_2)(\mathbf{k}) + (u_1v_3)(-\mathbf{j}) +$$
$$(u_2v_1)(-\mathbf{k}) + (u_2v_2)(0) + (u_2v_3)(\mathbf{i}) +$$
$$(u_3v_1)(\mathbf{j}) + (u_3v_2)(-\mathbf{i}) + (u_3v_3)(0)$$
$$= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

11.5 Concepts Review

1. a vector-valued function of a real variable

2. f and g are continuous at c; $f'(t)\mathbf{i}+g'(t)\mathbf{j}$

3. position

4. $\mathbf{r}'(t); \mathbf{r}''(t)$; tangent; concave

Problem Set 11.5

1. $\lim_{t\to 1}[2t\mathbf{i}-t^2\mathbf{j}] = \lim_{t\to 1}(2t)\mathbf{i}-\lim_{t\to 1}(t^2)\mathbf{j} = 2\mathbf{i}-\mathbf{j}$

2. $\lim_{t\to 3}[2(t-3)^2\mathbf{i}-7t^3\mathbf{j}]$
$= \lim_{t\to 3}[2(t-3)^2]\mathbf{i}-\lim_{t\to 3}(7t^3)\mathbf{j} = -189\mathbf{j}$

3. $\lim_{t\to 1}\left[\frac{t-1}{t^2-1}\mathbf{i}-\frac{t^2+2t-3}{t-1}\mathbf{j}\right]$
$= \lim_{t\to 1}\left(\frac{t-1}{(t-1)(t+1)}\right)\mathbf{i}-\lim_{t\to 1}\left(\frac{(t-1)(t+3)}{t-1}\right)\mathbf{j}$
$= \lim_{t\to 1}\left(\frac{1}{t+1}\right)\mathbf{i}-\lim_{t\to 1}(t+3)\mathbf{j} = \frac{1}{2}\mathbf{i}-4\mathbf{j}$

4. $\lim_{t\to -2}\left[\frac{2t^2-10t-28}{t+2}\mathbf{i}-\frac{7t^3}{t-3}\mathbf{j}\right]$
$= \lim_{t\to -2}\left(\frac{2t^2-10t-28}{t+2}\right)\mathbf{i}-\lim_{t\to -2}\left(\frac{7t^3}{t-3}\right)\mathbf{j}$
$= \lim_{t\to -2}(2t-14)\mathbf{i}-\frac{56}{5}\mathbf{j} = -18\mathbf{i}-\frac{56}{5}\mathbf{j}$

5. $\lim_{t\to 0}\left[\frac{\sin t\cos t}{t}\mathbf{i}-\frac{7t^3}{e^t}\mathbf{j}+\frac{t}{t+1}\mathbf{k}\right]$
$= \lim_{t\to 0}\left(\frac{\sin t\cos t}{t}\right)\mathbf{i}-\lim_{t\to 0}\left(\frac{7t^3}{e^t}\right)\mathbf{j}$
$+\lim_{t\to 0}\left(\frac{t}{t+1}\right)\mathbf{k} = \mathbf{i}$

6. $\lim_{t\to\infty}\left[\frac{t\sin t}{t^2}\mathbf{i}-\frac{7t^3}{t^3-3t}\mathbf{j}-\frac{\sin t}{t}\mathbf{k}\right]$
$= \lim_{t\to\infty}\left(\frac{t\sin t}{t^2}\right)\mathbf{i}-\lim_{t\to\infty}\left(\frac{7t^3}{t^3-3t}\right)\mathbf{j}-\lim_{t\to\infty}\left(\frac{\sin t}{t}\right)\mathbf{k}$
$= \lim_{t\to\infty}\left(\frac{\sin t}{t}\right)\mathbf{i}-7\mathbf{j}-\lim_{t\to\infty}\left(\frac{\sin t}{t}\right)\mathbf{k} = -7\mathbf{j}$

7. $\lim_{t\to 0^+}\left\langle \ln(t^3), t^2\ln t, t\right\rangle$ does not exist because $\lim_{t\to 0^+}\ln(t^3) = -\infty.$

8. $\lim_{t\to 0^-}\left\langle e^{-1/t^2}, \frac{t}{|t|}, |t|\right\rangle$
$= \left\langle \lim_{t\to 0^-} e^{-1/t^2}, \lim_{t\to 0^-}\frac{t}{|t|}, \lim_{t\to 0^-}|t|\right\rangle = \langle 0, -1, 0\rangle$

9. **a.** The domain of $f(t) = \frac{2}{t-4}$ is $(-\infty, 4)\cup(4,\infty)$. The domain of $g(t) = \sqrt{3-t}$ is $(-\infty, 3]$. The domain of $h(t) = \ln|4-t|$ is $(-\infty, 4)\cup(4,\infty)$. Thus, the domain of $\mathbf{r}$ is $(-\infty, 3]$ or $\{t\in\mathbb{R}: t\le 3\}$.

b. The domain of $f(t) = [\![t^2]\!]$ is $(-\infty,\infty)$. The domain of $g(t) = \sqrt{20-t}$ is $(-\infty, 20]$. The domain of $h(t) = 3$ is $(-\infty,\infty)$. Thus, the domain of $\mathbf{r}$ is $(-\infty, 20]$ or $\{t\in\mathbb{R}: t\le 20\}$.

c. The domain of $f(t) = \cos t$ is $(-\infty,\infty)$. The domain of $g(t) = \sin t$ is also $(-\infty,\infty)$. The domain of $h(t) = \sqrt{9-t^2}$ is $[-3,3]$. Thus, the domain of $\mathbf{r}$ is $[-3,3]$, or $\{t\in\mathbb{R}: -3\le t\le 3\}$.

10. **a.** The domain of $f(t) = \ln(t-1)$ is $(1,\infty)$. The domain of $g(t) = \sqrt{20-t}$ is $(-\infty, 20]$. Thus, the domain of $\mathbf{r}$ is $(1, 20]$ or $\{t\in\mathbb{R}: 1<t\le 20\}$.

b. The domain of $f(t) = \ln(t^{-1})$ is $(0,\infty)$). The domain of $g(t) = \tan^{-1} t$ is $(-\infty, \infty)$. The domain of $h(t) = t$ is $(-\infty,\infty)$. Thus, the domain of $\mathbf{r}$ is $(0,\infty)$ or $\{t\in\mathbb{R}: t>0\}$.

c. The domain of $g(t) = 1/\sqrt{1-t^2}$ is $(-1,1)$. The domain of $h(t) = 1/\sqrt{9-t^2}$ is $(-3,3)$. (The function f is $f(x) = 0$ which has domain $(-\infty,\infty)$.) Thus, domain of $\mathbf{r}$ is $(-1,1)$.

11. a. $f(t)=\dfrac{2}{t-4}$ is continuous on $(-\infty,4)\cup(4,\infty)$. $g(t)=\sqrt{3-t}$ is continuous on $(-\infty, 3]$. $h(t) = \ln|4-t|$ is continuous on $(-\infty,4)$ and on $(4,\infty)$. Thus, **r** is continuous on $(-\infty, 3]$ or $\{t\in\mathbb{R}: t\le 3\}$.

b. $f(t)=[\![t^2]\!]$ is continuous on $\left(-\sqrt{n+1},-\sqrt{n}\right)\cup\left(\sqrt{n},\sqrt{n+1}\right)$ where n is a non-negative integer.
$g(t)=\sqrt{20-t}$ is continuous on $\left(-\infty,20\right)$ or $\{t\in\mathbb{R}: t<20\}$. $h(t)=3$ is continuous on $(-\infty,\infty)$. Thus, **r** is continuous on $\left(-\sqrt{n+1},-\sqrt{n}\right)\cup\left(\sqrt{k},\sqrt{k+1}\right)$ where n and k are non-negative integers and $k < 400$ or
$\{t\in\mathbb{R}: t<20, t^2 \text{ not an integer}\}$.

c. $f(t)=\cos t$ and $f(t)=\sin t$ are continuous on $\left(-\infty,\infty\right)$.
$h(t)=\sqrt{9-t^2}$ is continuous on $\left[-3,3\right]$. Thus, **r** is continuous on $\left[-3,3\right]$.

12. a. $f(t) = \ln(t-1)$ is continuous on $\left(1,\infty\right)$.
$g(t)=\sqrt{20-t}$ is continuous on $(-\infty$, 20). Thus, **r** is continuous on (1, 20) or $\{t\in\mathbb{R}: 1<t<20\}$.

b. $f(t)=\ln(t^{-1})$ is continuous on $\left(0,\infty\right)$. $g(t)=\tan^{-1} t$ is continuous on $(-\infty, \infty)$. $h(t)=t$ is continuous on $(-\infty,\infty)$. Thus, **r** is continuous on $(0, \infty)$ or $\{t\in\mathbb{R}: t>0\}$.

c. $g(t)=1/\sqrt{1-t^2}$ is continuous on $(-1,1)$.
$h(t)=1/\sqrt{9-t^2}$ is continuous on $\left(-3,3\right)$. (The function f is $f(x)=0$ which is continuous on $(-\infty,\infty)$.) Thus, **r** is continuous on $\left(-1,1\right)$.

13. a. $D_t\mathbf{r}(t)=9(3t+4)^2\mathbf{i}+2te^{t^2}\mathbf{j}+0\mathbf{k}$

$D_t^2\mathbf{r}(t)=54(3t+4)\mathbf{i}+2(2t^2+1)e^{t^2}\mathbf{j}$

b. $D_t\mathbf{r}(t)=\sin 2t\mathbf{i}-3\sin 3t\mathbf{j}+2t\mathbf{k}$

$D_t^2\mathbf{r}(t)=2\cos 2t\mathbf{i}-9\cos 3t\mathbf{j}+2\mathbf{k}$

14. a. $\mathbf{r}'(t)=\left(e^t-2te^{-t^2}\right)\mathbf{i}+(\ln 2)2^t\mathbf{j}+\mathbf{k}$

$\mathbf{r}''(t)=\left(e^t+4t^2e^{-t^2}-2e^{-t^2}\right)\mathbf{i}+(\ln 2)^2 2^t\mathbf{j}$

b. $\mathbf{r}'(t)=2\sec^2 2t\mathbf{i}+\dfrac{1}{1+t^2}\mathbf{j}$

$\mathbf{r}''(t)=8\tan 2t\sec^2 2t\mathbf{i}-\dfrac{2t}{(1+t^2)^2}\mathbf{j}$

15. $\mathbf{r}'(t)=-e^{-t}\mathbf{i}-\dfrac{2}{t}\mathbf{j}$; $\mathbf{r}''(t)=e^{-t}\mathbf{i}+\dfrac{2}{t^2}\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}''(t)=e^{-2t}-\dfrac{2}{t^2}\ln\left(t^2\right)$

$$D_t[\mathbf{r}(t)\cdot\mathbf{r}''(t)]=-2e^{-2t}-\left[\frac{2}{t^2}\cdot\frac{1}{t^2}\cdot 2t-\frac{4}{t^3}\ln\left(t^2\right)\right]$$

$$=-2e^{-2t}-\frac{4}{t^3}+\frac{4\ln\left(t^2\right)}{t^3}$$

16. $\mathbf{r}'(t)=3\cos 3t\mathbf{i}+3\sin 3t\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$

$D_t[\mathbf{r}(t)\cdot\mathbf{r}'(t)]=0$

17. $h(t)\mathbf{r}(t)=e^{-3t}\sqrt{t-1}\mathbf{i}+e^{-3t}\ln(2t^2)\mathbf{j}$

$$D_t[h(t)\mathbf{r}(t)]=-\frac{e^{-3t}}{2}\left(\frac{6t-7}{\sqrt{t-1}}\right)\mathbf{i}+e^{-3t}\left(\frac{2}{t}-3\ln(2t^2)\right)\mathbf{j}$$

18. $h(t)\mathbf{r}(t) = \ln(3t-2)\sin 2t\mathbf{i} + \ln(3t-2)\cosh t\mathbf{j}$

$$D_t[h(t)\mathbf{r}(t)]=\left[2\ln(3t-2)\cos 2t+\frac{3\sin 2t}{3t-2}\right]\mathbf{i}+\left[\ln(3t-2)\sinh t+\frac{3\cosh t}{3t-2}\right]\mathbf{j}$$

19. $\mathbf{v}(t)=\mathbf{r}'(t)=4\mathbf{i}+10t\mathbf{j}+2\mathbf{k}$

$\mathbf{a}(t)=\mathbf{r}''(t)=10\mathbf{j}$

$\mathbf{v}(1)=4\mathbf{i}+10\mathbf{j}+2\mathbf{k}$; $\mathbf{a}(1)=10\mathbf{j}$;

$s(1)=\sqrt{16+100+4}=2\sqrt{30}\approx 10.954$

20. $\mathbf{v}(t)=\mathbf{i}+2(t-1)\mathbf{j}+3(t-3)^2\mathbf{k}$

$\mathbf{a}(t)=2\mathbf{j}+6(t-3)\mathbf{k}$

$\mathbf{v}(0)=\mathbf{i}-2\mathbf{j}+27\mathbf{k}$; $\mathbf{a}(0)=2\mathbf{j}-18\mathbf{k}$;

$s(0)=\sqrt{1+4+729}=\sqrt{734}\approx 27.092$

21. $\mathbf{v}(t)=-\frac{1}{t^2}\mathbf{i}-\frac{2t}{(t^2-1)^2}\mathbf{j}+5t^4\mathbf{k}$

$\mathbf{a}(t)=\frac{2}{t^3}\mathbf{i}+\frac{2+6t^2}{(t^2-1)^3}\mathbf{j}+20t^3\mathbf{k}$

$\mathbf{v}(2)=-\frac{1}{4}\mathbf{i}-\frac{4}{9}\mathbf{j}+80\mathbf{k}$;

$\mathbf{a}(2)=\frac{1}{4}\mathbf{i}+\frac{26}{27}\mathbf{j}+160\mathbf{k}$;

$s(2)=\sqrt{\frac{1}{16}+\frac{16}{81}+6400}=\frac{\sqrt{8,294,737}}{36}$

≈ 80.002

22. $\mathbf{v}(t)=6t^5\mathbf{i}+72t(6t^2-5)^5\mathbf{j}+\mathbf{k}$

$\mathbf{a}(t)=30t^4\mathbf{i}+72(66t^2-5)(6t^2-5)^4\mathbf{j}$

$\mathbf{v}(1)=6\mathbf{i}+72\mathbf{j}+\mathbf{k}$; $\mathbf{a}(1)=30\mathbf{i}+4392\mathbf{j}$;

$s(1)=\sqrt{36+5184+1}=\sqrt{5221}\approx 72.256$

23. $\mathbf{v}(t)=t^2\mathbf{j}+\frac{2}{3}t^{-1/3}\mathbf{k}$; $\mathbf{a}(t)=2t\mathbf{j}-\frac{2}{9}t^{-4/3}\mathbf{k}$

$\mathbf{v}(2)=4\mathbf{j}+\frac{2^{2/3}}{3}\mathbf{k}$; $\mathbf{a}(2)=4\mathbf{j}-\frac{2^{-1/3}}{9}\mathbf{k}$

$s(2)=\sqrt{16+\frac{2^{4/3}}{9}}\approx 4.035$

$\mathbf{v}(2)=4\mathbf{j}+\frac{2^{2/3}}{3}\mathbf{k}$; $\mathbf{a}(2)=4\mathbf{j}-\frac{1}{9\sqrt[3]{2}}\mathbf{k}$;

$s(2)=\sqrt{16+\frac{2^{4/3}}{9}}\approx 4.035$

24. $\mathbf{v}(t)=t^2\mathbf{i}+5(t-1)^3\mathbf{j}+\sin\pi t\mathbf{k}$

$\mathbf{a}(t)=2t\mathbf{i}+15(t-1)^2\mathbf{j}+\pi\cos\pi t\mathbf{k}$

$\mathbf{v}(2)=4\mathbf{i}+5\mathbf{j}$; $\mathbf{a}(2)=4\mathbf{i}+15\mathbf{j}+\pi\mathbf{k}$;

$s(2)=\sqrt{16+25}=\sqrt{41}\approx 6.403$

25. $\mathbf{v}(t)=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+\mathbf{k}$

$\mathbf{a}(t)=-\cos t\,\mathbf{i}-\sin t\,\mathbf{j}$

$\mathbf{v}(\pi)=-\mathbf{j}+\mathbf{k}$; $\mathbf{a}(\pi)=\mathbf{i}$; $s(\pi)=\sqrt{1+1}=\sqrt{2}\approx 1.414$

26. $\mathbf{v}(t)=2\cos 2t\,\mathbf{i}-3\sin 3t\,\mathbf{j}-4\sin 4t\,\mathbf{k}$

$\mathbf{a}(t)=-4\sin 2t\,\mathbf{i}-9\cos 3t\,\mathbf{j}-16\cos 4t\,\mathbf{k}$

$\mathbf{v}\left(\frac{\pi}{2}\right)=-2\mathbf{i}+3\mathbf{j}$; $\mathbf{a}\left(\frac{\pi}{2}\right)=-16\mathbf{k}$;

$s\left(\frac{\pi}{2}\right)=\sqrt{4+9}=\sqrt{13}\approx 3.606$

27. $\mathbf{v}(t)=\sec^2 t\mathbf{i}+3e^t\mathbf{j}-4\sin 4t\mathbf{k}$

$\mathbf{a}(t)=2\sec^2 t\tan t\mathbf{i}+3e^t\mathbf{j}-16\cos 4t\mathbf{k}$

$\mathbf{v}\left(\frac{\pi}{4}\right)=2\mathbf{i}+3e^{\pi/4}\mathbf{j}$; $\mathbf{a}\left(\frac{\pi}{4}\right)=4\mathbf{i}+3e^{\pi/4}\mathbf{j}+16\mathbf{k}$;

$s\left(\frac{\pi}{4}\right)=\sqrt{4+9e^{\pi/2}}\approx 6.877$

28. $\mathbf{v}(t)=-e^t\mathbf{i}-\sin\pi t\mathbf{j}+\frac{2}{3}t^{-1/3}\mathbf{k}$

$\mathbf{a}(t)=-e^t\mathbf{i}-\pi\cos\pi t\mathbf{j}-\frac{2}{9}t^{-4/3}\mathbf{k}$

$\mathbf{v}(2)=-e^2\mathbf{i}+\frac{2^{2/3}}{3}\mathbf{k}$; $\mathbf{a}(2)=-e^2\mathbf{i}-\pi\mathbf{j}-\frac{1}{9\sqrt[3]{2}}\mathbf{k}$;

$s(2)=\sqrt{e^4+\frac{2^{4/3}}{9}}\approx 7.408$

29. $\mathbf{v}(t)=(\pi t\cos\pi t+\sin\pi t)\mathbf{i}$

$+(\cos\pi t-\pi t\sin\pi t)\mathbf{j}-e^{-t}\mathbf{k}$

$\mathbf{a}(t)=(2\pi\cos\pi t-\pi^2 t\sin\pi t)\mathbf{i}$

$+(-2\pi\sin\pi t-\pi^2 t\cos\pi t)\mathbf{j}+e^{-t}\mathbf{k}$

$\mathbf{v}(2)=2\pi\mathbf{i}+\mathbf{j}-e^{-2}\mathbf{k}$; $\mathbf{a}(2)=2\pi\mathbf{i}-2\pi^2\mathbf{j}+e^{-2}\mathbf{k}$;

$s(2)=\sqrt{4\pi^2+1+e^{-4}}\approx 6.364$

30. $\mathbf{v}(t)=\frac{1}{t}\mathbf{i}+\frac{2}{t}\mathbf{j}+\frac{3}{t}\mathbf{k}$

$\mathbf{a}(t)=-\frac{1}{t^2}\mathbf{i}-\frac{2}{t^2}\mathbf{j}-\frac{3}{t^2}\mathbf{k}$

$\mathbf{v}(2)=\frac{1}{2}\mathbf{i}+\mathbf{j}+\frac{3}{2}\mathbf{k}$; $\mathbf{a}(2)=-\frac{1}{4}\mathbf{i}-\frac{1}{2}\mathbf{j}-\frac{3}{4}\mathbf{k}$;

$s(2)=\sqrt{\frac{1}{4}+1+\frac{9}{4}}=\frac{\sqrt{14}}{2}\approx 1.871$

31. If $\|\mathbf{v}\|=C$, then $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=C$ Differentiate implicitly to get $D_t(\mathbf{v}\cdot\mathbf{v})=2\mathbf{v}\cdot\mathbf{v}'=0.$ Thus, $\mathbf{v}\cdot\mathbf{v}'=\mathbf{v}\cdot\mathbf{a}=0,$ so $\mathbf{a}$ is perpendicular to $\mathbf{v}$.

32. If $\|\mathbf{r}(t)\| = C$, similar to Problem 31, $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$. Conversely, if $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, then $2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$. But since $2\mathbf{r}(t)\cdot\mathbf{r}'(t) = D_t[\mathbf{r}(t)\cdot\mathbf{r}(t)]$, this means that $\mathbf{r}(t)\cdot\mathbf{r}(t) = \|\mathbf{r}(t)\|^2$ is a constant, so $\|\mathbf{r}(t)\|$ is constant.

33. $s = \int_0^2 \sqrt{1^2 + \cos^2 t + (-\sin t)^2}\,dt$

$= \int_0^2 \sqrt{1 + \cos^2 t + \sin^2 t}\,dt = \int_0^2 \sqrt{1+1}\,dt$

$= \sqrt{2}\int_0^2 dt = \sqrt{2}(2-0) = 2\sqrt{2}$

34. $s = \int_0^2 \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2}\,dt$

$= \int_0^2 \sqrt{t^2 + 3}\,dt$

$= \left[\frac{t}{2}\sqrt{t^2+3} + \frac{3}{2}\ln\left|t + \sqrt{t^2+3}\right|\right]_0^2$

$= \sqrt{7} + \frac{3}{2}\ln\left(2 + \sqrt{7}\right) - \frac{3}{2}\ln\sqrt{3} \approx 4.126$

Use Formula 44 with $u = t$ and $a = \sqrt{3}$ for $\int \sqrt{t^2+3}\,dt.$

35. $s = \int_3^6 \sqrt{24t^2 + 4t^4 + 36}\,dt$

$= \int_3^6 2\sqrt{t^4 + 6t^2 + 9}\,dt$

$= \int_3^6 2(t^2+3)\,dt = 2\left[\frac{t^3}{3} + 3t\right]_3^6$

$= 2[72 + 18 - (9 + 9)] = 144$

36. $s = \int_0^1 \sqrt{4t^2 + 36t^4 + 324t^4}\,dt$

$= \int_0^1 2t\sqrt{1+90t^2}\,dt = \left[\frac{1}{135}(1+90t^2)^{3/2}\right]_0^1$

$= \frac{1}{135}(91^{3/2} - 1) \approx 6.423$

37. $s = \int_0^1 \sqrt{9t^4 + 36t^4 + 324t^4}\,dt$

$= \int_0^1 3\sqrt{41}t^2\,dt = \left[\sqrt{41}t^3\right]_0^1 = \sqrt{41} \approx 6.403$

38. $s = \int_0^1 \sqrt{343t^{12} + 98t^{12} + 1764t^{12}}\,dt$

$= \int_0^1 21\sqrt{5}t^6\,dt = \left[3\sqrt{5}t^7\right]_0^1 = 3\sqrt{5} \approx 6.708$

39. $\mathbf{f}'(u) = -\sin u\mathbf{i} + 3e^{3u}\mathbf{j};\ g'(t) = 6t$

$\mathbf{F}'(t) = \mathbf{f}'(g(t))g'(t)$

$= -6t\sin(3t^2 - 4)\mathbf{i} + 18te^{9t^2-12}\mathbf{j}$

40. $\mathbf{f}'(u) = 2u\mathbf{i} + \sin 2u\mathbf{j};\ g'(t) = \sec^2 t$

$\mathbf{F}'(t) = \mathbf{f}'(g(t))g'(t)$

$= 2\tan t\sec^2 t\mathbf{i} + \sec^2 t\sin(2\tan t)\mathbf{j}$

41. $\int_0^1 (e^t\mathbf{i} + e^{-t}\mathbf{j})dt = \left[e^t\mathbf{i} - e^{-t}\mathbf{j}\right]_0^1$

$= (e-1)\mathbf{i} + (1-e^{-1})\mathbf{j}$

42. $\int_{-1}^1 [(1+t)^{3/2}\mathbf{i} + (1-t)^{3/2}\mathbf{j}]dt$

$= \left[\frac{2}{5}(1+t)^{5/2}\mathbf{i} - \frac{2}{5}(1-t)^{5/2}\mathbf{j}\right]_{-1}^1$

$= \frac{8\sqrt{2}}{5}\mathbf{i} + \frac{8\sqrt{2}}{5}\mathbf{j}$

43. $\mathbf{r}(t) = 5\cos 6t\mathbf{i} + 5\sin 6t\mathbf{j}$

$\mathbf{v}(t) = -30\sin 6t\mathbf{i} + 30\cos 6t\mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{900\sin^2 6t + 900\cos^2 6t} = 30$

$\mathbf{a}(t) = -180\cos 6t\mathbf{i} - 180\sin 6t\mathbf{j}$

44. **a.** $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + (2t-3)\mathbf{k}$

$2t - 3 < 0$ for $t < \frac{3}{2}$, so the particle moves downward for $0 \le t < \frac{3}{2}$.

b. $|\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + (2t-3)^2}$

$= \sqrt{4t^2 - 12t + 10}$

$4t^2 - 12t + 10 = 0$ has no real-number solutions, so the particle never has speed 0, i.e., it never stops moving.

c. $t^2 - 3t + 2 = 12$ when $t^2 - 3t - 10 = (t+2)(t-5) = 0$, $t = -2, 5$. Since $t \ge 0$, the particle is 12 meters above the ground when $t = 5$.

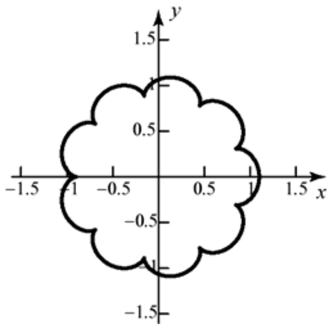
d. $\mathbf{v}(5) = \cos 5\mathbf{i} - \sin 5\mathbf{j} + 7\mathbf{k}$

45. a. The motion of the planet with respect to the sun can be given by
$x = R_p \cos t, \quad y = R_p \sin t.$

Assume that when $t = 0$, both the planet and the moon are on the x-axis. Since the moon orbits the planet 10 times for every time the planet orbits the sun, the motion of the moon with respect to the planet can be given by
$x = R_m \cos 10t, \quad y = R_m \sin 10t.$

Combining these equations, the motion of the moon with respect to the sun is given by $x = R_p \cos t + R_m \cos 10t$,
$y = R_p \sin t + R_m \sin 10t.$

b.



c. $x'(t) = -R_p \sin t - 10R_m \sin 10t$
$y'(t) = R_p \cos t + 10R_m \cos 10t$

The moon is motionless with respect to the sun when $x'(t)$ and $y'(t)$ are both 0.
Solve $x'(t) = 0$ for sin t and $y'(t) = 0$ for cos t to get $\sin t = -\frac{10R_m}{R_p}\sin 10t$,
$\cos t = -\frac{10R_m}{R_p}\cos 10t$.

Since $\sin^2 t + \cos^2 t = 1$

$1 = \frac{100R_m^2}{R_p^2}\sin^2 10t + \frac{100R_m^2}{R_p^2}\cos^2 10t$

$= \frac{100R_m^2}{R_p^2}$. Thus, $R_p^2 = 100R_m^2$ or $R_p = 10R_m$. Substitute this into $x'(t) = 0$ and $y'(t) = 0$ to get $-R_p(\sin t + \sin 10t) = 0$ and $R_p(\cos t + \cos 10t) = 0.$

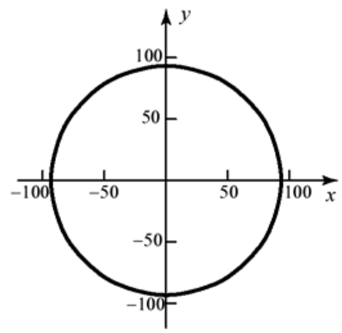
If $0 \le t \le \frac{\pi}{2}$, then to have sin t + sin $10t$ = 0 and cos t + cos $10t$ = 0 it must be that

$10t = \pi + t$ or $t = \frac{\pi}{9}$.

Thus, when the radius of the planet's orbit around the sun is ten times the radius of the moon's orbit around the planet and $t = \frac{\pi}{9}$, the moon is motionless with respect to the sun.

46. a. Years

b.



The moon's orbit is almost indistinguishable from a circle.

c. The sun orbits the earth once each year while the moon orbits the earth roughly 13 times each year.

d. $\|\mathbf{r}(0)\| = \|93.24\mathbf{i}\| = 93.24$ million mi, the sum of the orbital radii.

e. 93 – 0.24 = 92.76 million mi

f. No; since the moon orbits the earth 13 times for each time the earth orbits the sun, the moon could not be stationary with respect to the sun unless the radius of its orbit around the earth were $\frac{1}{13}$ th the radius of the earth's orbit around the sun.

g. $\mathbf{v}(t) = [-186\pi \sin(2\pi t) - 6.24\pi \sin(26\pi t)]\mathbf{i} + [186\pi \cos(2\pi t) + 6.24\pi \cos(26\pi t)]\mathbf{j}$

$\mathbf{a}(t) = [-372\pi^2 \cos(2\pi t) - 162.24\pi^2 \cos(26\pi t)]\mathbf{i}$
$\quad + [-372\pi^2 \sin(2\pi t) - 162.24\pi^2 \sin(26\pi t)]\mathbf{j}$

$\mathbf{v}\left(\frac{1}{2}\right) = 0\mathbf{i} + (-186\pi - 6.24\pi)\mathbf{j} = -192.24\pi\mathbf{j}$

$s\left(\frac{1}{2}\right) = 192.24\pi$ million mi/yr

$\mathbf{a}\left(\frac{1}{2}\right) = (372\pi^2 + 162.24\pi^2)\mathbf{i} + 0\mathbf{j} = 534.24\pi^2\mathbf{i}$

47. **a.** Winding upward around the right circular cylinder $x = \sin t$, $y = \cos t$ as t increases.

b. Same as part a, but winding faster/slower by a factor of $3t^2$.

c. With standard orientation of the axes, the motion is winding to the right around the right circular cylinder $x = \sin t$, $z = \cos t$.

d. Spiraling upward, with increasing radius, along the spiral $x = t\sin t$, $y = t\cos t$.

48. For this problem, keep in mind that $\mathbf{r}$, θ, $\mathbf{u}_1$, and $\mathbf{u}_2$ are all functions of t and that prime indicates differentiation with respect to t.

a. $\mathbf{u}_1 = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and $\mathbf{u}_2 = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$. Applying the Chain Rule to $\mathbf{u}_1$ gives

$$\mathbf{u}_1' = (-\sin\theta)\theta'\mathbf{i} + (\cos\theta)\theta'\mathbf{j}$$
$$= \theta'(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = \theta'\mathbf{u}_2$$

Similarly, applying the Chain Rule to $\mathbf{u}_2$ gives

$$\mathbf{u}_2' = (-\cos\theta)\theta'\mathbf{i} + (-\sin\theta)\theta'\mathbf{j}$$
$$= -\theta'(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = -\theta'\mathbf{u}_1$$

b. $$\mathbf{v}(t) = \mathbf{r}'(t) = D_t(r\,\mathbf{u}_1) = r\,\mathbf{u}_1' + r'\mathbf{u}_1$$
$$= r'\mathbf{u}_1 + r\theta'\mathbf{u}_2$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = D_t(r'\mathbf{u}_1 + r\theta'\mathbf{u}_2)$$
$$= r'\mathbf{u}_1' + r''\mathbf{u}_1 + r\theta'\mathbf{u}_2' + r\theta''\mathbf{u}_2 + \theta'r'\mathbf{u}_2$$
$$= \left(r'' - r(\theta')^2\right)\mathbf{u}_1 + (2r'\theta' + r\theta'')\mathbf{u}_2$$

c. The only force acting on the planet is the gravitational attraction of the sun, which is a force directed along the line from the sun to the planet. Thus, by Newton's Second Law,

$$m\mathbf{a} = \mathbf{F} = -c\mathbf{u}_1 + 0\mathbf{u}_2$$

From Newton's Law of Gravitation

$$\|\mathbf{F}\| = \frac{GMm}{r^2}$$

so from part (b)

$$m\mathbf{a} = -\frac{GMm}{r^2}\mathbf{u}_1$$

$$\left(r'' - r(\theta')^2\right)\mathbf{u}_1 + (2r'\theta' + r\theta'')\mathbf{u}_2 = \mathbf{a} = -\frac{GM}{r^2}\mathbf{u}_1$$

Equating the coefficients of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ gives

$$r'' - r(\theta')^2 = -\frac{GM}{r^2}$$
$$2r'\theta' r\theta'' = 0$$

d. $\mathbf{r}\times\mathbf{r}'$ is a constant vector by Example 8. Call it $\mathbf{D} = \mathbf{r}\times\mathbf{r}'$. Thus,

$$\mathbf{D} = \mathbf{r}\times\mathbf{r}' = \mathbf{r}\times(r_1\mathbf{u}_1 + r\theta'\mathbf{u}_2)$$
$$= r_1\,\mathbf{r}\times\mathbf{u}_1 + r\theta'\mathbf{r}\times\mathbf{u}_2$$
$$= \mathbf{0} + (r\theta')(r\,\mathbf{u}_1)\times\mathbf{u}_2$$
$$= r^2\theta'(\mathbf{u}_1\times\mathbf{u}_2)$$
$$= r^2\theta'\mathbf{k}$$

e. The speed at $t = 0$ is the distance from the sun times the angular velocity, that is, $v_0 = r_0\,\theta'(0)$. Thus, $\theta'(0) = v_0 / r_0$. Substituting these into the expression from part (d) gives

$$\left(r(0)\right)^2 \theta'(0)\mathbf{k} = \mathbf{D}$$

$$\left(r(0)\right)^2 \frac{v_0}{r_0}\mathbf{k} = \mathbf{D}$$

$$\mathbf{D} = r_0 v_0\,\mathbf{k}$$

Since **D** is a constant vector (i.e., constant for all *t*), we conclude that

$$r^2\theta'\mathbf{k} = \mathbf{D} = r_0 v_0\,\mathbf{k}$$

$$r^2\theta' = r_0 v_0$$

for all *t*.

f. Let $q = r'$. From (c)

$$q = r' = \frac{dr}{dt}$$

$$r'' = \frac{d}{dt}\frac{dr}{dt} = \frac{dq}{dt} = \frac{dq}{dr}\frac{dr}{dt} = q\frac{dq}{dr}$$

$$r'' - r\left(\theta'\right)^2 = -\frac{GM}{r^2}$$

$$q\frac{dq}{dr} - r\left(\frac{r_0 v_0}{r^2}\right)^2 = -\frac{GM}{r^2}$$

$$q\frac{dq}{dr} = r\left(\frac{r_0 v_0}{r^2}\right)^2 - \frac{GM}{r^2}$$

$$q\frac{dq}{dr} = \frac{r_0^2 v_0^2}{r^3} - \frac{GM}{r^2}$$

g. Integrating the result from (f) gives

$$\int q\frac{dq}{dr}\,dr = \int \left(\frac{r_0^2 v_0^2}{r^3} - \frac{GM}{r^2}\right)dr$$

$$\frac{1}{2}q^2 = \frac{r_0^2 v_0^2}{(-2)r^2} + \frac{GM}{r} + C$$

$$q^2 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} + C_1$$

When $t = 0$, $r'(0) = q(0) = 0$ since the rate of change of distance from the origin is 0 at the perihelion. Also, when $t = 0$, $r(0) = r_0$.

Thus

$$0 = -\frac{r_0^2 v_0^2}{r_0^2} + \frac{2GM}{r_0} + C_1$$

$$C_1 = -\frac{2GM}{r_0} + v_0^2$$

Thus,

$$q^2 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} + C_1 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} - \frac{2GM}{r_0} + v_0^2 = 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) + v_0^2\left(1 - \frac{r_0^2}{r^2}\right)$$

h. Let $p = 1/r$. Then $r = 1/p$ and $r' = -\frac{1}{p^2}p'$. Thus,

$$q^2 = (r')^2 = \left(-\frac{p'}{p^2}\right)^2 = \frac{(p')^2}{p^4}$$

Dividing both sides of the equation from (g) by $\left(r^2\theta'\right)^2$ and using the result from (e) that $r^2\theta' = r_0v_0$ gives

$$\frac{q^2}{\left(r^2\theta'\right)^2} = \frac{2GM}{\left(r^2\theta'\right)^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)+\frac{v_0^2}{\left(r^2\theta'\right)^2}\left(1-\frac{r_0^2}{r^2}\right)$$

$$\frac{q^2}{\left(r^2\theta'\right)^2} = \frac{2GM}{\left(r_0v_0\right)^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)+\frac{v_0^2}{\left(r_0v_0\right)^2}\left(1-\frac{r_0^2}{r^2}\right)$$

$$\frac{q^2}{\left(r^2\theta'\right)^2} = \frac{2GM}{r_0^2v_0^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)+\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

Now substitute the result from above to get

$$\frac{\frac{1}{p^4}(p')^2}{\frac{1}{p^4}(\theta')^2} = \frac{2GM}{v_0^2r_0^2}(p-p_0)+\left(p_0^2-p^2\right)$$

$$\frac{v_0^2r_0^2}{(\theta')^2}\left(\frac{dp}{dt}\right)^2 = 2GM\,(p-p_0)+v_0^2r_0^2\left(p_0^2-p^2\right)$$

$$\frac{v_0^2r_0^2}{(\theta')^2}\left(\frac{dp}{dt}\right)^2 = 2GM\,(p-p_0)+v_0^2\left(1-\frac{p^2}{p_0^2}\right)$$

i. Continuing with the equation from (h), and using the Chain Rule gives

$$v_0^2r_0^2\left(\frac{dp/dt}{d\theta/dt}\right)^2 = 2GM\,(p-p_0)+v_0^2\left(1-\frac{p^2}{p_0^2}\right)$$

$$\left(\frac{dp}{d\theta}\right)^2 = \frac{2GMp_0^2}{v_0^2}(p-p_0)+\left(p_0^2-p^2\right)$$

$$\left(\frac{dp}{d\theta}\right)^2 = p_0^2-\frac{2GMp_0^2}{v_0^2}p_0+\left(\frac{GMp_0^2}{v_0^2}\right)^2-\left(p^2-\frac{2GMp_0^2}{v_0^2}p+\left(\frac{GMp_0^2}{v_0^2}\right)^2\right)$$

$$\left(\frac{dp}{d\theta}\right)^2 = \left(p_0-\frac{GMp_0^2}{v_0^2}\right)^2-\left(p-\frac{GMp_0^2}{v_0^2}\right)^2$$

j. Taking the square root of both sides from part (i) gives

$$\frac{dp}{d\theta} = \pm\sqrt{\left(p_0-\frac{GMp_0^2}{v_0^2}\right)^2-\left(p-\frac{GMp_0^2}{v_0^2}\right)^2}$$

From (e) we have $r^2\theta' = r_0v_0$, so $\theta' = r_0v_0/r^2 > 0$. Recall that the planet is at its perihelion at time $t = 0$, so this is as close as it gets to the sun. Thus, for t near 0, the distance from the sun r must increase with t.

Thus, $\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{r'}{\theta'} > 0$ and from the beginning of (h) $r' = -\frac{1}{p^2}p'$. Thus, $\frac{dp}{d\theta} = \frac{dp/dt}{d\theta/dt} = \frac{-p^2r'}{d\theta/dt} < 0$

The minus sign is the correct sign to take.

k. Separating variables in this differential equation gives

$$\frac{-dp}{\sqrt{\left(p_0 - \dfrac{GMp_0^2}{v_0^2}\right)^2 - \left(p - \dfrac{GMp_0^2}{v_0^2}\right)^2}} = d\theta$$

$$\int 1/\sqrt{\left(p_0 - GMp_0^2 / v_0^2\right)^2 - \left(p - GMp_0^2 / v_0^2\right)^2}\, dp = \int d\theta$$

$$\cos^{-1}\left(\frac{p - GMp_0^2 / v_0^2}{p_0 - GMp_0^2 / v_0^2}\right) = \theta + C_2$$

The initial condition for this differential equation is $p = p_0$ when $t = 0$. Thus,

$$\cos^{-1}\left(\frac{p_0 - GMp_0^2 / v_0^2}{p_0 - GMp_0^2 / v_0^2}\right) = \theta(0) + C_2$$

$$\cos^{-1}(1) = 0 + C_2$$

$$0 = C_2$$

The solution is therefore $\theta = \cos^{-1}\left(\dfrac{p - GMp_0^2 / v_0^2}{p_0 - GMp_0^2 / v_0^2}\right)$

l. Finally (!)

$$\cos\theta = \frac{p - GMp_0^2 / v_0^2}{p_0 - GMp_0^2 / v_0^2}$$

Solving this for p gives $p = \left(p_0 - GMp_0^2 / v_0^2\right)(\cos\theta) + GMp_0^2 / v_0^2$

Recall that $p = 1/r$, so that

$$\frac{1}{r} = \left(p_0 - GMp_0^2 / v_0^2\right)(\cos\theta) + GMp_0^2 / v_0^2$$

$$r = \frac{1}{GMp_0^2 / v_0^2 + \left(p_0 - GMp_0^2 / v_0^2\right)(\cos\theta)} = \frac{r_0}{\dfrac{GM}{r_0 v_0^2} + \left(1 - \dfrac{GM}{r_0 v_0^2}\right)\cos\theta}$$

$$= \frac{r_0 \dfrac{r_0 v_0^2}{GM}}{1 + \left(\dfrac{r_0 v_0^2}{GM} - 1\right)\cos\theta} = \frac{r_0(1+e)}{1 + e\cos\theta}$$

where $e = \dfrac{r_0 v_0^2}{GM} - 1$ is the eccentricity. This is the polar equation of an ellipse.

11.6 Concepts Review

1. $1 + 4t$; $-3 - 2t$; $2 - t$

2. $\dfrac{x-1}{4} = \dfrac{y+3}{-2} = \dfrac{z-2}{-1}$

3. $2t\mathbf{i} - 3\mathbf{j} + 3t^2\mathbf{k}$

4. $\langle 2, -3, 3\rangle$; $\dfrac{x-1}{2} = \dfrac{y+3}{-3} = \dfrac{z-1}{3}$

Problem Set 11.6

1. A parallel vector is
$\mathbf{v} = \langle 4-1, 5+2, 6-3\rangle = \langle 3, 7, 3\rangle$.
$x = 1 + 3t, y = -2 + 7t, z = 3 + 3t$

2. A parallel vector is
$\mathbf{v} = \langle 7-2, -2+1, 3+5\rangle = \langle 5, -1, 8\rangle$
$x = 2 + 5t, y = -1 - t, z = -5 + 8t$

3. A parallel vector is
$\mathbf{v} = \langle 6-4, 2-2, -1-3\rangle = \langle 2, 0, -4\rangle$ or
$\langle 1, 0, -2\rangle$.
$x = 4 + t, y = 2, z = 3 - 2t$

4. A parallel vector is
$\mathbf{v} = \langle 5-5, 4+3, 2+3\rangle = \langle 0, 7, 5\rangle$
$x = 5, y = -3 + 7t, z = -3 + 5t$

5. $x = 4 + 3t, y = 5 + 2t, z = 6 + t$
$\dfrac{x-4}{3} = \dfrac{y-5}{2} = \dfrac{z-6}{1}$

6. $x = -1 - 2t, y = 3, z = -6 + 5t$
Since the second direction number is 0, the line does not have symmetric equations.

7. Another parallel vector is $\langle 1, 10, 100\rangle$.
$x = 1 + t, y = 1 + 10t, z = 1 + 100t$
$\dfrac{x-1}{1} = \dfrac{y-1}{10} = \dfrac{z-1}{100}$

8. $x = -2 + 7t, y = 2 - 6t, z = -2 + 3t$
$\dfrac{x+2}{7} = \dfrac{y-2}{-6} = \dfrac{z+2}{3}$

9. Set $z = 0$. Solving $4x + 3y = 1$ and $10x + 6y = 10$ yields $x = 4, y = -5$. Thus $P_1(4, -5, 0)$ is on the line. Set $y = 0$. Solving $4x - 7z = 1$ and $10x - 5z = 10$ yields $x = \dfrac{13}{10}, z = \dfrac{3}{5}$. Thus $P_2\left(\dfrac{13}{10}, 0, \dfrac{3}{5}\right)$ is on the line.
$\overrightarrow{P_1P_2} = \left\langle \dfrac{13}{10} - 4, 0 - (-5), \dfrac{3}{5} - 0\right\rangle = \left\langle -\dfrac{27}{10}, 5, \dfrac{3}{5}\right\rangle$ is a direction vector. Thus,
$\langle 27, -50, -6\rangle = -10\overrightarrow{P_1P_2}$ is also a direction vector. The symmetric equations are thus
$\dfrac{x-4}{27} = \dfrac{y+5}{-50} = \dfrac{z}{-6}$

10. With $x = 0$, $y - z = 2$ and $-2y + z = 3$ yield $(0, -5, -7)$.
With $y = 0$, $x - z = 2$ and $3x + z = 3$ yield $\left(\dfrac{5}{4}, 0, -\dfrac{3}{4}\right)$.
A vector parallel to the line is
$\left\langle \dfrac{5}{4}, 5, -\dfrac{3}{4} + 7\right\rangle = \left\langle \dfrac{5}{4}, 5, \dfrac{25}{4}\right\rangle$ or $\langle 1, 4, 5\rangle$.
$\dfrac{x}{1} = \dfrac{y+5}{4} = \dfrac{z+7}{5}$

11. $\mathbf{u} = \langle 1, 4, -2\rangle$ and $\mathbf{v} = \langle 2, -1, -2\rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -10, -2, -9\rangle$, and hence $\langle 10, 2, 9\rangle$ is parallel to the line.
With $y = 0$, $x - 2z = 13$ and $2x - 2z = 5$ yield $\left(-8, 0, -\dfrac{21}{2}\right)$. The symmetric equations are
$\dfrac{x+8}{10} = \dfrac{y}{2} = \dfrac{z + \frac{21}{2}}{9}$

12. $\mathbf{u} = \langle 1, -3, 1\rangle$ and $\mathbf{v} = \langle 6, -5, 4\rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -7, 2, 13\rangle$ is parallel to the line.
With $x = 0$, $-3y + z = -1$ and $-5y + 4z = 9$ yield $\left(0, \dfrac{13}{7}, \dfrac{32}{7}\right)$.
$\dfrac{x}{-7} = \dfrac{y - \frac{13}{7}}{2} = \dfrac{z - \frac{32}{7}}{13}$

13. $\langle 1, -5, 2\rangle$ is a vector in the direction of the line.
$\dfrac{x-4}{1} = \dfrac{y}{-5} = \dfrac{z-6}{2}$

14. $\langle 2,1,-3\rangle \times \langle 5,4,-1\rangle = \langle 11,-13,3\rangle$ is in the direction of the line.
$\frac{x+5}{11} = \frac{y-7}{-13} = \frac{z+2}{3}$

15. The point of intersection on the z-axis is (0, 0, 4). A vector in the direction of the line is $\langle 5-0,-3-0,4-4\rangle = \langle 5,-3,0\rangle$. Parametric equations are $x = 5t$, $y = -3t$, $z = 4$.

16. $\langle 3,1,-2\rangle \times \langle 2,3,-1\rangle = \langle 5,-1,7\rangle$ is in the direction of the line since the line is perpendicular to $\langle 3,1,-2\rangle$ and $\langle 2,3,-1\rangle$.
$\frac{x-2}{5} = \frac{y+4}{-1} = \frac{z-5}{7}$

17. Using $t = 0$ and $t = 1$, two points on the first line are (–2, 1, 2) and (0, 5, 1). Using $t = 0$, a point on the second line is (2, 3, 1). Thus, two nonparallel vectors in the plane are
$\langle 0+2,5-1,1-2\rangle = \langle 2,4,1\rangle$ and
$\langle 2+2,3-1,1-2\rangle = \langle 4,2,-1\rangle$
Hence, $\langle 2,4,-1\rangle \times \langle 4,2,-1\rangle = \langle -2,-2,-12\rangle$ is a normal to the plane, and so is $\langle 1, 1, 6\rangle$. An equation of the plane is
$1(x + 2) + 1(y - 1) + 6(z - 2) = 0$ or
$x + y + 6z = 11$.

18. Solve $\frac{x-1}{-4} = \frac{y-2}{3}$ and $\frac{x-2}{-1} = \frac{y-1}{1}$ simultaneously to get $x = 1$, $y = 2$. From the first line $\frac{1-1}{-4} = \frac{z-4}{-2}$, so $z = 4$ and (1, 2, 4) is on the first line. This point also satisfies the equations of the second line, so the lines intersect.
$\langle -4,3,-2\rangle$ and $\langle -1,1,6\rangle$ are parallel to the plane determined by the lines, so
$\langle -4,3,-2\rangle \times \langle -1,1,6\rangle = \langle 20,26,-1\rangle$ is a normal to the plane. An equation of the plane is
$20(x - 1) + 26(y - 2) - 1(z - 4) = 0$ or
$20x + 26y - z = 68$.

19. Using $t = 0$, another point in the plane is (1, –1, 4) and $\langle 2, 3, 1\rangle$ is parallel to the plane. Another parallel vector is
$\langle 1-1,-1+1,5-4\rangle = \langle 0, 0, 1\rangle$. Thus,
$\langle 2, 3, 1\rangle \times \langle 0, 0, 1\rangle = \langle 3,-2,0\rangle$ is a normal to the plane. An equation of the plane is
$3(x - 1) - 2(y + 1) + 0(z - 5) = 0$ or $3x - 2y = 5$.

20. Using $t = 0$, one point of the plane is (0, 1, 0).
$\langle 2,-1,1\rangle \times \langle 0, 1, 1\rangle = \langle -2,-2,2\rangle = -2\langle 1,1,-1\rangle$ is perpendicular to the normals of both planes, hence parallel to their line of intersection.
$\langle 3, 1, 2\rangle$ is parallel to the line in the plane we seek, thus $\langle 3, 1, 2\rangle \times \langle 1,1,-1\rangle = \langle -3,5,2\rangle$ is a normal to the plane. An equation of the plane is
$-3(x - 0) + 5(y - 1) + 2(z - 0) = 0$ or
$-3x + 5y + 2z = 5$.

21. **a.** With $t = 0$ in the first line, $x = 2 - 0 = 2$, $y = 3+4\cdot 0 = 3$, $z = 2\cdot 0 = 0$, so (2, 3, 0) is on the first line.

b. $\langle -1,4,2\rangle$ is parallel to the first line, while $\langle 1, 0, 2\rangle$ is parallel to the second line, so
$\langle -1,4,2\rangle \times \langle 1, 0, 2\rangle = \langle 8,4,-4\rangle = 4\langle 2,1,-1\rangle$
is normal to both. Thus, π has equation
$2(x - 2) + 1(y - 3) - 1(z - 0) = 0$ or
$2x + y - z = 7$, and contains the first line.

c. With $t = 0$ in the second line,
$x = -1 + 0 = -1$, $y = 2$, $z = -1+2\cdot 0 = -1$, so
$Q(-1, 2, -1)$ is on the second line.

d. From Example 10 of Section 11.3, the distance from Q to π is
$$\frac{|2(-1)+(2)-(-1)-7|}{\sqrt{4+1+1}} = \frac{6}{\sqrt{6}} = \sqrt{6} \approx 2.449.$$

22. With $t = 0$, (1, –3, –1) is on the first line.
$\langle 2,4,-1\rangle \times \langle -2,3,2\rangle = \langle 11,-2,14\rangle$ is perpendicular to both lines, so
$11(x - 1) - 2(y + 3) + 14(z + 1) = 0$ or
$11x - 2y + 14z = 3$ is parallel to both lines and contains the first line.
With $t = 0$, (4, 1, 0) is on the second line. The distance from (4, 1, 0) to $11x - 2y + 14z = 3$ is
$$\frac{|11(4)-2(1)+14(0)-3|}{\sqrt{121+4+196}} = \frac{39}{\sqrt{321}} \approx 2.1768.$$

23. $\mathbf{r}\left(\frac{\pi}{3}\right) = \mathbf{i} + 3\sqrt{3}\mathbf{j} + \frac{\pi}{3}\mathbf{k}$, so $\left(1, 3\sqrt{3}, \frac{\pi}{3}\right)$ is on the tangent line.
$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 6\cos t\,\mathbf{j} + \mathbf{k}$, so
$\mathbf{r}'\left(\frac{\pi}{3}\right) = -\sqrt{3}\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ is parallel to the tangent line at $t = \frac{\pi}{3}$. The symmetric equations of the line are $\frac{x-1}{-\sqrt{3}} = \frac{y-3\sqrt{3}}{3} = \frac{z-\frac{\pi}{3}}{1}$.

24. The curve is given by $\mathbf{r}(t) = 2t^2\mathbf{i} + 4t\mathbf{j} + t^3\mathbf{k}$.
$\mathbf{r}(1) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, so (2, 4, 1) is on the tangent line.
$\mathbf{r}'(t) = 4t\mathbf{i} + 4\mathbf{j} + 3t^2\mathbf{k}$, so $\mathbf{r}'(1) = 4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ is parallel to the tangent line. The parametric equations of the line are $x = 2 + 4t$, $y = 4 + 4t$, $z = 1 + 3t$.

25. The curve is given by $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j} + t^5\mathbf{k}$.
$\mathbf{r}(-1) = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, so (–3, 2, –1) is on the plane.
$\mathbf{r}'(t) = 3\mathbf{i} + 4t\mathbf{j} + 5t^4\mathbf{k}$, so $\mathbf{r}'(-1) = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ is in the direction of the curve at $t = -1$, hence normal to the plane. An equation of the plane is
$3(x + 3) - 4(y - 2) + 5(z + 1) = 0$ or
$3x - 4y + 5z = -22$.

26. $\mathbf{r}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\mathbf{i} + \frac{3\pi}{2}\mathbf{j}$, so $\left(\frac{\pi}{2}, \frac{3\pi}{2}, 0\right)$ is on the plane.
$\mathbf{r}'(t) = (t\cos t + \sin t)\mathbf{i} + 3\mathbf{j} + (2\cos t - 2t\sin t)\mathbf{k}$ so

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{i} + 3\mathbf{j} - \pi\mathbf{k}$ is in the direction of the curve at $\frac{\pi}{2}$, hence normal to the plane. An equation of the plane is
$1\left(x - \frac{\pi}{2}\right) + 3\left(y - \frac{3\pi}{2}\right) - \pi(z - 0) = 0$ or
$x + 3y - \pi z = 5\pi$.

27. a. $[x(t)]^2 + [y(t)]^2 + [z(t)]^2$

$$= (2t)^2 + \left(\sqrt{7t}\right)^2 + \left(\sqrt{9 - 7t - 4t^2}\right)^2$$
$$= 4t^2 + 7t + 9 - 7t - 4t^2 = 9$$

Thus, the curve lies on the sphere $x^2 + y^2 + z^2 = 9$ whose center is at the origin.

b. $\mathbf{r}(t) = 2t\,\mathbf{i} + \sqrt{7t}\,\mathbf{j} + \sqrt{9 - 7t - 4t^2}\,\mathbf{k}$

$$\mathbf{r}(1/4) = \frac{1}{2}\mathbf{i} + \sqrt{7/4}\,\mathbf{j} + \sqrt{9 - 7/4 - 1/4}\,\mathbf{k}$$
$$= \frac{1}{2}\mathbf{i} + \frac{\sqrt{7}}{2}\mathbf{j} + \sqrt{7}\,\mathbf{k}$$
$$\mathbf{r}'(t) = 2\mathbf{i} + \frac{\sqrt{7}}{2\sqrt{t}}\mathbf{j} + \frac{-7 - 8t}{2\sqrt{9 - 7t - 4t^2}}\mathbf{k}$$
$$\mathbf{r}'(1/4) = 2\mathbf{i} + \sqrt{7}\,\mathbf{j} - \frac{9}{2\sqrt{7}}\mathbf{k}$$

The tangent line is therefore

$$x = \frac{1}{2} + 2t$$
$$y = \frac{\sqrt{7}}{2} + \sqrt{7}\,t$$
$$z = \sqrt{7} - \frac{9}{2\sqrt{7}}t$$

This line intersects the xz-plane when $y = 0$, which occurs when $0 = \frac{\sqrt{7}}{2} + \sqrt{7}t$, that is, when $t = -\frac{1}{2}$. For this value of t, $x = -\frac{1}{2}$, $y = 0$, and $z = \frac{9}{4\sqrt{7}} + \sqrt{7} = \frac{37}{4\sqrt{7}}$. The point of intersection is therefore
$\left(-\frac{1}{2}, 0, \frac{37}{4\sqrt{7}}\right)$.

28. a. $[x(t)]^2 + [y(t)]^2 + [z(t)]^2$

$$= (\sin t\cos t)^2 + (\sin^2 t)^2 + \cos^2 t$$
$$= \sin^2 t\cos^2 t + \sin^4 t + \cos^2 t$$
$$= \sin^2 t(\cos^2 t + \sin^2 t) + \cos^2 t$$
$$= \sin^2 t + \cos^2 t = 1$$

Thus, the curve lies on the sphere
$x^2 + y^2 + z^2 = 1$
whose center is at the origin.

b. $\mathbf{r}\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$

$= \frac{\sqrt{3}}{4}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$, so $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ is on the tangent line.
$\mathbf{r}'(t) = (\cos^2 t - \sin^2 t)\mathbf{i} + 2\cos t\sin t\mathbf{j} - \sin t\mathbf{k}$
so $\mathbf{r}'\left(\frac{\pi}{6}\right) = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$ is parallel to the line. The line has equations
$x = \frac{\sqrt{3}}{4} + t,\ y = \frac{1}{4} + \sqrt{3}t,\ z = \frac{\sqrt{3}}{2} - t.$
The line intersects the xy-plane when $z = 0$, so $t = \frac{\sqrt{3}}{2}$, hence
$x = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4},\ y = \frac{1}{4} + \frac{3}{2} = \frac{7}{4}.$
The point is $\left(\frac{3\sqrt{3}}{4}, \frac{7}{4}, 0\right)$.

29. **a.** $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\mathbf{j} + (1-t^2)\mathbf{k}$. Notice that $x(t) + y(t) + z(t) = 2t + t^2 + 1 - t^2 = 2t + 1$. Since $x = 2t$, we have $x + y + z = x + 1.$, so this curve lies on the plane with equation $y + z = 1$.

b. Since $\mathbf{r}(2) = 4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{r}'(t) = 2\,\mathbf{i} + 2t\,\mathbf{j} - 2t\,\mathbf{k}$ and $\mathbf{r}'(2) = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, the equation of the tangent line is $x = 4 + 2t$, $y = 4 + 4t$, $z = -3 - 4t$. The line intersects the *xy*-plane when $z = 0$, that is, when $t = -\frac{3}{4}$, which gives $x = \frac{5}{2}$, $y = 1$, and $z = 0$. The point of intersection is therefore $\left(\frac{5}{2}, 1, 0\right)$.

30. In Figure 7, d is the magnitude of the scalar projection of $\overrightarrow{PQ}$ on $\mathbf{n}$. $\mathbf{pr_n}\,\overrightarrow{PQ} = \frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}$, so

$$\left|\mathbf{pr_n}\,\overrightarrow{PQ}\right| = \left|\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}\right| = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$$

The point (0, 0, 1) is on the plane $4x - 4y + 2z = 2$. With $P(0, 0, 1)$, $Q(4, -2, 3)$, and $\mathbf{n} = \langle 2, -2, 1\rangle = \frac{1}{2}\langle 4, -4, 2\rangle$,

$\overrightarrow{PQ} = \langle 4-0, -2-0, 3-1\rangle = \langle 4, -2, 2\rangle$ and

$$d = \frac{\left|\langle 4, -2, 2\rangle\cdot\langle 2, -2, 1\rangle\right|}{\sqrt{4+4+1}} = \frac{14}{3}.$$

From Example 10 of Section 11.3,

$$d = \frac{|4(4) - 4(-2) + 2(3) - 2|}{\sqrt{16+16+4}} = \frac{28}{6} = \frac{14}{3}.$$

31. Let $\overrightarrow{PR}$ be the scalar projection of $\overrightarrow{PQ}$ on $\mathbf{n}$.

Then $\left\|\overrightarrow{PQ}\right\|^2 = \left\|\overrightarrow{PR}\right\|^2 + d^2$ so

$$d^2 = \left\|\overrightarrow{PQ}\right\|^2 - \left\|\overrightarrow{PR}\right\|^2 = \left\|\overrightarrow{PQ}\right\|^2 - \left(\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|^2}\right)^2$$

$$= \frac{\left\|\overrightarrow{PQ}\right\|^2\|\mathbf{n}\|^2 - \left(\overrightarrow{PQ}\cdot\mathbf{n}\right)^2}{\|\mathbf{n}\|^2} = \frac{\left\|\overrightarrow{PQ}\times\mathbf{n}\right\|^2}{\|\mathbf{n}\|^2}$$

by Lagrange's Identity. Thus,

$$d = \frac{\left\|\overrightarrow{PQ}\times\mathbf{n}\right\|}{\|\mathbf{n}\|}.$$

a. $P(3, -2, 1)$ is on the line, so $\overrightarrow{PQ} = \langle 1-3, 0+2, -4-1\rangle = \langle -2, 2, -5\rangle$ while $\mathbf{n} = \langle 2, -2, 1\rangle$, so

$$d = \frac{\left\|\langle -2, 2, -5\rangle\times\langle 2, -2, 1\rangle\right\|}{\sqrt{4+4+1}} = \frac{\left\|\langle -8, -8, 0\rangle\right\|}{3} = \frac{8\sqrt{2}}{3} \approx 3.771$$

b. $P(1, -1, 0)$ is on the line, so $\overrightarrow{PQ} = \langle 2-1, -1+1, 3-0\rangle = \langle 1, 0, 3\rangle$ while $\mathbf{n} = \langle 2, 3, -6\rangle$.

$$d = \frac{\left\|\langle 1, 0, 3\rangle\times\langle 2, 3, -6\rangle\right\|}{\sqrt{4+9+36}} = \frac{\left\|\langle -9, 12, 3\rangle\right\|}{7} = \frac{3\sqrt{26}}{7} \approx 2.185$$

32. d is the distance between the parallel planes containing the lines. Since $\mathbf{n}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, it is normal to the planes. Thus, d is the magnitude of the scalar projection of $\overrightarrow{PQ}$ on $\mathbf{n}$, which is $\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$.

a. $P(3, -2, 1)$ is on the first line, $Q(-4, -5, 0)$ is on the second line, $\mathbf{n}_1 = \langle 1, 1, 2\rangle$, and $\mathbf{n}_2 = \langle 3, 4, 5\rangle$.

$\overrightarrow{PQ} = \langle -4-3, -5+2, 0-1\rangle = \langle -7, -3, -1\rangle$

$\mathbf{n} = \langle 1, 1, 2\rangle\times\langle 3, 4, 5\rangle = \langle -3, 1, 1\rangle$

$$d = \frac{\left|\langle -7, -3, -1\rangle\cdot\langle -3, 1, 1\rangle\right|}{\sqrt{9+1+1}} = \frac{17}{\sqrt{11}} \approx 5.126$$

b. $P(1, –2, 0)$ is on the first line, $Q(0, 1, 0)$ is on the second line, $\mathbf{n}_1 = \langle 2, 3, -4\rangle$, and

$\mathbf{n}_2 = \langle 3, 1, -5\rangle$.

$$\overrightarrow{PQ} = \langle 0-1, 1+2, 0-0\rangle = \langle -1, 3, 0\rangle$$

$$\mathbf{n} = \langle 2, 3, -4\rangle \times \langle 3, 1, -5\rangle = \langle -11, -2, -7\rangle$$

$$d = \frac{|\langle -1, 3, 0\rangle \cdot \langle -11, -2, -7\rangle|}{\sqrt{121+4+49}}$$

$$= \frac{5}{\sqrt{174}} \approx 0.379$$

11.7 Concepts Review

1. $\dfrac{d\mathbf{T}}{ds}$

2. $1/a$; 0

3. $\dfrac{d^2s}{dt^2}; \left(\dfrac{ds}{dt}\right)^2 \kappa$

4. 0

Problem Set 11.7

1. $\mathbf{r}(t) = \langle t, t^2\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t\rangle$

$\mathbf{v}(1) = \langle 1, 2\rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2\rangle$

$\mathbf{a}(1) = \langle 0, 2\rangle$

$$\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle$$

$$\mathbf{T}(1) = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

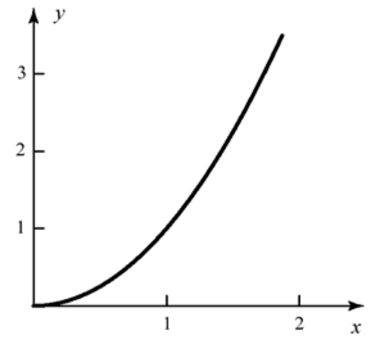
$$\mathbf{T}'(t) = \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle$$

$$\mathbf{T}'(1) = \left\langle -\frac{4}{5^{3/2}}, \frac{2}{5^{3/2}} \right\rangle$$

$$\|\mathbf{T}'(1)\| = \frac{2}{5}$$

$$\|\mathbf{v}(1)\| = \sqrt{5}$$

$$\kappa = \frac{\|\mathbf{T}'(1)\|}{\|\mathbf{v}(1)\|} = \frac{2}{5\sqrt{5}} = \frac{2}{5^{3/2}}$$



2. $\mathbf{r}(t) = \langle t^2, 1+2t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 2\rangle$

$\mathbf{v}(1) = \langle 2, 2\rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 0\rangle$

$\mathbf{a}(1) = \langle 2, 0\rangle$

$$\mathbf{T}(t) = \left\langle \frac{2t}{\sqrt{4+4t^2}}, \frac{2}{\sqrt{4+4t^2}} \right\rangle$$

$$= \left\langle \frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}} \right\rangle$$

$$\mathbf{T}(1) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

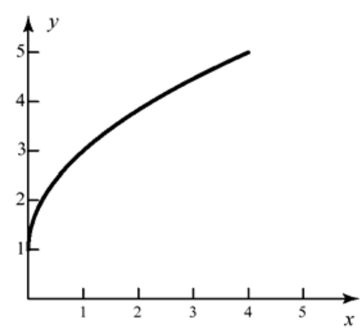
$$\mathbf{T}'(t) = \left\langle \frac{1}{(1+t^2)^{3/2}}, -\frac{t}{(1+t^2)^{3/2}} \right\rangle$$

$$\mathbf{T}'(1) = \left\langle \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}} \right\rangle$$

$$\|\mathbf{T}'(1)\| = \frac{1}{2}$$

$$\|\mathbf{v}(t)\| = 2\sqrt{2}$$

$$\kappa = \frac{\|\mathbf{T}'(1)\|}{\|\mathbf{v}(1)\|} = \frac{1/2}{2\sqrt{2}} = \frac{1}{4\sqrt{2}}$$



3. $\mathbf{r}(t) = \langle t, 2\cos t, 2\sin t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -2\sin t, 2\cos t\rangle$

$\mathbf{v}(\pi) = \langle 1, 0, -2\rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, -2\cos t, -2\sin t\rangle$

$\mathbf{a}(\pi) = \langle 0, 2, 0\rangle$

$$\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{5}}, -\frac{2\sin t}{\sqrt{5}}, \frac{2\cos t}{\sqrt{5}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \right\rangle$$

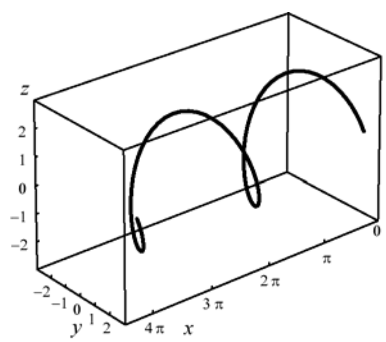
$$\mathbf{T}'(t) = \left\langle 0, -\frac{2\cos t}{\sqrt{5}}, -\frac{2\sin t}{\sqrt{5}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle 0, \frac{2}{\sqrt{5}}, 0 \right\rangle$$

$$\|\mathbf{T}'(\pi)\| = \frac{2}{\sqrt{5}}$$

$$\|\mathbf{v}(t)\| = \sqrt{5}$$

$$\kappa = \frac{\|\mathbf{T}'(\pi)\|}{\|\mathbf{v}(\pi)\|} = \frac{2/\sqrt{5}}{\sqrt{5}} = \frac{2}{5}$$



4. $\mathbf{r}(t) = \langle 5\cos t, 2t, 5\sin t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -5\sin t, 2, 5\cos t\rangle$

$\mathbf{v}(\pi) = \langle 0, 2, -5\rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -5\cos t, 0, -5\sin t\rangle$

$\mathbf{a}(\pi) = \langle 5, 0, 0\rangle$

$$\mathbf{T}(t) = \left\langle -\frac{5\sin t}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{5\cos t}{\sqrt{29}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle 0, \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}} \right\rangle$$

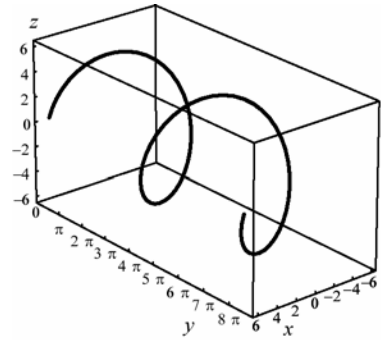
$$\mathbf{T}'(t) = \left\langle -\frac{5\cos t}{\sqrt{29}}, 0, -\frac{5\sin t}{\sqrt{29}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle \frac{5}{\sqrt{29}}, 0, 0 \right\rangle$$

$$\|\mathbf{T}'(\pi)\| = \frac{5}{\sqrt{29}}$$

$$\|\mathbf{v}(t)\| = \sqrt{29}$$

$$\kappa = \frac{\|\mathbf{T}'(\pi)\|}{\|\mathbf{v}(\pi)\|} = \frac{5/\sqrt{29}}{\sqrt{29}} = \frac{5}{29}$$



5. $$\mathbf{r}(t) = \left\langle \frac{t^2}{8}, 5\cos t, 5\sin t \right\rangle$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{t}{4}, -5\sin t, 5\cos t \right\rangle$$

$$\mathbf{v}(\pi) = \left\langle \frac{\pi}{4}, 0, -5 \right\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \left\langle \frac{1}{4}, -5\cos t, -5\sin t \right\rangle$$

$$\mathbf{a}(\pi) = \left\langle \frac{1}{4}, 5, 0 \right\rangle$$

$$\mathbf{T}(t) = \left\langle \frac{t}{\sqrt{400+t^2}}, -\frac{20\sin t}{\sqrt{400+t^2}}, \frac{20\cos t}{\sqrt{400+t^2}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle \frac{\pi}{\sqrt{400+\pi^2}}, 0, -\frac{20}{\sqrt{400+\pi^2}} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{400}{\left(400+t^2\right)^{3/2}}, \frac{-20\left((400+t^2)\cos t - t\sin t\right)}{\left(400+t^2\right)^{3/2}}, \frac{-20\left((400+t^2)\sin t + t\cos t\right)}{\left(400+t^2\right)^{3/2}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle \frac{400}{\left(400+\pi^2\right)^{3/2}}, \frac{-20\left((400+\pi^2)\cos\pi - \pi\sin\pi\right)}{\left(400+\pi^2\right)^{3/2}}, \frac{-20\left((400+\pi^2)\sin\pi + \pi\cos\pi\right)}{\left(400+\pi^2\right)^{3/2}} \right\rangle$$

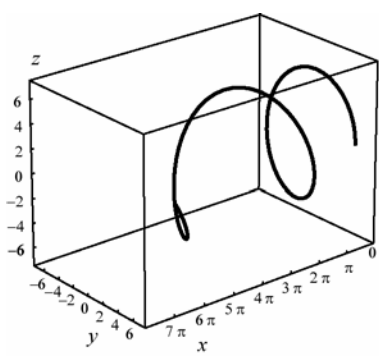
$$= \left\langle \frac{400}{\left(400+\pi^2\right)^{3/2}}, \frac{-20}{\left(400+\pi^2\right)^{1/2}}, \frac{20\pi}{\left(400+\pi^2\right)^{3/2}} \right\rangle$$

$$\|\mathbf{T}'(\pi)\| = \frac{400^2}{\left(400+\pi^2\right)^3} + \frac{400}{\left(400+\pi^2\right)} + \frac{400\pi^2}{\left(400+\pi^2\right)^3}$$

$$\approx 0.989091$$

$$\|\mathbf{v}(t)\| = \sqrt{\frac{\pi^2}{16}+25}$$

$$\kappa = \frac{\|\mathbf{T}'(\pi)\|}{\|\mathbf{v}(\pi)\|} \approx 0.195422$$



6. $\mathbf{r}(t) = \left\langle \frac{t^2}{4}, 2\cos t, 2\sin t \right\rangle$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{t}{2}, -2\sin t, 2\cos t \right\rangle$$

$$\mathbf{v}(\pi) = \left\langle \frac{\pi}{2}, 0, -2 \right\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \left\langle \frac{1}{2}, -2\cos t, -2\sin t \right\rangle$$

$$\mathbf{a}(\pi) = \left\langle \frac{1}{2}, 2, 0 \right\rangle$$

$$\mathbf{T}(t) = \left\langle \frac{t}{\sqrt{16+t^2}}, \frac{-4\sin t}{\sqrt{16+t^2}}, \frac{4\cos t}{\sqrt{16+t^2}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle \frac{\pi}{\sqrt{16+\pi^2}}, 0, \frac{-4}{\sqrt{16+\pi^2}} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{16}{\left(16+t^2\right)^{3/2}}, \frac{-4\left(16+t^2\right)\cos t + 4t\sin t}{\left(16+t^2\right)^{3/2}}, \frac{-4t\cos t + 4\left(16+t^2\right)\sin t}{\left(16+t^2\right)^{3/2}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle \frac{16}{\left(16+\pi^2\right)^{3/2}}, \frac{-4\left(16+\pi^2\right)\cos\pi}{\left(16+\pi^2\right)^{3/2}}, \frac{-4\pi\cos\pi}{\left(16+\pi^2\right)^{3/2}} \right\rangle$$

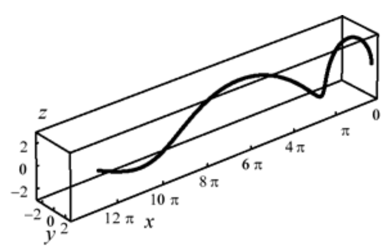
$$\|\mathbf{T}'(\pi)\| = \sqrt{\frac{16^2+4^2\left(16+\pi^2\right)^2+4^2\pi^2}{\left(16+\pi^2\right)^3}}$$

$$= \frac{4\sqrt{16+\pi^2+\left(16+\pi^2\right)^2}}{\left(16+\pi\right)^{3/2}}$$

$$\approx 0.801495$$

$$\|\mathbf{v}(t)\| = \sqrt{4+\frac{\pi^2}{4}}$$

$$\kappa = \frac{\|\mathbf{T}'(\pi)\|}{\|\mathbf{v}(\pi)\|} \approx 0.315164$$



7. $\mathbf{u}'(t) = 8t\mathbf{i} + 4\mathbf{j}$

$$\|\mathbf{u}'(t)\| = 4\sqrt{4t^2+1}$$

$$\mathbf{T}(t) = \frac{\mathbf{u}'(t)}{\|\mathbf{u}'(t)\|} = \frac{2t}{\sqrt{4t^2+1}}\mathbf{i} + \frac{1}{\sqrt{4t^2+1}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = 8t$ $\qquad$ $y'(t) = 4$

$x''(t) = 8$ $\qquad$ $y''(t) = 0$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{32}{64(4t^2+1)^{3/2}}$$

$$= \frac{1}{2(4t^2+1)^{3/2}}$$

$$\kappa\left(\frac{1}{2}\right) = \frac{1}{2(2)^{3/2}} = \frac{1}{4\sqrt{2}}$$

8. $\mathbf{r}'(t) = t^2\mathbf{i} + t\mathbf{j}$

$$\|\mathbf{r}'(t)\| = t\sqrt{t^2+1}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{t}{\sqrt{t^2+1}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = t^2$ $\qquad$ $y'(t) = t$

$x''(t) = 2t$ $\qquad$ $y''(t) = 1$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{t^2}{t^3(t^2+1)^{3/2}} = \frac{1}{t(t^2+1)^{3/2}}$$

$$\kappa(1) = \frac{1}{1(2)^{3/2}} = \frac{1}{2\sqrt{2}}$$

9. $\mathbf{z}'(t) = -3\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$$\|\mathbf{z}'(t)\| = \sqrt{9\sin^2 t + 16\cos^2 t}$$

$$\mathbf{T}(t) = \frac{\mathbf{z}'(t)}{\|\mathbf{z}'(t)\|}$$

$$= -\frac{3\sin t}{\sqrt{9\sin^2 t + 16\cos^2 t}}\mathbf{i} + \frac{4\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$x'(t) = -3\sin t$ $\qquad$ $y'(t) = 4\cos t$

$x''(t) = -3\cos t$ $\qquad$ $y''(t) = -4\sin t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{12}{\left(9\sin^2 t + 16\cos^2 t\right)^{3/2}}$$

$$\kappa\left(\frac{\pi}{4}\right) = \frac{12}{\left(\frac{25}{2}\right)^{3/2}} = \frac{24\sqrt{2}}{125}$$

10. $\mathbf{r}'(t) = e^t\mathbf{i} + e^t\mathbf{j}$

$$\|\mathbf{r}'(t)\| = e^t\sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(\ln 2) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = e^t$ $\qquad$ $y'(t) = e^t$

$x''(t) = e^t$ $\qquad$ $y''(t) = e^t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = 0$$

$\kappa(\ln 2) = 0$

11. $x = 1 - t^2, \quad y = 1 - t^3$

$x'(t) = -2t, \quad y'(t) = -3t^2$

$x''(t) = -2, \quad y''(t) = -6t$

$$\mathbf{r}(t) = \left(1-t^2\right)\mathbf{i} + \left(1-t^3\right)\mathbf{j}$$

$$\mathbf{r}'(t) = -2t\,\mathbf{i} - 3t^2\,\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-2t\,\mathbf{i} - 3t^2\,\mathbf{j}}{\sqrt{4t^2+9t^4}}$$

$$\mathbf{T}(1) = \frac{-2(1)\mathbf{i} - 3(1)^2\,\mathbf{j}}{\sqrt{4(1)^2 + 9(1)^4}} = -\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}$$

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{3/2}}$$

$$= \frac{\left|(-2t)(-6t) - (-2)\left(-3t^2\right)\right|}{\left((-2t)^2 + \left(-3t^2\right)^2\right)^{3/2}}$$

$$= \frac{\left|12t^2 - 6t^2\right|}{\left(4t^2+9t^4\right)^{3/2}} = \frac{6t^2}{\left(4t^2+9t^4\right)^{3/2}}$$

When $t = 1$, the curvature is

$$\kappa = \frac{6}{(4+9)^{3/2}} = \frac{6}{13^{3/2}} \approx 0.128008$$

12. $\mathbf{r}'(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\cosh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{i} + \frac{\sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{j}$$

$$\mathbf{T}(\ln 3) = \frac{5}{\sqrt{41}}\mathbf{i} + \frac{4}{\sqrt{41}}\mathbf{j}$$

$x'(t) = \cosh t \qquad y'(t) = \sinh t$

$x''(t) = \sinh t \qquad y''(t) = \cosh t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}};$$

$$\kappa(\ln 3) = \frac{1}{\left(\frac{41}{9}\right)^{3/2}} = \frac{27}{41\sqrt{41}}$$

13. $\mathbf{r}'(t) = -(\cos t + \sin t)e^{-t}\mathbf{i} + (\cos t - \sin t)e^{t}\mathbf{j}$

$|\mathbf{r}'(t)| = \sqrt{2}e^{-t}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\frac{\cos t + \sin t}{\sqrt{2}}\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(0) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = -(\cos t - \sin t)e^{-t} \quad y'(t) = (\cos t - \sin t)e^{-t}$

$x''(t) = (2\sin t)e^{-t} \qquad y''(t) = (-2\cos t)e^{-t}$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{2e^{-2t}}{2\sqrt{2}e^{-3t}} = \frac{e^t}{\sqrt{2}}$$

$$\kappa(0) = \frac{1}{\sqrt{2}}$$

14. $\mathbf{r}'(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{t^2 + 1}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\cos t - t\sin t}{\sqrt{t^2+1}}\mathbf{i} + \frac{\sin t + t\cos t}{\sqrt{t^2+1}}\mathbf{j}$$

$$\mathbf{T}(1) = \frac{\cos 1 - \sin 1}{\sqrt{2}}\mathbf{i} + \frac{\sin 1 + \cos 1}{\sqrt{2}}\mathbf{j}$$

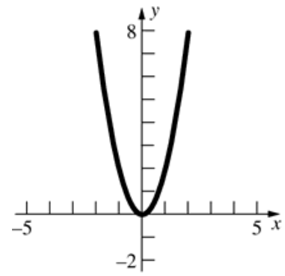
$x'(t) = \cos t - t\sin t \qquad y'(t) = \sin t + t\cos t$

$x''(t) = -2\sin t - t\cos t \quad y''(t) = 2\cos t - t\sin t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{t^2 + 2}{(t^2 + 1)^{3/2}}$$

$$\kappa(1) = \frac{3}{2\sqrt{2}}$$

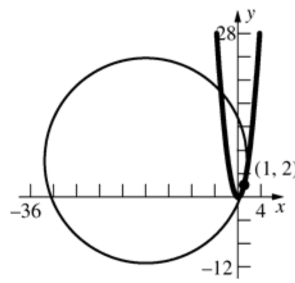
15.



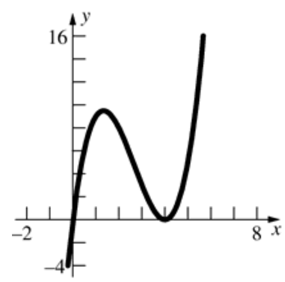
$y' = 4x,\ y'' = 4$

$$\kappa = \frac{4}{(1 + 16x^2)^{3/2}}$$

At (1, 2), $\kappa = \frac{4}{17\sqrt{17}}$ and $R = \frac{17\sqrt{17}}{4}$.



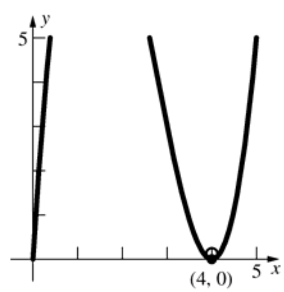
16.



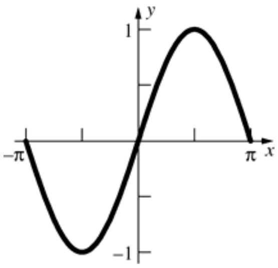
$y' = 3x^2 - 16x + 16,\ y'' = 6x - 16$

$$\kappa = \frac{|6x - 16|}{\left[1 + (3x^2 - 16x + 16)^2\right]^{3/2}}$$

At (4, 0), $\kappa = \frac{8}{1} = 8$ and $R = \frac{1}{8}$.



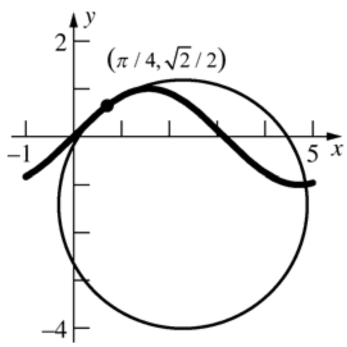
17.



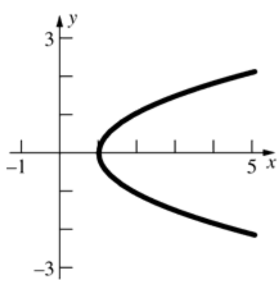
$y' = \cos x,\ y'' = -\sin x$

$$\kappa = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$$

At $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right), \kappa = \frac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{3/2}} = \frac{2}{3\sqrt{3}}$ and $R = \frac{3\sqrt{3}}{2}.$



18.

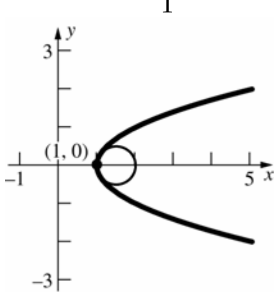


$2yy' = 1$

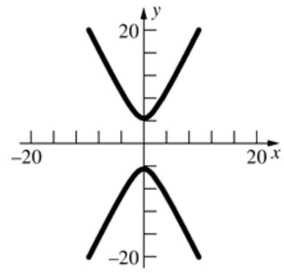
$$y' = \frac{1}{2y},\ y'' = -\frac{y'}{2y^2} = -\frac{1}{4y^3}$$

$$\kappa = \frac{\left|\frac{1}{4y^3}\right|}{\left(1+\frac{1}{4y^2}\right)^{3/2}} = \frac{2}{(4y^2+1)^{3/2}}$$

At (1, 0), $\kappa = \frac{2}{1} = 2$ and $R = \frac{1}{2}.$



19.

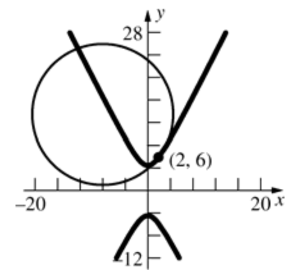


$2yy' - 8x = 0$

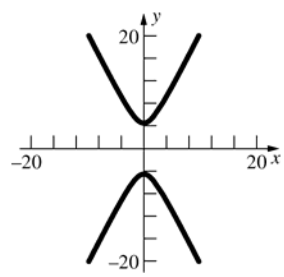
$$y' = \frac{4x}{y},\ y'' = \frac{4(y - xy')}{y^2} = \frac{4(y^2-4x^2)}{y^3}$$

$$\kappa = \frac{\frac{4y^2-16x^2}{y^3}}{\left(1+\frac{16x^2}{y^2}\right)^{3/2}} = \frac{4(y^2-4x^2)}{(y^2+16x^2)^{3/2}}$$

At (2, 6), $\kappa = \frac{80}{1000} = \frac{2}{25}$ and $R = \frac{25}{2}.$



20.

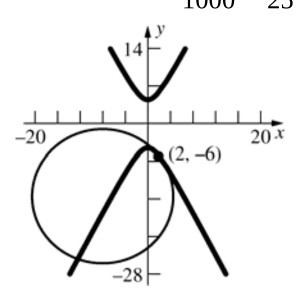


$2yy' - 8x = 0$

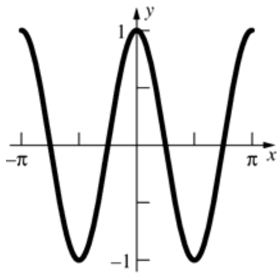
$$y' = \frac{4x}{y},\ y'' = \frac{4(y - xy')}{y^2} = \frac{4(y^2-4x^2)}{y^3}$$

$$\kappa = \frac{\frac{4(y^2-4x^2)}{y^3}}{\left(1+\frac{16x^2}{y^2}\right)^{3/2}} = \frac{4(y^2-4x^2)}{(y^2+16x^2)^{3/2}}$$

At (2, –6), $\kappa = \frac{80}{1000} = \frac{2}{25}$ and $R = \frac{25}{2}.$



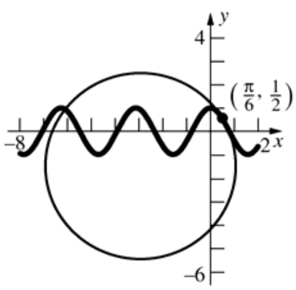
21.



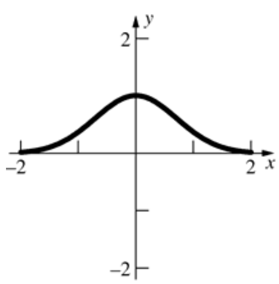
$y' = -2\sin 2x,\ y'' = -4\cos 2x$

$$\kappa = \frac{|4\cos 2x|}{(1+4\sin^2 2x)^{3/2}}$$

At $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\kappa = \frac{2}{8} = \frac{1}{4}$ and $R = 4$.



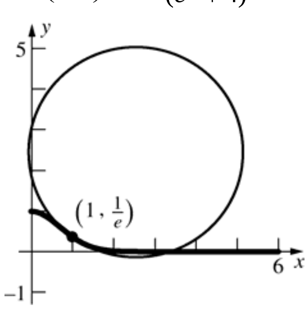
22



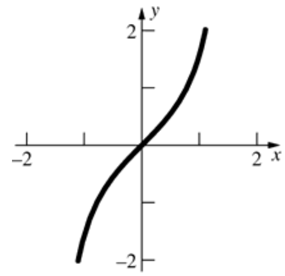
$y' = -2xe^{-x^2},\ y'' = (4x^2-2)e^{-x^2}$

$$\kappa = \frac{\left|(4x^2-2)e^{-x^2}\right|}{(1+4x^2e^{-2x^2})^{3/2}} = \frac{e^{2x^2}\left|4x^2-2\right|}{(e^{2x^2}+4x^2)^{3/2}}$$

At $\left(1, \frac{1}{e}\right)$, $\kappa = \frac{2e^2}{(e^2+4)^{3/2}}$ and $R = \frac{(e^2+4)^{3/2}}{2e^2}$.



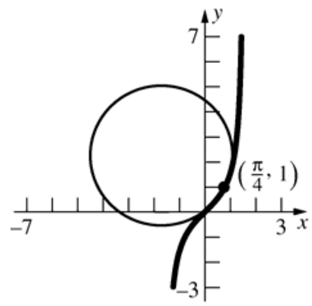
23



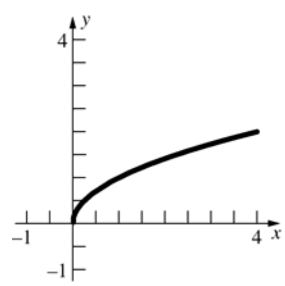
$y' = \sec^2 x,\ y'' = 2\sec^2 x \tan x$

$$\kappa = \frac{\left|2\sec^2 x \tan x\right|}{(1+\sec^4 x)^{3/2}}$$

At $\left(\frac{\pi}{4}, 1\right)$, $\kappa = \frac{4}{5\sqrt{5}}$ and $R = \frac{5\sqrt{5}}{4}$.



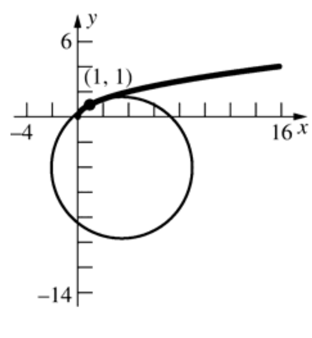
24



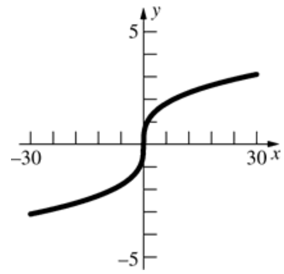
$y' = \frac{1}{2\sqrt{x}},\ y'' = -\frac{1}{4x^{3/2}}$

$$\kappa = \frac{\left|\frac{1}{4x^{3/2}}\right|}{\left(1+\frac{1}{4x}\right)^{3/2}} = \frac{2}{(4x+1)^{3/2}}$$

At (1, 1), $\kappa = \frac{2}{5\sqrt{5}}$ and $R = \frac{5\sqrt{5}}{2}$.



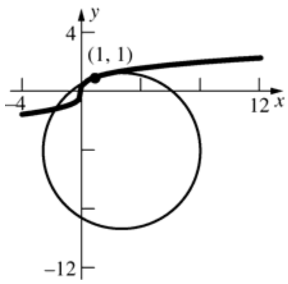
25



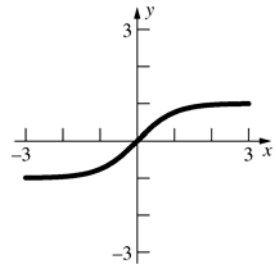
$$y' = \frac{1}{3x^{2/3}},\ y'' = -\frac{2}{9x^{5/3}}$$

$$\kappa = \frac{\left|\frac{2}{9x^{5/3}}\right|}{\left(1+\frac{1}{9x^{4/3}}\right)^{3/2}} = \frac{6x^{1/3}}{(9x^{4/3}+1)^{3/2}}$$

At (1, 1), $\kappa = \dfrac{6}{10\sqrt{10}} = \dfrac{3}{5\sqrt{10}}$ and $R = \dfrac{5\sqrt{10}}{3}$.



26

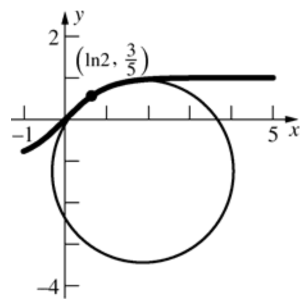


$$y' = \operatorname{sech}^2 x,\ y'' = -2\operatorname{sech}^2 x \tanh x$$

$$\kappa = \frac{\left|2\operatorname{sech}^2 x \tanh x\right|}{(1+\operatorname{sech}^4 x)^{3/2}}$$

At $\left(\ln 2, \dfrac{3}{5}\right)$, $\kappa = \dfrac{\frac{96}{125}}{\left(\frac{881}{625}\right)^{3/2}} = \dfrac{12,000}{881\sqrt{881}}$ and

$R = \dfrac{881\sqrt{881}}{12,000}$.



27. $\mathbf{r}'(t) = t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{i} + 2t\mathbf{k}$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{2\mathbf{i}+\mathbf{j}+4\mathbf{k}}{\sqrt{4+1+16}} = \frac{2}{\sqrt{21}}\mathbf{i} + \frac{1}{\sqrt{21}}\mathbf{j} + \frac{4}{\sqrt{21}}\mathbf{k}$$

$$a_T(2) = \frac{\mathbf{r}'(2)\cdot\mathbf{r}''(2)}{\|\mathbf{r}'(2)\|} = \frac{(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+4\mathbf{k})}{\sqrt{21}} = \frac{18}{\sqrt{21}}$$

$$a_N(2) = \frac{\|\mathbf{r}'(2)\times\mathbf{r}''(2)\|}{\|\mathbf{r}'(2)\|} = \frac{\|(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\times(\mathbf{i}+4\mathbf{k})\|}{\sqrt{21}} = \frac{1}{\sqrt{21}}\|4\mathbf{i}-4\mathbf{j}-\mathbf{k}\| = \frac{\sqrt{33}}{\sqrt{21}} = \sqrt{\frac{11}{7}}$$

$$\mathbf{N}(2) = \frac{\mathbf{r}''(2) - a_T(2)\mathbf{T}(2)}{a_N(2)} = \frac{(\mathbf{i}+4\mathbf{k}) - \frac{18}{\sqrt{21}}\left(\frac{2}{\sqrt{21}}\mathbf{i} + \frac{1}{\sqrt{21}}\mathbf{j} + \frac{4}{\sqrt{21}}\mathbf{k}\right)}{\sqrt{\frac{11}{7}}} = \sqrt{\frac{7}{11}}\left(-\frac{15}{21}\mathbf{i} - \frac{18}{21}\mathbf{j} + \frac{12}{21}\mathbf{k}\right)$$

$$= \sqrt{\frac{7}{11}}\left(-\frac{5}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{4}{7}\mathbf{k}\right) = -\frac{5}{\sqrt{77}}\mathbf{i} - \frac{6}{\sqrt{77}}\mathbf{j} + \frac{4}{\sqrt{77}}\mathbf{k}$$

$$\kappa(2) = \frac{\|\mathbf{r}'(2)\times\mathbf{r}''(2)\|}{\|\mathbf{r}'(2)\|^3} = \frac{\sqrt{33}}{\left(\sqrt{21}\right)^3} = \sqrt{\frac{33}{9261}} = \sqrt{\frac{11}{3087}} = \frac{\sqrt{11}}{21\sqrt{7}}$$

$$\mathbf{B}(2) = \mathbf{T}(2)\times\mathbf{N}(2) = \left(\frac{2}{\sqrt{21}}\mathbf{i} + \frac{1}{\sqrt{21}}\mathbf{j} + \frac{4}{\sqrt{21}}\mathbf{k}\right)\times\left(-\frac{5}{\sqrt{77}}\mathbf{i} - \frac{6}{\sqrt{77}}\mathbf{j} + \frac{4}{\sqrt{77}}\mathbf{k}\right) = \frac{4}{\sqrt{33}}\mathbf{i} - \frac{4}{\sqrt{33}}\mathbf{j} - \frac{1}{\sqrt{33}}\mathbf{k}$$

28. $\mathbf{r}(t) = \langle \sin 3t, \cos 3t, t\rangle$

$\mathbf{r}'(t) = \langle 3\cos 3t, -3\sin 3t, 1\rangle$

$\mathbf{r}''(t) = \langle -9\sin 3t, -9\cos 3t, 0\rangle$

$$\mathbf{T}\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{9}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|} = \frac{\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1\right\rangle}{\sqrt{\frac{9}{4}+\frac{27}{4}+1}} = \left\langle \frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle$$

$$a_T\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{9}\right)\cdot\mathbf{r}''\left(\frac{\pi}{9}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|} = \frac{\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1\right\rangle\cdot\left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0\right\rangle}{\sqrt{10}} = 0$$

$$a_N\left(\frac{\pi}{9}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\times\mathbf{r}''\left(\frac{\pi}{9}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|} = \frac{\left|\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1\right\rangle\times\left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0\right\rangle\right|}{\sqrt{10}} = \frac{1}{\sqrt{10}}\left|\left\langle \frac{9}{2}, -\frac{9\sqrt{3}}{2}, -27\right\rangle\right| = \frac{1}{\sqrt{10}}\left(9\sqrt{10}\right) = 9$$

$$\mathbf{N}\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{9}\right) - a_T\left(\frac{\pi}{9}\right)\mathbf{T}\left(\frac{\pi}{9}\right)}{a_N} = \frac{1}{9}\left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0\right\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right\rangle$$

$$\kappa\left(\frac{\pi}{9}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\times\mathbf{r}''\left(\frac{\pi}{9}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|^3} = \frac{9\sqrt{10}}{\left(\sqrt{10}\right)^3} = \frac{9}{10}$$

$$\mathbf{B}\left(\frac{\pi}{9}\right) = \mathbf{T}\left(\frac{\pi}{9}\right)\times\mathbf{N}\left(\frac{\pi}{9}\right) = \left\langle \frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle\times\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right\rangle = \left\langle \frac{1}{2\sqrt{10}}, -\frac{\sqrt{3}}{2\sqrt{10}}, -\frac{3}{\sqrt{10}}\right\rangle$$

29. $\mathbf{r}(t) = \langle 7\sin 3t, 7\cos 3t, 14t\rangle$

$\mathbf{r}'(t) = \langle 21\cos 3t, -21\sin 3t, 14\rangle$

$\mathbf{r}''(t) = \langle -63\sin 3t, -63\cos 3t, 0\rangle$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{\langle -21, 0, 14\rangle}{\sqrt{441+196}} = \frac{1}{7\sqrt{13}}\langle -21, 0, 14\rangle = \left\langle -\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\right\rangle$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)\cdot\mathbf{r}''\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{\langle -21, 0, 14\rangle\cdot\langle 0, -63, 0\rangle}{7\sqrt{13}} = 0$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{\left|\langle -21, 0, 14\rangle\times\langle 0, -63, 0\rangle\right|}{7\sqrt{13}} = \frac{\left|\langle 882, 0, -1323\rangle\right|}{7\sqrt{13}} = \frac{441\sqrt{13}}{7\sqrt{13}} = 63$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{3}\right) - a_T\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N\left(\frac{\pi}{3}\right)} = \frac{1}{63}\langle 0, 63, 0\rangle = \langle 0, 1, 0\rangle$$

$$\kappa\left(\frac{\pi}{3}\right) = \frac{\left\|\mathbf{r}\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|^3} = \frac{441\sqrt{13}}{\left(7\sqrt{13}\right)^3} = \frac{9}{91}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right) = \left\langle -\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\right\rangle\times\langle 0, 1, 0\rangle = \left\langle -\frac{2}{\sqrt{13}}, 0, -\frac{3}{\sqrt{13}}\right\rangle$$

30. $\mathbf{r}'(t) = -3\cos^2 t\sin t\mathbf{i} + 3\sin^2 t\cos t\mathbf{k}$

$\mathbf{r}''(t) = (6\cos t\sin^2 t - 3\cos^3 t)\mathbf{i} + (6\cos^2 t\sin t - 3\sin^3 t)\mathbf{k}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{0}$ so the object is motionless at $t_1 = \frac{\pi}{2}$.

κ, **T**, **N**, and **B** do not exist.

31. $\mathbf{r}'(t) = \sinh\frac{t}{3}\mathbf{i} + \mathbf{j}$

$\mathbf{r}''(t) = \frac{1}{3}\cosh\frac{t}{3}\mathbf{i}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}}{\sqrt{\sinh^2\frac{1}{3} + 1}} = \frac{\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}}{\cosh\frac{1}{3}} = \tanh\frac{1}{3}\mathbf{i} + \operatorname{sech}\frac{1}{3}\mathbf{j}$$

$$a_T(1) = \frac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{\|\mathbf{r}'(1)\|} = \frac{\left(\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}\right)\cdot\left(\frac{1}{3}\cosh\frac{1}{3}\mathbf{i}\right)}{\cosh\frac{1}{3}} = \frac{1}{3}\sinh\frac{1}{3}$$

$$a_N(1) = \frac{\|\mathbf{r}'(1)\times\mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|} = \frac{\left\|\left(\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}\right)\times\left(\frac{1}{3}\cosh\frac{1}{3}\mathbf{i}\right)\right\|}{\cosh\frac{1}{3}} = \frac{\left\|-\frac{1}{3}\cos\frac{1}{3}\mathbf{k}\right\|}{\cosh\frac{1}{3}} = \frac{1}{3}$$

$$\mathbf{N}(1) = \frac{\mathbf{r}''(1) - a_T(1)\mathbf{T}(1)}{a_N(1)} = 3\left[\frac{1}{3}\cosh\frac{1}{3}\mathbf{i} - \frac{1}{3}\sinh\frac{1}{3}\left(\tanh\frac{1}{3}\mathbf{i} + \operatorname{sech}\frac{1}{3}\mathbf{j}\right)\right] = \left(\cosh\frac{1}{3} - \frac{\sinh^2\frac{1}{3}}{\cosh\frac{1}{3}}\right)\mathbf{i} - \tanh\frac{1}{3}\mathbf{j}$$

$$= \operatorname{sech}\frac{1}{3}\mathbf{i} - \tanh\frac{1}{3}\mathbf{j}$$

$$\kappa(1) = \frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{\frac{1}{3}\cosh\frac{1}{3}}{\cosh^3\frac{1}{3}} = \frac{1}{3}\operatorname{sech}^2\frac{1}{3}$$

$$\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \left(\tanh\frac{1}{3}\mathbf{i} + \operatorname{sech}\frac{1}{3}\mathbf{j}\right)\times\left(\operatorname{sech}\frac{1}{3}\mathbf{i} - \tanh\frac{1}{3}\mathbf{j}\right) = \left(-\operatorname{sech}^2\frac{1}{3} - \tanh^2\frac{1}{3}\right)\mathbf{k} = -\mathbf{k}$$

32. $\mathbf{r}'(t) = e^{7t}(7\cos 2t - 2\sin 2t)\mathbf{i} + e^{7t}(7\sin 2t + 2\cos 2t)\mathbf{j} + 7e^{7t}\mathbf{k}$

$\mathbf{r}''(t) = e^{7t}(45\cos 2t - 28\sin 2t)\mathbf{i} + e^{7t}(45\sin 2t + 28\cos 2t)\mathbf{j} + 49e^{7t}\mathbf{k}$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{e^{7\pi/3}\left(-\frac{7}{2} - \sqrt{3}\right)\mathbf{i} + e^{7\pi/3}\left(\frac{7\sqrt{3}}{2} - 1\right)\mathbf{j} + 7e^{7\pi/3}\mathbf{k}}{e^{7\pi/3}\sqrt{\left(-\frac{7}{2} - \sqrt{3}\right)^2 + \left(\frac{7\sqrt{3}}{2} - 1\right)^2 + 49}} = \left(-\frac{7 + 2\sqrt{3}}{2\sqrt{102}}\right)\mathbf{i} + \left(\frac{7\sqrt{3} - 2}{2\sqrt{102}}\right)\mathbf{j} + \frac{7}{\sqrt{102}}\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)\cdot\mathbf{r}''\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{714e^{14\pi/3}}{e^{7\pi/3}\sqrt{102}} = 7\sqrt{102}e^{7\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|} = \frac{\left\|e^{14\pi/3}\left(49 + 14\sqrt{3}\right)\mathbf{i} + e^{14\pi/3}\left(14 - 49\sqrt{3}\right)\mathbf{j} + 106e^{14\pi/3}\mathbf{k}\right\|}{e^{7\pi/3}\sqrt{102}} = 2\sqrt{53}e^{7\pi/3}$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{3}\right) - a_T\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N} = \frac{2 - 7\sqrt{3}}{2\sqrt{53}}\mathbf{i} - \frac{7 + 2\sqrt{3}}{2\sqrt{53}}\mathbf{j};\quad \kappa\left(\frac{\pi}{3}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|^3} = \frac{2\sqrt{5406}e^{14\pi/3}}{\sqrt{102^3}e^{7\pi}} = \frac{\sqrt{53}}{51}e^{-7\pi/3};$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{42 + 49\sqrt{3}}{6\sqrt{1802}}\mathbf{i} + \frac{14\sqrt{3} - 147}{6\sqrt{1802}}\mathbf{j} + \sqrt{\frac{53}{102}}\mathbf{k}$$

33. $\mathbf{r}'(t) = -2e^{-2t}\mathbf{i} + 2e^{2t}\mathbf{j} + 2\sqrt{2}\mathbf{k}$

$\mathbf{r}''(t) = 4e^{-2t}\mathbf{i} + 4e^{2t}\mathbf{j}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{-2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}}{\sqrt{4+4+8}} = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

$$a_T(0) = \frac{\mathbf{r}'(0)\cdot\mathbf{r}''(0)}{\|\mathbf{r}'(0)\|} = \frac{\left(-2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}\right)\cdot(4\mathbf{i} + 4\mathbf{j})}{4} = 0$$

$$a_N(0) = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|} = \frac{\left|-8\sqrt{2}\mathbf{i} + 8\sqrt{2}\mathbf{j} - 16\mathbf{k}\right|}{4} = \frac{16\sqrt{2}}{4} = 4\sqrt{2}$$

$$\mathbf{N}(0) = \frac{\mathbf{r}''(0) - a_T(0)\mathbf{T}(0)}{a_N(0)} = \frac{4\mathbf{i} + 4\mathbf{j}}{4\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\kappa(0) = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|^3} = \frac{16\sqrt{2}}{64} = \frac{\sqrt{2}}{4}$$

$$\mathbf{B}(0) = \mathbf{T}(0)\times\mathbf{N}(0) = \left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}\right)\times\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$$

34. $\mathbf{r}(t) = \left\langle \ln t,\ 3t,\ t^2 \right\rangle$

$$\mathbf{r}'(t) = \left\langle \frac{1}{t},\ 3,\ 2t \right\rangle$$

$$\mathbf{r}''(t) = \left\langle -\frac{1}{t^2},\ 0,\ 2 \right\rangle$$

$$T(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{\left\langle \frac{1}{2}, 3, 4\right\rangle}{\sqrt{\frac{1}{4}+9+16}} = \frac{\left\langle \frac{1}{2}, 3, 4\right\rangle}{\frac{\sqrt{101}}{2}} = \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}} \right\rangle$$

$$a_T(2) = \frac{\mathbf{r}'(2)\cdot\mathbf{r}''(2)}{\|\mathbf{r}'(2)\|} = \frac{2}{\sqrt{101}}\left(\left\langle \frac{1}{2}, 3, 4\right\rangle\cdot\left\langle -\frac{1}{4}, 0, 2\right\rangle\right) = \frac{2}{\sqrt{101}}\left(\frac{63}{8}\right) = \frac{63}{4\sqrt{101}}$$

$$a_N(2) = \frac{\|\mathbf{r}'(2)\times\mathbf{r}''(2)\|}{\|\mathbf{r}'(2)\|} = \frac{2}{\sqrt{101}}\left|\left\langle 6, -2, \frac{3}{4}\right\rangle\right| = \frac{2}{\sqrt{101}}\left(\frac{\sqrt{649}}{4}\right) = \frac{\sqrt{649}}{2\sqrt{101}}$$

$$\mathbf{N}(2) = \frac{\mathbf{r}''(2) - a_T(2)\mathbf{T}(2)}{a_N(2)} = \frac{2\sqrt{101}}{\sqrt{649}}\left(\left\langle -\frac{1}{4}, 0, 2\right\rangle - \frac{63}{4\sqrt{101}}\left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right\rangle\right)$$

$$= \frac{2\sqrt{101}}{\sqrt{649}}\left\langle -\frac{41}{101}, -\frac{189}{202}, \frac{76}{101}\right\rangle = \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}}\right\rangle$$

$$\kappa(2) = \frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \left(\frac{\sqrt{649}}{4}\right)\left(\frac{2}{\sqrt{101}}\right)^3 = \frac{2\sqrt{649}}{101\sqrt{101}}$$

$$\mathbf{B}(2) = \mathbf{T}(2)\times\mathbf{N}(2) = \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right\rangle \times \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}}\right\rangle = \left\langle \frac{24}{\sqrt{649}}, -\frac{8}{\sqrt{649}}, \frac{3}{\sqrt{649}}\right\rangle$$

35. $y' = \frac{1}{x},\ y'' = -\frac{1}{x^2}$

$$\kappa = \frac{\left|\frac{1}{x^2}\right|}{\left(1+\frac{1}{x^2}\right)^{3/2}} = \frac{|x|}{(x^2+1)^{3/2}}$$

Since $0 < x < \infty$, $\kappa = \frac{x}{(x^2+1)^{3/2}}$.

$$\kappa' = \frac{(x^2+1)^{3/2} - 3x^2(x^2+1)^{1/2}}{(x^2+1)^3} = \frac{-2x^2+1}{(x^2+1)^{5/2}}$$

$\kappa' = 0$ when $x = \frac{1}{\sqrt{2}}$. Since $\kappa' > 0$ on $\left(0, \frac{1}{\sqrt{2}}\right)$ and $\kappa' < 0$ on $\left(\frac{1}{\sqrt{2}}, \infty\right)$, so κ is maximum when $x = \frac{1}{\sqrt{2}},\ y = \ln\frac{1}{\sqrt{2}} = -\frac{\ln 2}{2}$. The point of maximum curvature is $\left(\frac{1}{\sqrt{2}}, -\frac{\ln 2}{2}\right)$.

36 $y' = \cos x,\ y'' = -\sin x$

$$\kappa = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$$

$$\kappa' = \frac{\frac{|\sin x|}{\sin x}\cos x(1+\cos^2 x)^{3/2} + 3|\sin x|\cos x \sin x(1+\cos^2 x)^{1/2}}{(1+\cos^2 x)^3} = \frac{2|\sin x|\cot x(2+\cos^2 x)}{(1+\cos^2 x)^{5/2}}$$

$k' = 0$ when $x = -\frac{\pi}{2}, \frac{\pi}{2}$. κ' is not defined when $x = -\pi, 0$. Since $\kappa' > 0$ on $\left(-\pi, -\frac{\pi}{2}\right) \cup \left(0, \frac{\pi}{2}\right)$ and $\kappa' < 0$ on $\left(-\frac{\pi}{2}, 0\right) \cup \left(\frac{\pi}{2}, \pi\right)$, so κ has local maxima when $x = -\frac{\pi}{2}, y = -1$ and $x = \frac{\pi}{2}, y = 1$.

$$\kappa\left(-\frac{\pi}{2}\right) = \kappa\left(\frac{\pi}{2}\right) = 1$$

The points of maximum curvature are $(0,1)$ and $\left(\frac{\pi}{2}, 1\right)$.

37. $y' = \sinh x,\ y'' = \cosh x$

$$\kappa = \frac{\cosh x}{(1+\sinh^2 x)^{3/2}} = \operatorname{sech}^2 x$$

$\kappa' = -2\operatorname{sech}^2 x \tanh x$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $(-\infty, 0)$ and $\kappa' < 0$ on $(0, \infty)$, so κ is maximum when $x = 0$, $y = 1$. The point of maximum curvature is $(0,1)$.

38. $y' = \cosh x,\ y'' = \sinh x$

$$\kappa = \frac{|\sinh x|}{(1+\cosh^2 x)^{3/2}}$$

$$\kappa' = \frac{\frac{|\sinh x|}{\sinh x}\cosh x(1+\cosh^2 x)^{3/2} - 3|\sinh x|\cosh x \sinh x(1+\cosh^2 x)^{1/2}}{(1+\cosh^2 x)^3} = \frac{2|\sinh x|\coth x(2-\cosh^2 x)}{(1+\cosh^2 x)^{5/2}}$$

κ' is not defined when $x = 0$ and $\kappa' = 0$ when $\cosh x = \sqrt{2}$ or $x = \pm\ln\left(\sqrt{2}+1\right)$. Since $\kappa' > 0$ on $\left(-\infty, -\ln\left(\sqrt{2}+1\right)\right) \cup \left(0, \ln\left(\sqrt{2}+1\right)\right)$ and $\kappa' < 0$ on $\left(-\ln\left(\sqrt{2}+1\right), 0\right) \cup \left(\ln\left(\sqrt{2}+1\right), \infty\right)$, κ has local maxima when $x = -\ln\left(\sqrt{2}+1\right),\ y = -1$ and $x = \ln\left(\sqrt{2}+1\right),\ y = 1$.

$$\kappa\left(-\ln\left(\sqrt{2}+1\right)\right) = \kappa\left(\ln\left(\sqrt{2}+1\right)\right) = \frac{1}{3\sqrt{3}}$$

The points of maximum curvature are $\left(-\ln\left(\sqrt{2}+1\right), -1\right)$ and $\left(\ln\left(\sqrt{2}+1\right), 1\right)$.

39. $y' = e^x,\ y'' = e^x$

$$\kappa = \frac{e^x}{(1+e^{2x})^{3/2}}$$

$$\kappa' = \frac{e^x(1+e^{2x})^{3/2} - 3e^{3x}(1+e^{2x})^{1/2}}{(1+e^{2x})^3}$$

$$= \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{5/2}}$$

$\kappa' = 0$ when $x = -\frac{1}{2}\ln 2$. Since $\kappa' > 0$ on $\left(-\infty, -\frac{1}{2}\ln 2\right)$ and $\kappa' < 0$ on $\left(-\frac{1}{2}\ln 2, \infty\right)$, so κ is maximum when $x = -\frac{1}{2}\ln 2,\ y = \frac{1}{\sqrt{2}}$. The point of maximum curvature is $\left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

40. $y' = -\tan x,\ \ y'' = -\sec^2 x$

$$\kappa = \frac{|\sec^2 x|}{(1+\tan^2 x)^{3/2}} = |\cos x|$$

Since $-\frac{\pi}{2} < x < \frac{\pi}{2}, \kappa = \cos x$.

$\kappa' = -\sin x$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $\left(-\frac{\pi}{2}, 0\right)$ and $\kappa' < 0$ on $\left(0, \frac{\pi}{2}\right)$, κ is maximum when $x = 0$, $y = 0$. The point of maximum curvature is (0, 0).

41. $\mathbf{r}'(t) = 3\mathbf{i} + 6t\mathbf{j}$

$\mathbf{r}''(t) = 6\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 3\sqrt{1+4t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{12t}{\sqrt{1+4t^2}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 36 - \frac{144t^2}{1+4t^2} = \frac{36}{1+4t^2}$$

$$a_N = \frac{6}{\sqrt{1+4t^2}}$$

At $t_1 = \frac{1}{3}, a_T = \frac{12}{\sqrt{13}}$ and $a_N = \frac{18}{\sqrt{13}}$.

42. $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{i}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{4t^2+1}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{4t}{\sqrt{4t^2+1}}$$

$$a_N^2 = \|\mathbf{r}''(t)\|^2 - a_T^2 = 4 - \frac{16t^2}{4t^2+1} = \frac{4}{4t^2+1}$$

$$a_N = \frac{2}{\sqrt{4t^2+1}}$$

At $t_1 = 1, a_T = \frac{4}{\sqrt{5}}$ and $a_N = \frac{2}{\sqrt{5}}$.

43. $\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = 2\sqrt{1+t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2t}{\sqrt{1+t^2}}$$

$$a_N^2 = \|\mathbf{r}''(t)\|^2 - a_T^2 = 4 - \frac{4t^2}{1+t^2} = \frac{4}{1+t^2}$$

$$a_N = \frac{2}{\sqrt{1+t^2}}$$

At $t_1 = -1, a_T = -\sqrt{2}$ and $a_N = \sqrt{2}$.

44. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$

$\mathbf{r}''(t) = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = a$$

$$a_T = \frac{d^2s}{dt^2} = 0$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = a^2$$

$$a_N = a$$

At $t_1 = \frac{\pi}{6}, a_T = 0$ and $a_N = a$.

45. $\mathbf{r}'(t) = a\sinh t\mathbf{i} + a\cosh t\mathbf{j}$

$\mathbf{r}''(t) = a\cosh t\mathbf{i} + a\sinh t\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = a\sqrt{\sinh^2 t + \cosh^2 t}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2a\cosh t\sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

$$a_N^2 = \|\mathbf{r}''(t)\|^2\ a_T^2$$

$$= a^2(\cosh^2 t + \sinh^2 t) - \frac{4a^2\cosh^2 t\sinh^2 t}{\sinh^2 t + \cosh^2 t}$$

$$= \frac{a^2(\cosh^2 t - \sinh t^2)^2}{\sinh^2 t + \cosh^2 t}$$

$$a_N = \frac{a(\cosh^2 t - \sinh^2 t)}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

At $t_1 = \ln 3, \cosh t = \frac{5}{3}, \sinh t = \frac{4}{3}$, and

$\sqrt{\sinh^2 t + \cosh^2 t} = \frac{\sqrt{41}}{3}$, so $a_T = \frac{40a}{3\sqrt{41}}$ and

$a_N = \frac{3a}{\sqrt{41}}$.

46. $x'(t) = 3$ $\quad y'(t) = -6$

$x''(t) = 0$ $\quad y''(t) = 0$

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = 3\sqrt{5}$$

$$a_T = \frac{d^2s}{dt^2} = 0$$

$$a_N^2 = [x''(t)^2 + y''(t)^2] - a_T^2 = 0;\ a_N = 0$$

At $t_1 = 0, a_T = 0$ and $a_N = 0$.

47. $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{k}$

$$a_T(t) = \frac{(\mathbf{i}+3\mathbf{j}+2t\mathbf{k})\cdot(2\mathbf{k})}{\sqrt{1+9+4t^2}} = \frac{4t}{\sqrt{10+4t^2}}$$

$$a_T(1) = \frac{4}{\sqrt{14}}$$

$$a_N(t) = \frac{|(\mathbf{i}+3\mathbf{j}+2t\mathbf{k})\times(2\mathbf{k})|}{\sqrt{10+4t^2}} = \frac{|6\mathbf{i}-2\mathbf{j}|}{\sqrt{10+4t^2}}$$

$$= \frac{\sqrt{36+4}}{\sqrt{10+4t^2}} = 2\sqrt{\frac{10}{10+4t^2}} = 2\sqrt{\frac{5}{5+2t^2}}$$

$$a_N(1) = 2\sqrt{\frac{5}{7}}$$

48. $\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle$

$\mathbf{r}'(t) = \left\langle 1, 2t, 3t^2 \right\rangle$

$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$

$$a_T(t) = \frac{\left\langle 1, 2t, 3t^2\right\rangle \cdot \langle 0, 2, 6t\rangle}{\sqrt{1+4t^2+9t^4}} = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$$

$$a_T(2) = \frac{152}{\sqrt{161}}$$

$$a_N(t) = \frac{\left|\left\langle 1, 2t, 3t^2\right\rangle \times \langle 0, 2, 6t\rangle\right|}{\sqrt{1+4t^2+9t^4}} = \frac{\left|\left\langle 6t^2, -6t, 2\right\rangle\right|}{\sqrt{1+4t^2+9t^4}}$$

$$= \sqrt{\frac{36t^4+36t^2+4}{1+4t^2+9t^4}} = 2\sqrt{\frac{9t^4+9t^2+1}{1+4t^2+9t^4}}$$

$$a_N(2) = 2\sqrt{\frac{181}{161}}$$

49. $\mathbf{r}(t) = \left\langle e^{-t}, 2t, e^t \right\rangle$; $\mathbf{r}'(t) = \left\langle -e^{-t}, 2, e^t \right\rangle$

$\mathbf{r}''(t) = \left\langle e^{-t}, 0, e^t \right\rangle$; $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -e^{-2t} + e^{2t}$

$\|\mathbf{r}'(t)\| = \sqrt{e^{-2t} + 4 + e^{2t}}$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \left|\left\langle 2e^t, 2, -2e^{-t} \right\rangle\right| = \sqrt{4e^{2t} + 4 + 4e^{-2t}} = 2\sqrt{e^{2t} + 1 + e^{-2t}}$$

$$a_T(t) = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + 4 + e^{-2t}}}$$

$a_T(0) = 0$

$$a_N(t) = 2\sqrt{\frac{e^{2t} + 1 + e^{-2t}}{e^{2t} + 4 + e^{-2t}}}$$

$$a_N(0) = 2\sqrt{\frac{3}{6}} = \sqrt{2}$$

50. $\mathbf{r}'(t) = 2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{i} - 2\mathbf{j}$

$$a_T(t) = \frac{[2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \cdot (2\mathbf{i} - 2\mathbf{j})}{\sqrt{4(t-2)^2 + 4t^2 + 1}} = \frac{4(t-2) + 4t}{\sqrt{8t^2 - 16t + 17}} = \frac{8t - 8}{\sqrt{8t^2 - 16t + 17}}$$

$$a_T(2) = \frac{8}{\sqrt{17}}$$

$$a_N(t) = \frac{\|[2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \times (2\mathbf{i} - 2\mathbf{j})\|}{\sqrt{8t^2 + -16t + 17}} = \frac{\|2\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}\|}{\sqrt{8t^2 - 16t + 17}} = \frac{6\sqrt{2}}{\sqrt{8t^2 - 16t + 17}}$$

$$a_N(2) = \frac{6\sqrt{2}}{\sqrt{17}}$$

51. $\mathbf{r}'(t) = (1 - t^2)\mathbf{i} - (1 + t^2)\mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = -2t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -2t(1 - t^2) + 2t(1 + t^2) = 4t^3$

$\|\mathbf{r}'(t)\| = \sqrt{(1 - t^2)^2 + (1 + t^2)^2 + 1} = \sqrt{2t^4 + 3}$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|2t\mathbf{i} - 2t\mathbf{j} - 4t\mathbf{k}\| = \sqrt{4t^2 + 4t^2 + 16t^2} = 2\sqrt{6}\,|t|$$

$$a_T(t) = \frac{4t^3}{\sqrt{2t^4 + 3}}; \qquad a_T(3) = 36\sqrt{\frac{3}{55}}$$

$$a_N(t) = \frac{2\sqrt{6}\,|t|}{\sqrt{2t^4 + 3}}; \qquad a_N(3) = 6\sqrt{\frac{2}{55}}$$

52. $\mathbf{r}'(t) = \mathbf{i} + t^2\mathbf{j} - \dfrac{1}{t^2}\mathbf{k}, t > 0$

$\mathbf{r}''(t) = 2t\mathbf{j} + \dfrac{2}{t^3}\mathbf{k}$

$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 2t^3 - \dfrac{2}{t^5} = \dfrac{2}{t^5}(t^8 - 1)$

$\|\mathbf{r}'(t)\| = \sqrt{1 + t^4 + \dfrac{1}{t^4}} = \dfrac{1}{t^2}\sqrt{t^4 + t^8 + 1}$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \left\|\frac{4}{t}\mathbf{i} - \frac{2}{t^3}\mathbf{j} + 2t\mathbf{k}\right\| = \sqrt{\frac{4}{t^6} + \frac{16}{t^2} + 4t^2} = \frac{2}{t^3}\sqrt{1 + 4t^4 + t^8}$$

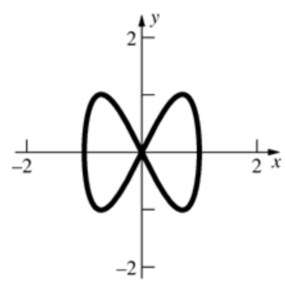
$$a_T(t) = \frac{\frac{2}{t^5}(t^8 - 1)}{\frac{1}{t^2}\sqrt{t^4 + t^8 + 1}} = \frac{2(t^8 - 1)}{t^3\sqrt{t^8 + t^4 + 1}}$$

$a_T(1) = 0$

$$a_N(t) = \frac{\frac{2}{t^3}\sqrt{1 + 4t^4 + t^8}}{\frac{1}{t^2}\sqrt{t^4 + t^8 + 1}} = \frac{2}{t}\sqrt{\frac{t^8 + 4t^4 + 1}{t^8 + t^4 + 1}}$$

$$a_N(1) = 2\sqrt{\frac{6}{3}} = 2\sqrt{2}$$

53.



$\mathbf{v}(t) = \langle \cos t, 2\cos 2t \rangle$, $\mathbf{a}(t) = \langle -\sin t, -4\sin 2t \rangle$

$\mathbf{a}(t) = 0$ if and only if $-\sin t = 0$ and $-4 \sin 2t = 0$, which occurs if and only if $t = 0, \pi, 2\pi$, so it occurs only at the origin.

$\mathbf{a}(t)$ points to the origin if and only if $\mathbf{a}(t) = -k\mathbf{r}(t)$ for some k and $\mathbf{r}(t)$ is not $\mathbf{0}$. This occurs if and only if $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$, so it occurs only at (1, 0) and (–1, 0).

54. $\mathbf{v}(t) = \langle -\sin t + t\cos t + \sin t, \cos t + t\sin t - \cos t\rangle$
$= t\cos t\mathbf{i} + t\sin t\mathbf{j}$
$\mathbf{a}(t) = \langle -t\sin t + \cos t, t\cos t + \sin t\rangle$

a. $\dfrac{ds}{dt} = \|\mathbf{v}(t)\| = \left|t(\cos^2 t + \sin^2 t)^{1/2}\right| = t$
(since $t \ge 0$)

b. $a_T = \dfrac{d^2s}{dt^2} = \left(\dfrac{d}{dt}\right)(t) = 1$
$a_N^2 = |a|^2 - a_T^2$
$= [t^2(\sin^2 t + \cos^2 t) + (\cos^2 t + \sin^2 t)] - 1 = t^2$
Therefore, $a_N = t$.

55. $s''(t) = a_T = 0 \Rightarrow \text{speed} = s'(t) = c$ (a constant)
$\kappa\left(\dfrac{ds}{dt}\right)^2 = a_N = 0 \Rightarrow \kappa = 0$ or $\dfrac{ds}{dt} = 0 \Rightarrow \kappa = 0$

56. $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$;
$\mathbf{v}(t) = -a\omega\sin\omega t\,\mathbf{i} + b\omega\cos\omega t\,\mathbf{j}$
$\mathbf{a}(t) = \left\langle -a\omega^2\cos\omega t, -b\omega^2\sin\omega t\right\rangle = -\omega^2\mathbf{r}(t)$

$\mathbf{T} = \dfrac{\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{1/2}}$;

$\dfrac{d\mathbf{T}}{dt} = \dfrac{(\mathbf{v}\cdot\mathbf{v})\mathbf{a} - (\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{3/2}}$

$= \dfrac{-ab\omega}{(a^2\sin^2\omega t + b^2\cos\omega t)^{3/2}}(b\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j})$

$\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{ab\omega(b^2\cos^2\omega t + a^2\sin^2\omega t)^{1/2}}{(a^2\sin^2\omega t + b^2\cos\omega t)^{3/2}}$

$= \dfrac{ab\omega}{a^2\sin^2\omega t + b^2\cos^2\omega t}$

Then
$\dfrac{\frac{d\mathbf{T}}{dt}}{\left\|\frac{d\mathbf{T}}{dt}\right\|} = \dfrac{-1}{(a^2\sin^2\omega t + b^2\cos^2\omega t)^{1/2}}$
$\times(b\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j})$

Note that this was done assuming $ab > 0$; if $ab < 0$, drop the negative sign in the numerator.

57. $\mathbf{v}(5)$ is tangent to the helix at the point where the particle is 12 meters above the ground. Its path is described by $\mathbf{v}(5) = \cos 5\mathbf{i} - \sin 5\mathbf{j} + 7\mathbf{k}$.

58. $a_N = 0$ wherever $\kappa = 0$ or $\dfrac{ds}{dt} = 0$. κ, the curvature, is 0 at the inflection points, which occur at multiples of $\dfrac{\pi}{2}$. However, $\dfrac{ds}{dt} \ne 0$ on this curve. Therefore, $a_N = 0$ at multiples of $\dfrac{\pi}{2}$.

59. It is given that at (–12, 16), $s'(t) = 10$ ft/s and $s''(t) = 5$ ft/s^2. From Example 2, $\kappa = \dfrac{1}{20}$.

Therefore, $a_T = 5$ and $a_N = \left(\dfrac{1}{20}\right)(10)^2 = 5$, so
$\mathbf{a} = 5\mathbf{T} + 5\mathbf{N}$.
Let $\mathbf{r}(t) = \langle 20\cos t, 20\sin t\rangle$ describe the circle.
$\mathbf{r}(t) = \langle -12, 16\rangle \Rightarrow \cos t = -\dfrac{3}{5}$ and $\sin t = \dfrac{4}{5}$.
$\mathbf{v}(t) = \langle -20\sin t, 20\cos t\rangle$, so $|\mathbf{v}(t)| = 20$.
Then $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle -\sin t, \cos t\rangle$.

Thus, at (–12, 16), $\mathbf{T} = \left\langle -\dfrac{4}{5}, -\dfrac{3}{5}\right\rangle$ and

$\mathbf{N} = \left\langle \dfrac{3}{5}, -\dfrac{4}{5}\right\rangle$ since $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$ and pointing to the concave side of the curve.
Therefore,
$\mathbf{a} = 5\left\langle -\dfrac{4}{5}, -\dfrac{3}{5}\right\rangle + 5\left\langle \dfrac{3}{5}, -\dfrac{4}{5}\right\rangle = -\mathbf{i} - 7\mathbf{j}$.

60. $s'(t) = 4$ and $s''(t) = 0$.
$\kappa = \dfrac{|y''|}{[1 + (y')^2]^{3/2}} = \dfrac{2}{(1 + 4x^2)^{3/2}}$
Therefore,
$\mathbf{a} = (0)\mathbf{T} + (4)^2\dfrac{2}{(1 + 4x^2)^{3/2}}\mathbf{N} = \dfrac{32}{(1 + 4x^2)^{3/2}}\mathbf{N}$.

61. Let $\mu mg = \dfrac{mv_R^2}{R}$. Then $v_R = \sqrt{\mu gR}$. At the values given,
$v_R = \sqrt{(0.4)(32)(400)} = \sqrt{5120} \approx 71.55$ ft/s
(about 49.79 mi/h).

62. a. $\dfrac{R\|\mathbf{F}\|\sin\theta}{v_R^2} = \dfrac{\|\mathbf{F}\|\cos\theta}{g}$ (from the given equations, equating m in each.)
Therefore, $v_R = \sqrt{Rg\tan\theta}$.

b. For the values given,
$v_R = \sqrt{(400)(32)(\tan 10°)} \approx 47.51$ ft/s.

63. $\tan\phi = y'$
$(1 + y'^2)^{3/2} = (1 + \tan^2\phi)^{3/2} = \sec^3\phi$
$\kappa = \dfrac{|y''|}{\left[1 + y'^2\right]^{3/2}} = \dfrac{|y''|}{\left|\sec^3\phi\right|} = \left|y''\cos^3\phi\right|$

64. $\mathbf{N} = \dfrac{\frac{d\mathbf{T}}{ds}}{\left\|\frac{d\mathbf{T}}{ds}\right\|} = \dfrac{\langle -\sin\phi, \cos\phi\rangle \left(\frac{d\phi}{ds}\right)}{(\sin^2\phi + \cos^2\phi)^{1/2}\left|\frac{d\phi}{ds}\right|}$

$= \dfrac{\frac{d\phi}{ds}}{\left|\frac{d\phi}{ds}\right|}\langle -\sin\phi, \cos\phi\rangle$

If $\dfrac{d\phi}{ds} > 0, \mathbf{N} = \langle -\sin\phi, \cos\phi\rangle$ and if $\dfrac{d\phi}{ds} < 0, \mathbf{N} = \langle \sin\phi, -\cos\phi\rangle$, so **N** points to the concave side of the curve in either case.

65. $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. Left-multiply by $\mathbf{T}$ and use Theorem 11.4C

$$\begin{aligned}\mathbf{T}\times\mathbf{B} &= \mathbf{T}\times(\mathbf{T}\times\mathbf{N})\\ &= (\mathbf{T}\bullet\mathbf{N})\mathbf{T} - (\mathbf{T}\bullet\mathbf{T})\mathbf{N}\\ &= 0\mathbf{T} - 1\mathbf{N}\\ &= -\mathbf{N}\end{aligned}$$

Thus, $\mathbf{N} = -\mathbf{T}\times\mathbf{B} = \mathbf{B}\times\mathbf{T}$.
To derive a result for **T** in terms of **N** and **B**, begin with $\mathbf{N} = \mathbf{B}\times\mathbf{T}$ and left multiply by **B**:

$$\begin{aligned}\mathbf{B}\times\mathbf{N} &= \mathbf{B}\times(\mathbf{B}\times\mathbf{T})\\ &= (\mathbf{B}\bullet\mathbf{T})\mathbf{B} - (\mathbf{B}\bullet\mathbf{B})\mathbf{T}\\ &= 0\mathbf{B} - 1\mathbf{T}\\ &= -\mathbf{T}\end{aligned}$$

Thus, $\mathbf{T} = -\mathbf{B}\times\mathbf{N} = \mathbf{N}\times\mathbf{B}$.

66. Since $\lim\limits_{x\to 0^-} y = \lim\limits_{x\to 0^+} y = 0 = y(0)$, y is continuous.

$$y'(x) = \begin{cases} 0 & \text{if } x \le 0\\ 3x^2 & \text{if } x > 0\end{cases}$$

is continuous since
$\lim\limits_{x\to 0^-} y' = \lim\limits_{x\to 0^+} y' = 0 = y'(0).$

$$y''(x) = \begin{cases} 0 & \text{if } x \le 0\\ 6x & \text{if } x > 0\end{cases}$$

is continuous since
$\lim\limits_{x\to 0^-} y'' = \lim\limits_{x\to 0^+} y'' = 0 = y''(0).$ Thus,

$\kappa = \dfrac{|y''|}{(1+y'^2)^{3/2}}$ is continuous also. If $x \ne 0$ then y' and κ are continuous as elementary functions.

67. Let
$P_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5.$
$P_5(0) = 0 \Rightarrow a_0 = 0$
$P_5'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4$, so
$P_5'(0) = 0 \Rightarrow a_1 = 0.$
$P''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3$, so
$P_5''(0) = 0 \Rightarrow a_2 = 0.$
Thus, $P_5(x) = a_3x^3 + a_4x^4 + a_5x^5$,
$P_5'(x) = 3a_3x^2 + 4a_4x^3 + 5a_5x^4$, and
$P_5''(x) = 6a_3x + 12a_4x^2 + 20a_5x^3.$
$P_5(1) = 1, P_5'(1) = 0,$ and $P_5''(1) = 0 \Rightarrow$
$a_3 + a_4 + a_5 = 1$
$3a_3 + 4a_4 + 5a_5 = 0$
$6a_3 + 12a_4 + 20a_5 = 0$
The simultaneous solution to these equations is
$a_3 = 10, a_4 = -15, a_5 = 6$, so
$P_5(x) = 10x^3 - 15x^4 + 6x^5.$

68. Let
$P_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5.$
Then $P_5(x)$ must satisfy
$P_5(0) = 0$; $P_5(1) = 1$; $P_5'(0) = 0$; $P_5'(1) = 1$;
$P_5''(0) = 0$; $P_5''(1) = 0$
As in Problem 67, the three conditions at 0 imply $a_0 = a_1 = a_2 = 0$. The three conditions at 0 lead to the system of equations
$a_3 + a_4 + a_5 = 1$
$3a_3 + 4a_4 + 5a_5 = 1$
$6a_3 + 12a_4 + 20a_5 = 0$
The solution to this system is
$a_3 = 6,\ a_4 = -8,\ a_5 = 3$. Thus, the required polynomial is $P_5(x) = 6x^3 - 8x^4 + 3x^5$.

69. Let the polar coordinate equation of the curve be $r = f(\theta)$. Then the curve is parameterized by $x = r\cos\theta$ and $y = r\sin\theta$.

$x' = -r\sin\theta + r'\cos\theta \qquad y' = r\cos\theta + r'\sin\theta$

$x'' = -r\cos\theta - 2r'\sin\theta + r''\cos\theta \qquad y'' = -r\sin\theta + 2r'\cos\theta + r''\sin\theta$

By Theorem A, the curvature is

$$\kappa = \frac{|x'y'' - y'x''|}{\left|x'^2 + y'^2\right|^{3/2}}$$

$$= \frac{|(-r\sin\theta + r'\cos\theta)(-r\sin\theta + 2r'\cos\theta + r''\sin\theta) - (r\cos\theta + r'\sin\theta)(-r\cos\theta - 2r'\sin\theta + r''\cos\theta)|}{[(-r\sin\theta + r'\cos\theta)^2 + (r\cos\theta + r'\sin\theta)^2]^{3/2}}$$

$$= \frac{\left|r^2 + 2r'^2 - rr''\right|}{(r^2 + r'^2)^{3/2}}.$$

70. $r' = -4\sin\theta, r'' = -4\cos\theta$

$$\kappa = \frac{\left|16\cos^2\theta + 32\sin^2\theta + 16\cos^2\theta\right|}{(16\cos^2\theta + 16\sin^2\theta)^{3/2}} = \frac{32}{64} = \frac{1}{2}$$

71. $r' = -\sin\theta, r'' = -\cos\theta$

At $\theta = 0$, $r = 2$, $r' = 0$, and $r'' = -1$.

$$\kappa = \frac{|4+0+2|}{(4+0)^{3/2}} = \frac{6}{8} = \frac{3}{4}$$

72. $r' = 1, r'' = 0$

At $\theta = 1$, $r = 1$, $r' = 1$, and $r'' = 0$.

$$\kappa = \frac{|1+2-0|}{(1+1)^{3/2}} = \frac{3}{2\sqrt{2}}$$

73. $r' = -4\sin\theta, r'' = -4\cos\theta$

At $\theta = \frac{\pi}{2}, r = 4, r' = -4,$ and $r'' = 0$.

$$\kappa = \frac{|16+32-0|}{(16+16)^{3/2}} = \frac{48}{128\sqrt{2}} = \frac{3}{8\sqrt{2}}$$

74. $r' = 3e^{3\theta}, r'' = 9e^{3\theta}$

At $\theta = 1$, $r = e^3, r' = 3e^3,$ and $r'' = 9e^3$.

$$\kappa = \frac{\left|e^6 + 18e^6 - 9e^6\right|}{(e^6 + 9e^6)^{3/2}} = \frac{10e^6}{10\sqrt{10}e^9} = \frac{1}{\sqrt{10}e^9}$$

75. $r' = 4\cos\theta, r'' = -4\sin\theta$

At $\theta = \frac{\pi}{2}, r = 8, r' = 0, r'' = -4.$

$$\kappa = \frac{|64+0+32|}{(64+0)^{3/2}} = \frac{96}{512} = \frac{3}{16}$$

76. $r' = 6e^{6\theta}, r'' = 36e^{6\theta}$

$$\kappa = \frac{\left|e^{12\theta} + 72e^{12\theta} - 36e^{12\theta}\right|}{(e^{12\theta} + 36e^{12\theta})^{3/2}} = \frac{37e^{12\theta}}{37\sqrt{37}e^{18\theta}}$$

$$= \frac{1}{\sqrt{37}}\left(\frac{1}{e^{6\theta}}\right) = \frac{1}{\sqrt{37}}\left(\frac{1}{r}\right)$$

77. $r = \sqrt{\cos 2\theta}$; $r' = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}, r'' = -\frac{\cos^2 2\theta + 1}{(\cos 2\theta)^{3/2}}$

$$\kappa = \frac{\left|\cos 2\theta + \frac{2\sin^2 2\theta}{\cos 2\theta} + \frac{\cos^2 2\theta + 1}{\cos 2\theta}\right|}{\left(\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}\right)^{3/2}} = \frac{\left|\frac{3}{\cos 2\theta}\right|}{\left(\frac{1}{\cos 2\theta}\right)^{3/2}}$$

$$= 3\left(\sqrt{\cos 2\theta}\right) = 3r$$

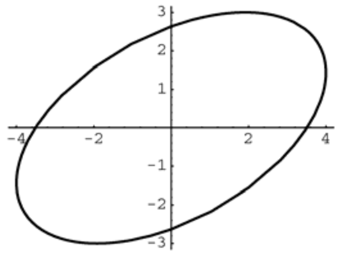
78. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $x = f(t)$ and $y = g(t)$; $\mathbf{v}(t) = \mathbf{r}'(t)$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{x'}{\sqrt{x'^2 + y'^2}}\mathbf{i} + \frac{y'}{\sqrt{x'^2 + y'^2}}\mathbf{j} = \frac{x''(x'^2 + y'^2) - x'(x'x'' + y'y'')}{(x'^2 + y'^2)^{3/2}}\mathbf{i} + \frac{y''(x'^2 + y'^2) - y'(x'x'' + y'y'')}{(x'^2 + y'^2)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(t) = \frac{(x''y' - x'y'')y'}{(x'^2 + y'^2)^{3/2}}\mathbf{i} + \frac{(y''x' - y'x'')x'}{(x'^2 + y'^2)^{3/2}}\mathbf{j} = \frac{(x'y'' - y'x'')}{(x'^2 + y'^2)^{3/2}}(-y'\mathbf{i} + x'\mathbf{j})$$

$$\|\mathbf{T}'(t)\| = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}\sqrt{x'^2 + y'^2} = \frac{|x'y'' - y'x''|}{x'^2 + y'^2} \quad \Rightarrow \quad \kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$$

79.



Maximum curvature ≈ 0.7606,
minimum curvature ≈ 0.1248

80. $\mathbf{B}\cdot\mathbf{B}=1$

$$\frac{d}{ds}(\mathbf{B}\cdot\mathbf{B})=2\mathbf{B}\cdot\frac{d\mathbf{B}}{ds}=0$$

Thus, $\frac{d\mathbf{B}}{ds}$ is perpendicular to **B**.

81. $\frac{d\mathbf{B}}{ds}=\frac{d}{ds}(\mathbf{T}\times\mathbf{N})=\frac{d\mathbf{T}}{ds}\times\mathbf{N}+\mathbf{T}\times\frac{d\mathbf{N}}{ds}$

Since $\mathbf{N}=\frac{\frac{d\mathbf{T}}{ds}}{\left\|\frac{d\mathbf{T}}{ds}\right\|}$, $\frac{d\mathbf{T}}{ds}\times\mathbf{N}=0$, so $\frac{d\mathbf{B}}{ds}=\mathbf{T}\times\frac{d\mathbf{N}}{ds}$.

Thus $\mathbf{T}\cdot\frac{d\mathbf{B}}{ds}=\mathbf{T}\cdot\left(\mathbf{T}\times\frac{d\mathbf{N}}{ds}\right)=(\mathbf{T}\times\mathbf{T})\cdot\frac{d\mathbf{N}}{ds}=0$, so $\frac{d\mathbf{B}}{ds}$ is perpendicular to **T**.

82. **N** is perpendicular to **T**, and **B** = **T** × **N** is perpendicular to both **T** and **N**. Thus, since $\frac{d\mathbf{B}}{ds}$ is perpendicular to both **T** and **B**, it is parallel to **N**, and hence there is some number $\tau(s)$ such that $\frac{d\mathbf{B}}{ds}=-\tau(s)\mathbf{N}$.

83. Let $ax + by + cz + d = 0$ be the equation of the plane containing the curve. Since **T** and **N** lie in the plane $\mathbf{B}=\pm\frac{a\mathbf{i}+b\mathbf{j}+c\mathbf{k}}{\sqrt{a^2+b^2+c^2}}$. Thus, **B** is a constant vector and $\frac{d\mathbf{B}}{ds}=\mathbf{0}$, so $\tau(s) = 0$, since **N** will not necessarily be **0** everywhere.

84. $\mathbf{r}'(t)=a_0\mathbf{i}+b_0\mathbf{j}+c_0\mathbf{k}$

$\mathbf{r}''(t)=\mathbf{0}$

Thus, $\mathbf{r}'(t)\times\mathbf{r}''(t)=\mathbf{0}$ and since

$\kappa=\frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$, $\kappa=0$.

To show that $\tau=0$, note that the curve is confined to a plane. This means that the curve is two-dimensional and thus $\tau=0$.

85. $\mathbf{r}(t) = 6\cos\pi t\mathbf{i} + 6\sin\pi t\mathbf{j}, + 2t\mathbf{k}, t > 0$

Let $(6\cos\pi t)^2+(6\sin\pi t)^2+(2t)^2=100.$

Then $36(\cos^2\pi t+\sin^2\pi t)+4t^2=100;$

$4t^2=64;\ t=4.$

$\mathbf{r}(4) = 6\mathbf{i} + 8\mathbf{k}$, so the fly will hit the sphere at the point (6, 0, 8).

$\mathbf{r}'(t)=-6\pi\sin\pi t\mathbf{i}+6\pi\cos\pi t\mathbf{j}+2\mathbf{k}$, so the fly will have traveled

$$\int_0^4\sqrt{(-6\pi\sin\pi t)^2+(6\pi\cos\pi t)^2+(2)^2}\,dt$$

$$=\int_0^4\sqrt{36\pi^2+4}\,dt$$

$$=\sqrt{36\pi^2+4}(4-0)=8\sqrt{9\pi^2+1}\approx 75.8214$$

86. $\mathbf{r}(t)=\left\langle 10\cos t, 10\sin t, \left(\frac{34}{2\pi}\right)t\right\rangle$

Using the result of Example 1 with $a = 10$ and $c=\frac{34}{2\pi}$, the length of one complete turn is

$$2\pi\sqrt{(10)^2+\left(\frac{34}{2\pi}\right)^2}\text{ angstroms}$$

$=10^{-8}\sqrt{400\pi^2+34^2}$ cm. Therefore, the total length of the helix is

$(2.9)(10^8)(10^{-8})\sqrt{400\pi^2+34^2}\approx 207.1794$ cm.

11.8 Concepts Review

1. traces; cross sections

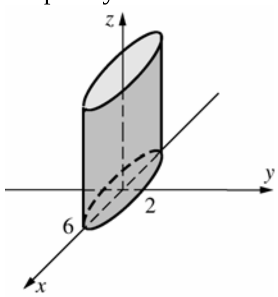
2. cylinders; z-axis

3. ellipsoid

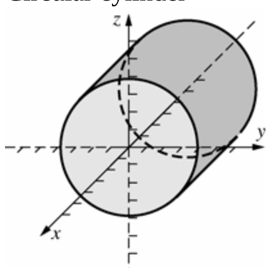
4. elliptic paraboloid

Problem Set 11.8

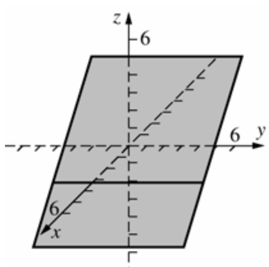
1. $\frac{x^2}{36}+\frac{y^2}{4}=1$
Elliptic cylinder



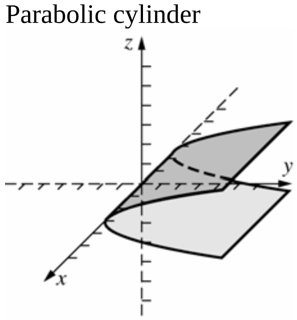
2. Circular cylinder



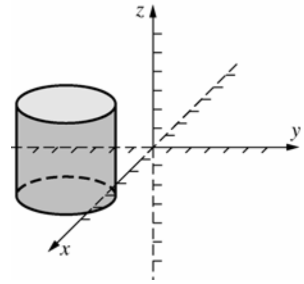
3. Plane



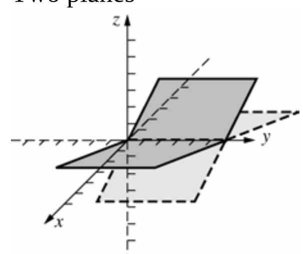
4. Parabolic cylinder



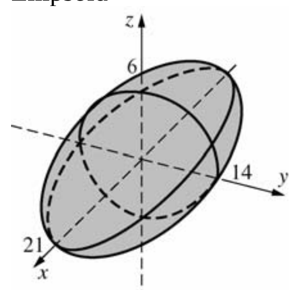
5. $(x-4)^2+(y+2)^2=7$
Circular cylinder



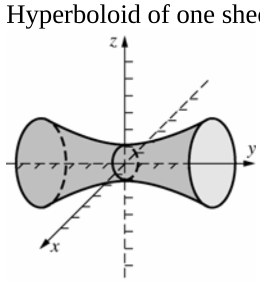
6. Two planes



7. $\frac{x^2}{441}+\frac{y^2}{196}+\frac{z^2}{36}=1$
Ellipsoid

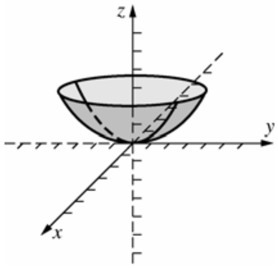


8. $\frac{x^2}{1}-\frac{y^2}{9}+\frac{z^2}{1}=1$
Hyperboloid of one sheet

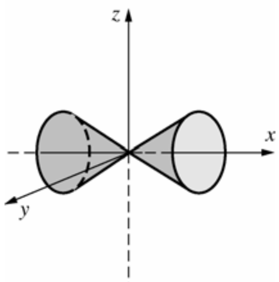


9. $z = \frac{x^2}{8} + \frac{y^2}{2}$

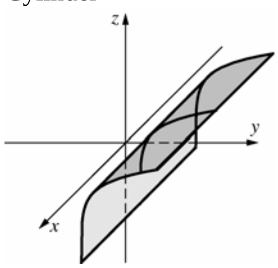
Elliptic paraboloid



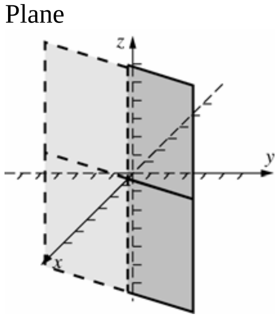
10. Circular cone



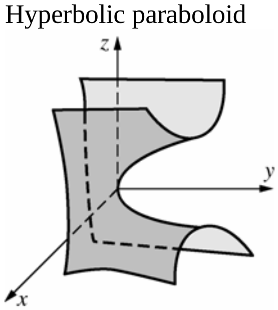
11. Cylinder



12. Plane

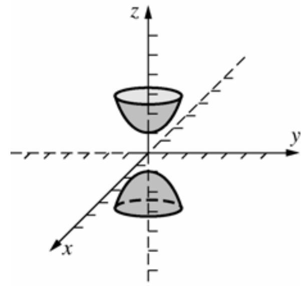


13. Hyperbolic paraboloid



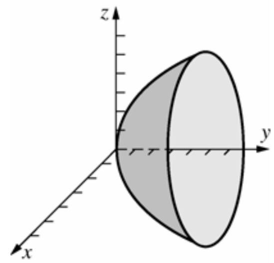
14. $-\frac{x^2}{4} - \frac{y^2}{4} + \frac{z^2}{1} = 1$

Hyperboloid of two sheets



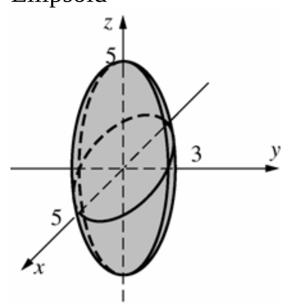
15. $y = \frac{x^2}{4} + \frac{z^2}{9}$

Elliptic Paraboloid

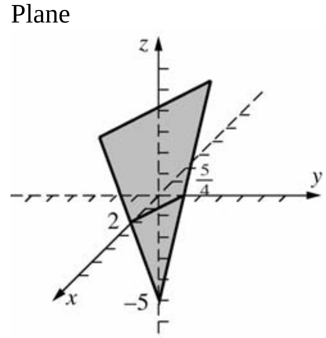


16. $\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{25} = 1$

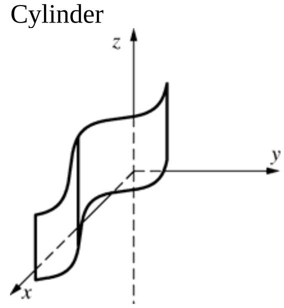
Ellipsoid



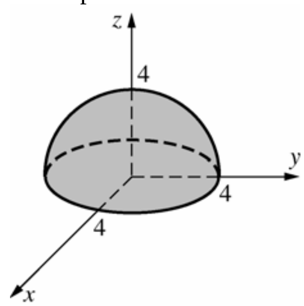
17. Plane



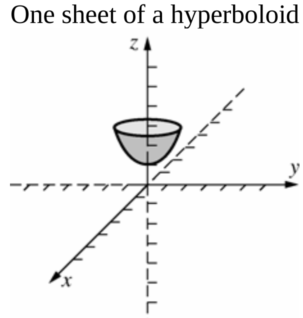
18. Cylinder



19. Hemisphere



20. One sheet of a hyperboloid of two sheets.



21. a. Replacing x by $-x$ results in an equivalent equation.

b. Replacing x by $-x$ and y by $-y$ results in an equivalent equation.

c. Replacing x by $-x$, y by $-y$, and z with $-z$, results in an equivalent equation.

22. a. Replacing y by $-y$ results in an equivalent equation.

b. Replacing x with $-x$ and z with $-z$ results in an equivalent equation.

c. Replacing y by $-y$ and z with $-z$ results in an equivalent equation.

23. All central ellipsoids are symmetric with respect to (a) the origin, (b) the x-axis, and the (c) xy-plane.

24. All central hyperboloids of one sheet are symmetric with respect to (a) the origin, (b) the y-axis, and (c) the xy-plane.

25. All central hyperboloids of two sheets are symmetric with respect to (a) the origin, (b) the z-axis, and (c) the yz-plane.

26. a. 1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 18

b. 1, 2, 6, 7, 8, 9, 10, 14, 16, 19, 20

27. At $y = k$, the revolution generates a circle of radius $x=\sqrt{\frac{y}{2}}=\sqrt{\frac{k}{2}}$. Thus, the cross section in the plane $y = k$ is the circle $x^2+z^2=\frac{k}{2}$ or $2x^2+2z^2=k.$ The equation of the surface is $y=2x^2+2z^2.$

28. At $z = k$, the revolution generates a circle of radius $y=\frac{z}{2}=\frac{k}{2}$. Thus, the cross section in the plane

$z = k$ is the circle $x^2+y^2=\frac{k^2}{4}$ or $4x^2+4y^2=k^2.$ The equation of the surface is $z^2=4x^2+4y^2.$

29. At $y = k$, the revolution generates a circle of radius $x=\sqrt{3-\frac{3}{4}y^2}=\sqrt{3-\frac{3}{4}k^2}.$ Thus, the cross section in the plane $y = k$ is the circle $x^2+z^2=3-\frac{3}{4}k^2$ or $12-4x^2-4z^2=3k^2.$ The equation of the surface is $4x^2+3y^2+4z^2=12.$

30. At $x = k$, the revolution generates a circle of radius $y=\sqrt{\frac{4}{3}x^2-4}=\sqrt{\frac{4}{3}k^2-4}.$ Thus, the cross section in the plane $x = k$ is the circle $y^2+z^2=\frac{4}{3}k^2-4$ or $12+3y^2+3z^2=4k^2.$ The equation of the surface is $4x^2=12+3y^2+3z^2.$

31. When $z = 4$ the equation is $4=\frac{x^2}{4}+\frac{y^2}{9}$ or $1=\frac{x^2}{16}+\frac{y^2}{36},$ so $a^2=36, b^2=16,$ and $c^2=a^2-b^2=20,$ hence $c=\pm 2\sqrt{5}.$ The major axis of the ellipse is on the y-axis so the foci are at $\left(0, \pm 2\sqrt{5}, 4\right).$

32. When $x = 4$, the equation is $z = \frac{16}{4} + \frac{y^2}{9}$ or

$y^2 = 9(z-4) = 4 \cdot \frac{9}{4}(z-4)$, hence $p = \frac{9}{4}$. The vertex is at (4, 0, 4) so the focus is

$\left(4,\ 0,\ 4+\frac{9}{4}\right) = \left(4,\ 0,\ \frac{25}{4}\right)$.

33. When $z = h$, the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{h^2}{c^2} = 1$ or

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{c^2 - h^2}{c^2}$ which is equivalent to

$\frac{x^2}{\frac{a^2(c^2-h^2)}{c^2}} + \frac{y^2}{\frac{b^2(c^2-h^2)}{c^2}} = 1$, which is $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

with $A = \frac{a}{c}\sqrt{c^2 - h^2}$ and $B = \frac{b}{c}\sqrt{c^2 - h^2}$. Thus, the area is

$\pi\left(\frac{a}{c}\sqrt{c^2-h^2}\right)\left(\frac{b}{c}\sqrt{c^2-h^2}\right) = \frac{\pi ab(c^2-h^2)}{c^2}$

34. The equation of the elliptical cross section is

$\frac{x^2}{a^2(h-z)} + \frac{y^2}{b^2(h-z)} = 1$, for each z in $[0, h)$.

Therefore, $\Delta V \approx \pi\left(a\sqrt{h-z}\right)\left(b\sqrt{h-z}\right)\Delta z$

$= \pi ab(h-z)\Delta z$, using the area formula mentioned in Problem 33.

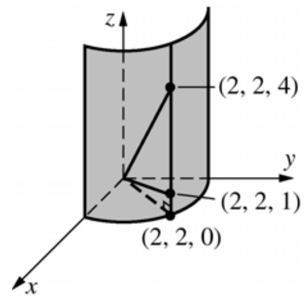
Therefore, $V = \int_0^h \pi ab(h-z)dz = \pi ab\left[hz - \frac{z^2}{2}\right]_0^h$

$= \pi ab\left[\left(h - \frac{h^2}{2}\right) - 0\right] = \frac{\pi abh^2}{2}$, which is the height times one half the area of the base ($z = 0$),

$\pi\left(a\sqrt{h}\right)\left(b\sqrt{h}\right) = \pi abh.$

35. Equating the expressions for y, $4 - x^2 = x^2 + z^2$ or $1 = \frac{x^2}{2} + \frac{z^2}{4}$ which is the equation of an ellipse in the xz-plane with major diameter of $2\sqrt{4} = 4$ and minor diameter $2\sqrt{2}$.

36.



$y = x$ intersects the cylinder when $x = y = 2$. Thus, the vertices of the triangle are (0, 0, 0), (2, 2, 1), and (2, 2, 4). The area of the triangle with sides represented by $\langle 2, 2, 1\rangle$ and $\langle 2, 2, 4\rangle$ is

$\frac{1}{2}\|\langle 2,\ 2,\ 1\rangle \times \langle 2,\ 2,\ 4\rangle\| = \frac{1}{2}\|\langle 6, -6,\ 0\rangle\|$

$= \frac{1}{2}\left(6\sqrt{2}\right) = 3\sqrt{2}$.

37. $(t\cos t)^2 + (t\sin t)^2 - t^2$

$= t^2(\cos^2 t + \sin^2 t) - t^2 = t^2 - t^2 = 0,$

hence every point on the spiral is on the cone. For $\mathbf{r} = 3t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}$, every point satisfies $x^2 + 9y^2 - 9z^2 = 0$ so the spiral lies on the elliptical cone.

38. It is clear that $x = y$ at each point on the curve. Thus, the curve lies in the plane $x = y$. Since $z = t^2 = y^2$, the curve is the intersection of the plane $x = y$ with the parabolic cylinder $z = y^2$. Let the line $y = x$ in the xy-plane be the u-axis, then the curve determined by $\mathbf{r}$ is in the uz-plane. The u-coordinate of a point on the curve, (y, y, z) is the signed distance of the point $(y,\ y,\ 0)$ from the origin, i.e., $u = \sqrt{2}y$. Thus, $u^2 = 2y^2 = 2z$ or $u^2 = 4\left(\frac{1}{2}\right)z$. This is a parabola in the uz-plane with vertex at $u = 0$, $z = 0$ and focus at $u = 0$, $z = \frac{1}{2}$. The focus is at $\left(0,\ 0,\ \frac{1}{2}\right)$.

11.9 Concepts Review

1. circular cylinder; sphere

2. plane; cone

3. $\rho^2 = r^2 + z^2$

4. $x^2 + y^2 + z^2 = 4z$, so
$x^2 + y^2 + z^2 - 4z + 4 = 4$ or
$x^2 + y^2 + (z-2)^2 = 4.$

Problem Set 11.9

1. Cylindrical to Spherical:
$\rho = \sqrt{r^2 + z^2}$
$\cos\phi = \dfrac{z}{\sqrt{r^2+z^2}}$
$\theta = \theta$
Spherical to Cylindrical:
$r = \rho \sin\phi$
$z = \rho\cos\phi$
$\theta = \theta$

2. **a.** $(\rho,\theta,\phi) = \left(\sqrt{2}, \dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$

b. Note: $\left(-2, \dfrac{\pi}{4}, 2\right) = \left(2, \dfrac{5\pi}{4}, 2\right)$
$(\rho,\theta,\phi) = \left(2\sqrt{2}, \dfrac{5\pi}{4}, \dfrac{\pi}{4}\right)$

3. **a.** $x = 6\cos\left(\dfrac{\pi}{6}\right) = 3\sqrt{3}$
$y = 6\sin\left(\dfrac{\pi}{6}\right) = 3$
$z = -2$

b. $x = 4\cos\left(\dfrac{4\pi}{3}\right) = -2$
$y = 4\sin\left(\dfrac{4\pi}{3}\right) = -2\sqrt{3}$
$z = -8$

4. **a.** $x = 8\sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$
$y = 8\sin\left(\dfrac{\pi}{6}\right)\sin\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$
$z = 8\cos\left(\dfrac{\pi}{6}\right) = 4\sqrt{3}$

b. $x = 4\sin\left(\dfrac{3\pi}{4}\right)\cos\left(\dfrac{\pi}{3}\right) = \sqrt{2}$
$y = 4\sin\left(\dfrac{3\pi}{4}\right)\sin\left(\dfrac{\pi}{3}\right) = \sqrt{6}$
$z = 4\cos\left(\dfrac{3\pi}{4}\right) = -2\sqrt{2}$

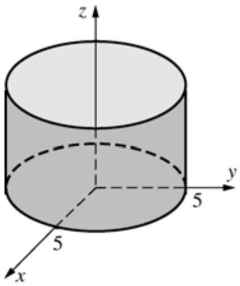
5. **a.** $\rho = \sqrt{x^2+y^2+z^2}$
$= \sqrt{4+12+16} = 4\sqrt{2}$
$\tan\theta = \dfrac{y}{x} = \dfrac{-2\sqrt{3}}{2} = -\sqrt{3}$ and (x, y) is in the 4th quadrant so $\theta = \dfrac{5\pi}{3}.$
$\cos\phi = \dfrac{z}{\rho} = \dfrac{4}{4\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ so $\phi = \dfrac{\pi}{4}.$
Spherical: $\left(4\sqrt{2}, \dfrac{5\pi}{3}, \dfrac{\pi}{4}\right)$

b. $\rho = \sqrt{2+2+12} = 4$
$\tan\theta = \dfrac{\sqrt{2}}{-\sqrt{2}} = -1$ and (x, y) is in the 2nd quadrant so $\theta = \dfrac{3\pi}{4}.$
$\cos\phi = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}$ so $\dfrac{\pi}{6}.$
Spherical: $\left(4, \dfrac{3\pi}{4}, \dfrac{\pi}{6}\right)$

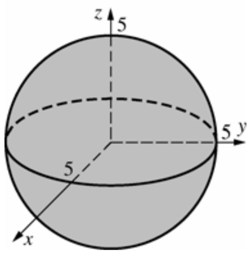
6. **a.** $r = \sqrt{4+4} = 2\sqrt{2}$
$\tan\theta = \dfrac{2}{2} = 1$, $x > 0$, $y > 0$, so $\theta = \dfrac{\pi}{4}.$ $z = 3$

b. $r = \sqrt{48+16} = 8$
$\tan\theta = -\dfrac{4}{4\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$, $x > 0$, $y < 0$, so
$\theta = \dfrac{11\pi}{6}$. $z = 6$

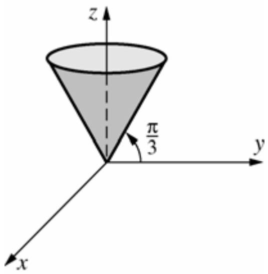
7. $r = 5$
Cylinder



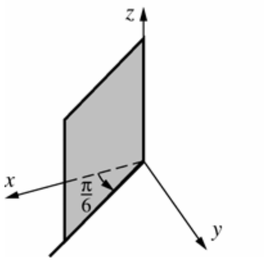
8. $\rho = 5$
Sphere



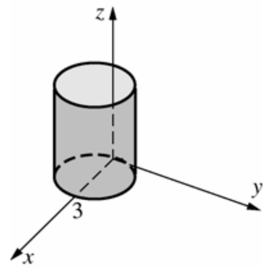
9. $\phi = \frac{\pi}{6}$
Cone



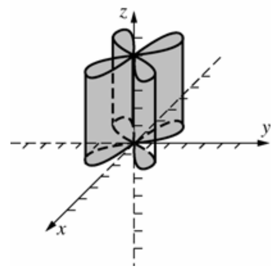
10. $\theta = \frac{\pi}{6}$
Plane



11. $r = 3\cos\theta$
Circular cylinder



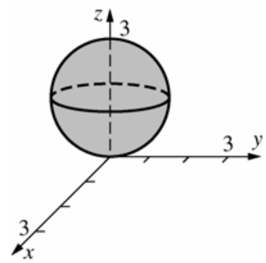
12. $r = 2\sin 2\theta$
4-leaved cylinder



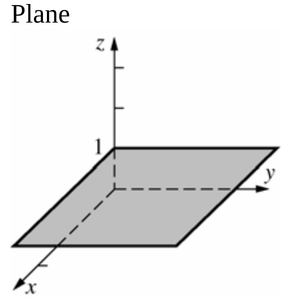
13. $\rho = 3\cos\phi$

$$x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = \frac{9}{4}$$

Sphere



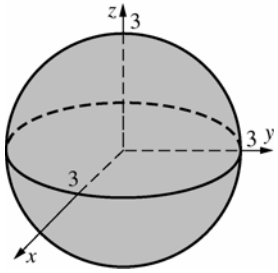
14. $\rho = \sec\phi$
$\rho\cos\phi = 1$
$z = 1$
Plane



15. $r^2 + z^2 = 9$

$x^2 + y^2 + z^2 = 9$

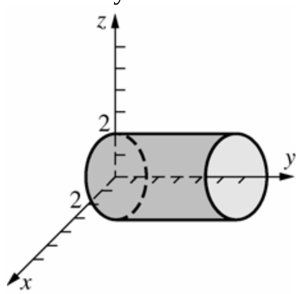
Sphere



16. $r^2 \cos^2\theta + z^2 = 4$

$x^2 + z^2 = 4$

Circular cylinder



17. $x^2 + y^2 = 9; r^2 = 9;\ r = 3$

18. $r^2\cos^2\theta - r^2\sin^2\theta = 25; r^2\cos 2\theta = 25;$

$r^2 = 25\sec^2\theta; r = 5\sec\theta$

19. $r^2 + 4z^2 = 10$

20. $(x^2 + y^2 + z^2) + 3z^2 = 10; \rho^2 + 3\rho^2\cos^2\phi = 10;$

$$\rho^2 = \frac{10}{1 + 3\cos^2\phi}$$

21. $(x^2 + y^2 + z^2) - 3z^2 = 0; \rho^2 - 3\rho^2\cos^2\phi = 0;$

$\cos^2\phi = \frac{1}{3}$ (pole is not lost); $\cos^2\phi = \frac{1}{3}$ (or

$\sin^2\phi = \frac{2}{3}$ or $\tan^2\phi = 2$)

22. $\rho^2[\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - \cos^2\phi] = 1;$

$\rho^2[\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - 1 + \sin^2\phi] = 1;$

$\rho^2[\sin^2\phi\cos^2\theta - 1 + \sin^2\phi(1 - \sin^2\theta)] = 1;$

$\rho^2[\sin^2\phi\cos^2\theta - 1 + \sin^2\phi\cos^2\theta] = 1;$

$\rho^2[2\sin^2\phi\cos^2\theta - 1] = 1;\ \ \rho^2 = \dfrac{1}{2\sin^2\phi\cos^2\theta - 1}$

23. $(r^2 + z^2) + z^2 = 4; \rho^2 + \rho^2\cos^2\phi = 4;$

$$\rho^2 = \frac{4}{1 + \cos^2\phi}$$

24. $\rho^2 = 2\rho\cos\phi; r^2 + z^2 = 2z; r^2 = 2z - z^2;$

$r = \sqrt{2z - z^2}$

25. $r\cos\theta + r\sin\theta = 4;\ r = \dfrac{4}{\sin\theta + \cos\theta}$

26. $\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi = 1;$

$$\rho = \frac{1}{\sin\phi(\sin\theta + \cos\theta) + \cos\phi}$$

27. $(x^2 + y^2 + z^2) - z^2 = 9;\ \ \rho^2 - \rho^2\cos^2\phi = 9;$

$\rho^2(1 - \cos^2\phi) = 9; \rho^2\sin^2\phi = 9$

$\rho\sin\phi = 3$

28. $r^2 = 2r\sin\theta;\ x^2 + y^2 = 2y;\ x^2 + (y-1)^2 = 1$

29. $r^2\cos 2\theta = z; r^2(\cos^2\theta - \sin^2\theta) = z;$

$(r\cos\theta)^2 - (r\sin\theta)^2 = z;\ \ x^2 - y^2 = z$

30. $\rho\sin\phi = 1$ (spherical); $r = 1$ (cylindrical);

$x^2 + y^2 = 1$ (Cartesian)

31. $z = 2x^2 + 2y^2 = 2(x^2 + y^2)$ (Cartesian); $z = 2r^2$ (cylindrical)

32. $2x^2 + 2y^2 - z^2 = 2$ (Cartesian); $2r^2 - z^2 = 2$ (cylindrical)

33. For St. Paul:

$\rho = 3960$,

$\theta = 360° - 93.1° = 266.9° \approx 4.6583$ rad

$\phi = 90° - 45° = 45° = \frac{\pi}{4}$ rad

$$x = 3960 \sin\frac{\pi}{4}\cos 4.6583 \approx -151.4$$

$$y = 3960 \sin\frac{\pi}{4}\sin 4.6583 \approx -2796.0$$

$$z = 3960 \cos\frac{\pi}{4} \approx 2800.1$$

For Oslo:

$\rho = 3960$, $\theta = 10.5° \approx 0.1833$ rad,

$\phi = 90° - 59.6° = 30.4° \approx 0.5306$ rad

$x = 3960 \sin 0.5306 \cos 0.1833 \approx 1970.4$

$y = 3960 \sin 0.5306 \sin 0.1833 \approx 365.3$

$z = 3960 \cos 0.5306 \approx 3415.5$

As in Example 7,

$$\cos\gamma \approx \frac{(-151.4)(1970.4)+(-2796.0)(365.3)+(2800.1)(3415.5)}{3960^2} \approx 0.5257$$

so $\gamma \approx 1.0173$ and the great-circle distance is $d \approx 3960(1.0173) \approx 4029$ mi

34. For New York:

$\rho = 3960$, $\theta = 360° - 74° = 286° \approx 4.9916$ rad

$\phi = 90° - 40.4° = 49.6° \approx 0.8657$ rad

$x = 3960 \sin 0.8657 \cos 4.9916 \approx 831.1$

$y = 3960 \sin 0.8657 \sin 4.9916 \approx -2898.9$

$z = 3960 \cos 0.8657 \approx 2566.5$

For Greenwich:

$\rho = 3960$, $\theta = 0$, $\phi = 90° - 51.3° = 38.7° \approx 0.6754$ rad

$x = 3960 \sin 0.6754 \cos 0 \approx 2475.8$

$y = 0$

$z = 3960 \cos 0.6754 \approx 3090.6$

$$\cos\gamma = \frac{(831.1)(2475.8)+(-2898.9)(0)+(2566.5)(3090.6)}{3960^2} \approx 0.6370$$

so $\gamma \approx 0.8802$ and the great-circle distance is $d \approx 3960(0.8802) \approx 3485$ mi

35. From Problem 33, the coordinates of St. Paul are $P(-151.4, -2796.0, 2800.1)$.

For Turin:

$\rho = 3960$, $\theta = 7.4° \approx 0.1292$ rad, $\phi = \frac{\pi}{4}$ rad

$$x = 3960 \sin\frac{\pi}{4}\cos 0.1292 \approx 2776.8$$

$$y = 3960 \sin\frac{\pi}{4}\sin 0.1292 \approx 360.8$$

$$z = 3960 \cos\frac{\pi}{4} \approx 2800.1$$

$$\cos\gamma \approx \frac{(-151.4)(2776.8)+(-2796.0)(360.8)+(2800.1)(2800.1)}{3960^2} \approx 0.4088$$

so $\gamma \approx 1.1497$ and the great-circle distance is $d \approx 3960(1.1497) \approx 4553$ mi

36. The circle inscribed on the earth at 45° parallel ($\phi = 45°$) has radius $3960\cos\frac{\pi}{4}$. The longitudinal angle between St. Paul and Turin is 93.1° + 7.4° = 100.5° $\approx$ 1.7541 rad

Thus, the distance along the 45° parallel is $\left(3960\cos\frac{\pi}{4}\right)(1.7541) \approx 4912$ mi

37. Let St. Paul be at $P_1(-151.4, -2796.0, 2800.1)$ and Turin be at $P_2(2776.8, 360.8, 2800.1)$ and O be the center of the earth. Let β be the angle between the z-axis and the plane determined by O, P_1, and P_2. $\overrightarrow{OP_1} \times \overrightarrow{OP_2}$ is normal to the plane. The angle between the z-axis and $\overrightarrow{OP_1} \times \overrightarrow{OP_2}$ is complementary to β. Hence

$$\beta = \frac{\pi}{2} - \cos^{-1}\left(\frac{\left(\overrightarrow{OP_1} \times \overrightarrow{OP_2}\right)\cdot \mathbf{k}}{\left\|\overrightarrow{OP_1} \times \overrightarrow{OP_2}\right\|\|\mathbf{k}\|}\right) \approx \frac{\pi}{2} - \cos^{-1}\left(\frac{7.709\times 10^6}{1.431\times 10^7}\right) \approx 0.5689 .$$

The distance between the North Pole and the St. Paul-Turin great-circle is 3960(0.5689) $\approx$ 2253 mi

38. $x_i = \rho_i \sin\phi_i \cos\theta_i,\ y_i = \rho_i \sin\phi_i \sin\theta_i,\ z_i = \rho_i \cos\phi_i$ for i = 1, 2.

$$d^2 = (\rho_2 \sin\phi_2 \cos\theta_2 - \rho_1 \sin\phi_1 \cos\theta_1)^2 + (\rho_2 \sin\phi_2 \sin\theta_2 - \rho_1 \sin\phi_1 \sin\theta_1)^2$$
$$+(\rho_2 \cos\phi_2 - \rho_1\cos\phi_1)^2$$
$$= \rho_2^2 \sin^2\phi_2(\cos^2\theta_2 + \sin^2\theta_2) + \rho_1^2 \sin^2\phi_1(\cos^2\theta_1 + \sin^2\theta_1) + \rho_2^2\cos^2\phi_2 + \rho_1^2\cos^2\phi_1$$
$$-2\rho_1\rho_2 \sin\phi_1 \sin\phi_2(\cos\theta_1\cos\theta_2 + \sin\theta_1 \sin\theta_2) - 2\rho_1\rho_2\cos\phi_1\cos\phi_2$$
$$= \rho_2^2\sin^2\phi_2 + \rho_1^2\sin^2\phi_1 + \rho_2^2\cos^2\phi_2 + \rho_1^2\cos^2\phi_1 - 2\rho_1\rho_2\sin\phi_1\sin\phi_2[\cos(\theta_1-\theta_2)]$$
$$-2\rho_1\rho_2\cos\phi_1\cos\phi_2$$
$$= \rho_2^2(\sin^2\phi_2 + \cos^2\phi_2) + \rho_1^2(\sin^2\phi_1 + \cos^2\phi_1) + 2\rho_1\rho_2[-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$$
$$= \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2[-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$$
$$= \rho_1^2 - 2\rho_1\rho_2 + \rho_2^2 + 2\rho_1\rho_2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$$
$$= (\rho_1-\rho_2)^2 + 2\rho_1\rho_2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$$

Hence, $d = \{(\rho_1-\rho_2)^2 + 2\rho_1\rho_2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]\}^{1/2}$

39. Let P_1 be (a_1, θ_1, ϕ_1) and P_2 be (a_2, θ_2, ϕ_2). If γ is the angle between $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ then the great-circle distance between P_1 and P_2 is $a\gamma$. $|OP_1| = |OP_2| = a$ while the straight-line distance between P_1 and P_2 is (from Problem 38) $d^2 = (a-a)^2 + 2a^2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 - \cos\phi_1\cos\phi_2]$

$= 2a^2\{1-[\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2]\}$.

Using the Law Of Cosines on the triangle OP_1P_2,

$d^2 = a^2 + a^2 - 2a^2\cos\gamma \ = 2a^2(1-\cos\gamma)$.

Thus, γ is the central angle and $\cos\gamma = \cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2$.

40. The longitude/latitude system (α, β) is related to a spherical coordinate system (ρ, θ, ϕ) by the following relations: $\rho = 3960$; the trigonometric function values of α and θ are identical but

$-\pi \le \alpha \le \pi$ rather than $0 \le \theta \le 2\pi$, and $\beta = \frac{\pi}{2} - \phi$ so $\sin\beta = \cos\phi$ and $\cos\beta = \sin\phi$.

From Problem 39, the great-circle distance between $(3960, \theta_1, \phi_1)$ and $(3960, \theta_2, \phi_2)$ is 3960γ where $0 \le \gamma \le \pi$ and $\cos\gamma = \cos(\theta_1 - \theta_2)\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 = \cos(\alpha_1 - \alpha_2)\cos\beta_1 \cos\beta_2 + \sin\beta_1 \sin\beta_2$.

41. a. New York (–74°, 40.4°); Greenwich (0°, 51.3°)
$\cos\gamma = \cos(-74° - 0°)\cos(40.4°)\cos(51.3°) + \sin(40.4°)\sin(51.3°) \approx 0.637$
Then $\gamma \approx 0.880$ rad, so $d \approx 3960(0.8801) \approx 3485$ mi.

b. St. Paul (–93.1°, 45°); Turin (7.4°, 45°)
$\cos\gamma = \cos(-93.1° - 7.4°)\cos(45°)\cos(45°) + \sin(45°)\sin(45°) \approx 0.4089$
Then $\gamma \approx 1.495$ rad, so $d \approx 3960(1.1495) \approx 4552$ mi.

c. South Pole (7.4°, –90°); Turin (7.4°, 45°)
Note that any value of α can be used for the poles.
$\cos\gamma = \cos 0° \cos(-90°)\cos(45°) + \sin(-90°)\sin(45°) = -\frac{1}{\sqrt{2}}$
thus $\gamma = 135° = \frac{3\pi}{4}$ rad, so $d = 3960\left(\frac{3\pi}{4}\right) \approx 9331$ mi.

d. New York (–74°, 40.4°); Cape Town (18.4°, –33.9°)
$\cos\gamma = \cos(-74° - 18.4°)\cos(40.4°)\cos(-33.9°) + \sin(40.4°)\sin(-33.9°) \approx -0.3880$
Then $\gamma \approx 1.9693$ rad, so $d \approx 3960(1.9693) \approx 7798$ mi.

e. For these points $\alpha_1 = 100°$ and $\alpha_2 = -80°$ while $\beta_1 = \beta_2 = 0$, hence
$\cos\gamma = \cos 180°$ and $\gamma = \pi$ rad, so $d = 3960\pi \approx 12{,}441$ mi.

42. $\rho = 2a\sin\phi$ is independent of θ so the cross section in each half-plane, $\theta = k$, is a circle tangent to the origin and with radius $2a$. Thus, the graph of $\rho = 2a\sin\phi$ is the surface of revolution generated by revolving about the z-axis a circle of radius $2a$ and tangent to the z-axis at the origin.

11.10 Chapter Review

Concepts Test

1. True: The coordinates are defined in terms of distances from the coordinate planes in such a way that they are unique.

2. False: The equation is $(x-2)^2 + y^2 + z^2 = -5$, so the solution set is the empty set.

3. True: See Section 11.2.

4. False: See previous problem. It represents a plane if A and B are not both zero.

5. False: The distance between (0, 0, 3) and (0, 0, –3) (a point from each plane) is 6, so the distance between the planes is less than or equal to 6 units.

6. False: It is normal to the plane.

7. True: Let $t = \frac{1}{2}$.

8. True: Direction cosines are $\frac{a}{\|\mathbf{u}\|}, \frac{b}{\|\mathbf{u}\|}, \frac{c}{\|\mathbf{u}\|}$

9. True: $(2\mathbf{i} - 3\mathbf{j}) \cdot (6\mathbf{i} + 4\mathbf{j}) = 0$ if and only if the vectors are perpendicular.

10. True: Since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \mathbf{u}\cdot\mathbf{v}$.

11. False: The dot product for three vectors $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ is not defined.

12. True: $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta| \le \|\mathbf{u}\|\|\mathbf{v}\|$ since $|\cos\theta| \le 1$.

13. True: If $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$, then $|\cos\theta| = 1$ since $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta|$. Thus, **u** is a scalar multiple of **v**. If **u** is a scalar multiple of **v**, $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\cdot k\mathbf{u}\| = k\|\mathbf{u}\|^2$
$= \|\mathbf{u}\|\|k\mathbf{u}\| = \|\mathbf{u}\|\|\mathbf{v}\|$.

14. False: If **u** = –**i** and $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$, then $\mathbf{u}+\mathbf{v} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ and $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\| = 1$.

15. True: $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$ so $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$ or $|\mathbf{u}| = |\mathbf{v}|$.

16. True: $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$
$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}$

17. True: Theorem 11.5A

18. True: $D_t[\mathbf{F}(t)\cdot\mathbf{F}(t)]$
$= \mathbf{F}(t)\cdot\mathbf{F}'(t) + \mathbf{F}(t)\cdot\mathbf{F}'(t)$
$= 2\mathbf{F}(t)\cdot\mathbf{F}'(t)$

19. True: $\|\,\|\mathbf{u}\|\mathbf{u}\,\| = \|\mathbf{u}\|\|\mathbf{u}\| = \|\mathbf{u}\|^2$

20. False: The dot product of a scalar and a vector is not defined.

21. True: $\|\mathbf{u}\times\mathbf{v}\| = \|-\mathbf{v}\times\mathbf{u}\| = |-1|\|\mathbf{v}\times\mathbf{u}\|$
$= \|\mathbf{v}\times\mathbf{u}\|$

22. True: $(k\mathbf{v})\times\mathbf{v} = k(\mathbf{v}\times\mathbf{v}) = k(\mathbf{0}) = \mathbf{0}$

23. False: Obviously not true if **u** = **v**. (More generally, it is only true when **u** and **v** are also perpendicular.)

24. False: It multiplies **v** by *a*; it multiplies the length of **v** by $|a|$.

25. True: $\frac{\|\mathbf{u}\times\mathbf{v}\|}{(\mathbf{u}\cdot\mathbf{v})} = \frac{\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta}{\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta} = \tan\theta$

26. True: The vectors are both parallel and perpendicular, so one or both must be **0**.

27. True: $|(2\mathbf{i}\times 2\mathbf{j})\cdot(\mathbf{j}\times\mathbf{i})| = 4|(\mathbf{k})\cdot(-\mathbf{k})|$
$= 4(\mathbf{k}\cdot\mathbf{k}) = 4$

28. False: Let **u** = **v** = **i**, **w** = **j**.
Then (**u** × **v**) × **w** = **0** × **w** = **0**; but
u × (**v** × **w**) = **i** × **k** = –**j**.

29. True: Since $\langle b_1, b_2, b_3\rangle$ is normal to the plane.

30. False: Each line can be represented by parametric equations, but lines with any zero direction number cannot be represented by symmetric equations.

31. True: $\kappa = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} = 0 \Rightarrow \|\mathbf{r}'\times\mathbf{r}''\|$
$= \|\mathbf{r}'\|\|\mathbf{r}''\|\sin\theta = 0$. Thus, either $\mathbf{r}'$ and $\mathbf{r}''$ are parallel or either $\mathbf{r}'$ or $\mathbf{r}''$ is **0**, which implies that the path is a straight line.

32. True: An ellipse bends the sharpest at points on the major axis.

33. False: κ depends only on the shape of the curve.

34. True: $x' = 3$ $\quad y' = 2$
$x'' = 0$ $\quad y'' = 0$
Thus, $\kappa = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = 0.$

35. False: $x' = -2\sin t$ $\quad y' = 2\cos t$
$x'' = -2\cos t$ $\quad y'' = -2\sin t$
Thus, $\kappa = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{4}{8} = \frac{1}{2}.$

36. True: $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$;

$$\mathbf{T}'(t) = -\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}\mathbf{r}'(t) + \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}''(t);$$

$\mathbf{T}(t)\cdot\mathbf{T}'(t)$

$$= \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)\cdot\left[-\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}\mathbf{r}'(t) + \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}''(t)\right]$$

$$= -\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2} + \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2} = 0$$

37. False: Consider uniform circular motion: $|dv/dt| = 0$ but $\|\mathbf{v}\| = a\omega$.

38. True: $\kappa = \dfrac{|y''|}{[1+y'^{\,2}]^{3/2}} = 0$

39. False: If $y'' = k$ then $y' = kx + C$ and $\kappa = \dfrac{|y''|}{[1+y'^2]^{3/2}} = \dfrac{k}{[1+(kx+C)^2]^{3/2}}$ is not constant.

40. False: For example, if **u** = **i** and **v** = **j**, then $\mathbf{u}\cdot\mathbf{v} = 0$.

41. False: For example, if $\mathbf{r}(t) = \cos t^2\mathbf{i} + \sin t^2\mathbf{j}$, then $\mathbf{r}'(t) = -2t\sin t^2\mathbf{i} + 2t\cos t^2\mathbf{j}$, so $\|\mathbf{r}(t)\| = 1$ but $\|\mathbf{r}'(t)\| = 2t$.

42. True: If $\mathbf{v}\cdot\mathbf{v} = \text{constant}$, differentiate both sides to get $\mathbf{v}\cdot\mathbf{v}' + \mathbf{v}\cdot\mathbf{v}' = 2\mathbf{v}\cdot\mathbf{v}' = 0$, so $\mathbf{v}\cdot\mathbf{v}' = 0$.

43. True If $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j} + ct\mathbf{k}$, then $\mathbf{r}'(t) = -a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j} + c\mathbf{k}$ so

$$\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\| = \frac{-a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j} + c\mathbf{k}}{\sqrt{a^2\omega^2 + c^2}}$$

$\mathbf{T}'(t) = \dfrac{-a\omega^2\cos\omega t\,\mathbf{i} - a\omega^2\sin\omega t\,\mathbf{j}}{\sqrt{a\omega^2 + c^2}}$ which points directly to the *z*-axis. Therefore $\mathbf{N}(t) = \mathbf{T}'(t)/\|\mathbf{T}'(t)\|$ points directly to the *z*-axis.

44. False: Suppose **v**(*t*) = cos *t***i** + sin *t***j**, then $\|\mathbf{v}(t)\| = 1$ but **a**(*t*) = –sin *t***i** + cos *t***j** which is non-zero.

45. True: **T** depends only upon the shape of the curve, hence **N** and **B** also.

46. True: If **v** is perpendicular to **a**, then **T** is also perpendicular to **a**, so $\dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = a_T = \mathbf{T}\cdot\mathbf{a} = 0.$ Thus speed $= \dfrac{ds}{dt}$ is a constant.

47. False: If **r**(*t*) = *a* cos *t***i** + *a* sin *t***j** + *ct***k** then **v** is perpendicular to **a**, but the path of motion is a circular helix, not a circle.

48. False: The circular helix (see Problem 27) has constant curvature.

49. True: The curves are identical, although the motion of an object moving along the curves would be different.

50. False: At any time $0 < t < 1$, $\mathbf{r}_1(t) \ne \mathbf{r}_2(t)$

51. True: The parameterization affects only the rate at which the curve is traced out.

52. True: If a curve lies in a plane, then **T** and **N** will lie in the plane, so **T** × **N** = **B** will be a unit vector normal to the plane.

53. False: For **r**(*t*) = cos *t***i** + sin *t* **j**, $\|\mathbf{r}(t)\| = 1$, but $\mathbf{r}'(t) \ne \mathbf{0}$.

54. True: The plane passes through the origin so its intersection with the sphere is a great circle. The radius of the circle is 1, so is curvature is $\dfrac{1}{1} = 1$.

55. False: The graph of $\rho = 0$ is the origin.

56. False: It is a parabolic cylinder.

57. False: The origin, $\rho = 0$, has infinitely many spherical coordinates, since any value of θ and ϕ can be used.

Sample Test Problems

1. The center of the sphere is the midpoint $\left(\frac{-2+4}{2}, \frac{3+1}{2}, \frac{3+5}{2}\right) = (1, 2, 4)$ of the diameter. The radius is $r = \sqrt{(1+2)^2 + (2-3)^2 + (4-3)^2}$ $= \sqrt{9+1+1} = \sqrt{11}$. The equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-4)^2 = 11$

2. $(x^2 - 6x + 9) + (y^2 + 2y + 1) + (z^2 - 8z + 16)$
$= 9 + 1 + 16;\ (x-3)^2 + (y+1)^2 + (z-4)^2 = 26$
Center: (3, –1, 4); radius: $\sqrt{26}$

3. a. $3\langle 2,-5\rangle - 2\langle 1,1\rangle = \langle 6,-15\rangle - \langle 2,2\rangle = \langle 4,-17\rangle$

b. $\langle 2, -5\rangle \cdot \langle 1, 1\rangle = 2 + (-5) = -3$

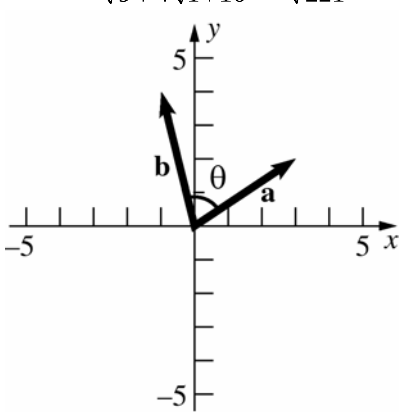
c. $\langle 2, -5\rangle \cdot (\langle 1, 1\rangle + \langle -6, 0\rangle) = \langle 2, -5\rangle \cdot \langle -5, 1\rangle$
$= -10 + (-5) = -15$

d. $(4\langle 2, -5\rangle + 5\langle 1, 1\rangle) \cdot 3\langle -6, 0\rangle$
$= \langle 13, -15\rangle \cdot \langle -18, 0\rangle = -234 + 0 = -234$

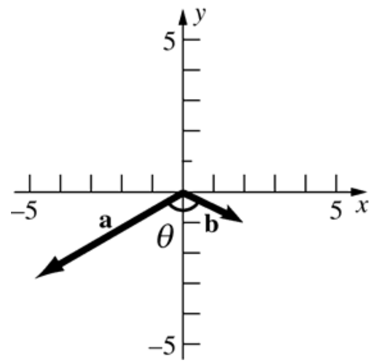
e. $\sqrt{36+0}\,\langle -6, 0\rangle \cdot \langle 1,-1\rangle = 6(-6+0) = -36$

f. $\langle -6, 0\rangle \cdot \langle -6, 0\rangle - \sqrt{36+0}$
$= (36+0) - 6 = 30$

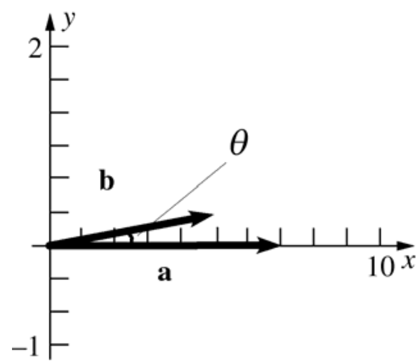
4. a. $\cos\theta = \frac{(3)(-1)+(2)(4)}{\sqrt{9+4}\sqrt{1+16}} = \frac{5}{\sqrt{221}} \approx 0.3363$



b. $\cos\theta = \frac{(-5)(2)+(-3)(-1)}{\sqrt{25+9}\sqrt{4+1}} = -\frac{7}{\sqrt{170}}$
≈ -0.5369



c. $\cos\theta = \frac{(7)(5)+(0)(1)}{\sqrt{49+0}\sqrt{25+1}} = \frac{5}{\sqrt{26}} \approx 0.9806$



5. a. $\mathbf{a} + \mathbf{b} + \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

b. $\mathbf{b} \bullet \mathbf{c} = (0)(3) + (1)(-1) + (-2)(4) = -9$

c. $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k})$
$= -2 - 6 - 6 = -14$

d. $\mathbf{b} \cdot \mathbf{c}$ is a scalar, and $\mathbf{a}$ crossed with a scalar doesn't exist.

e. $\|\mathbf{a} - \mathbf{b}\| = \|-\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}\|$
$= \sqrt{1^2 + 0^2 + 4^2} = \sqrt{17}$

f. From part (c), $\mathbf{b} \times \mathbf{c} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ so $\|\mathbf{b} \times \mathbf{c}\| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$.

6. **a.** $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{0+5-3}{(3\sqrt{3})(\sqrt{10})}$

≈ 0.121716

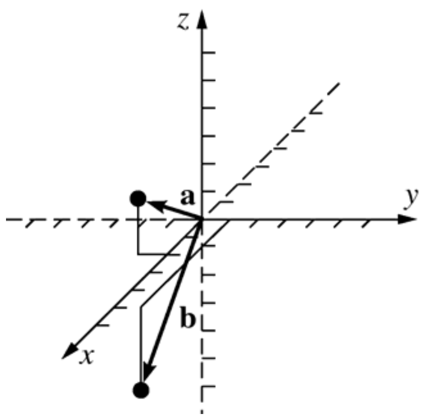
$\theta = \cos^{-1} 0.121716 \approx 83.009^\circ$

b. $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{-1-0+6}{(\sqrt{5})(\sqrt{11})}$

≈ 0.67200

$\theta = \cos^{-1} 0.67200 \approx 47.608^\circ$

7.



a. $\|\mathbf{a}\| = \sqrt{4+1+4} = 3;$

$\|\mathbf{b}\| = \sqrt{25+1+9} = \sqrt{35}$

b. $\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, direction cosines $\frac{2}{3}, -\frac{1}{3}$, and $\frac{2}{3}$.

$\frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{5}{\sqrt{35}}\mathbf{i} + \frac{1}{\sqrt{35}}\mathbf{j} - \frac{3}{\sqrt{35}}\mathbf{k}$, direction cosines $\frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}$

c. $\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

d. $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{10-1-6}{3\sqrt{35}}$

$= \frac{3}{3\sqrt{35}} = \frac{1}{\sqrt{35}}$

$\theta = \cos^{-1}\frac{1}{\sqrt{35}} \approx 1.4010 \approx 80.27^\circ$

8. **a.** $\langle -5,-5,5\rangle = -5\langle 1,1,-1\rangle$

b. $\langle 2,-1,1\rangle \times \langle 0, 5, 1\rangle = \langle 6,-2,10\rangle$

c. $\langle 2,-1,1\rangle \cdot \langle -7,1,-5\rangle = -20$

d. $\langle 2,-1,1\rangle \times \langle -7,1,-5\rangle = \langle 4,3,-5\rangle$

9. $c\langle 3,3,-1\rangle \times \langle -1,-2,4\rangle = c\langle 10,-11,-3\rangle$ for any c in $\mathbb{R}$.

10. Two vectors determined by the points are $\langle -1,7,-3\rangle$ and $\langle 3,-1,-3\rangle$. Then $\langle -1,7,-3\rangle \times \langle 3,-1,-3\rangle = -4\langle 6, 3, 5\rangle$ and $\langle 6, 3, 5\rangle$ are normal to the plane.
$\frac{\pm\langle 6, 3, 5\rangle}{\sqrt{36+9+25}} = \frac{\pm\langle 6, 3,5\rangle}{\sqrt{70}}$ are the unit vectors normal to the plane.

11. **a.** $y = 7$, since y must be a constant.

b. $x = -5$, since it is parallel to the yz-plane.

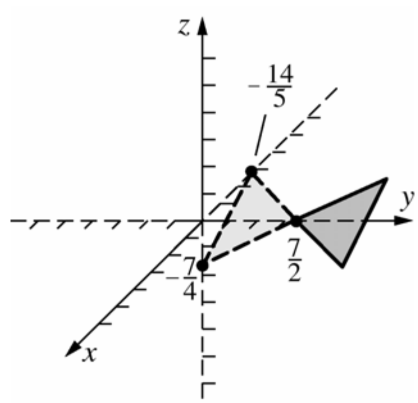
c. $z = -2$, since it is parallel to the xy-plane.

d. $3x - 4y + z = -45$, since it can be expressed as $3x - 4y + z = D$ and D must satisfy $3(-5) - 4(7) + (-2) = D$, so $D = -45$.

12. **a.** $\langle 4+1,1-5,1+7\rangle = \langle 5,-4,8\rangle$ is along the line, hence normal to the plane, which has equation $\langle x-2,\ y+4,\ z+5\rangle\cdot\langle 5,\ -4,\ 8\rangle = 0$.

b. $5(x-2) - 4(y+4) + 8(z+5) = 0$ or $5x - 4y + 8z = -14$

c.



13. If the planes are perpendicular, their normals will also be perpendicular. Thus $0 = \langle 1, 5, C\rangle \cdot \langle 4, -1, 1\rangle = 4 - 5 + C$, so $C = 1$.

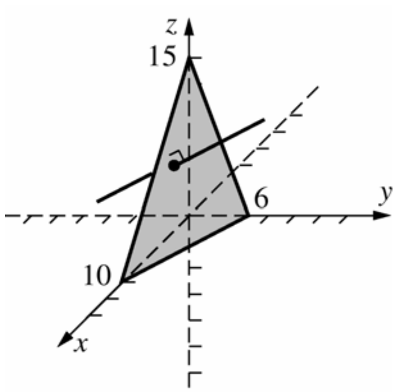
14. Two vectors in the same plane are $\langle 3, -2, -3\rangle$ and $\langle 3, 7, 0\rangle$. Their cross product, $3\langle 7, -3, 9\rangle$, is normal to the plane. An equation of the plane is $7(x-2) - 3(y-3) + 9(z+1) = 0$ or $7x - 3y + 9z = -4$.

15. A vector in the direction of the line is $\langle 8, 1, -8\rangle$. Parametric equations are $x = -2 + 8t$, $y = 1 + t$, $z = 5 - 8t$.

16. In the yz-plane, $x = 0$. Solve $-2y + 4z = 14$ and $2y - 5z = -30$, obtaining $y = 25$ and $z = 16$.
In the xz-plane, $y = 0$. Solve $x + 4z = 14$ and $-x - 5z = -30$, obtaining $x = -50$ and $z = 16$. Therefore, the points are $(0, 25, 16)$ and $(-50, 0, 16)$.

17. $(0, 25, 16)$ and $(-50, 0, 16)$ are on the line, so $\langle 50, 25, 0\rangle = 25\langle 2, 1, 0\rangle$ is in the direction of the line. Parametric equations are $x = 0 + 2t$, $y = 25 + 1t$, $z = 16 + 0t$ or $x = 2t$, $y = 25 + t$, $z = 16$.

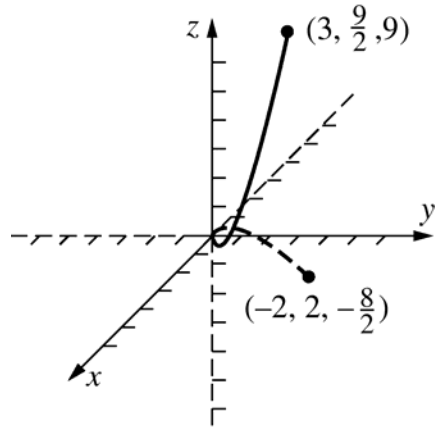
18. $\langle 3, 5, 2\rangle$ is normal to the plane, so is in the direction of the line. Symmetric equations of the line are $\frac{x-4}{3} = \frac{y-5}{5} = \frac{z-8}{2}$.



19. $\langle 5, -4, -3\rangle$ is a vector in the direction of the line, and $\langle 2, -2, 1\rangle$ is a position vector to the line. Then a vector equation of the line is $\mathbf{r}(t) = \langle 2, -2, 1\rangle + t\langle 5, -4, -3\rangle$.

20.

t	x	y	z
–2	–2	2	–8/3
–1	–1	1/2	–1/3
0	0	0	0
1	1	1/2	1/3
2	2	2	8/3
3	3	9/2	9



21. $\mathbf{r}'(t) = \langle 1, t, t^2\rangle$, $\mathbf{r}'(2) = \langle 1, 2, 4\rangle$ and $\mathbf{r}(2) = \left\langle 2, 2, \frac{8}{3}\right\rangle$. Symmetric equations for the tangent line are $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z-\frac{8}{3}}{4}$. Normal plane is $1(x-2) + 2(y-2) + 4\left(z - \frac{8}{3}\right) = 0$ or $3x + 6y + 12z = 50$.

22. $\mathbf{r}(t) = \langle t\cos t, t\sin t, 2t\rangle$;

$\mathbf{r}'(t) = \langle -t\sin t + \cos t,\ t\cos t + \sin t,\ 2\rangle$;

$\mathbf{r}''(t) = \langle -t\cos t - 2\sin t, -t\sin t + 2\cos t,\ 0\rangle$

$\mathbf{r}'\left(\frac{\pi,}{2}\right) = \left\langle -\frac{\pi}{2}, 1, 2\right\rangle; \mathbf{r}''\left(\frac{\pi}{2}\right) = \left\langle -2, -\frac{\pi}{2}, 0\right\rangle$

$\left|\mathbf{r}'\left(\frac{\pi}{2}\right)\right| = \frac{\sqrt{\pi^2+20}}{2}$;

$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{2}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{2}\right)\right\|} = \frac{\langle -\pi, 2, 4\rangle}{\sqrt{\pi^2+20}}$

23. $\mathbf{r}'(t) = e^t\langle \cos t + \sin t, -\sin t + \cos t, 1\rangle$

$\|\mathbf{r}'(t)\| = \sqrt{3}e^t$

Length is $\int_1^5 \sqrt{3}e^t\,dt = \left[\sqrt{3}e^t\right]_1^5$

$= \sqrt{3}(e^5 - e) \approx 252.3509.$

24. $-(\mathbf{F}_1 + \mathbf{F}_2) = -5\mathbf{i} - 9\mathbf{j}$

25. Let the wind vector be

$\mathbf{w} = \langle 100\cos 30°, 100\sin 30°\rangle$

$= \langle 50\sqrt{3}, 50\rangle.$

Let $\mathbf{p} = \langle p_1, p_2\rangle$ be the plane's air velocity vector.

We want $\mathbf{w} + \mathbf{p} = 450\,\mathbf{j} = \langle 0, 450\rangle$.

$\langle 50\sqrt{3}, 50\rangle + \langle p_1, p_2\rangle = \langle 0, 450\rangle$

$\Rightarrow 50\sqrt{3} + p_1 = 0, 50 + p_2 = 450$

$\Rightarrow p_1 = -50\sqrt{3}, p_2 = 400$

Therefore, $\mathbf{p} = \langle -50\sqrt{3}, 400\rangle$. The angle β formed with the vertical satisfies

$\cos\beta = \frac{\mathbf{p}\cdot\mathbf{j}}{\|\mathbf{p}\|\|\mathbf{j}\|} = \frac{400}{\sqrt{167,500}}$; $\beta \approx 12.22°$. Thus, the heading is N12.22°W. The air speed is $\|\mathbf{p}\| = \sqrt{167,500} \approx 409.27$ mi/h.

26. $\mathbf{r}(t) = \langle e^{2t}, e^{-t}\rangle; \mathbf{r}'(t) = \langle 2e^{2t}, -e^{-t}\rangle$

a. $\lim_{t\to 0}\langle e^{2t}, e^{-t}\rangle = \left\langle \lim_{t\to 0} e^{2t}, \lim_{t\to 0} e^{-t}\right\rangle = \langle 1, 1\rangle$

b. $\lim_{h\to 0}\frac{\mathbf{r}(0+h) - \mathbf{r}(0)}{h} = \mathbf{r}'(0) = \langle 2, -1\rangle$

c. $\int_0^{\ln 2}\langle e^{2t}, e^{-t}\rangle\,dt$

$= \left[\left\langle \left(\frac{1}{2}\right)e^{2t}, -e^{-t}\right\rangle\right]_0^{\ln 2}$

$= \left\langle 2, -\frac{1}{2}\right\rangle - \left\langle \frac{1}{2}, -1\right\rangle = \left\langle \frac{3}{2}, \frac{1}{2}\right\rangle$

d. $D_t[t\mathbf{r}(t)] = t\mathbf{r}'(t) + \mathbf{r}(t)$

$= t\langle 2e^{2t}, -e^{-t}\rangle + \langle e^{2t}, e^{-t}\rangle$

$= \langle e^{2t}(2t+1), e^{-t}(1-t)\rangle$

e. $D_t[\mathbf{r}(3t+10)] = [\mathbf{r}'(3t+10)](3)$

$= 3\langle 2e^{6t+20}, -e^{-3t-10}\rangle$

$= \langle 6e^{6t+20}, -3e^{-3t-10}\rangle$

f. $D_t[\mathbf{r}(t)\cdot\mathbf{r}'(t)] = D_t[2e^{4t} - e^{-2t}]$

$= 8e^{4t} + 2e^{-2t}$

27. a. $\mathbf{r}'(t) = \left\langle \frac{1}{t}, -6t\right\rangle; \mathbf{r}''(t) = \langle -t^{-2}, -6\rangle$

b. $\mathbf{r}'(t) = \langle \cos t, -2\sin 2t\rangle$;

$\mathbf{r}''(t) = \langle -\sin t, -4\cos 2t\rangle$

c. $\mathbf{r}'(t) = \langle \sec^2 t, -4t^3\rangle$;

$\mathbf{r}''(t) = \langle 2\sec^2 t\tan t, -12t^2\rangle$

28. $\mathbf{v}(t) = \langle e^t, -e^{-t}, 2\rangle$

$\mathbf{a}(t) = \langle e^t, e^{-t}, 0\rangle$; $\mathbf{v}(\ln 2) = \left\langle 2, -\frac{1}{2}, 2\right\rangle$

$\mathbf{a}(\ln 2) = \left\langle 2, \frac{1}{2}, 0\right\rangle$

$\kappa(\ln 2) = \frac{|\mathbf{v}(\ln 2)\times\mathbf{a}(\ln 2)|}{|\mathbf{v}(\ln 2)|^3} = \frac{|\langle -1, 4, 2\rangle|}{\left(\sqrt{\frac{33}{4}}\right)^3}$

$= \frac{8}{33^{3/2}}\sqrt{21} = 8\sqrt{\frac{21}{35937}} = \frac{8\sqrt{7}}{\sqrt{11979}}$

≈ 0.1934

29. $\mathbf{v}(t) = \left\langle 1,\ 2t,\ 3t^2 \right\rangle;\ \mathbf{a}(t) = \langle 0,\ 2,\ 6t \rangle$

$\mathbf{v}(1) = \langle 1,\ 2,\ 3 \rangle$

$\|\mathbf{v}(1)\| = \sqrt{14}$

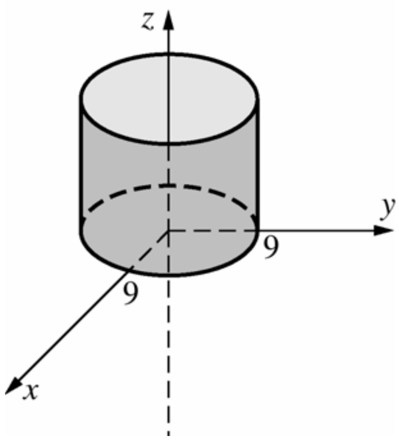
$\mathbf{a}(1) = \langle 0,\ 2,\ 6 \rangle$

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} = \frac{0+4+18}{\sqrt{14}} = \frac{22}{\sqrt{14}} \approx 5.880;$$

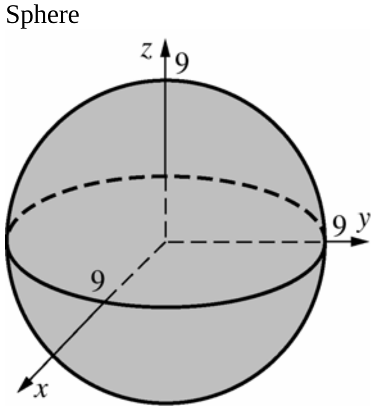
$$a_N = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \frac{|\langle 6,\ -6,\ 2 \rangle|}{\sqrt{14}}$$

$$= \frac{2\sqrt{19}}{\sqrt{14}} \approx 2.330$$

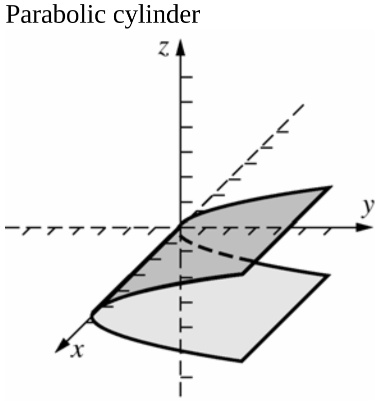
30. Circular cylinder



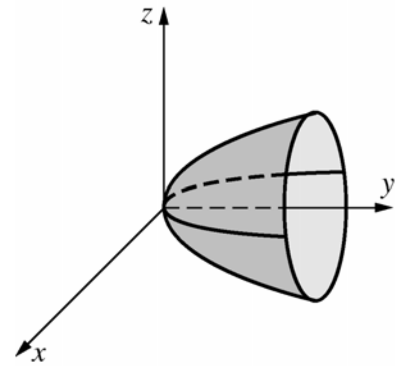
31. Sphere



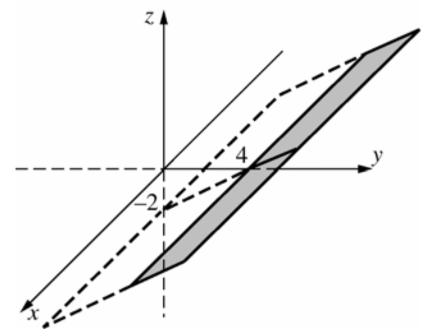
32. Parabolic cylinder



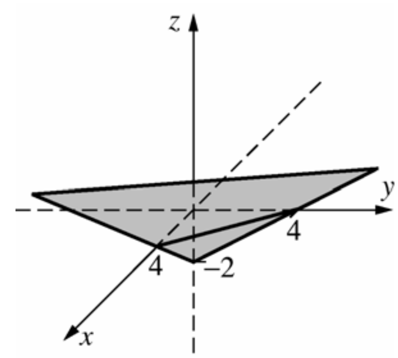
33. Circular paraboloid



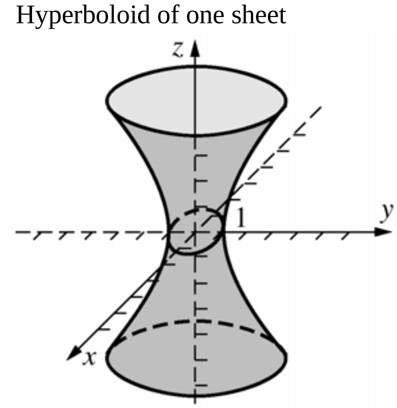
34. Plane



35. Plane

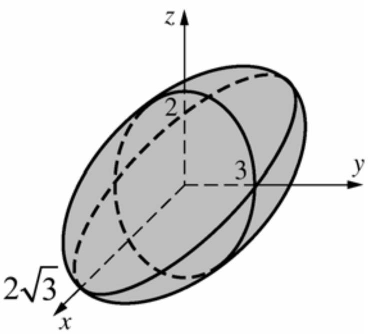


36. Hyperboloid of one sheet



37. $\dfrac{x^2}{12}+\dfrac{y^2}{9}+\dfrac{z^2}{4}=1$

Ellipsoid



38. The graph of $3x^2+4y^2+9z^2=-36$ is the empty set.

39. **a.** $r^2=9;\ r=3$

b. $(x^2+y^2)+3y^2=16$

$r^2+3r^2\sin^2\theta=16,\ r^2=\dfrac{16}{1+3\sin^2\theta}$

c. $r^2=9z$

d. $r^2+4z^2=10$

40. **a.** $x^2+y^2+z^2=9$

b. $x^2+z^2=4$

c. $r^2(\cos^2\theta-\sin^2\theta)+z^2=1;$

$x^2-y^2+z^2=1$

41. **a.** $\rho^2=4;\ \rho=2$

b. $x^2+y^2+z^2-2z^2=0;$

$\rho^2-2\rho^2\cos^2\phi=0;\quad \rho^2(1-2\cos^2\phi)=0;$

$1-2\cos^2\phi=0;\ \cos^2\phi=\dfrac{1}{2};\ \phi=\dfrac{\pi}{4}$ or

$\phi=\dfrac{3\pi}{4}.$

Any of the following (as well as others) would be acceptable:

$\left(\phi-\dfrac{\pi}{4}\right)\left(\phi-\dfrac{3\pi}{4}\right)=0$

$\cos^2\phi=\dfrac{1}{2}$

$\sec^2\phi=2$

$\tan^2\phi=1$

c. $2x^2-(x^2+y^2+z^2)=1;$

$2\rho^2\sin^2\phi\cos^2\theta-\rho^2=1;$

$\rho^2=\dfrac{1}{2\sin^2\phi\cos^2\theta-1}$

d. $x^2+y^2=z;$

$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta=\rho\cos\phi$

$\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)=\rho\cos\phi;$

$\rho\sin^2\phi=\cos\phi;\ \ \rho=\cot\phi\csc\phi$

(Note that when we divided through by ρ in part c and d we did not lose the pole since it is also a solution of the resulting equations.)

42. Cartesian coordinates are $\left(2\sqrt{2},\,2\sqrt{2},\,4\sqrt{3}\right)$ and $\left(\sqrt{2},\,\sqrt{6},\,-2\sqrt{2}\right)$. Distance

$\left[2+\left(2\sqrt{2}-\sqrt{6}\right)^2+\left(4\sqrt{3}+2\sqrt{2}\right)^2\right]^{1/2}\approx 9.8659.$

43. (2, 0, 0) is a point of the first plane. The distance between the planes is

$\dfrac{\left|2(2)-3(0)+\sqrt{3}(0)-9\right|}{\sqrt{4+9+3}}=\dfrac{5}{\sqrt{16}}=1.25$

44. $\langle 2,-4,1\rangle$ and $\langle 3,2,-5\rangle$ are normal to the respective planes. The acute angle between the two planes is the same as the acute angle θ between the normal vectors.

$\cos\theta=\dfrac{|6-8-5|}{\sqrt{21}\sqrt{38}}=\dfrac{7}{\sqrt{798}},$

so $\theta\approx 1.3204$ rad $\approx 75.65°$

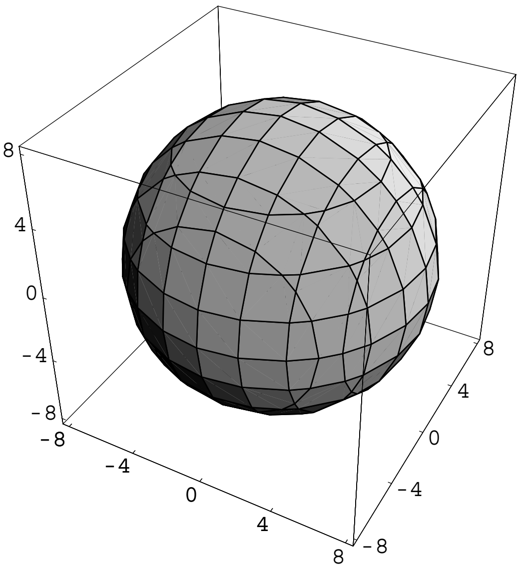
45. If speed $=\dfrac{ds}{dt}=c,$ a constant, then

$\mathbf{a}=\dfrac{d^2s}{dt^2}\mathbf{T}+\left(\dfrac{ds}{dt}\right)^2\kappa\mathbf{N}=c^2\kappa\mathbf{N}$ since $\dfrac{d^2s}{dt^2}=0.$

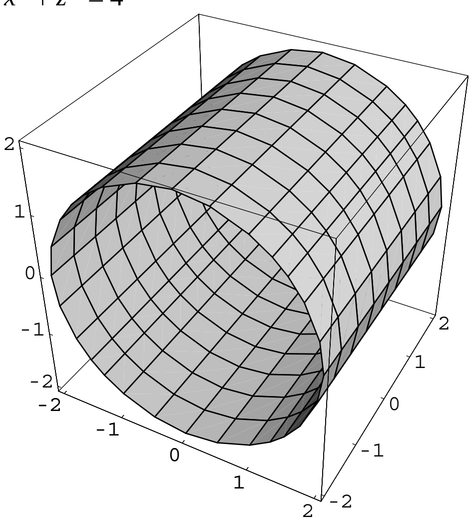
T is in the direction of **v**, while **N** is perpendicular to **T** and hence to **v** also. Thus, $\mathbf{a}=c^2\kappa\mathbf{N}$ is perpendicular to **v**.

Review and Preview

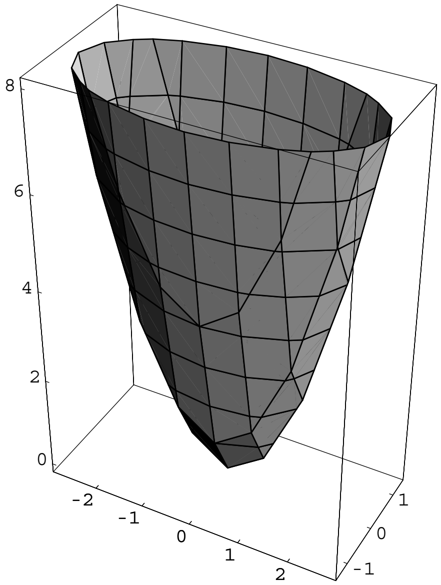
1. $x^2 + y^2 + z^2 = 64$



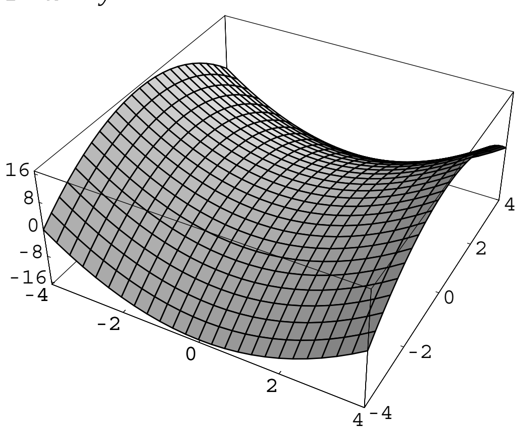
2. $x^2 + z^2 = 4$



3. $z = x^2 + 4y^2$



4. $z = x^2 - y^2$



5. **a.** $\frac{d}{dx}2x^3 = 6x^2$

b. $\frac{d}{dx}5x^3 = 15x^2$

c. $\frac{d}{dx}kx^3 = 3kx^2$

d. $\frac{d}{dx}ax^3 = 3ax^2$

6. **a.** $\frac{d}{dx}\sin 2x = 2\cos 2x$

b. $\frac{d}{dt}\sin 17t = 17\cos 17t$

c. $\frac{d}{dt}\sin at = a\cos at$

d. $\frac{d}{dt}\sin bt = b\cos bt$

7. **a.** $\frac{d}{da}\sin 2a = 2\cos 2a$

b. $\frac{d}{da}\sin 17a = 17\cos 17a$

c. $\frac{d}{da}\sin ta = t\cos ta$

d. $\frac{d}{da}\sin sa = s\cos sa$

8. **a.** $\frac{d}{dt}e^{4t+1} = 4e^{4t+1}$

b. $\frac{d}{dx}e^{-7x+4} = -7e^{-7x+4}$

c. $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$

d. $\frac{d}{dx}e^{tx+s} = te^{tx+s}$

9. $f(x) = \frac{1}{x^2-1}$ is both continuous and differentiable at $x = 2$ since rational functions are continuous and differentiable at every real number in their domain.

10. $f(x) = \tan x$ is not continuous at $x = \pi/2$ since $f(\pi/2)$ is undefined.

11. $f(x) = |x-4|$ is continuous at $x = 4$ since $\lim_{x\to 4}|x-4| = f(4) = 0.$ f is not differentiable at $x = 4$.

12. As x approaches 0 from the right, $1/x$ approaches $+\infty$, so $\sin(1/x)$ oscillates between -1 and 1 and $\lim_{x\to 0} f(x)$ does not exist. Since the limit at $x = 0$ does not exist, f is not continuous at $x = 0$. Consequently, f is not differentiable at $x = 0$.

13. $f'(x) = 3-3(x-1)^2$; $f'(x) = 0$ when $x = 0, 2$; $f''(x) = -6(x-1)$; Since $f''(2) = -6$, a local maximum occurs at $x = 2$. Since $f(0) = -1$, $f(2) = 5$, and $f(4) = -15$, the maximum value of f on [0, 4] is 5 while the minimum value is -15.

14. $$f'(x) = 4x^3 - 54x^2 + 226x - 288$$
$$= 2(2x-9)\left(x^2 - 9x + 16\right)$$

$f'(x) = 0$ when $x = \frac{9}{2}$ or $x = \frac{9\pm\sqrt{17}}{2}$ (using the quadratic formula).

In [2, 6], $f'(x) = 0$ when $x = \frac{9-\sqrt{17}}{2} \approx 2.438$ or $x = \frac{9}{2} = 4.5.$ $f''(x) = 12x^2 - 108x + 226.$ Since $f''(2.438) > 0,$ a local minimum occurs at $x = \frac{9-\sqrt{17}}{2}.$ Since $f''(4.5) < 0,$ a local maximum occurs at $x = 4.5$.

$f(2) = f(6) = 0,$ $f\left(\frac{9-\sqrt{17}}{2}\right) = -4,$ and $f(4.5) = 14.0625.$ Thus, the minimum value of f on [2, 6] is -4 and the maximum value of f on [2, 6] is 14.0625.

15. $S = 2\pi r^2 + 2\pi rh$

Since the volume is to be 8 cubic feet, we have

$$V = 8$$
$$\pi r^2 h = 8$$
$$h = \frac{8}{\pi r^2}$$

Substituting for h in our surface area equation gives us

$$S = 2\pi r^2 + 2\pi r\left(\frac{8}{\pi r^2}\right) = 2\pi r^2 + \frac{16}{r}$$

Thus, we can write S as a function of r:

$$S(r) = 2\pi r^2 + \frac{16}{r}$$

16. The area of two of the sides of the box will be $l \cdot h.$ Two other sides will have area $w \cdot h.$ The area of the base is $l \cdot w.$ Thus, the total cost, C, of the box will be $C = 2lh + 2wh + 3lw,$ where C is in dollars. Since $h = \frac{27}{lw}$, $C = \frac{54}{w} + \frac{54}{l} + 3lw.$

CHAPTER 12 Derivatives for Functions of Two or More Variables

12.1 Concepts Review

1. real-valued function of two real variables

2. level curve; contour map

3. concentric circles

4. parallel lines

Problem Set 12.1

1. **a.** 5

   b. 0

   c. 6

   d. $a^6 + a^2$

   e. $2x^2, \quad x \neq 0$

   f. Undefined

   The natural domain is the set of all (x, y) such that y is nonnegative.

2. **a.** 4

   b. 17

   c. $\dfrac{17}{16}$

   d. $1 + a^2, a \neq 0$

   e. $x^3 + x, x \neq 0$

   f. Undefined

   The natural domain is the set of all (x, y) such that x is nonzero.

3. **a.** $\sin(2\pi) = 0$

   b. $4\sin\left(\dfrac{\pi}{6}\right) = 2$

   c. $16\sin\left(\dfrac{\pi}{2}\right) = 16$

   d. $\pi^2 \sin(\pi^2) \approx -4.2469$

4. **a.** 6

   b. 12

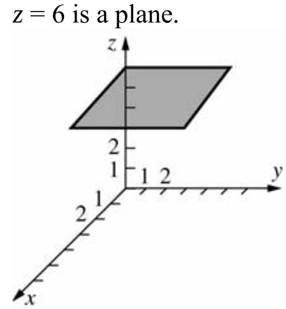
   c. 2

   d. $(3\cos 6)^{1/2} + 1.44 \approx 3.1372$

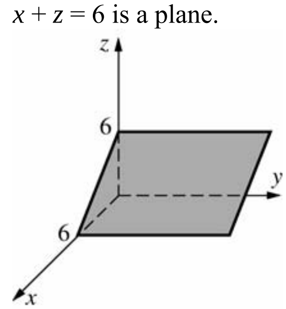
5. $F(t\cos t, \sec^2 t) = t^2 \cos^2 t \sec^2 t = t^2, \quad \cos\ t \neq 0$

6. $F(f(t), g(t)) = F(\ln t^2, e^{t/2})$
   $= \exp(\ln t^2) + (e^{t/2})^2 = t^2 + e^t, t \neq 0$

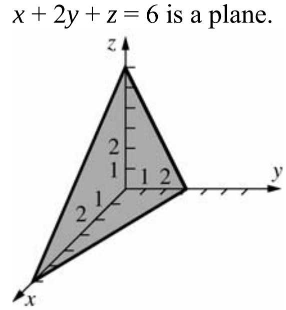
7. $z = 6$ is a plane.



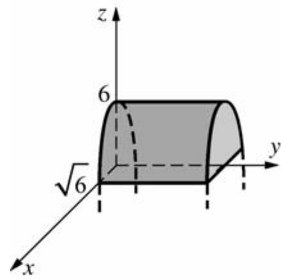
8. $x + z = 6$ is a plane.



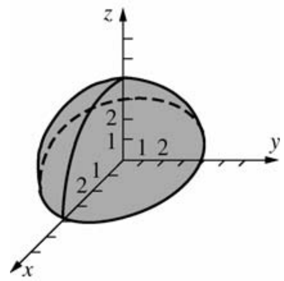
9. $x + 2y + z = 6$ is a plane.



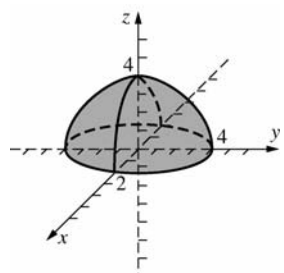
10. $z = 6 - x^2$ is a parabolic cylinder.



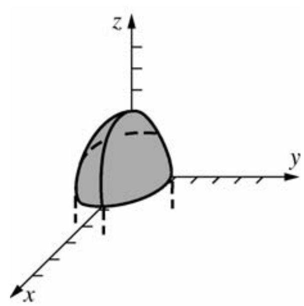
11. $x^2 + y^2 + z^2 = 16$, $z \geq 0$ is a hemisphere.



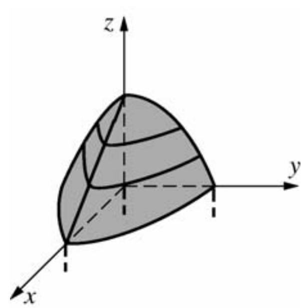
12. $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{16} = 1$, $z \geq 0$ is a hemi-ellipsoid.



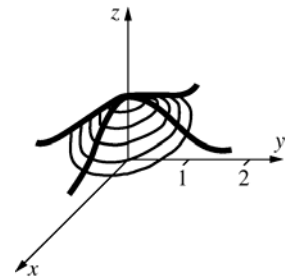
13. $z = 3 - x^2 - y^2$ is a paraboloid.



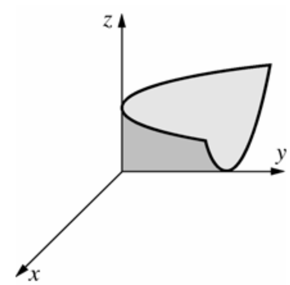
14. $z = 2 - x - y^2$



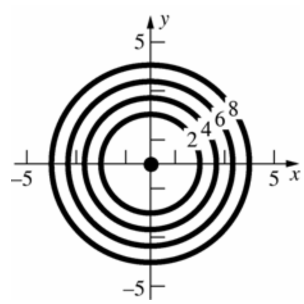
15. $z = \exp[-(x^2 + y^2)]$



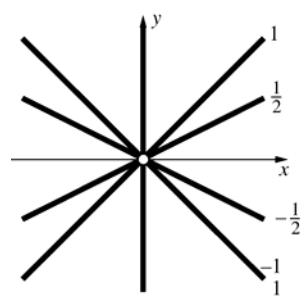
16. $z = \dfrac{x^2}{y}$, $y > 0$



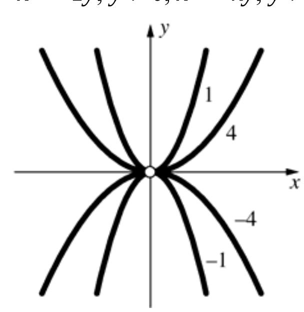
17. $x^2 + y^2 = 2z; x^2 + y^2 = 2k$



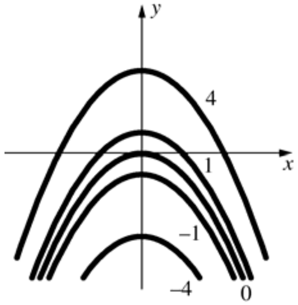
18. $x = zy, y \neq 0$; $x = ky$, $y \neq 0$



19. $x^2 = zy, y \neq 0; x^2 = ky, y \neq 0$



20. $x^2 = -(y - z); x^2 = -(y - k)$



21. $z = \dfrac{x^2 + 1}{x^2 + y^2}, k = 1, 2, 4$

$k = 1$: $y^2 = 1$ or $y = \pm 1$;
two parallel lines

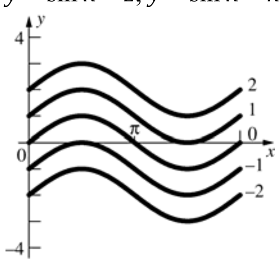
$k = 2$: $2x^2 + 2y^2 = x^2 + 1$

$\dfrac{x^2}{1} + \dfrac{y^2}{\frac{1}{2}} = 1$; ellipse

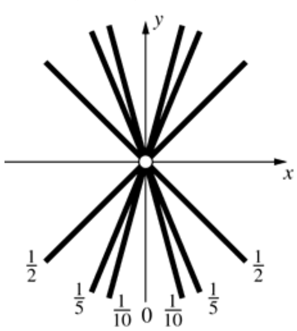
$k = 4$: $4x^2 + 4y^2 = x^2 + 1$

$\dfrac{x^2}{\frac{1}{3}} + \dfrac{y^2}{\frac{1}{4}} = 1$; ellipse

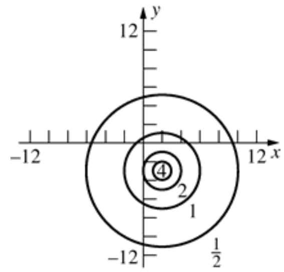
22. $y = \sin x + z; y = \sin x + k$



23. $x = 0$, if $T = 0$:
$y^2 = \left(\dfrac{1}{T} - 1\right)x^2$, if $y \neq 0$.



24. $(x - 2)^2 + (y + 3)^2 = \dfrac{16}{V^2}$



25. a. San Francisco and St. Louis had a temperature between 70 and 80 degrees Fahrenheit.

b. Drive northwest to get to cooler temperatures, and drive southeast to get warmer temperatures.

c. Since the level curve for 70 runs southwest to northeast, you could drive southwest or northeast and stay at about the same temperature.

26. a. The lowest barometric pressure, 1000 millibars and under, occurred in the region of the Great Lakes, specifically near Wisconsin. The highest barometric pressure, 1025 millibars and over, occurred on the east coast, from Massachusetts to South Carolina.

b. Driving northwest would take you to lower barometric pressure, and driving southeast would take you to higher barometric pressure.

c. Since near St. Louis the level curves run southwest to northeast, you could drive southwest or northeast and stay at about the same barometric pressure.

27. $x^2 + y^2 + z^2 \geq 16$; the set of all points on and outside the sphere of radius 4 that is centered at the origin

28. The set of all points inside (the part containing the *z*-axis) and on the hyperboloid of one sheet; $\dfrac{x^2}{9} + \dfrac{y^2}{9} - \dfrac{z^2}{9} = 1.$

29. $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{1} \leq 1$; points inside and on the ellipsoid

30. Points inside (the part containing the z-axis) or on the hyperboloid of one sheet, $\frac{x^2}{9}+\frac{y^2}{9}-\frac{z^2}{16}=1$, excluding points on the coordinate planes

31. Since the argument to the natural logarithm function must be positive, we must have $x^2+y^2+z^2>0$. This is true for all (x,y,z) except $(x,y,z)=(0,0,0)$. The domain consists all points in $\mathbb{R}^3$ except the origin.

32. Since the argument to the natural logarithm function must be positive, we must have $xy>0$. This occurs when the ordered pair (x,y) is in the first quadrant or the third quadrant of the xy-plane. There is no restriction on z. Thus, the domain consists of all points (x,y,z) such that x and y are both positive or both negative.

33. $x^2+y^2+z^2=k,\ k>0$; set of all spheres centered at the origin

34. $100x^2+16y^2+25z^2=k,\ k>0$;
$\frac{x^2}{\frac{k}{100}}+\frac{y^2}{\frac{k}{16}}+\frac{z^2}{\frac{k}{25}}=1$; set of all ellipsoids centered at origin such that their axes have ratio $\left(\frac{1}{10}\right):\left(\frac{1}{4}\right):\left(\frac{1}{5}\right)$ or 2:5:4.

35. $\frac{x^2}{\frac{1}{16}}+\frac{y^2}{\frac{1}{4}}-\frac{z^2}{1}=k$; the elliptic cone
$\frac{x^2}{9}+\frac{y^2}{9}=\frac{z^2}{16}$ and all hyperboloids (one and two sheets) with z-axis for axis such that a:b:c is $\left(\frac{1}{4}\right):\left(\frac{1}{4}\right):\left(\frac{1}{3}\right)$ or 3:3:4.

36. $\frac{x^2}{\frac{1}{9}}-\frac{y^2}{\frac{1}{4}}-\frac{z^2}{1}=k$; the elliptical cone
$\frac{y^2}{9}+\frac{z^2}{36}=\frac{x^2}{4}$ and all hyperboloids (one and two sheets) with x-axis for axis such that a:b:c is $\left(\frac{1}{3}\right):\left(\frac{1}{2}\right):1$ or 2:3:6

37. $4x^2-9y^2=k,\ k$ in R; $\frac{x^2}{\frac{k}{4}}-\frac{y^2}{\frac{k}{9}}=1$, if $k\neq 0$;
planes $y=\pm\frac{2x}{3}$ (for $k=0$) and all hyperbolic cylinders parallel to the z-axis such that the ratio a:b is $\left(\frac{1}{2}\right):\left(\frac{1}{3}\right)$ or 3:2 (where a is associated with the x-term)

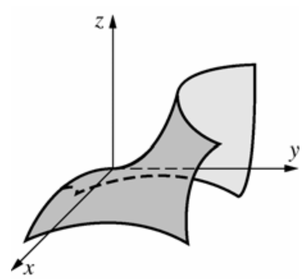
38. $e^{x^2+y^2+z^2}=k,\ k>0$
$x^2+y^2+z^2=\ln k$
concentric circles centered at the origin.

39. **a.** All (w,x,y,z) except $(0,0,0,0)$, which would cause division by 0.

b. All $(x_1,x_2,\ldots,x_n)$ in n-space.

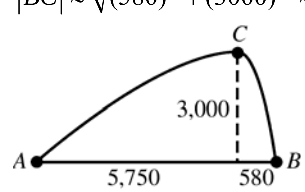
c. All $(x_1,x_2,\ldots,x_n)$ that satisfy $x_1^2+x_2^2+\cdots+x_n^2\le 1$; other values of $(x_1,x_2,\ldots,x_n)$ would lead to the square root of a negative number.

40. If $z=0$, then $x=0$ or $x=\pm\sqrt{3}y$.



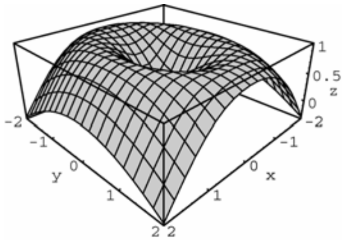
41. **a.** AC is the least steep path and BC is the most steep path between A and C since the level curves are farthest apart along AC and closest together along BC.

b. $|AC|\approx\sqrt{(5750)^2+(3000)^2}\approx 6490$ ft
$|BC|\approx\sqrt{(580)^2+(3000)^2}\approx 3060$ ft

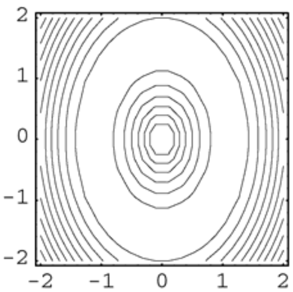


42. Completing the squares on x and y yields the equivalent equation
$f(x,y)+25.25=(x-0.5)^2+3(y+2)^2$, an elliptic paraboloid.

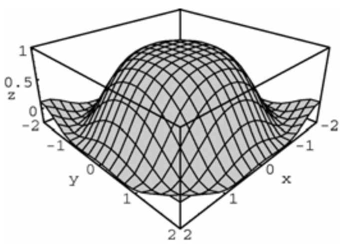
43.



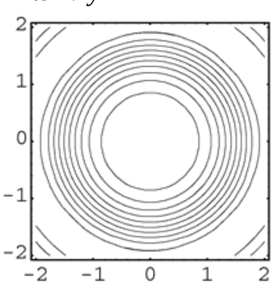
$\sin\sqrt{2x^2 + y^2}$



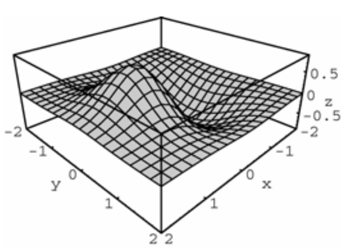
44.



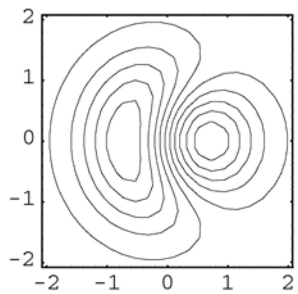
$\dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$



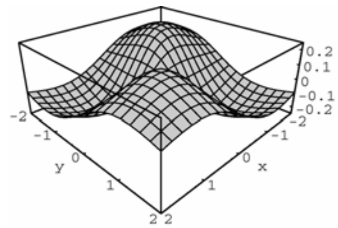
45.



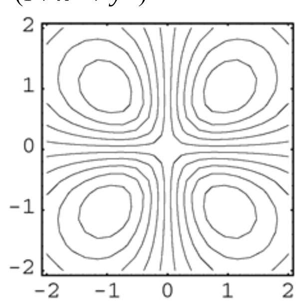
$(2x - y^2)\exp(-x^2 - y^2)$



46.



$\dfrac{\sin x \sin y}{(1 + x^2 + y^2)}$



12.2 Concepts Review

1. $\lim\limits_{h \to 0} \dfrac{[(f(x_0 + h,\ y_0) - f(x_0,\ y_0)]}{h}$; partial derivative of f with respect to x

2. 5; 1

3. $\dfrac{\partial^2 f}{\partial y \partial x}$

4. 0

Problem Set 12.2

1. $f_x(x,\ y) = 8(2x - y)^3$; $f_y(x,\ y) = -4(2x - y)^3$

2. $f_x(x,\ y) = 6(4x - y^2)^{1/2}$;
$f_y(x,\ y) = -3y(4x - y^2)^{1/2}$

3. $f_x(x, y) = \frac{(xy)(2x) - (x^2 - y^2)(y)}{(xy)^2} = \frac{x^2 + y^2}{x^2 y}$

$f_y(x, y) = \frac{(xy)(-2y) - (x^2 - y^2)(x)}{(xy)^2}$

$= -\frac{(x^2 + y^2)}{xy^2}$

4. $f_x(x,y) = e^x \cos y$; $f_y(x, y) = -e^x \sin y$

5. $f_x(x, y) = e^y \cos x$; $f_y(x, y) = e^y \sin x$

6. $f_x(x, y) = \left(-\frac{1}{3}\right)(3x^2 + y^2)^{-4/3}(6x)$

$= -2x(3x^2 + y^2)^{-4/3}$;

$f_y(x, y) = \left(-\frac{1}{3}\right)(3x^2 + y^2)^{-4/3}(2y)$

$= \left(-\frac{2y}{3}\right)(3x^2 + y^2)^{-4/3}$

7. $f_x(x, y) = x(x^2 - y^2)^{-1/2}$;

$f_y(x, y) = -y(x^2 - y^2)^{-1/2}$

8. $f_u(u, v) = ve^{uv}$; $f_v(u, v) = ue^{uv}$

9. $g_x(x, y) = -ye^{-xy}$; $g_y(x, y) = -xe^{-xy}$

10. $f_s(s, t) = 2s(s^2 - t^2)^{-1}$;

$f_t(s, t) = -2t(s^2 - t^2)^{-1}$

11. $f_x(x, y) = 4[1 + (4x - 7y)^2]^{-1}$;

$f_y(x, y) = -7[1 + (4x - 7y)^2]^{-1}$

12. $F_w(w, z) = w\frac{1}{\sqrt{1 - \left(\frac{w}{z}\right)^2}}\left(\frac{1}{z}\right) + \sin^{-1}\left(\frac{w}{z}\right)$

$= \frac{\frac{w}{z}}{\sqrt{1 - \left(\frac{w}{z}\right)^2}} + \sin^{-1}\left(\frac{w}{z}\right)$;

$F_z = (w, z) = w\frac{1}{\sqrt{1 - \left(\frac{w}{z}\right)^2}}\left(-\frac{w}{z^2}\right) = \frac{-\left(\frac{w}{z}\right)^2}{\sqrt{1 - \left(\frac{w}{z}\right)^2}}$

13. $f_x(x, y) = -2xy\sin(x^2 + y^2)$;

$f_y(x, y) = -2y^2 \sin(x^2 + y^2) + \cos(x^2 + y^2)$

14. $f_s(s, t) = -2se^{t^2 - s^2}$; $f_s(s, t) = 2te^{t^2 - s^2}$

15. $F_x(x, y) = 2\cos x \cos y$; $F_y(x, y) = -2\sin x \sin y$

16. $f_r(r, \theta) = 9r^2 \cos 2\theta$; $f_\theta(r, \theta) = -6r^3 \sin 2\theta$

17. $f_x(x, y) = 4xy^3 - 3x^2 y^5$;

$f_{xy}(x, y) = 12xy^2 - 15x^2 y^4$

$f_y(x, y) = 6x^2 y^2 - 5x^3 y^4$;

$f_{yx}(x, y) = 12xy^2 - 15x^2 y^4$

18. $f_x(x, y) = 5(x^3 + y^2)^4(3x^2)$;

$f_{xy}(x, y) = 60x^2(x^3 + y^2)^3(2y)$

$= 120x^2 y(x^3 + y^2)^3$

$f_y(x, y) = 5(x^3 + y^2)^4(2y)$;

$f_{yx}(x, y) = 40y(x^3 + y^2)^3(3x^2)$

$= 120x^2 y(x^3 + y^2)^3$

19. $f_x(x, y) = 6e^{2x} \cos y$; $f_{xy}(x, y) = -6e^{2x} \sin y$

$f_y(x, y) = -3e^{2x} \sin y$; $f_{yx}(x, y) = -6e^{2x} \sin y$

20. $f_x(x, y) = y(1 + x^2 y^2)^{-1}$;

$f_{xy}(x, y) = (1 - x^2 y^2)(1 + x^2 y^2)^{-2}$

$f_x(x, y) = x(1 + x^2 y^2)^{-1}$;

$f_{xy}(x, y) = (1 - x^2 y^2)(1 + x^2 y^2)^{-2}$

21. $F_x(x, y) = \frac{(xy)(2) - (2x - y)(y)}{(xy)^2} = \frac{y^2}{x^2 y^2} = \frac{1}{x^2}$;

$F_x(3, -2) = \frac{1}{9}$

$F_y(x, y) = \frac{(xy)(-1) - (2x - y)(x)}{(xy)^2} = \frac{-2x^2}{x^2 y^2} = -\frac{2}{x^2}$;

$F_y(3, -2) = -\frac{1}{2}$

22. $F_x(x, y) = (2x + y)(x^2 + xy + y^2)^{-1}$;

$F_x(-1, 4) = \frac{2}{13} \approx 0.1538$

$F_y(x, y) = (x + 2y)(x^2 + xy + y^2)^{-1}$;

$F_y(-1, 4) = \frac{7}{13} \approx 0.5385$

23. $f_x(x, y) = -y^2(x^2 + y^4)^{-1};$

$f_x\left(\sqrt{5}, -2\right) = -\frac{4}{21} \approx -0.1905$

$f_y(x, y) = 2xy(x^2 + y^4)^{-1};$

$f_y\left(\sqrt{5}, -2\right) = -\frac{4\sqrt{5}}{21} \approx -0.4259$

24. $f_x(x, y) = e^y \sinh x;$

$f_x(-1, 1) = e \sinh(-1) \approx -3.1945$

$f_y(x, y) = e^y \cosh x;$

$f_y(-1, 1) = e \cosh(-1) \approx 4.1945$

25. Let $z = f(x, y) = \frac{x^2}{9} + \frac{y^2}{4}.$

$f_y(x, y) = \frac{y}{2}$

The slope is $f_y(3, 2) = 1.$

26. Let $z = f(x, y) = (1/3)(36 - 9x^2 - 4y^2)^{1/2}.$

$f_y(x, y) = \left(-\frac{4}{3}\right) y(36 - 9x^2 - y^2)^{-1/2}$

The slope is $f_y(1, -2) = \frac{8}{3\sqrt{11}} \approx 0.8040.$

27. $z = f(x, y) = \left(\frac{1}{2}\right)(9x^2 + 9y^2 - 36)^{1/2}$

$f_x(x, y) = \frac{9x}{2(9x^2 + 9y^2 - 36)^{1/2}}$

$f_x(2, 1) = 3$

28. $z = f(x, y) = \left(\frac{5}{4}\right)(16 - x^2)^{1/2}.$

$f_x(x, y) = \left(-\frac{5}{4}\right) x(16 - x^2)^{-1/2}$

$f_x(2, 3) = -\frac{5}{4\sqrt{3}} \approx -0.7217$

29. $V_r(r, h) = 2\pi rh;$

$V_r(6, 10) = 120\pi \approx 376.99$ in.^{2}

30. $T_y(x, y) = 3y^2;$ $T_y(3, 2) = 12$ degrees per ft

31. $P(V, T) = \frac{kT}{V}$

$P_T(V, T) = \frac{k}{V};$

$P_T(100, 300) = \frac{k}{100}$ lb/in.^{2} per degree

32. $V[P_V(V, T)] + T[P_T(V, T)]$

$= V(-kTV^{-2}) + T(kV^{-1}) = 0$

$P_V V_T T_P = \left(-\frac{kT}{V^2}\right)\left(\frac{k}{P}\right)\left(\frac{V}{k}\right) = -\frac{kT}{PV} = -\frac{PV}{PV} = -1$

33. $f_x(x, y) = 3x^2 y - y^3;$ $f_{xx}(x, y) = 6xy;$

$f_y(x, y) = x^3 - 3xy^2;$ $f_{yy}(x, y) = -6xy$

Therefore, $f_{xx}(x, y) + f_{yy}(x, y) = 0.$

34. $f_x(x, y) = 2x(x^2 + y^2)^{-1};$

$f_{xx}(x, y) = -2(x^2 - y^2)(x^2 + y^2)^{-1}$

$f_y(x, y) = 2y(x^2 + y^2)^{-1};$

$f_{yy}(x, y) = 2(x^2 - y^2)(x^2 + y^2)^{-1}$

35. $F_y(x, y) = 15x^4 y^4 - 6x^2 y^2;$

$F_{yy}(x, y) = 60x^4 y^3 - 12x^2 y;$

$F_{yyy}(x, y) = 180x^4 y^2 - 12x^2$

36. $f_x(x, y) = [-\sin(2x^2 - y^2)](4x)$

$= -4x \sin(2x^2 - y^2)$

$f_{xx}(x, y) = (-4x)[\cos(2x^2 - y^2)](4x)$

$+ [\sin(2x^2 - y^2)](-4)$

$f_{xxy}(x, y) = -16x^2[-\sin(2x^2 - y^2)](-2y)$

$- 4[\cos(2x^2 - y^2)](-2y)$

$= -32x^2 y \sin(2x^2 - y^2) + 8y \cos(2x^2 - y^2)$

37. **a.** $\frac{\partial^3 f}{\partial y^3}$

b. $\frac{\partial^3 y}{\partial y \partial x^2}$

c. $\frac{\partial^4 y}{\partial y^3 \partial x}$

38. a. f_{yxx}

b. f_{yyxx}

c. f_{yyxxx}

39. a. $f_x(x, y, z) = 6xy - yz$

b. $f_y(x, y, z) = 3x^2 - xz + 2yz^2$;
$f_y(0, 1, 2) = 8$

c. Using the result in a, $f_{xy}(x, y, z) = 6x - z$.

40. a. $12x^2(x^3 + y^2 + z)^3$

b. $f_y(x, y, z) = 8y(x^3 + y^2 + z)^3$;
$f_y(0, 1, 1) = 64$

c. $f_z(x, y, z) = 4(x^3 + y^2 + z)^3$;
$f_{zz}(x, y, z) = 12(x^2 + y^2 + z)^2$

41. $f_x(x, y, x) = -yze^{-xyz} - y(xy - z^2)^{-1}$

42. $f_x(x, y, z) = \left(\frac{1}{2}\right)\left(\frac{xy}{z}\right)^{-1/2}\left(\frac{y}{z}\right)$;

$f_x(-2, -1, 8) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^{-1/2}\left(-\frac{1}{8}\right) = -\frac{1}{8}$

43. If $f(x, y) = x^4 + xy^3 + 12$, $f_y(x, y) = 3xy^2$; $f_y(1, -2) = 12$. Therefore, along the tangent line $\Delta y = 1 \Rightarrow \Delta z = 12$, so $\langle 0, 1, 12 \rangle$ is a tangent vector (since $\Delta x = 0$). Then parametric equations of the tangent line are $\left\{\begin{array}{c} x = 1 \\ y = -2 + t \\ z = 5 + 12t \end{array}\right\}$. Then the point of *xy*-plane at which the bee hits is (1, 0, 29) [since $y = 0 \Rightarrow t = 2 \Rightarrow x = 1, z = 29$].

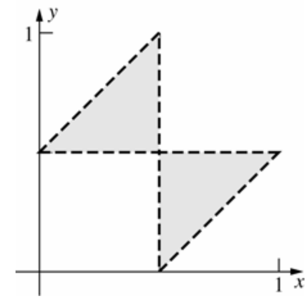
44. The largest rectangle that can be contained in the circle is a square of diameter length 20. The edge of such a square has length $10\sqrt{2}$, so its area is 200. Therefore, the domain of *A* is $\{(x, y) : 0 \le x^2 + y^2 < 400\}$, and the range is (0, 200].

45. Domain: (Case $x < y$)
The lengths of the sides are then x, $y - x$, and $1 - y$. The sum of the lengths of any two sides must be greater than the length of the remaining side, leading to three inequalities:

$x + (y - x) > 1 - y \Rightarrow y > \frac{1}{2}$

$(y - x) + (1 - y) > x \Rightarrow x < \frac{1}{2}$

$x + (1 - y) > y - x \Rightarrow y < x + \frac{1}{2}$



The case for $y < x$ yields similar inequalities (x and y interchanged). The graph of D_A, the domain of *A* is given above. In set notation it is

$$D_A = \left\{(x, y) : x < \frac{1}{2}, y > \frac{1}{2}, y < x + \frac{1}{2}\right\}$$
$$\cup \left\{(x, y) : x > \frac{1}{2}, y < \frac{1}{2}, x < y + \frac{1}{2}\right\}.$$

Range: The area is greater than zero but can be arbitrarily close to zero since one side can be arbitrarily small and the other two sides are bounded above. It seems that the area would be largest when the triangle is equilateral. An equilateral triangle with sides equal to $\frac{1}{3}$ has area $\frac{\sqrt{3}}{36}$. Hence the range of *A* is $\left(0, \frac{\sqrt{3}}{36}\right]$. (In Sections 8 and 9 of this chapter methods will be presented which will make it easy to prove that the largest value of *A* will occur when the triangle is equilateral.)

46. a. $u = \cos(x)\cos(ct)$:
$u_x = -\sin(x)\cos(ct)$; $u_t = -c\cos(x)\sin(ct)$
$u_{xx} = -\cos(x)\cos(ct)$
$u_{tt} = -c^2\cos(x)\cos(ct)$
Therefore, $c^2 u_{xx} = u_{tt}$.

$u = e^x\cosh(ct)$:
$u_x = e^x\cosh(ct), u_t = ce^x\sinh(ct)$
$u_{xx} = e^x\cosh(ct), u_{tt} = c^2e^x\cosh(ct)$
Therefore, $c^2 u_{xx} = u_{tt}$.

b. $u = e^{-ct}\sin(x)$:

$u_x = e^{-ct}\cos x$

$u_{xx} = -e^{-ct}\sin x$

$u_t = -ce^{ct}\sin x$

Therefore, $cu_{xx} = u_t$.

$u = t^{-1/2}e^{-x^2/4ct}$:

$$u_x = t^{-1/2}e^{-x^2/4ct}\left(-\frac{x}{2ct}\right)$$

$$u_{xx} = \frac{(x^2 - 2ct)}{(4c^2t^{5/2}e^{x^2/4ct})}$$

$$u_t = \frac{(x^2 - 2ct)}{(4ct^{5/2}e^{x^2/4ct})}$$

Therefore, $cu_{xx} = u_t$.

47. a. Moving parallel to the y-axis from the point (1, 1) to the nearest level curve and approximating $\frac{\Delta z}{\Delta y}$, we obtain

$$f_y(1, 1) = \frac{4-5}{1.25-1} = -4.$$

b. Moving parallel to the x-axis from the point (–4, 2) to the nearest level curve and approximating $\frac{\Delta z}{\Delta x}$, we obtain

$$f_x(-4, 2) \approx \frac{1-0}{-2.5-(-4)} = \frac{2}{3}.$$

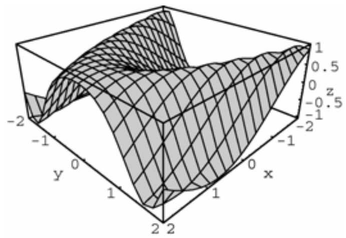
c. Moving parallel to the x-axis from the point (–5, –2) to the nearest level curve and approximately $\frac{\Delta z}{\Delta x}$, we obtain

$$f_x(-4, -5) \approx \frac{1-0}{-2.5-(-5)} = \frac{2}{5}.$$

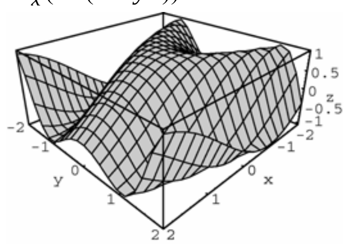
d. Moving parallel to the y-axis from the point (0, –2) to the nearest level curve and approximating $\frac{\Delta z}{\Delta y}$, we obtain

$$f_y(0, 2) \approx \frac{0-1}{\frac{-19}{8}-(-2)} = \frac{8}{3}.$$

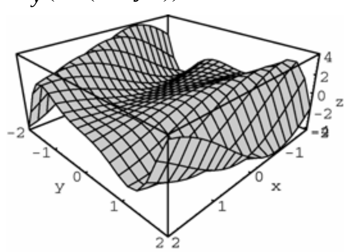
48. a. $\sin(x + y^2)$



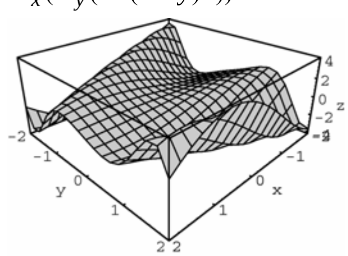
b. $D_x(\sin(x + y^2))$



c. $D_y(\sin(x + y^2))$



d. $D_x(D_y(\sin(x + y)^2))$



49. a. $f_y(x, y, z)$

$$= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

b. $f_z(x, y, z)$

$$= \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

c. $G_x(w, x, y, z)$

$$= \lim_{\Delta x \to 0} \frac{G(w, x + \Delta x, y, z) - G(w, x, y, z)}{\Delta x}$$

d. $\frac{\partial}{\partial z}\lambda(x,y,z,t)$

$$= \lim_{\Delta z \to 0} \frac{\lambda(x,y,z+\Delta z,t)-\lambda(x,y,z,t)}{\Delta z}$$

e. $\frac{\partial}{\partial b_2} S(b_0, b_1, b_2, \ldots, b_n) =$

$$= \lim_{\Delta b_2 \to 0} \left(\frac{S(b_0, b_1, b_2 + \Delta b_2, \ldots, b_n) - S(b_0, b_1, b_2, \ldots, b_n)}{\Delta b_2} \right)$$

50. a. $\frac{\partial}{\partial w}(\sin w \sin x \cos y \cos z)$

$= \cos w \sin x \cos y \cos z$

b. $\frac{\partial}{\partial x}\left[x \ln(wxyz)\right] = x \cdot \frac{wyz}{wxyz} + 1 \cdot \ln(wxyz)$

$= 1 + \ln(wxyz)$

c. $\lambda_t(x,y,z,t)$

$$= \frac{(1+xyzt)\cos x - t(\cos x)xyz}{(1+xyzt)^2}$$

$$= \frac{\cos x}{(1+xyzt)^2}$$

12.3 Concepts Review

1. 3; (x, y) approaches $(1, 2)$.

2. $\lim_{(x,\,y)\to(1,\,2)} f(x,\,y) = f(1,\,2)$

3. contained in S

4. an interior point of S; boundary points

Problem Set 12.3

1. –18

2. 3

3. $\lim_{(x,\,y)\to(2,\,\pi)} \left[x\cos^2 xy - \sin\left(\frac{xy}{3}\right)\right]$

$$= 2\cos^2 2\pi - \sin\left(\frac{2\pi}{3}\right) = 2 - \frac{\sqrt{3}}{2} \approx 1.1340$$

4. The limit does not exist because of Theorem A. The function is a rational function, but the limit of the denominator is 0, while the limit of the numerator is -1.

5. $-\frac{5}{2}$

6. -1

7. 1

8. $\lim_{(x,\,y)\to(0,\,0)} \frac{\tan(x^2+y^2)}{(x^2+y^2)}$

$$= \lim_{(x,\,y)\to(0,\,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} \frac{1}{\cos(x^2+y^2)}$$

$= (1)(1) = 1$

9. The limit does not exist since the function is not defined anywhere along the line $y = x$. That is, there is no neighborhood of the origin in which the function is defined everywhere except possibly at the origin.

10. $\lim_{(x,\,y)\to(0,\,0)} \frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2}$

$$= \lim_{(x,\,y)\to(0,\,0)} (x^2-y^2) = 0$$

11. Changing to polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta \cdot r\sin\theta}{r}$$

$$= \lim_{r\to 0} r\cos\theta \cdot \sin\theta = 0$$

12. If (x, y) approaches (0, 0) along the line $y = x$,

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{(x^2+x^2)^2} = \lim_{(x,x)\to(0,0)} \frac{1}{4x^2} = +\infty$$

Thus, the limit does not exist.

13. Use polar coordinates.

$$\frac{x^{7/3}}{x^2+y^2} = \frac{r^{7/3}(\cos\theta)^{7/3}}{r^2} = r^{1/3}(\cos\theta)^{7/3}$$

$r^{1/3}(\cos\theta)^{7/3} \to 0$ as $r \to 0$, so the limit is 0.

14. Changing to polar coordinates,

$$\lim_{r\to 0} r^2 \cos\theta \sin\theta \cdot \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0} r^2 \cos\theta \sin\theta \cos 2\theta = 0$$

15. $f(x,\ y) = \dfrac{r^4 \cos^2\theta \sin^2\theta}{r^2\cos^2\theta + r^4 \sin^4\theta}$

$$= r^2\left(\frac{\cos^2\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}\right)$$

If $\cos\theta = 0$, then $f(x,y) = 0$. If $\cos\theta \neq 0$, hen this converges to 0 as $r \to 0$. Thus the limit is 0.

16. As (x, y) approaches (0,0) along x = y^2,

$\lim\limits_{(x,x)\to(0,0)} \dfrac{y^4}{y^4 + y^4} = \dfrac{1}{2}$. Along the x-axis,

however, $\lim\limits_{(x,0)\to(0,0)} \dfrac{0}{x^2} = 0.$ Thus, the limit does not exist.

17. $f(x, y)$ is continuous for all (x, y) since for all (x, y), $x^2 + y^2 + 1 \neq 0.$

18. $f(x, y)$ is continuous for all (x, y) since for all (x, y), $x^2 + y^2 + 1 > 0.$

19. Require $1 - x^2 - y^2 > 0; x^2 + y^2 < 1.$ S is the interior of the unit circle centered at the origin.

20. Require $1 + x + y > 0;\ y > -x - 1.$ S is the set of all (x, y) above the line $y = -x - 1.$

21. Require $y - x^2 \neq 0.$ S is the entire plane except the parabola $y = x^2$.

22. The only points at which f might be discontinuous occur when $xy = 0$.

$\lim\limits_{(x,\ y)\to(a,\ 0)} \dfrac{\sin(xy)}{xy} = 1 = f(a,\ 0)$ for all nonzero a in $\mathbb{R}$, and then

$\lim\limits_{(x,\ y)\to(0,\ b)} \dfrac{\sin(xy)}{xy} = 1 = f(0,\ b)$ for all b in $\mathbb{R}$.

Therefore, f is continuous on the entire plane.

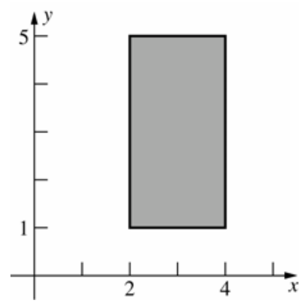
23. Require $x - y + 1 \geq 0; y \leq x + 1.$ S is the region below and on the line $y = x + 1$.

24. Require $4 - x^2 - y^2 > 0; x^2 + y^2 < 4.$ S is the interior of the circle of radius 2 centered at the origin.

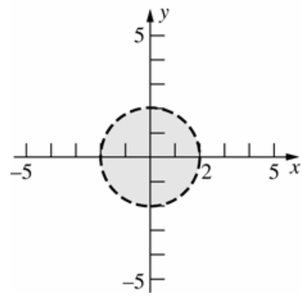
25. $f(x, y, z)$ is continuous for all $(x, y, z) \neq (0,0,0)$ since for all $(x, y, z) \neq (0,0,0)$, $x^2 + y^2 + z^2 > 0.$

26. Require $4 - x^2 - y^2 - z^2 > 0;$ $x^2 + y^2 + z^2 < 4.$ S is the space in the interior of the sphere centered at the origin with radius 2.

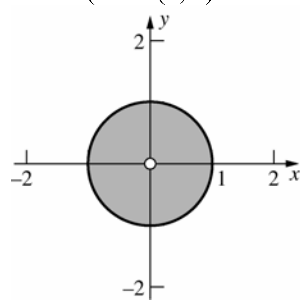
27. The boundary consists of the points that form the outer edge of the rectangle. The set is closed.



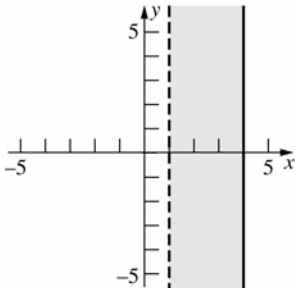
28. The boundary consists of the points of the circle shown. The set is open.



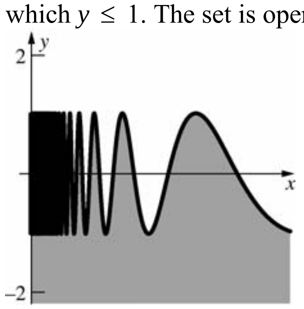
29. The boundary consists of the circle and the origin. The set is neither open (since, for example, (1, 0) is not an interior point), nor closed (since (0, 0) is not in the set).



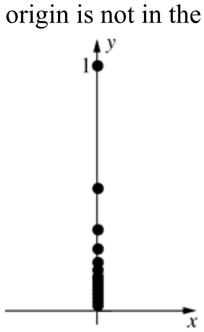
30. The boundary consists of the points on the line $x = 1$ along with the points on the line $x = 4$. The set is neither closed nor open.



31. The boundary consists of the graph of $y = \sin\left(\frac{1}{x}\right)$ along with the part of the y-axis for which $y \le 1$. The set is open.



32. The boundary is the set itself along with the origin. The set is neither open (since none of its points are interior points) nor closed (since the origin is not in the set).



33. $\dfrac{x^2-4y^2}{x-2y} = \dfrac{(x+2y)(x-2y)}{x-2y} = x+2y$ (if $x \ne 2y$)

If $x = 2y$, $x+2y = 2x$. Take $g(x) = 2x$.

34. Let L and M be the latter two limits.

$[f(x, y) + g(x, y)] - [L + M]|$

$\le \left|f(x, y) - L\right| + \left|f(x, y) - M\right| \le \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}$

for (x, y) in some δ-neighborhood of (a, b). Therefore,

$$\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = L + M.$$

35. Along the x-axis ($y = 0$): $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{0}{x^2+0} = 0$.

Along $y = x$:

$$\lim_{(x, y)\to(0, 0)} \frac{x^2}{2x^2} = \lim_{(x, y)\to(0, 0)} \frac{1}{2} = \frac{1}{2}.$$

Hence, the limit does not exist because for some points near the origin $f(x, y)$ is getting closer to 0, but for others it is getting closer to $\frac{1}{2}$.

36. Along $y = 0$: $\displaystyle\lim_{x\to 0} \frac{0}{x^2+0} = 0.$ Along $y = x$:

$$\lim_{x\to 0} \frac{x^2+x^3}{x^2+x^2} = \lim_{x\to 0} \frac{1+x}{2} = \frac{1}{2}.$$

37. a. $\displaystyle\lim_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0} \frac{mx^3}{x^4+m^2x^2}$

$\displaystyle= \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$

b. $\displaystyle\lim_{x\to 0} \frac{x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

c. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2y}{x^4+y}$ does not exist.

38. f is discontinuous at each overhang. More interesting, f is discontinuous along the Continental Divide.

39. a. $\{(x, y, z): x^2+y^2 = 1, z \text{ in } [1, 2]\}$ [For $x^2+y^2 < 1$, the particle hits the hemisphere and then slides to the origin (or bounds toward the origin); for $x^2+y^2 = 1$, it bounces up; for $x^2+y^2 > 1$, it falls straight down.]

b. $\{(x, y, z): x^2+y^2 = 1, z = 1\}$ (As one moves at a level of $z = 1$ from the rim of the bowl toward any position away from the bowl there is a change from seeing all of the interior of the bowl to seeing none of it.)

c. $\{(x, y, z): z = 1\}$ [$f(x, y, z)$ is undefined (infinite) at $(x, y, 1)$.]

d. ϕ (Small changes in points of the domain result in small changes in the shortest path from the points to the origin.)

40. f is continuous on an open set D and P_0 is in D implies that there is neighborhood of P_0 with radius r on which f is continuous. f is continuous at $P_0 \Rightarrow \lim_{P \to P_0} f(P) = f(P_0)$. Now let $\varepsilon = f(P_0)$ which is positive. Then there is a δ such that $0 < \delta < r$ and $|f(p) - f(P_0)| < f(P_0)$ if P is in the δ-neighborhood of P_0. Therefore, $-f(P_0) < f(p) - f(P_0) < f(P_0)$, so $0 < f(p)$ (using the left-hand inequality) in that δ-neighborhood of P_0.

41. a. $f(x, y) = \begin{cases} (x^2 + y^2)^{1/2} + 1 & \text{if } y \neq 0 \\ |x - 1| & \text{if } y = 0 \end{cases}$. Check discontinuities where $y = 0$.

As $y = 0$, $(x^2 + y^2)^{1/2} + 1 = |x| + 1$, so f is continuous if $|x| + 1 = |x - 1|$. Squaring each side and simplifying yields $|x| = -x$, so f is continuous for $x \le 0$. That is, f is discontinuous along the positive x-axis.

b. Let $P = (u, v)$ and $Q = (x, y)$.

$$f(u, v, x, y) = \begin{cases} |OP| + |OQ| & \text{if } P \text{ and } Q \text{ are not on same ray from the origin and neither is the origin} \\ |\mathrm{PQ}| & \text{otherwise} \end{cases}.$$

This means that in the first case one travels from P to the origin and then to Q; in the second case one travels directly from P to Q without passing through the origin, so f is discontinuous on the set $\{(u, v, x, y) : \langle u, v \rangle = k\langle x, y \rangle$ for some $k > 0, \langle u, v \rangle \neq \mathbf{0}, \langle x, y \rangle \neq \mathbf{0}\}$.

42. a. $$f_x(0, y) = \lim_{h \to 0} \left(\frac{\frac{hy(h^2 - y^2)}{h^2 + y^2} - 0}{h} \right) = \lim_{h \to 0} \frac{y(h^2 - y^2)}{h^2 + y^2} = -y$$

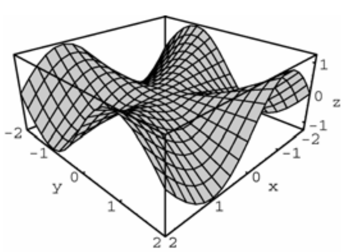
b. $$f_y(x, 0) = \lim_{h \to 0} \left(\frac{\frac{xh(x^2 - h^2)}{x^2 + h^2} - 0}{h} \right) = \lim_{h \to 0} \frac{y(x^2 - h^2)}{x^2 + y^2} = x$$

c. $$f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(0 + h, y) - f_y(0, y)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

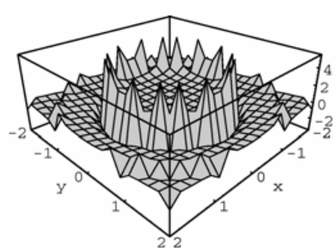
d. $$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_x(x, 0 + h) - f_x(x, 0)}{h} = \lim_{h \to 0} \frac{-h - 0}{h} = -1$$

Therefore, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

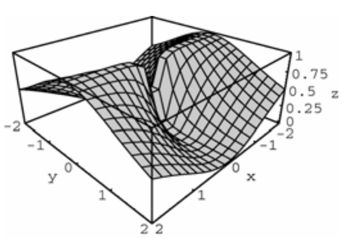
43.



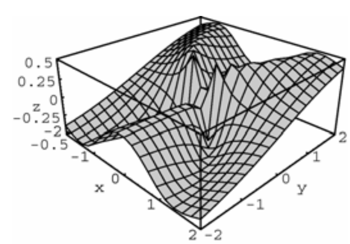
b.



44. a.



45.



46. A function *f* of three variables is continuous at a point (a,b,c) if $f(a,b,c)$ is defined and equal to the limit of $f(x,y,z)$ as (x,y,z) approaches (a,b,c). In other words,

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c).$$

A function of three variables is continuous on an open set *S* if it is continuous at every point in the interior of the set. The function is continuous at a boundary point *P* of *S* if $f(Q)$ approaches $f(P)$ as *Q* approaches *P* along any path through points in *S* in the neighborhood of *P*.

47. If we approach the point $(0,0,0)$ along a straight path from the point (x,x,x), we have

$$\lim_{(x,x,x)\to(0,0,0)} \frac{x(x)(x)}{x^3+x^3+x^3} = \lim_{(x,x,x)\to(0,0,0)} \frac{x^3}{3x^3} = \frac{1}{3}$$

Since the limit does not equal to $f(0,0,0)$, the function is not continuous at the point $(0,0,0)$.

48. If we approach the point $(0,0,0)$ along the x-axis, we get

$$\lim_{(x,0,0)\to(0,0,0)} (0+1)\frac{(x^2-0^2)}{(x^2+0^2)} = \lim_{(x,0,0)\to(0,0,0)} \frac{x^2}{x^2} = 1$$

Since the limit does not equal $f(0,0,0)$, the function is not continuous at the point $(0,0,0)$.

12.4 Concepts Review

1. gradient

2. locally linear

3. $\frac{\partial f}{\partial x}(\mathbf{p})\mathbf{i} + \frac{\partial f}{\partial y}(\mathbf{p})\mathbf{j}$; $\mathbf{y}^2\mathbf{i} + 2xy\mathbf{j}$

4. tangent plane

Problem Set 12.4

1. $\left\langle 2xy+3y,\ x^2+3x \right\rangle$

2. $\left\langle 3x^2y,\ x^3-3y^2 \right\rangle$

3. $\nabla f(x,\ y) = \left\langle (x)(e^{xy}y)+(e^{xy})(1),\ xe^{xy}x \right\rangle = e^{xy}\left\langle xy+1,\ x^2 \right\rangle$

4. $\left\langle 2xy\cos y,\ x^2(\cos y - y\sin y) \right\rangle$

5. $x(x+y)^{-2}\left\langle y(x+2),\ x^2 \right\rangle$

6. $\nabla f(x,\ y) = \left\langle 3[\sin^2(x^2y)][\cos(x^2y)](2xy),\ 3[\sin^2(x^2y)][\cos(x^2y)](x^2) \right\rangle = 3x\sin^2(x^2y)\cos(x^2y)\left\langle 2y,\ x \right\rangle$

7. $(x^2+y^2+z^2)^{-1/2}\langle x,\ y,\ z\rangle$

8. $\left\langle 2xy+z^2, x^2+2yz, y^2+2xz\right\rangle$

9. $\nabla f(x,\ y)=\left\langle (x^2y)(e^{x-z})+(e^{x-z})(2xy),\ x^2e^{x-z},\ x^2ye^{x-z}(-1)\right\rangle = xe^{x-z}\left\langle y(x+2), x, -xy\right\rangle$

10. $\left\langle xz(x+y+z)^{-1}+z\ln(x+y+z),\ xz(x+y+z)^{-1},\ xz(x+y+z)^{-1}+x\ln(x+y+z)\right\rangle$

11. $\nabla f(x,\ y)=\left\langle 2xy-y^2,\ x^2-2xy\right\rangle;\ \ \nabla f(-2, 3)=\langle -21, 16\rangle$

$z = f(-2, 3)+\langle -21, 16\rangle\cdot\langle x+2,\ y-3\rangle = 30+(-21x-42+16y-48)$

$z = -21x + 16y - 60$

12. $\nabla f(x,\ y)=\left\langle 3x^2y+3y^2,\ x^3+6xy\right\rangle$, so $\nabla f(2, -2)=(-12, -16)$.

Tangent plane:

$z = f(2, -2)+\nabla(2, -2)\cdot\langle x-2,\ y+2\rangle \ = 8+\langle -12, -16\rangle\cdot\langle x-2,\ y+2\rangle = 8 + (-12x + 24 - 16y - 32)$

$z = -12x - 16y$

13. $\nabla f(x,\ y)=\langle -\pi\sin(\pi x)\sin(\pi y),\ \pi\cos(\pi x)\cos(\pi y)+2\pi\cos(2\pi y)\rangle$

$\nabla f\left(-1, \frac{1}{2}\right)=\langle 0, -2\pi\rangle$

$z = f\left(-1, \frac{1}{2}\right)+\langle 0, -2\pi\rangle\cdot\left\langle x+1,\ y-\frac{1}{2}\right\rangle = -1+(0-2\pi y+\pi);$

$z = -2\pi y + (\pi - 1)$

14. $\nabla f(x,\ y)=\left\langle \frac{2x}{y}, -\frac{x^2}{y^2}\right\rangle; \nabla f(2, -1)=\langle -4, -4\rangle$

$z = f(2, -1)+\langle -4, -4\rangle\cdot\langle x-2,\ y+1\rangle$

$= -4 + \ (-4x + 8 - 4y - 4)$

$z = -4x - 4y$

15. $\nabla f(x,\ y,\ z)=\left\langle 6x+z^2, -4y, 2xz\right\rangle$, so $\nabla f(1, 2, -1)=\langle 7, -8, -2\rangle$

Tangent hyperplane:

$w = f(1, 2, -1)+\nabla f(1, 2, -1)\cdot\langle x-1,\ y-2,\ z+1\rangle = -4+\langle 7, -8, -2\rangle\cdot\langle x-1,\ y-2,\ z+1\rangle$

$= -4 + (7x - 7 - 8y + 16 - 2z - 2)$

$w = 7x - 8y - 2z + 3$

16. $\nabla f(x,\ y,\ z)=\langle yz+2x,\ xz,\ xy\rangle;\ \ \nabla f(2, 0, -3)=\langle 4, -6, 0\rangle$

$w = f(2, 0, -3)+\langle 4, -6, 0\rangle\cdot\langle x-2,\ y,\ z+3\rangle = 4 + (4x - 8 - 6y + 0)$

$w = 4x - 6y - 4$

17. $\nabla\left(\frac{f}{g}\right)=\frac{\left\langle gf_x-fg_x, gf_y-fg_y, gf_z-fg_z\right\rangle}{g^2}=\frac{g\left\langle f_x,\ f_y,\ f_z\right\rangle - f\left\langle g_x,\ g_y,\ g_z\right\rangle}{g^2}=\frac{g\nabla f-f\nabla g}{g^2}$

18. $\nabla(f^r)=\left\langle rf^{r-1}f_x, rf^{r-1}f_y, rf^{r-1}f_z\right\rangle = rf^{r-1}\left\langle f_x,\ f_y,\ f_z\right\rangle = rf^{r-1}\nabla f$

19. Let $F(x,y,z) = x^2 - 6x + 2y^2 - 10y + 2xy - z = 0$

$\nabla F(x,y,z) = \langle 2x - 6 + 2y, 4y - 10 + 2x, -1 \rangle$

The tangent plane will be horizontal if $\nabla F(x,y,z) = \langle 0,0,k \rangle$, where $k \neq 0$. Therefore, we have the following system of equations:

$2x + 2y - 6 = 0$

$2x + 4y - 10 = 0$

Solving this system yields $x = 1$ and $y = 2$. Thus, there is a horizontal tangent plane at $(x,y) = (1,2)$.

20. Let $F(x,y,z) = x^3 - z = 0$

$\nabla F(x,y,z) = \langle 3x^2, 0, -1 \rangle$

The tangent plane will be horizontal if $\nabla F(x,y,z) = \langle 0,0,k \rangle$, where $k \neq 0$. Therefore, we need only solve the equation $3x^2 = 0$. There is a horizontal tangent plane at $(x,y) = (0,y)$. (Note: there are infinitely many points since y can take on any value).

21. a. The point $(2,1,9)$ projects to $(2,1,0)$ on the *xy* plane. The equation of a plane containing this point and parallel to the *x*-axis is given by $y = 1$. The tangent plane to the surface at the point $(2,1,9)$ is given by

$z = f(2,1) + \nabla f(2,1) \cdot \langle x-2, y-1 \rangle$

$= 9 + \langle 12,10 \rangle \langle x-2, y-1 \rangle$

$= 12x + 10y - 25$

The line of intersection of the two planes is the tangent line to the surface, passing through the point $(2,1,9)$, whose projection in the *xy* plane is parallel to the x-axis. This line of intersection is parallel to the cross product of the normal vectors for the planes. The normal vectors are $\langle 12,10,-1 \rangle$ and $\langle 0,1,0 \rangle$ for the tangent plane and vertical plane respectively. The cross product is given by

$\langle 12,10,-1 \rangle \times \langle 0,1,0 \rangle = \langle 1,0,12 \rangle$

Thus, parametric equations for the desired tangent line are $x = 2 + t$

$y = 1$

$z = 9 + 12t$

b. Using the equation for the tangent plane from the previous part, we now want the vertical plane to be parallel to the *y*-axis, but still pass through the projected point $(2,1,0)$. The vertical plane now has equation $x = 2$. The normal equations are given by $\langle 12,10,-1 \rangle$ and $\langle 1,0,0 \rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:

$\langle 12,10,-1 \rangle \times \langle 1,0,0 \rangle = \langle 0,10,10 \rangle$

Thus, parametric equations for the desired tangent line are $x = 2$

$y = 1 + 10t$

$z = 9 + 10t$

c. Using the equation for the tangent plane from the first part, we now want the vertical plane to be parallel to the line $y = x$, but still pass through the projected point $(2,1,0)$. The vertical plane now has equation $y - x + 1 = 0$. The normal equations are given by $\langle 12,10,-1 \rangle$ and $\langle 1,-1,0 \rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:

$\langle 12,10,-1 \rangle \times \langle 1,-1,0 \rangle = \langle -1,-1,-22 \rangle$

Thus, parametric equations for the desired tangent line are $x = 2 - t$

$y = 1 - t$

$z = 9 - 22t$

22. a. The point $(3,2,72)$ on the surface is the point $(3,2,0)$ when projected into the *xy* plane. The equation of a plane containing this point and parallel to the x-axis is given by $y = 2$. The tangent plane to the surface at the point $(3,2,72)$ is given by

$z = f(3,2) + \nabla f(3,2) \cdot \langle x-3, y-2 \rangle$

$= 72 + \langle 48,108 \rangle \langle x-3, y-2 \rangle$

$= 48x + 108y - 288$

The line of intersection of the two planes is the tangent line to the surface, passing through the point $(3,2,72)$, whose projection in the *xy* plane is parallel to the x-axis. This line of intersection is parallel to the cross product of the normal vectors for the planes. The normal vectors are $\langle 48,108,-1 \rangle$ and $\langle 0,2,0 \rangle$ for the tangent plane and vertical plane respectively. The cross product is given by

$\langle 48,108,-1 \rangle \times \langle 0,2,0 \rangle = \langle 2,0,96 \rangle$

Thus, parametric equations for the desired tangent line are

$x = 3 + 2t$

$y = 2$

$z = 72 + 96t$

b. Using the equation for the tangent plane from the previous part, we now want the vertical plane to be parallel to the y-axis, but still pass through the projected point $(3, 2, 72)$. The vertical plane now has equation $x = 3$. The normal equations are given by $\langle 48, 108, -1\rangle$ and $\langle 3, 0, 0\rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:
$\langle 48, 108, -1\rangle \times \langle 3, 0, 0\rangle = \langle 0, -3, -324\rangle$
Thus, parametric equations for the desired tangent line are
$x = 3$
$y = 2 - 3t$
$z = 72 - 324t$

c. Using the equation for the tangent plane from the first part, we now want the vertical plane to be parallel to the line $x = -y$, but still pass through the projected point $(3, 2, 72)$. The vertical plane now has equation $y + x - 5 = 0$. The normal equations are given by $\langle 48, 108, -1\rangle$ and $\langle 1, 1, 0\rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:
$\langle 48, 108, -1\rangle \times \langle 1, 1, 0\rangle = \langle 1, -1, -60\rangle$
Thus, parametric equations for the desired tangent line are
$x = 3 + t$
$y = 2 - t$
$z = 72 - 60t$

23. $\nabla f(x, y) = \left\langle -10\left(\frac{1}{2\sqrt{|xy|}}\frac{|xy|}{xy}y\right), -10\left(\frac{1}{2\sqrt{|xy|}}\frac{|xy|}{xy}x\right)\right\rangle = \frac{-5xy}{|xy|^{3/2}}\langle y, x\rangle$ $\left[\text{Note that } \frac{|a|}{a} = \frac{a}{|a|}.\right]$

$\nabla f(1, -1) = \langle -5, 5\rangle$

Tangent plane:
$z = f(1, -1) + \nabla f(1, -1)\cdot\langle x-1, y+1\rangle = -10 + \langle -5, 5\rangle\cdot\langle x-1, y+1\rangle = -10 + (-5x + 5 + 5y + 5)$
$z = -5x + 5y$

24. Let **a** be any point of S and let **b** be any other point of S. Then for some c on the line segment between **a** and **b**:
$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(c)\cdot(\mathbf{b}-\mathbf{a}) = 0\cdot(\mathbf{b}-\mathbf{a}) = 0,$ so $f(\mathbf{b}) = f(\mathbf{a})$ (for all **b** in S).

25. $f(\mathbf{a}) - f(\mathbf{b}) = f\langle 2,1\rangle - f\langle 0,0\rangle = 4 - 9 = -5$
$\nabla f(x, y) = \langle -2x, -2y\rangle$; $\mathbf{b} - \mathbf{a} = \langle 2,1\rangle$
The value $\mathbf{c} = \langle c_x, c_y\rangle$ will be a solution to
$-5 = \langle -2c_x, -2c_y\rangle\langle 2,1\rangle$
$\mathbf{c} \in \{\langle c_x, c_y\rangle : 4c_x + 2c_y = 5\}$
In order for **c** to be between **a** and **b**, **c** must lie on the line $y = \frac{1}{2}x$. Consequently, **c** will be the solution to the following system of equations: $4c_x + 2c_y = 5$ and $c_y = \frac{1}{2}c_x$. The solution is $\mathbf{c} = \langle 1, \frac{1}{2}\rangle$.

26. $f(\mathbf{b}) - f(\mathbf{a}) = f\langle 2,6\rangle - f\langle 0,0\rangle = 0 - 2 = -2$
$\nabla f(x, y) = \left\langle \frac{-x}{\sqrt{4-x^2}}, 0\right\rangle$; $\mathbf{b} - \mathbf{a} = \langle 2,6\rangle$
The value $\mathbf{c} = \langle c_x, c_y\rangle$ will be the solution to
$-2 = \left\langle \frac{-c_x}{\sqrt{4-c_x^2}}, 0\right\rangle\langle 2,6\rangle$
$-2 = \frac{-2c_x}{\sqrt{4-c_x^2}} \Rightarrow c_x = \sqrt{2}$
Since **c** must be between **a** and **b**, **c** must lie on the line y = 3x. Since $c_x = \sqrt{2}$, $c_y = 3\sqrt{2}$.
Thus, $\mathbf{c} = \langle \sqrt{2}, 3\sqrt{2}\rangle$.

27. $\nabla f(\mathbf{p}) = \nabla g(\mathbf{p}) \Rightarrow \nabla[f(\mathbf{p}) - g(\mathbf{p})] = \mathbf{0}$
$\Rightarrow f(\mathbf{p}) - g(\mathbf{p})$ is a constant.

28. $\nabla f(\mathbf{p}) = \mathbf{p} \Rightarrow \nabla f(x, y) = \langle x, y \rangle$

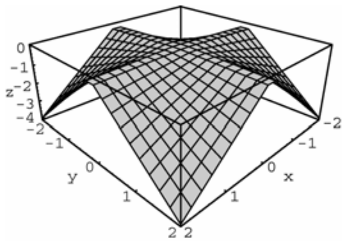
$\Rightarrow f_x(x, y) = x,\ f_y(x, y) = y$

$\Rightarrow f(x, y) = \frac{1}{2}x^2 + \alpha(y)$ for any function of y,

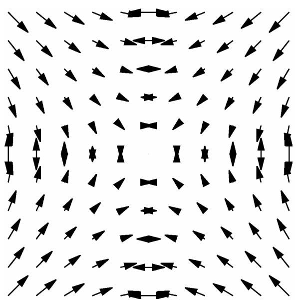
and $f(x, y) = \frac{1}{2}y^2 + \beta(x)$ for any function of x.

$\Rightarrow f(x, y) = \frac{1}{2}(x^2 + y^2) + C$ for any C in $\mathbb{R}$.

29.



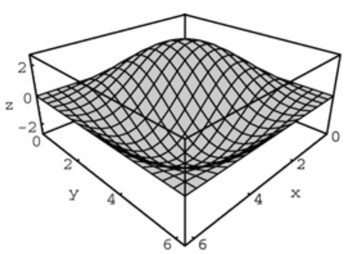
$-|xy|$



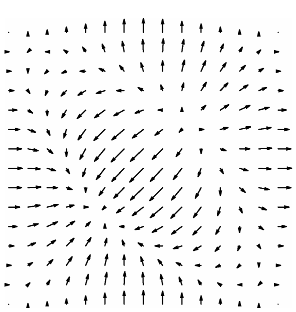
a. The gradient points in the direction of greatest increase of the function.

b. No. If it were, $|0+h| - |0| = 0 + |h|\delta(h)$ where $\delta(h) \to 0$ as $h \to 0$, which is possible.

30.



$\sin(x) + \sin(y) - \sin(x + y)$



31. a. (i)

$$\nabla[f+g] = \frac{\partial(f+g)}{\partial x}\mathbf{i} + \frac{\partial(f+g)}{\partial y}\mathbf{j} + \frac{\partial(f+g)}{\partial z}\mathbf{k}$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} + \frac{\partial g}{\partial z}\mathbf{k}$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} + \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}$$

$$= \nabla f + \nabla g$$

(ii) $$\nabla[\alpha f] = \frac{\partial[\alpha f]}{\partial x}\mathbf{i} + \frac{\partial[\alpha f]}{\partial y}\mathbf{j} + \frac{\partial[\alpha f]}{\partial z}\mathbf{k}$$

$$= \alpha\frac{\partial[f]}{\partial x}\mathbf{i} + \alpha\frac{\partial[f]}{\partial y}\mathbf{j} + \alpha\frac{\partial[f]}{\partial z}\mathbf{k}$$

$$= \alpha\nabla f$$

(iii)

$$\nabla[fg] = \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k}$$

$$= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\mathbf{j}$$

$$+ \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\mathbf{k}$$

$$= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)$$

$$+ g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)$$

$$= f\nabla g + g\nabla f$$

b. (i)

$$\nabla[f+g] = \frac{\partial(f+g)}{\partial x_1}\mathbf{i}_1 + \frac{\partial(f+g)}{\partial x_2}\mathbf{i}_2$$

$$+ \cdots + \frac{\partial(f+g)}{\partial x_n}\mathbf{i}_n$$

$$= \frac{\partial f}{\partial x_1}\mathbf{i}_1 + \frac{\partial g}{\partial x_1}\mathbf{i}_1 + \frac{\partial f}{\partial x_2}\mathbf{i}_2 + \frac{\partial g}{\partial x_2}\mathbf{i}_2$$

$$+ \cdots + \frac{\partial f}{\partial x_n}\mathbf{i}_n + \frac{\partial g}{\partial x_n}\mathbf{i}_n$$

$$= \frac{\partial f}{\partial x_1}\mathbf{i}_1 + \frac{\partial f}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{i}_n$$

$$+ \frac{\partial g}{\partial x_1}\mathbf{i}_1 + \frac{\partial g}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial g}{\partial x_n}\mathbf{i}_n$$

$$= \nabla f + \nabla g$$

(ii)

$$\nabla[\alpha f]=\frac{\partial[\alpha f]}{\partial x_1}\mathbf{i}_1+\frac{\partial[\alpha f]}{\partial x_2}\mathbf{i}_2+\cdots+\frac{\partial[\alpha f]}{\partial x_n}\mathbf{i}_n$$
$$=\alpha\frac{\partial[f]}{\partial x_1}\mathbf{i}_1+\alpha\frac{\partial[f]}{\partial x_2}\mathbf{i}_2+\cdots+\alpha\frac{\partial[f]}{\partial x_n}\mathbf{i}_n$$
$$=\alpha\nabla f$$

(iii)

$$\nabla[fg]=\frac{\partial(fg)}{\partial x_1}\mathbf{i}_1+\frac{\partial(fg)}{\partial x_2}\mathbf{i}_2+\cdots+\frac{\partial(fg)}{\partial x_n}\mathbf{i}_n$$
$$=\left(f\frac{\partial g}{\partial x_1}+g\frac{\partial f}{\partial x_1}\right)\mathbf{i}_1+\left(f\frac{\partial g}{\partial x_2}+g\frac{\partial f}{\partial x_2}\right)\mathbf{i}_2$$
$$+\cdots+\left(f\frac{\partial g}{\partial x_n}+g\frac{\partial f}{\partial x_n}\right)\mathbf{i}_n$$
$$=f\left(\frac{\partial g}{\partial x_1}\mathbf{i}_1+\frac{\partial g}{\partial x_2}\mathbf{i}_2+\cdots+\frac{\partial g}{\partial x_n}\mathbf{i}_n\right)$$
$$+g\left(\frac{\partial f}{\partial x_1}\mathbf{i}_1+\frac{\partial f}{\partial x_2}\mathbf{i}_2+\cdots+\frac{\partial f}{\partial x_n}\mathbf{i}_n\right)$$
$$=f\nabla g+g\nabla f$$

12.5 Concepts Review

1. $\dfrac{[f(\mathbf{p}+h\mathbf{u})-f(\mathbf{p})]}{h}$

2. $u_1f_x(x,y)+u_2f_y(x,y)$

3. greatest increase

4. level curve

Problem Set 12.5

1. $D_{\mathbf{u}}f(x,y)=\left\langle 2xy,\ x^2\right\rangle\cdot\left\langle\frac{3}{5},-\frac{4}{5}\right\rangle;\ D_{\mathbf{u}}f(1,2)=\frac{8}{5}$

2. $D_{\mathbf{u}}f(x,y)=\left\langle x^{-1}y^2,\ 2y\ln x\right\rangle\cdot\left[\left(\frac{1}{\sqrt{2}}\right)\langle 1,-1\rangle\right];$

$D_{\mathbf{u}}f(1,4)=8\sqrt{2}\approx 11.3137$

3. $D_{\mathbf{u}}f(x,y)=f(x,y)\cdot\mathbf{u}\quad\left(\text{where } u=\frac{\mathbf{a}}{|\mathbf{a}|}\right)$

$=\langle 4x+y,\ x-2y\rangle\cdot\frac{\langle 1,-1\rangle}{\sqrt{2}};$

$D_{\mathbf{u}}f(3,-2)=\langle 10,7\rangle\cdot\frac{\langle 1,-1\rangle}{\sqrt{2}}=\frac{3}{\sqrt{2}}\approx 2.1213$

4. $D_{\mathbf{u}}f(x,y)$

$=\langle 2x-3y,\ -3x+4y\rangle\cdot\left[\left(\frac{1}{\sqrt{5}}\right)\langle 2,-1\rangle\right];$

$D_{\mathbf{u}}f(-1,2)=-\frac{27}{\sqrt{5}}\approx -12.0748$

5. $D_{\mathbf{u}}f(x,y)=e^x\langle\sin y,\cos y\rangle\cdot\left[\left(\frac{1}{2}\right)\langle 1,\sqrt{3}\rangle\right];$

$D_{\mathbf{u}}f\left(0,\frac{\pi}{4}\right)=\frac{\left(\sqrt{2}+\sqrt{6}\right)}{4}\approx 0.9659$

6. $D_{\mathbf{u}}f(x,y)=\left\langle -ye^{-xy}-xe^{-xy}\right\rangle\cdot\frac{\langle -1,\sqrt{3}\rangle}{2}$

$D_{\mathbf{u}}f(1,-1)=\langle e,-e\rangle\cdot\frac{\langle -1,\sqrt{3}\rangle}{2}=\frac{-e-e\sqrt{3}}{2}$

≈ -3.7132

7. $D_{\mathbf{u}}f(x,y,z)=$

$=\left\langle 3x^2y,\ x^3-2yz^2,\ -2y^2z\right\rangle\cdot\left[\left(\frac{1}{3}\right)\langle 1,-2,2\rangle\right];$

$D_{\mathbf{u}}f(-2,1,3)=\frac{52}{3}$

8. $D_{\mathbf{u}}f(x,y,z)=\langle 2x,2y,2z\rangle\cdot\left[\left(\frac{1}{2}\right)\langle\sqrt{2},-1,-1\rangle\right];$

$D_{\mathbf{u}}f(1,-1,2)=\sqrt{2}-1\approx 0.4142$

9. f increases most rapidly in the direction of the gradient. $\nabla f(x,y)=\left\langle 3x^2,-5y^4\right\rangle;$

$\nabla f(2,-1)=\langle 12,-5\rangle$

$\frac{\langle 12,-5\rangle}{13}$ is the unit vector in that direction. The rate of change of $f(x, y)$ in that direction at that point is the magnitude of the gradient.

$|\langle 12,-5\rangle|=13$

10. $\nabla f(x,y)=\left\langle e^y\cos x,\ e^y\sin x\right\rangle;$

$\nabla f\left(\frac{5\pi}{6},0\right)=\left\langle -\frac{\sqrt{3}}{2},\frac{1}{2}\right\rangle,$ which is a unit vector.

The rate of change in that direction is 1.

11. $\nabla f(x, y, z) = \left\langle 2xyz, x^2z, x^2y \right\rangle$;

$f(1, -1, 2) = \langle -4, 2, -1 \rangle$

A unit vector in that direction is $\left(\frac{1}{\sqrt{21}}\right)\langle -4, 2, -1 \rangle$. The rate of change in that direction is $\sqrt{21} \approx 4.5826$.

12. f increases most rapidly in the direction of the gradient. $\nabla f(x, y, z) = \left\langle e^{yz}, xze^{yz}, xye^{yz} \right\rangle$;

$\nabla f(2, 0, -4) = \langle 1, -8, 0 \rangle$

$\frac{\langle 1, -8, 0 \rangle}{\sqrt{65}}$ is a unit vector in that direction.

$|\langle 1, -8, 0 \rangle| = \sqrt{65} \approx 8.0623$ is the rate of change of $f(x, y, z)$ in that direction at that point.

13. $-\nabla f(x, y) = 2\langle x, y \rangle$; $-\nabla f(-1, 2) = 2\langle -1, 2 \rangle$ is the direction of most rapid decrease. A unit vector in that direction is $\mathbf{u} = \left(\frac{1}{\sqrt{5}}\right)\langle -1, 2 \rangle$.

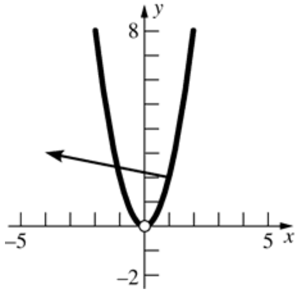
14. $-\nabla f(x, y) = \langle -3\cos(3x - y), \cos(3x - y) \rangle$;

$-\nabla f\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)\langle -3, 1 \rangle$ is the direction of most rapid decrease. A unit vector in that direction is $\left(\frac{1}{\sqrt{10}}\right)\langle -3, 1 \rangle$.

15. The level curves are $\frac{y}{x^2} = k$. For $\mathbf{p} = (1, 2)$, $\mathbf{k} = 2$, so the level curve through (1, 2) is $\frac{y}{x^2} = 2$ or $y = 2x^2$ $(x \neq 0)$.

$\nabla f(x, y) = \left\langle -2yx^{-3}, x^{-2} \right\rangle$

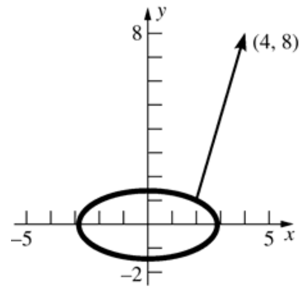
$\nabla f(1, 2) = \langle -4, 1 \rangle$, which is perpendicular to the parabola at (1, 2).



16. At (2, 1), $x^2 + 4y^2 = 8$ is the level curve.

$\nabla f(x, y) = \langle 2x, 8y \rangle$

$\nabla f(2, 1) = 4\langle 1, 2 \rangle$, which is perpendicular to the level curve at (2, 1).



17. $u = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

$D_{\mathbf{u}} f(x, y, z) = \langle y, x, 2z \rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

$D_{\mathbf{u}} f(1, 1, 1) = \frac{2}{3}$

18. $\left(0, \frac{\pi}{3}\right)$ is on the y-axis, so the unit vector toward the origin is $-\mathbf{j}$.

$D_{\mathbf{u}}(x, y) = \left\langle -e^{-x}\cos y, -e^{-x}\sin y \right\rangle \cdot \langle 0, -1 \rangle$

$= e^{-x}\sin y$;

$D_{\mathbf{u}}\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

19. a. Hottest if denominator is smallest; i.e., at the origin.

b. $\nabla T(x, y, z) = \frac{-200\langle 2x, 2y, 2z \rangle}{(5 + x^2 + y^2 + z^2)^2}$;

$\nabla T(1, -1, 1) = \left(-\frac{25}{4}\right)\langle 1, -1, 1 \rangle$

$\langle -1, 1, -1 \rangle$ is one vector in the direction of greatest increase.

c. Yes

20. $-\nabla V(x, y, z)$

$= -100e^{-(x^2+y^2+z^2)}\langle -2x, -2y, -2z \rangle$

$= 200e^{-(x^2+y^2+z^2)}\langle x, y, z \rangle$ is the direction of greatest decrease at (x, y, z), and it points away from the origin.

21. $\nabla f(x, y, z)$

$$= \left\langle x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2},\right.$$

$$y\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2},$$

$$\left. z\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2}\right\rangle$$

$$= \left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2}\right)\langle x, y, z\rangle$$

which either points towards or away from the origin.

22. Let $D = \sqrt{x^2+y^2+z^2}$ be the distance. Then we have

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right\rangle = \left\langle \frac{dT}{dD}\frac{\partial D}{\partial x}, \frac{dT}{dD}\frac{\partial D}{\partial y}, \frac{dT}{dD}\frac{\partial D}{\partial z}\right\rangle$$

$$= \left\langle \frac{dT}{dD} x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}, \frac{dT}{dD} y\left(x^2+y^2+z^2\right)^{-\frac{1}{2}},\right.$$

$$\left. \frac{dT}{dD} z\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right\rangle$$

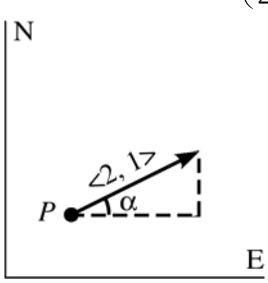
$$= \left(\frac{dT}{dD}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)\langle x, y, z\rangle$$

which either points towards or away from the origin.

23. He should move in the direction of

$$-\nabla f(\mathbf{p}) = -\langle f_x(\mathbf{p}),\ f_y(\mathbf{p})\rangle = -\left\langle -\frac{1}{2}, -\frac{1}{4}\right\rangle$$

$= \left(\frac{1}{4}\right)\langle 2, 1\rangle$. Or use $\langle 2, 1\rangle$. The angle α formed with the East is $\tan^{-1}\left(\frac{1}{2}\right) \approx 26.57°$ (N63.43°E).



24. The unit vector from (2, 4) toward (5, 0) is $\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$. Then

$$D_{\mathbf{u}} f(2, 4) = \langle -3, 8\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = -8.2.$$

25. The climber is moving in the direction of $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\langle -1, 1\rangle$. Let

$$f(x, y) = 3000e^{-(x^2+2y^2)/100}.$$

$$\nabla f(x, y) = 3000e^{-(x^2+2y^2)/100}\left\langle -\frac{x}{50}, -\frac{y}{25}\right\rangle;$$

$$f(10, 10) = -600e^{-3}\langle 1, 2\rangle$$

She will move at a slope of

$$D_u(10, 10) = -600e^{-3}\langle 1, 2\rangle \cdot \left(\frac{1}{\sqrt{2}}\right)\langle -1, 1\rangle$$

$$= \left(-300\sqrt{2}\right)e^{-3} \approx -21.1229\,.$$

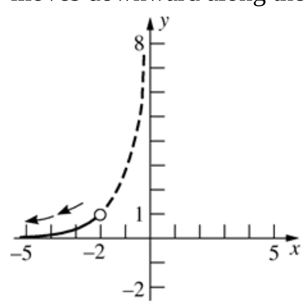
She will descend. Slope is about –21.

26. $\dfrac{\frac{dx}{dt}}{2x} = \dfrac{\frac{dy}{dt}}{-2y};\ \dfrac{dx}{x} = \dfrac{dy}{-y};\ \ln|x| = -\ln|y| + C$

At $t = 0$: $\ln|-2| = -\ln|1| + C \Rightarrow C = \ln 2.$

$$\ln|x| = -\ln|y| + \ln 2 = \ln\left|\frac{2}{y}\right|;\ |x| = \left|\frac{2}{y}\right|;\ |xy| = 2$$

Since the particle starts at (–2, 1) and neither x nor y can equal 0, the equation simplifies to $xy = -2$. $\nabla T(-2, 1) = \langle -4, -2\rangle$, so the particle moves downward along the curve.



27. $\nabla T(x, y) = \langle -4x, -2y\rangle$

$$\frac{dx}{dt} = -4x, \frac{dy}{dt} = -2y$$

$\dfrac{\frac{dx}{dt}}{-4x} = \dfrac{\frac{dy}{dt}}{-2y}$ has solution $|x| = 2y^2$. Since the particle starts at (–2, 1), this simplifies to $x = -2y^2$.

28. $f(1, -1) = 5\langle -1, 1\rangle$

$$D_{\langle u_1, u_2\rangle} f(1, -1) = \langle u_1, u_2\rangle \cdot \langle -5, 5\rangle = -5u_1 + 5u_2$$

a. $\langle -1, 1\rangle$ (in the direction of the gradient); $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\langle -1, 1\rangle$.

b. $\pm\langle 1, 1\rangle$ (direction perpendicular to gradient); $\mathbf{u} = \left(\pm\frac{1}{\sqrt{2}}\right)\langle 1, 1\rangle$

c. Want $D_\mathbf{u} f(1, -1) = 1$ where $|\mathbf{u}| = 1$. That is, want $-5u_1 + 5u_2 = 1$ and $u_1^2 + u_2^2 = 1$. Solutions are $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$ and $\left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$.

29. a. $\nabla T(x, y, z) = \left\langle -\frac{10(2x)}{(x^2+y^2+z^2)^2}, -\frac{10(2y)}{(x^2+y^2+z^2)^2}, -\frac{10(2z)}{(x^2+y^2+z^2)^2}\right\rangle$

$= -\frac{20}{(x^2+y^2+z^2)^2}\langle x, y, z\rangle$

$r(t) = \langle t\cos \pi t, t \sin \pi t, t\rangle$, so $r(1) = \langle -1, 0, 1\rangle$. Therefore, when $t = 1$, the bee is at $(-1, 0, 1)$, and $\nabla T(-1, 0, 1) = -5\langle -1, 0, 1\rangle$.

$\mathbf{r}'(t) = \langle \cos \pi t - \pi t \sin \pi t, \sin \pi t + \pi t \cos \pi t, 1\rangle$, so $\mathbf{r}'(1) = \langle -1, -\pi, 1\rangle$.

$U = \frac{r'(1)}{|r'(1)|} = \frac{\langle -1, -\pi, 1\rangle}{2+\pi^2}$ is the unit tangent vector at $(-1, 0, 1)$.

$D_\mathbf{u}T(-1, 0, 1) = \mathbf{u} \cdot \nabla T(-1, 0, 1)$

$= \frac{\langle -1, -\pi, 1\rangle \cdot \langle 5, 0, -5\rangle}{\sqrt{2+\pi^2}} = -\frac{10}{\sqrt{2+\pi^2}} \approx -2.9026$

Therefore, the temperature is decreasing at about 2.9°C per meter traveled when the bee is at $(-1, 0, 1)$; i.e., when $t = 1$ s.

b. Method 1: (First express T in terms of t.)

$T = \frac{10}{x^2+y^2+z^2} = \frac{10}{(t\cos \pi t)^2 + (t \sin \pi t)^2 + (t)^2} = \frac{10}{2t^2} = \frac{5}{t^2}$

$T(t) = 5t^{-2}; T'(t) = -10t^{-3}; t'(1) = -10$

Method 2: (Use Chain Rule.)

$D_tT(t) = \frac{dT}{ds}\frac{ds}{dt} = (D_\mathbf{u}T)(|\mathbf{r}'(t)|)$, so $D_tT(t) = [D_\mathbf{u}T(-1, 0, 1)](|\mathbf{r}'(1)|) = -\frac{10}{\sqrt{2+\pi^2}}\left(\sqrt{2+\pi^2}\right) = -10$

Therefore, the temperature is decreasing at about 10°C per second when the bee is at $(-1, 0, 1)$; i.e., when $t = 1$ s.

30. a. $D_\mathbf{u} f = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle \cdot \langle f_x \cdot f_y\rangle = -6$, so

$3f_x - 4f_y = -30.$

$D_\mathbf{v} f = \left\langle \frac{4}{5}, \frac{3}{5}\right\rangle \cdot \langle f_x, f_y\rangle = 17$, so

$4f_x + 3f_y = 85.$

The simultaneous solution is

$f_x = 10$, $f_y = 15$, so $\nabla f = \langle 10, 15\rangle$.

b. Without loss of generality, let $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$. If θ and ϕ are the angles between u and ∇f, and between $\mathbf{v}$ and ∇f, then:

1. $\theta + \phi = \frac{\pi}{2}$ (if ∇f is in the 1st quadrant).

2. $\theta = \frac{\pi}{2} + \phi$ (if ∇f is in the 2nd quadrant).

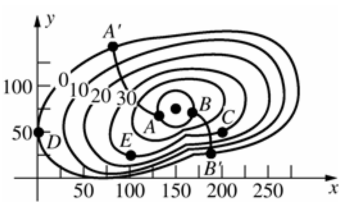
3. $\phi + \theta = \frac{3\pi}{2}$ (if ∇f is in the 3rd quadrant).

4. $\phi = \frac{\pi}{2} + \theta$ (if ∇f is in the 4th quadrant).

In each case cos ϕ = sin θ or cos ϕ = –sin θ, so $\cos^2 \phi = \sin^2 \theta$. Thus,

$$(D_{\mathbf{u}} f)^2 + (D_{\mathbf{v}} f)^2 = (\mathbf{u} \cdot \nabla f)^2 + (\mathbf{v} \cdot \nabla f)^2$$
$$= |\nabla f|^2 \cos^2 \theta + |\nabla f|^2 \cos^2 \phi$$
$$= |\nabla f|^2 (\cos^2 \theta + \cos^2 \phi)$$
$$= |\nabla f|^2 \cos^2 \theta + \sin^2 \theta = |\nabla f|^2.$$

31.



a. $A'(100, 120)$

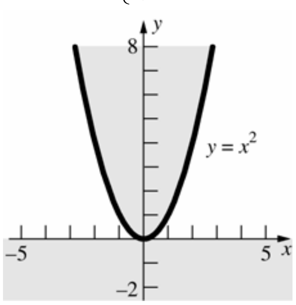
b. $B'(190, 25)$

c. $f_x(C) \approx \frac{20-30}{230-200} = -\frac{1}{3};\ f_y(D) = 0;$

$D_{\mathbf{u}} f(E) \approx \frac{40-30}{25} = \frac{2}{5}$

32. Graph of domain of f

$$f(x, y) = \begin{Bmatrix} 0, \text{ in shaded region} \\ 1, \text{ elsewhere} \end{Bmatrix}$$



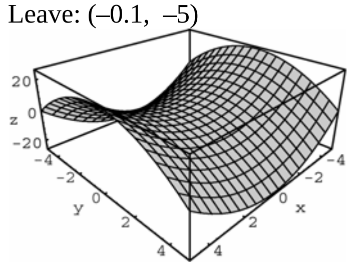
$\lim\limits_{(x,\, y)\to(0,\, 0)} f(x, y)$ does not exist since $(x, y) \to (0, 0)$:

along the y-axis, $f(x, y) = 0$, but along the $y = x^4$ curve, $f(x, y) = 1$.

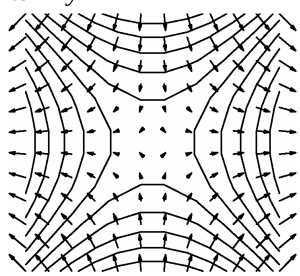
Therefore, f is not differentiable at the origin. But $D_{\mathbf{u}} f(0, 0)$ exists for all u since

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h}$$
$$= \lim_{h\to 0} (0) = 0, \text{ and}$$
$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h}$$
$$= \lim_{h\to 0} (0) = 0, \text{ so } \nabla f(0, 0) = \langle 0, 0 \rangle = \mathbf{0}. \text{ Then}$$
$$D_{\mathbf{u}} f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u} = \mathbf{0} \cdot \mathbf{u} = 0.$$

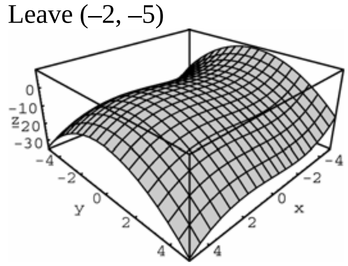
33. Leave: (–0.1, –5)



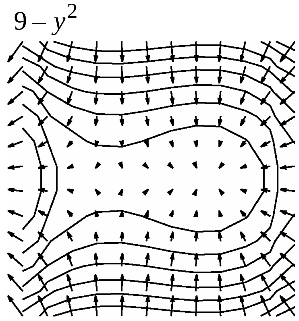
$x^2 - y^2$



34. Leave (–2, –5)



$\frac{x - x^3}{9 - y^2}$



35. Leave: (3, 5)

36. (4.2, 4.2)

12.6 Concepts Review

1. $\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$

2. $y^2\cos t+2xy(-\sin t)$
$=\cos^3 t-2\sin^2 t\cos t$

3. $\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

4. 12

Problem Set 12.6

1. $\frac{dw}{dt}=(2xy^3)(3t^2)+(3x^2y^2)(2t)$
$=(2t^9)(3t^2)+(3t^{10})(2t)=12t^{11}$

2. $\frac{dw}{dt}=(2xy-y^2)(-\sin t)+(x^2-2xy)(\cos t)$
$=(\sin t+\cos t)(1-3\sin t\cos t)$

3. $\frac{dw}{dt}=(e^x\sin y+e^y\cos x)(3)+(e^x\cos y+e^y\sin x)(2)=3e^{3t}\sin 2t+3e^{2t}\cos 3t+2e^{3t}\cos 2t+2e^{2t}\sin 3t$

4. $\frac{dw}{dt}=\left(\frac{1}{x}\right)\sec^2 t+\left(-\frac{1}{y}\right)(2\sec^2 t\tan t)=\frac{\sec^2 t}{\tan t}-2\tan t=\frac{\sec^2 t-2\tan^2 t}{\tan t}=\frac{1-\tan^2 t}{\tan t}$

5. $\frac{dw}{dt}=[yz^2(\cos(xyz^2)](3t^2)+[xz^2\cos(xyz^2)](2t)+[2xyz\cos(xyz^2)](1)$
$=(3yz^2t^2+2xz^2t+2xyz)\cos(xyz^2)=(3t^6+2t^6+2t^6)\cos(t^7)=7t^6\cos(t^7)$

6. $\frac{dw}{dt}=(y+z)(2t)+(x+z)(-2t)+(y+x)(-1)=2t(2-t-t^2)-2t(1-t+t^2)-(1)=-4t^3+2t-1$

7. $\frac{\partial w}{\partial t}=(2xy)(s)+(x^2)(-1)=2st(s-t)s-s^2t^2=s^2t(2s-3t)$

8. $\frac{\partial w}{\partial t}=(2x-x^{-1}y)(-st^{-2})+(-\ln x)(s^2)=s^2\left[1-2t^{-3}-\ln\left(\frac{s}{t}\right)\right]$

9. $\frac{\partial w}{\partial t}=e^{x^2+y^2}(2x)(s\cos t)+e^{x^2+y^2}(2y)(\sin s)=2e^{x^2+y^2}(xs\cos t+y\sin s)$
$=2(s^2\sin t\cos t+t\sin^2 s)\exp(s^2\sin^2 t+t^2\sin^2 s)$

10. $\frac{\partial w}{\partial t}=[(x+y)^{-1}-(x-y)^{-1}](e^s)+[(x+y)^{-1}+(x-y)^{-1}](se^{st})=\frac{2e^{s(t+1)}(st-1)}{t^2e^{2s}-e^{2st}}$

11. $\frac{\partial w}{\partial t}=\frac{x(-s\sin st)}{(x^2+y^2+z^2)^{1/2}}+\frac{y(s\cos st)}{(x^2+y^2+z^2)^{1/2}}+\frac{z(s^2)}{(x^2+y^2+z^2)^{1/2}} = s^4t(1+s^4t^2)^{-1/2}$

12. $\frac{\partial w}{\partial t}=(e^{xy+z}y)(1)+(e^{xy+z}x)(-1)+(e^{xy+z})(2t) = e^{xy+z}(y-x+2t)=e^{s^2}(0)=0$

13. $\frac{\partial z}{\partial t}=(2xy)(2)+(x^2)(-2st) = 4(2t+s)(1-st^2)-2st(2t+s)^2;\left.\left(\frac{\partial z}{\partial t}\right)\right|_{(1,\ -2)} = 72$

14. $\frac{\partial z}{\partial s}=(y+1)(1)+(x+1)(rt) = 1+rt(1+2s+r+t);\left.\left(\frac{\partial z}{\partial s}\right)\right|_{(1,\ -1,\ 2)} = 5$

15. $\frac{dw}{dx}=(2u-\tan v)(1)+(-u\sec^2 v)(\pi) = 2x-\tan\pi x-\pi x\sec^2\pi x$

$\left.\frac{dw}{dx}\right|_{x=\frac{1}{4}}=\left(\frac{1}{2}\right)-1-\left(\frac{\pi}{2}\right)=-\frac{1+\pi}{2}\approx -2.0708$

16. $\frac{\partial w}{\partial \theta}=(2xy)(-\rho\sin\theta\sin\phi)+(x^2)(\rho\cos\theta\sin\phi)+(2z)(0) = \rho^3\cos\theta\sin^3\phi(-2\sin^2\theta+\cos^2\theta);$

$\left.\left(\frac{\partial w}{\partial\theta}\right)\right|_{(2,\ \pi,\ \frac{\pi}{2})} = -8$

17. $V(r,\ h)=\pi r^2h,\ \frac{dr}{dt}=0.5\ \text{in./yr},$

$\frac{dh}{dt}=8\ \text{in./yr}$

$\frac{dV}{dt}=(2\pi rh)\left(\frac{dr}{dt}\right)+(\pi r^2)\left(\frac{dh}{dt}\right);$

$\left.\left(\frac{dV}{dt}\right)\right|_{(20,\ 400)} = 11200\pi\ \text{in.}^3/\text{yr}$

$=\frac{11200\pi\ \text{in.}^3}{1\ \text{yr}}\times\frac{1\ \text{board ft}}{144\ \text{in.}^3}\approx 244.35\ \text{board ft/yr}$

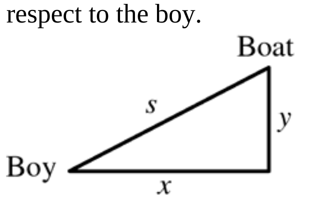
18. Let $T=e^{-x-3y}$.

$\frac{dT}{dt}=e^{-x-3y}(-1)\frac{dx}{dt}+e^{-x-3y}(-3)\frac{dy}{dt}$

$=e^{-x-3y}(-1)(2)+e^{-x-3y}(-3)(2) = -8e^{-x-3y}$

$\left.\frac{dT}{dt}\right|_{(0,\ 0)} = -8,$ so the temperature is decreasing at 8°/min.

19. The stream carries the boat along at 2 ft/s with respect to the boy.



$\frac{dx}{dt}=2,\ \frac{dy}{dt}=4,\ s^2=x^2+y^2$

$2s\left(\frac{ds}{dt}\right)=2x\left(\frac{dx}{dt}\right)+2y\left(\frac{dy}{dt}\right)$

$\frac{ds}{dt}=\frac{(2x+4y)}{s}$

When $t=3$, $x=6$, $y=12$, $s=6\sqrt{5}$. Thus,

$\left.\left(\frac{ds}{dt}\right)\right|_{t=3} = \sqrt{20}\approx 4.47\ \text{ft/s}$

20. $V(r,\ h)=\left(\frac{1}{3}\right)\pi r^2h,\ \frac{dh}{dt}=3\ \text{in./min},$

$\frac{dr}{dt}=2\ \text{in./min}$

$\frac{dV}{dt}=\left(\frac{2}{3}\right)\pi rh\left(\frac{dr}{dt}\right)+\left(\frac{1}{3}\right)\pi r^2\left(\frac{dh}{dt}\right);$

$\left.\left(\frac{dV}{dt}\right)\right|_{(40,\ 100)} = \frac{20{,}800\pi}{3}\approx 21{,}782\ \text{in.}^3/\text{min}$

21. Let $F(x,\ y) = x^3 + 2x^2y - y^3 = 0$.

Then $\dfrac{dy}{dx} = \dfrac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\dfrac{(3x^2+4xy)}{2x^2-3y^2} = \dfrac{3x^2+4xy}{3y^2-2x^2}$.

22. Let $F(x,\ y) = ye^{-x} + 5x - 17 = 0$.

$$\frac{dy}{dx} = -\frac{(-ye^{-x}+5)}{e^{-x}} = y - 5e^x$$

23. Let $F(x, y) = x \sin y + y \cos x = 0$.

$$\frac{dy}{dx} = -\frac{(\sin y - y\sin x)}{x\cos y + \cos x} = \frac{y\sin x - \sin y}{x\cos y + \cos x}$$

24. Let $F(x,\ y) = x^2 \cos y - y^2 \sin x = 0$.

Then $\dfrac{dy}{dx} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \dfrac{-(2x\cos y - y^2\cos x)}{-x^2\sin y - 2y\sin x}$

$= \dfrac{2x\cos y - y^2\cos x}{x^2\sin y + 2y\sin x}$.

25. Let $F(x,\ y,\ z) = 3x^2z + y^3 - xyz^3 = 0$.

$$\frac{\partial z}{\partial x} = -\frac{(6xz - yz^3)}{3x^2 - 3xyz^2} = \frac{yz^3 - 6xz}{3x^2 - 3xyz^2}$$

26. Let $f(x,\ y,\ z) = ye^{-x} + z\sin x = 0$.

$$\frac{\partial x}{\partial z} = \frac{-\sin x}{-ye^{-x} + z\cos x} = \frac{\sin x}{ye^{-x} - z\cos x}$$

27. $\dfrac{\partial T}{\partial s} = \dfrac{\partial T}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial T}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial T}{\partial z}\dfrac{\partial z}{\partial s} + \dfrac{\partial T}{\partial w}\dfrac{\partial w}{\partial s}$

28. We use the z_r notation for $\dfrac{\partial z}{\partial r}$.

$z_r = z_x x_r + z_y y_r = z_x \cos\theta + z_y \sin\theta$

$z_\theta = z_x x_\theta + z_y y_\theta = z_x(-r\sin\theta) + z_y(r\cos\theta)$, so

$r^{-1}z_\theta = -z_x\sin\theta + z_y\cos\theta$. Thus,

$(z_r)^2 + (r^{-2})(z_\theta)^2 = (z_x\cos\theta + z_y\sin\theta)^2$

$\qquad + (-z_x\sin\theta + z_y\cos\theta)^2$

$= (z_x)^2 + (z_y)^2$ (expanding and using

$\sin^2\theta + \cos^2\theta = 1$).

29. $y = \left(\dfrac{1}{2}\right)[f(u) + f(v)]$,

where $u = x - ct$, $v = x + ct$.

$y_x = \left(\dfrac{1}{2}\right)[f'(u)(1) + f'(v)(1)] = \left(\dfrac{1}{2}\right)[f'(u) + f'(v)]$

$y_{xx} = \left(\dfrac{1}{2}\right)[f''(u)(1) + f''(v)(1)]$

$= \left(\dfrac{1}{2}\right)[f''(u) + f''(v)]$

$y_t = \left(\dfrac{1}{2}\right)[f'(u)(-c) + f'(v)(c)]$

$= \left(-\dfrac{c}{2}\right)[f'(u) - f'(u)]$

$y_{tt} = \left(-\dfrac{c}{2}\right)[f''(u)(-c) - f''(v)(c)]$

$= \left(\dfrac{c^2}{2}\right)[f''(u) + f''(v)] = c^2 y_{xx}$

30. Let $w = f(x, y, z)$ where $x = r - s$, $y = s - t$, $z = t - r$. Then

$w_r + w_s + w_t = (w_x x_r + w_x x_s) + (w_y y_s + w_y y_t)$

$\qquad + (w_z z_t + w_z z_r)$

$= [w_x(1) + w_x(-1)] + [w_y(1) + w_y(-1)]$

$\qquad + [w_z(1) + w_z(-1)]$

$= 0$

31. Let $w = \int_x^y f(u)du = -\int_y^x f(u)du$, where $x = g(t)$, $y = h(t)$.

Then

$\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = -f(x)g'(t) + f(y)h'(t)$

$= f(h(t))h'(t) - f(g(t))g'(t)$.

Thus, for the particular function given

$F'(t) = \sqrt{9 + (t^2)^4}\,(2t)$

$\qquad - \sqrt{9 + \left(\sin\sqrt{2}\pi t\right)^4}\left(\sqrt{2}\pi\cos\sqrt{2}\pi t\right)$;

$F'\left(\sqrt{2}\right) = (5)\left(2\sqrt{2}\right) - (3)\left(\sqrt{2}\pi\right)$

$= 10\sqrt{2} - 3\sqrt{2}\pi \approx 0.8135$.

32. If $f(tx,\ ty) = tf(x,\ y)$, then

$\dfrac{d}{dt}[f(tx, ty)] = \dfrac{d}{dt}[tf(x, y)]$.

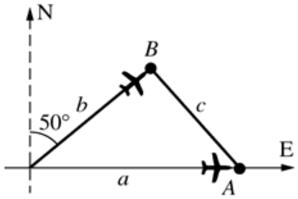
That is,

$[f_{tx}(tx,\ ty)][x] + [f_{ty}(tx,\ ty)][y] = f(x,\ y)$.

Letting $t = 1$ yields the desired result.

33. $c^2 = a^2 + b^2 - 2ab\cos 40°$ (Law of Cosines) where a, b, and c are functions of t.

$2cc' = 2aa' + 2bb' - 2(a'b + ab')\cos 40°$ so $c' = \dfrac{aa' + bb' - (a'b + ab')\cos 40°}{c}$.



When $a = 200$ and $b = 150$, $c^2 = (200)^2 + (150)^2 - 2(200)(150)\cos 40° = 62{,}500 - 60{,}000 \cos 40°$.

It is given that $a' = 450$ and $b' = 400$, so at that instant,

$$c' = \frac{(200)(450) + (150)(400) - [(450)(150) + (200)(400)]\cos 40°}{\sqrt{62{,}500 - 60{,}000\cos 40°}} \approx 288.$$

Thus, the distance between the airplanes is increasing at about 288 mph.

34. $r = \langle x, y, z\rangle$, so $r^2 = |r|^2 = x^2 + y^2 + z^2$.

$F = \dfrac{GMm}{x^2 + y^2 + z^2}$, so

$$F'(t) = F_m m'(t) + F_x x'(t) + F_y y'(t) + F_z z'(t)$$

$$= \frac{GMm'(t)}{x^2 + y^2 + z^2} - \frac{2GMmxx'(t)}{(x^2 + y^2 + z^2)^2} - \frac{2GMmyy'(t)}{(x^2 + y^2 + z^2)^2} + \frac{2GMmzz'(t)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{GM[(x^2 + y^2 + z^2)m'(t) - 2m(xx'(t) + yy'(t) + zz'(t))]}{(x^2 + y^2 + z^2)^2}.$$

12.7 Concepts Review

1. perpendicular

2. $\langle 3, 1, -1\rangle$

3. $x - +4(y-1) + 6(z-1) = 0$

4. $\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

Problem Set 12.7

1. $\nabla F(x, y, z) = 2\langle x, y, z\rangle$;

$\nabla F\left(2, 3, \sqrt{3}\right) = 2\left\langle 2, 3, \sqrt{3}\right\rangle$

Tangent Plane:

$2(x-2) + 3(y-3) + \sqrt{3}\left(z - \sqrt{3}\right) = 0$, or

$2x + 3y + \sqrt{3}z = 16$

2. $\nabla F(x, y, z) = 2\langle 8x, y, 8z\rangle$;

$\nabla F\left(1, 2, \dfrac{\sqrt{2}}{2}\right) = 4\left\langle 4, 1, 2\sqrt{2}\right\rangle$

Tangent Plane:

$4(x-1) + 1(y-2) + 2\sqrt{2}\left(z - \dfrac{\sqrt{2}}{2}\right)$, or

$4x + y + 2\sqrt{2}z = 8$.

3. Let $F(x, y, z) = x^2 - y^2 + z^2 + 1 = 0$.

$\nabla F(x, y, z) = \langle 2x, -2y, 2z\rangle = 2\langle x, -y, z\rangle$

$\nabla F\left(1, 3, \sqrt{7}\right) = 2\left\langle 1, -3, \sqrt{7}\right\rangle$, so $\left\langle 1, -3, \sqrt{7}\right\rangle$ is normal to the surface at the point. Then the tangent plane is

$1(x-1) - 3(y-3) + \sqrt{7}\left(z - \sqrt{7}\right) = 0$, or more simply, $x - 3y + \sqrt{7}z = -1$.

4. $\nabla f(x, y, z) = 2\langle x, y, -z\rangle$;
$\nabla f(2, 1, 1) = 2\langle 2, 1, -1\rangle$
Tangent plane:
$2(x-2)+1(y-1)-1(z-1)=0$, or $2x+y-z=4$.

5. $\nabla f(x, y) = \left(\frac{1}{2}\right)\langle x, y\rangle; \nabla f(2, 2) = \langle 1, 1\rangle$
Tangent plane: $z-2=1(x-2)+1(y-2)$, or
$x+y-z=2$.

6. Let $f(x, y) = xe^{-2y}$.
$\nabla f(x, y) = \left\langle e^{-2y}, -2xe^{-2y}\right\rangle$
$\nabla f(1,0) = \langle 1, -2\rangle$
Then $\langle 1, -2, -1\rangle$ is normal to the surface at
(1, 0, 1), and the tangent plane is
$1(x-1)-2(y-0)-1(z-1)=0$, or $x-2y-z=0$.

7. $\nabla f(x, y) = \left\langle -4e^{3y}\sin 2x, 6e^{3y}\cos 2x\right\rangle$;
$\nabla f\left(\frac{\pi}{3}, 0\right) = \left\langle -2\sqrt{3}, -3\right\rangle$
Tangent plane: $z+1=-2\sqrt{3}\left(x-\frac{\pi}{3}\right)-3(y-0)$,
or $2\sqrt{3}x+3y+z=\frac{\left(2\sqrt{3}\pi-3\right)}{3}$.

8. $\nabla f(x, y) = \left(\frac{1}{2}\right)\left\langle \frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}}\right\rangle$; $\nabla f(1, 4) = \left\langle \frac{1}{2}, \frac{1}{4}\right\rangle$
Tangent plane: $z-3=\left(\frac{1}{2}\right)(x-1)+\left(\frac{1}{4}\right)(y-4)$,
or $\frac{1}{2}x+\frac{1}{4}y-z=-\frac{3}{2}$.

9. Let $z = f(x, y) = 2x^2y^3$;
$dz = 4xy^3dx + 6x^2y^2dy$. For the points given,
$dx = -0.01$, $dy = 0.02$,
$dz = 4(-0.01)+6(0.02) = 0.08$.
$\Delta z = f(0.99, 1.02) - f(1, 1)$
$= 2(0.99)^2(1.02)^3 - 2(1)^2(1)^3 \approx 0.08017992$

10. $dz = (2x-5y)dx + (-5x+1)dy$
$= (-11)(0.03)+(-9)(-0.02) = -0.15$
$\Delta z = f(2.03, 2.98) - f(2,3) = -0.1461$

11. $dz = 2x^{-1}dx + y^{-1}dy = (-1)(0.02)+\left(\frac{1}{4}\right)(-0.04)$
$= -0.03$
$\Delta z = f(-1.98, 3.96) - f(-2.4)$
$= \ln[(-1.98)^2(3.96)] - \ln 16 \approx -0.030151$

12. Let $z = f(x, y) = \tan^{-1} xy$;
$dz = \frac{y}{1+x^2y^2}dx + \frac{x}{1+x^2y^2}dy$;
$= \frac{(-0.5)(-0.03)+(-2)(-0.01)}{1+(4)(0.25)} = 0.0175.$
$\Delta z = f(-2.03, -0.51) - f(-2, -0.5) \approx 0.017342$

13. Let
$F(x, y, z) = x^2 - 2xy - y^2 - 8x + 4y - z = 0$;
$\nabla F(x, y, z) = \langle 2x-2y-8, -2x-2y+4, -1\rangle$
Tangent plane is horizontal if $\nabla F = \langle 0, 0, k\rangle$ for any $k \neq 0$.
$2x - 2y - 8 = 0$ and $-2x - 2y + 4 = 0$ if $x = 3$ and $y = -1$. Then $z = -14$. There is a horizontal tangent plane at (3, –1, –14).

14. $\langle 8, -3, -1\rangle$ is normal to $8x - 3y - z = 0$.
Let $F(x, y, z) = 2x^2 + 3y^2 - z$.
$\nabla F(x, y, z) = \langle 4x, 6y, -1\rangle$ is normal to
$z = 2x^2 + 3y^2$ at (x, y, z). $4x = 8$ and
$6y = -3$, if $x = 2$ and $y = -\frac{1}{2}$; then
$z = 8.75$ at $\left(2, -\frac{1}{2}, 8.75\right)$.

15. For $F(x, y, z) = x^2 + 4y + z^2 = 0$,
$\nabla F(x, y, z) = \langle 2x, 4, 2z\rangle = 2\langle x, 2, z\rangle$.
$F(0, -1, 2) = 0$, and
$\nabla F(0, -1, 2) = 2\langle 0, 2, 2\rangle = 4\langle 0, 1, 1\rangle$.
For $G(x, y, z) = x^2 + y^2 + z^2 - 6z + 7 = 0$,
$\nabla G(x, y, z) = \langle 2x, 2y, 2z-6\rangle = 2\langle x, y, z-3\rangle$.
$G(0, -1, 2) = 0$, and
$\nabla G(0, -1, 2) = 2\langle 0, -1, -1\rangle = -2\langle 0, 1, 1\rangle$.
$\langle 0, 1, 1\rangle$ is normal to both surfaces at
(0, –1, 2) so the surfaces have the same tangent plane; hence, they are tangent to each other at (0, –1, 2).

16. (1, 1, 1) satisfies each equation, so the surfaces intersect at (1, 1, 1). For

$z = f(x, y) = x^2 y : \nabla f(x, y) = \left\langle 2xy, x^2 \right\rangle;$

$\nabla f(1, 1) = \langle 2, 1\rangle$, so $\langle 2, 1, -1\rangle$ is normal at (1, 1, 1).

For $f(x, y, z) = x^2 - 4y + 3 = 0;$

$\nabla f(x, y, z) = \langle 2, -4, 0\rangle;$

$\nabla f(1, 1, 1) = \langle 2, -4, 0\rangle$ so $\langle 2, -4, 0\rangle$ is normal at (1, 1, 1).

$\langle 2, 1, -1\rangle \cdot \langle 2, -4, 0\rangle = 0$, so the normals, hence tangent planes, and hence the surfaces, are perpendicular at (1, 1, 1).

17. Let $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 12 = 0;$

$\nabla F(x, y, z) = 2\langle x, 2y, 3z\rangle$ is normal to the plane.

A vector in the direction of the line, $\langle 2, 8, -6\rangle = 2\langle 1, 4, -3\rangle$, is normal to the plane.

$\langle x, 2y, 3z\rangle = k\langle 1, 4, \text{-}3\rangle$ and (x, y, z) is on the surface for points (1, 2, –1) [when k =1] and (–1, –2, 1) [when k = –1].

18. Let $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

$$\nabla F(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

$$\nabla F(x_0, y_0, z_0) = 2\left\langle \frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2} \right\rangle$$

The tangent plane at (x_0, y_0, z_0) is

$$\frac{x_0(x - x_0)}{a^2} + \frac{y_0(y - y_0)}{b^2} + \frac{z_0(z - z_0)}{c^2} = 0.$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \right) = 0$$

Therefore, $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} + \dfrac{z_0 z}{c^2} = 1$, since

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1.$$

19. $\nabla f(x, y, z) = 2\langle 9x, 4y, 4z\rangle;$

$\nabla f(1, 2, 2) = 2\langle 9, 8, 8\rangle$

$\nabla g(x, y, z) = 2\langle 2x, -y, 3z\rangle;$

$\nabla f(1, 2, 2) = 4\langle 1, -1, 3\rangle$

$\langle 9, 8, 8\rangle \times \langle 1, -1, 3\rangle = \langle 32, -19, -17\rangle$

Line: $x = 1 + 32t, y = 2 \text{-} 19t, z = 2 \text{-} 17t$

20. Let $f(x, y, z) = x - z^2$, and $g(x, y, z) = y - z^3$.

$\nabla f(x, y, z) = \langle 1, 0, -2z\rangle$ and

$\nabla g(x, y, z) = \left\langle 0, 1, -3z^2 \right\rangle$

$\nabla f(1, 1, 1) = \langle 1, 0, -2\rangle$ and

$\nabla g(1, 1, 1) = \langle 0, 1, -3\rangle$

$\langle 1, 0, -2\rangle \times \langle 0, 1, -3\rangle = \langle 2, 3, 1\rangle$

Line: $x = 1 + 2t, y = 1 + 3t, z = 1 + t$

21. $dS = S_A dA + S_W dW$

$$= -\frac{W}{(A-W)^2}dA + \frac{A}{(A-W)^2}dW = \frac{-WdA + AdW}{(A-W)^2}$$

At W = 20, A = 36:

$$dS = \frac{-20dA + 36dW}{256} = \frac{-5dA + 9dW}{64}.$$

Thus, $|dS| \le \dfrac{5|dA| + 9|dW|}{64} \le \dfrac{5(0.02) + 9(0.02)}{64}$

$= 0.004375$

22. $V = lwh$, $dl = dw = \dfrac{1}{2}$, $dh = \dfrac{1}{4}$, l = 72, w = 48, h = 36

$dV = whdl + lhdw + lwdh = 3024$ in.^{3} (1.75 ft^3)

23. $V = \pi r^2 h, dV = 2\pi rh\,dr + \pi r^2 dh$

$|dV| \le 2\pi rh|dr| + \pi r^2 |dh| \le 2\pi rh(0.02r) + \pi r^2(0.03h)$

$= 0.04\pi r^2 h + 0.03\pi r^2 h = 0.07V$

Maximum error in V is 7%.

24. $T = f(L, g) = 2\pi\sqrt{\dfrac{L}{g}}$

$dT = f_L dL + f_g dg$

$$= 2\pi\left(\frac{1}{2\sqrt{\frac{L}{g}}}\right)\left(\frac{1}{g}\right)dL + 2\pi\left(\frac{1}{2\sqrt{\frac{L}{g}}}\right)\left(-\frac{L}{g^2}\right)dg$$

$= \dfrac{\pi(gdL - Ldg)}{g^2\sqrt{\frac{L}{g}}}$, so

$$\frac{dT}{T} = \frac{\pi(gdL - Ldg)}{\left(2\pi\sqrt{\frac{L}{g}}\right)\left(g^2\sqrt{\frac{L}{g}}\right)} = \frac{gdL - Ldg}{2gL}$$

$$= \frac{1}{2}\left(\frac{dL}{L} - \frac{dg}{g}\right).$$

Therefore,

$$\left|\frac{dT}{T}\right| \le \frac{1}{2}\left(\left|\frac{dL}{L}\right| + \left|\frac{dg}{g}\right|\right) = \frac{1}{2}(0.5\% + 0.3\%) = 0.4\%.$$

25. Solving for R, $R = \dfrac{R_1R_2}{R_1+R_2}$, so

$$\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1+R_2)^2} \text{ and } \frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1+R_2)^2}.$$

Therefore, $dR = \dfrac{R_2^2 dR_1 + R_1^2 dR_2}{(R_1+R_2)^2}$;

$|dR| \le \dfrac{R_2^2|dR_1| + R_1^2|dR_2|}{(R_1+R_2)^2}$. Then at $R_1 = 25$,

$R_2 = 100, dR_1 = dR_2 = 0.5$, $R = \dfrac{(25)(100)}{25+100} = 20$

and $|dR| \le \dfrac{(100)^2(0.5)+(25)^2(0.5)}{(125)^2} = 0.34.$

26. Let $F(x, y, z) = x^2 + y^2 + 2z^2$.

$\nabla F(x, y, z) = \langle 2x, 2y, 4z\rangle$;

$\nabla F(1, 2, 1) = 2\langle 1, 2, 2\rangle$; $\dfrac{\nabla F}{|\nabla F|} = \left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right\rangle$

Thus, $\mathbf{u} = \left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right\rangle$ is the unit vector in the direction of flight, and

$\langle x, y, z\rangle = \langle 1, 2, 1\rangle + 4t\left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right\rangle$ is the location of the bee along its line of flight t seconds after takeoff. Using the parametric form of the line of flight to substitute into the equation of the plane yields $t = 3$ as the time of intersection with the plane. Then substituting this value of t into the equation of the line yields $x = 5$, $y = 10$, $z = 9$ so the point of intersection is (5, 10, 9).

27. Let $F(x, y, z) = xyz = k$; let (a, b, c) be any point on the surface of F.

$\nabla F(x, y, z) = \langle yz, xz, xy\rangle = \left\langle \dfrac{k}{x}, \dfrac{k}{y}, \dfrac{k}{z}\right\rangle$

$= k\left\langle \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}\right\rangle$

$\nabla F(a, b, c) = k\left\langle \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}\right\rangle$

An equation of the tangent plane at the point is

$\left(\dfrac{1}{a}\right)(x-a) + \left(\dfrac{1}{b}\right)(x-b) + \left(\dfrac{1}{c}\right)(x-c) = 0$, or

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 3.$

Points of intersection of the tangent plane on the coordinate axes are $(3a, 0, 0)$, $(0, 3b, 0)$, and $(0, 0, 3c)$.

The volume of the tetrahedron is

$\left(\dfrac{1}{3}\right)$(area of base)(altitude)$= \dfrac{1}{3}\left(\dfrac{1}{2}|3a||3b|\right)(|3c|)$

$= \dfrac{9|abc|}{2} = \dfrac{9|k|}{2}$ (a constant).

28. If $F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$, then $\nabla F(x, y, z) = 0.5\left\langle \dfrac{1}{\sqrt{x}}, \dfrac{1}{\sqrt{y}}, \dfrac{1}{\sqrt{z}}\right\rangle$. The equation of the tangent is

$$0.5\left\langle \frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}}\right\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0, \text{ or } \frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = a.$$

Intercepts are $a\sqrt{x_0}, a\sqrt{y_0}, a\sqrt{z_0}$; so the sum is $a\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = a^2$.

29. $f(x, y) = (x^2 + y^2)^{1/2}$; $f(3, 4) = 5$

$f_x(x, y) = x(x^2 + y^2)^{-1/2}$; $f_x(3, 4) = \frac{3}{5} = 0.6$; $f_y = (x, y) = y(x^2 + y^2)^{-1/2}$; $f_x(3, 4) = \frac{4}{5} = 0.8$

$f_{xx}(x, y) = y^2(x^2 + y^2)^{-3/2}$; $f_x(3, 4) = \frac{16}{125} = 0.128$; $f_{xy}(x, y) = -xy(x^2 + y^2)^{-3/2}$;

$f_{xy}(3, 4) = -\frac{12}{125} = -0.096$

$f_{yy} = x^2(x^2 + y^2)^{-3/2}$; $f_{xx}(3, 4) = \frac{9}{125} = 0.072$

Therefore, the second order Taylor approximation is

$$f(x, y) = 5 + 0.6(x-3) + 0.8(y-4) + 0.5[0.128(x-3)^2 + 2(-0.096)(x-3)(y-4) + 0.072(y-4)^2]$$

a. First order Taylor approximation: $f(x, y) = 5 + 0.6(x - 3) + 0.8(y - 4)$.
Thus, $f(3.1, 3.9) \approx 5 + 0.6(0.1) + 0.8(-0.1) = 4.98$.

b. $f(3.1, 3.9) \approx 5 + 0.6(-0.1) + 0.8(0.1) + 0.5[0.128(0.1)^2 + 2(-0.096)(0.1)(-0.1) + 0.072(-0.1)^2] = 4.98196$

c. $f(3.1,\ 3.9) \approx 4.9819675$

30. $f(x, y) = \tan\left(\frac{x^2 + y^2}{64}\right)$; $f(0, 0) = 0$

$f_x(x, y) = \frac{x}{32} \cdot \sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_x(0, 0) = 0$

$f_y(x, y) = \frac{y}{32} \cdot \sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_y(0, 0) = 0$

$f_{xx}(x, y) = \frac{2x^2}{32^2} \sec^2\left(\frac{x^2 + y^2}{64}\right) \tan\left(\frac{x^2 + y^2}{64}\right) + \frac{1}{32} \sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_{xx}(0, 0) = \frac{1}{32}$

$f_{yy}(x, y) = \frac{2y^2}{32^2} \sec^2\left(\frac{x^2 + y^2}{64}\right) \tan\left(\frac{x^2 + y^2}{64}\right) + \frac{1}{32} \sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_{yy}(0, 0) = \frac{1}{32}$

When computed, each term of $f_{xy}(x, y)$ will contain either x or y, resulting in $f_{xy}(0, 0) = 0$. Therefore, the second-order Taylor approximation is

$$f(x, y) = 0 + 0 \cdot x + 0 \cdot y + \frac{1}{2}\left[\frac{1}{32}x^2 + 2 \cdot 0 \cdot x \cdot y + \frac{1}{32}y^2\right] = \frac{1}{64}x^2 + \frac{1}{64}y^2$$

a. The first-order Taylor approximation is $f(x, y) = 0 + 0 \cdot x + 0 \cdot y = 0$; Thus, $f(0.2, -0.3) \approx 0$.

b. $f(0.2, -0.3) \approx \frac{1}{64}(0.2)^2 + \frac{1}{64}(-0.3)^3 = 0.00203125$

c. $f(0.2, -0.3) \approx 0.0020312528$

12.8 Concepts Review

1. closed bounded

2. boundary; stationary; singular

3. $\nabla f(x_0, y_0) = \mathbf{0}$

4. $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$

Problem Set 12.8

1. $\nabla f(x, y) = \langle 2x-4, 8y\rangle = \langle 0, 0\rangle$ at (2, 0), a stationary point.
$D = f_{xx} f_{yy} - f_{xy}^2 = (2)(8)-(0)^2 = 16 > 0$ and $f_{xx} = 2 > 0.$ Local minimum at (2, 0).

2. $\nabla f(x, y) = \langle 2x-2, 8y+8\rangle = \langle 0, 0\rangle$ at (1, –1), a stationary point. $D = f_{xx} f_{yy} - f_{xy}^2 = (2)(8)-(0)2 = 16 > 0$ and $f_{xx} = 2 > 0.$ Local minimum at (1, –1).

3. $\nabla f(x, y) = \left\langle 8x^3 - 2x, 6y\right\rangle = \left\langle 2x(4x^2-1), 6y\right\rangle$
$= \langle 0,0\rangle,$ at $(0,0), (0.5,0), (-0.5,0)$ all stationary points.
$f_{xx} = 24x^2 - 2;$ $D = f_{xx} f_{yy} - f_{xy}^2 = (24x^2-2)(6)-(0)^2 = 12(12x^2-1).$
At $(0,0):$ $D = -12$, so (0, 0) is a saddle point.
At $(0.5,0)$ and $(-0.5,0):$ $D = 24$ and $f_{xx} = 6$, so local minima occur at these points.

4. $\nabla f(x, y) = \left\langle y^2 - 12x, 2xy-6y\right\rangle = \langle 0, 0\rangle$ at stationary points $(0,0), (3,-6)$ and $(3,6).$
$D = f_{xx} f_{yy} - f_{xy}^2 = (-12)(2x-6)-(2y)^2 = -4(y^2+6x-18),$ $f_{xx} = -12$
At (0, 0): D = 72, and $f_{xx} = -12,$ so local maximum at (0, 0).
At $(3, \pm 6):$ $D = -144,$ so $(3, \pm y)$ are saddle points.

5. $\nabla f(x, y) = \langle y, x\rangle = \langle 0, 0\rangle$ at (0, 0), a stationary point.
$D = f_{xx} f_{yy} - f_{xy}^2 = (0)(0)-(1)^2 = -1,$ so (0, 0) is a saddle point.

6. Let $\nabla f(x, y) = \left\langle 3x^2 - 6y, 3y^2 - 6x\right\rangle = \langle 0, 0\rangle.$ Then $3x^2 - 6y = 0$ and $3y^2 - 6x = 0.$
$3x^2 - 6y = 0 \to 3x^2 = 6y \to x^2 = 2y \to x^4 = 4y^2 \to \frac{1}{4}x^4 = y^2$
$3y^2 - 6x = 0 \to 3\left(\frac{1}{4}x^4\right) - 6x = 0 \to \frac{3}{4}x^4 - 6x = 0 \to \frac{3}{4}x\left(x^3 - 8\right) = 0 \to \frac{3}{4}x(x-2)\left(x^2+2x+4\right) = 0 \to x = 0, x = 2$
$x = 0:$ $3x^2 - 6y = 0 \to 3(0) - 6y = 0 \to -6y = 0 \to y = 0$
$x = 2:$ $3x^2 - 6y = 0 \to 3(2)^2 - 6y \to 12 - 6y = 0 \to 12 = 6y \to 2 = y$
Solving simultaneously, we obtain the solutions (0, 0) and (2, 2).
$f_{xx} = 6x;$ $D = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - (-6)^2 = 36\,(xy-1)$; At (0, 0): $D = -36 < 0$, so (0, 0) is a saddle point.
At (2, 2): $D = 108 > 0$, $f_{xx} > 0,$ so a local minimum occurs at $(2,2)$.

7. $\nabla f(x, y) = \left\langle \frac{x^2 y - 2}{x^2}, \frac{xy^2 - 4}{y^2} \right\rangle = \langle 0, 0\rangle$ at (1, 2).

$D = f_{xx} f_{yy} - f_{xy}^2 = (4x-3)(8y-3) - (1)^2 = 32x^{-3}y^{-3} - 1,\ f_{xx} = 4x^{-3}$

At (1, 2): $D = 3 > 0$, and $f_{xx} > 0,$ so a local minimum at (1, 2).

8. $\nabla f(x, y) = -2\exp(-x^2 - y^2 + 4y)\langle x, y-2\rangle = \langle 0, 0\rangle$ at $(0, 2)$.

$D = f_{xx} f_{yy} - f_{xy}^2 = \exp 2(-x^2 - y^2 + 4y)[(4x^2 - 2)(4y^2 - 16y + 14) - (4xy - 8x)^2],$

$f_{xx} = (4x^2 - 2)\exp(-x^2 - y^2 + 4y)$

At $(0, 2)$: $D > 0$, and $f_{xx} < 0,$ so local maximum at (0, 2).

9. Let $\nabla f(x, y)$

$= \langle -\sin x - \sin(x+y), -\sin y - \sin(x+y)\rangle = \langle 0, 0\rangle$

Then $\begin{pmatrix} -\sin x - \sin(x+y) = 0 \\ \sin y + \sin(x+y) = 0 \end{pmatrix}$. Therefore,

$\sin x = \sin y$, so $x = y = \frac{\pi}{4}$. However, these values satisfy neither equation. Therefore, the gradient is defined but never zero in its domain, and the boundary of the domain is outside the domain, so there are no critical points.

10. $\nabla f(x, y) = \langle 2x - 2a\cos y, 2ax\sin y\rangle = \langle 0, 0\rangle$ at

$\left(0, \pm\frac{\pi}{2}\right), (a, 0)$

$D = f_{xx} f_{yy} - f_{xy}^2 = (2)(2ax\cos y) - (2a\sin y)^2,$

$f_{xx} = 2$

At $\left(0, \pm\frac{\pi}{2}\right)$: $D = -4a^2 < 0,$ so $\left(0, \pm\frac{\pi}{2}\right)$ are saddle points. At $(a, 0)$: $D = 4a^2 > 0$ and $f_{xx} > 0,$ so local minimum at $(a, 0)$.

11. We do not need to use calculus for this one. $3x$ is minimum at 0 and $4y$ is minimum at –1. (0, –1) is in S, so $3x + 4y$ is minimum at (0, –1); the minimum value is –4. Similarly, $3x$ and $4y$ are each maximum at 1. (1, 1) is in S, so $3x + 4y$ is maximum at (1, 1); the maximum value is 7. (Use calculus techniques and compare.)

12. We do not need to use calculus for this one. Each of x^2 and y^2 is minimum at 0 and (0, 0) is in S, so $x^2 + y^2$ is minimum at (0, 0); the minimum value is 0. Similarly, x^2 and y^2 are maximum at $x = 3$ and $y = 4$, respectively, and (3, 4) is in S, so $x^2 + y^2$ is maximum at (3, 4); the maximum value is 25. (Use calculus techniques and compare.)

13. $\nabla f(x, y) = \langle 2x, -2y\rangle = \langle 0, 0\rangle$ at (0, 0).

$D = f_{xx} f_{yy} - f_{xy}^2 = (2)(-2) - (0)2 < 0,$ so (0, 0) is a saddle point. A parametric representation of the boundary of S is $x = \cos t$, $y = \sin t$, t in $[0, 2\pi]$.

$f(x, y) = f(x(t), y(t)) = \cos^2 t - \sin^2 t + 1$

$= \cos 2t - 1$

$\cos 2t - 1$ is maximum if $\cos 2t = 1$, which occurs for $t = 0,\ \pi,\ 2\pi$. The points of the curve are $(\pm 1, 0)$. $f(\pm 1, 0) = 2$

$f(x, y) = \cos 2t - 1$ is minimum if $\cos 2t = -1$, which occurs for $t = \frac{\pi}{2}, \frac{3\pi}{2}$. The points of the curve are $(0, \pm 1)$. $f(0, \pm 1) = 0$. Global minimum of 0 at (0, ± 1); global maximum of 2 at (±1, 0).

14. $\nabla f(x, y) = \langle 2x - 6, 2y - 8\rangle = \langle 0, 0\rangle$ at (3, 4), which is outside S, so there are no stationary points. There are also no singular points. $x = \cos t$, $y = \sin t$, t in $[0, 2\pi]$ is a parametric representation of the boundary of S.

$f(x, y) = f(x(t), y(t))$

$= \cos^2 t - 6\cos t + \sin^2 t - 8\sin t + 7$

$= 8 - 6\cos t - 8\sin t = F(t)$

$F'(t) = 6\sin t - 8\cos t = 0$ if $\tan t = \frac{4}{3}$. t can be in the 1st or 3rd quadrants. The corresponding points of the curve are $\left(\pm\frac{3}{5}, \pm\frac{4}{5}\right)$.

$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = 18;\ f\left(\frac{3}{5}, \frac{4}{5}\right) = -2$

Global minimum of –2 at $\left(\frac{3}{5}, \frac{4}{5}\right)$; global maximum of 18 at $\left(-\frac{3}{5}, -\frac{4}{5}\right)$.

15. Let x, y, z denote the numbers, so $x + y + z = N$. Maximize

$P = xyz = xy(N - x - y) = Nxy - x^2y - xy^2.$

Let $\nabla P(x, y) = \left\langle Ny - 2xy - y^2, Nx - x^2 - 2xy \right\rangle$

$= \langle 0, 0 \rangle.$

Then $\begin{pmatrix} Ny - 2xy - y^2 = 0 \\ Nx - x^2 - 2xy = 0 \end{pmatrix}.$

$N(x, -y) = x^2 - y^2 = (x+y)(x-y).$ $x = y$ or $N = x + y.$

Therefore, $x = y$ (since $N = x + y$ would mean that $P = 0$, certainly not a maximum value).

Then, substituting into $Nx - x^2 - 2xy = 0,$ we obtain $Nx - x^2 - 2x^2 = 0,$ from which we obtain $x(N - 3x) = 0$, so $x = \frac{N}{3}$ (since $x = 0 \Rightarrow P = 0$).

$P_{xx} = -2y;$

$D = P_{xx}P_{yy} - P_{xy}^2$

$= (-2y)(-2x) - (N - 2x - 2y)^2$

$= 4xy - (N - 2x - 2y)^2$

At $x = y = \frac{N}{3}: D = \frac{N^2}{3} > 0, P_{xx} = -\frac{2N}{3} < 0$ (so local maximum)

If $x = y = \frac{N}{3},$ then $z = \frac{N}{3}.$

Conclusion: Each number is $\frac{N}{3}$. (If the intent is to find three distinct numbers, then there is no maximum value of P that satisfies that condition.)

16. Let s be the distance from the origin to (x, y, z) on the plane. $s^2 = x^2 + y^2 + z^2$ and $x + 2y + 3z = 12.$ Minimize

$s^2 = f(y, z) = (12 - 2y - 3z)^2 + y^2 + z^2.$

$\nabla f(y, z) = \langle -48 + 12x + 10y, -72 + 12y + 20z \rangle$

$= \langle 0, 0 \rangle$ at $\left(\frac{12}{7}, \frac{18}{7}\right).$

$D = f_{yy}f_{zz} - f_{yz}^2 = 56 > 0$ and $f_{yy} = 10 > 0;$

local maximum at $\left(\frac{12}{7}, \frac{18}{7}\right)$

$s^2 = \frac{504}{49},$ so the shortest distance is

$s = \frac{6\sqrt{14}}{7} \approx 3.2071.$

17. Let S denote the surface area of the box with dimensions x, y, z.
$S = 2xy + 2xz + 2yz$ and $V_0 = xyz,$ so

$S = 2(xy + V_0y^{-1} + V_0x^{-1}).$

Minimize $f(x, y) = xy + V_0y^{-1} + V_0x^{-1}$ subject to $x > 0, y > 0.$

$\nabla f(x, y) = \left\langle y - V_0x^{-2}, x - V_0y^{-2} \right\rangle = \langle 0, 0 \rangle$ at $(V_0^{1/3}, V_0^{1/3}).$

$D = f_{xx}f_{yy} - f_{xy}^2 = 4V_0^2x^{-3}y^{-3} - 1,$

$f_{xx} = 2V_0x^{-3}.$

At $\left(V_0^{1/3}, V_0^{1/3}\right)$: $D = 3 > 0,$ $f_{xx} = 2 > 0,$ so local minimum.

Conclusion: The box is a cube with edge $V_0^{1/3}.$

18. Let L denote the sum of edge lengths for a box of dimensions x, y, z. Minimize $L = 4x + 4y + 4z$, subject to $V_0 = xyz.$

$L(x, y) = 4x + 4y + \frac{4V_0}{xy}, x > 0, y > 0$

Let

$\nabla L(x, y) = 4x^{-1}y^{-1}\left\langle x^{-1}(x^2y - V_0), y^{-1}(xy^2 - V_0) \right\rangle$

$= \langle 0,0 \rangle.$

Then $x^2y = V_0$ and $xy^2 = V_0,$ from which it follows that $x = y$. Therefore $x = y = z = V_0^{1/3}.$

$L_{xx} = \frac{8V_0}{x^3y};$

$D = L_{xx}L_{yy} - L_{xy}^2 = \left(\frac{8V_0}{x^3y}\right)\left(\frac{8V_0}{xy^3}\right) - \left(\frac{4V_0}{x^2y^2}\right)^2$

At $\left(V_0^{1/3}, V_0^{1/3}\right)$: $D > 0,$ $L_{xx} > 0$ (so local minimum).

There are no other critical points, and as $(x, y) \to$ boundary, $L \to \infty$. Hence, the optimal box is a cube with edge $V_0^{1/3}.$

19. Let S denote the area of the sides and bottom of the tank with base l by w and depth h.
$S = lw + 2lh + 2wh$ and $lwh = 256.$

$S(l, w) = lw + 2l\left(\frac{256}{lw}\right) + 2w\left(\frac{256}{lw}\right),$ $w > 0, l > 0.$

$S(l\ w) = \left\langle w - 5121^{-2}, l - 512w^{-2} \right\rangle = \langle 0, 0 \rangle$ at $(8, 8)$. $h = 4$ there. At $(8, 8)$ $D > 0$ and $S_{11} > 0,$ so local minimum. Dimensions are 8’ × 8’ × 4’.

20. Let V denote the volume of the box and (x, y, z) denote its 1st octant vertex.

$V = (2x)(2y)(2z) = 8xyz$ and $24x^2 + y^2 + z^2 = 9$.

$$V^2 = 64\left[\left(\frac{1}{24}\right)(9 - y^2 - z^2)\right]y^2z^2$$

Maximize $f(y, z) = (9 - y^2 - z^2)y^2z^2$, $y > 0$,
$z > 0$.

$\nabla f(y, z) = 2\left\langle yz^2(9 - 2y^2 - z^2),\ y^2z(9 - y^2 - 2z^2)\right\rangle = \langle 0, 0\rangle$ at $(\sqrt{3}, \sqrt{3})$. $x = \frac{\sqrt{2}}{4}$

At $\left(\sqrt{3}, \sqrt{3}\right)$, $D = f_{yy}f_{zz} - f_{yz}^2 > 0$ and $f_{yy} < 0$, so local maximum. The greatest possible volume is

$$8\left(\frac{\sqrt{2}}{4}\right)\left(\sqrt{3}\right)\left(\sqrt{3}\right) = 6\sqrt{2}.$$

21. Let $\langle x, y, z\rangle$ denote the vector; let S be the sum of its components.

$x^2 + y^2 + z^2 = 81$, so $z = (81 - x^2 - y^2)^{1/2}$.

Maximize $S(x, y) = x + y + (81 - x^2 - y^2)^{1/2}$, $0 \le x^2 + y^2 \le 9$.

Let $\nabla S(x, y) = \left\langle 1 - x(81 - x^2 - y^2)^{-1/2}, 1 - y(81 - x^2 - y^2)^{-1/2}\right\rangle = \langle 0, 0\rangle$.

Therefore, $x = (81 - x^2 - y^2)^{1/2}$ and $y = (81 - x^2 - y^2)^{1/2}$. We then obtain $x = y = 3\sqrt{3}$ as the only stationary point. For these values of x and y, $z = 3\sqrt{3}$ and $S = 9\sqrt{3} \approx 15.59$.

The boundary needs to be checked. It is fairly easy to check each edge of the boundary separately. The largest value of S at a boundary point occurs at three places and turns out to be $\frac{18}{\sqrt{2}} \approx 12.73$.

Conclusion: the vector is $3\sqrt{3}\langle 1, 1, 1\rangle$.

22. Let $P(x, x, z)$ be any point in the plane $2x + 4y + 3z = 12$. The square of the distance between the origin and P is $d^2 = x^2 + y^2 + z^2$. Consequently, $d^2 = f(x, y) = x^2 + y^2 + (12 - 2x - 4y)^2/9$. To find the critical points, set $f_x(x, y) = 2x + \frac{2}{9}(12 - 2x - 4y)(-2) = 0$ and $f_y(x, y) = 2y + \frac{2}{9}(12 - 2x - 4y)(-4) = 0$ The resulting system of equations is $13x + 8y = 24$ and $8x + 25y = 48$, which leads to a critical point of $\left(\frac{24}{29}, \frac{48}{29}\right)$. Since $f_{xx}(x, y) = \frac{26}{9}$, $f_{yy}(x, y) = \frac{50}{9}$, and $f_{xy}(x, y) = \frac{16}{9}$, $D\left(\frac{24}{29}, \frac{48}{29}\right) = \frac{116}{9}$ Since $D\left(\frac{24}{29}, \frac{48}{29}\right) > 0$ and $f_{xx}\left(\frac{24}{29}, \frac{48}{29}\right) > 0$, $\left(\frac{24}{29}, \frac{48}{29}\right)$ yields a minimum distance. The point on the plane $2x + 4y + 3z = 12$ that is closest to the origin is $\left(\frac{24}{29}, \frac{48}{29}, \frac{36}{29}\right)$ and this minimum distance is approximately 2.2283.

23. Let $P(x, y, z)$ be any point on $z = x^2 + y^2$. The square of the distance between the point $(1, 2, 0)$ and P can be expressed as $d^2 = f(x, y) = (x - 1)^2 + (y - 2)^2 + z^2$. To find the critical points, set $f_x(x, y)$ $= 4x^3 + 2x + 4xy^2 - 2 = 0$ and $f_y(x, y) = 4y^3 + 2y + 4x^2y - 4 = 0$. Multiplying the first equation by y and the second equation by x and summing the results leads to the equation $-2y + 4x = 0$. Thus, $y = 2x$. Substituting into the first equation yields $10x^3 + x - 1 = 0$, whose solution is $x \approx 0.393$. Consequently, $y \approx 0.786$. $f_{xx}(x, y) = 2 + 12x^2 + 4y^2$, $f_{yy}(x, y) = 2 + 12y^2 + 4x^2$, and $f_{xy}(x, y) = 8xy$. The value of D for the critical point (0.393, 0.786) is approximately 57 and since $f_{xx}(0.393, 0.786) > 0$, (0.393, 0.786) yields a minimum distance. The point on the surface $z = x^2 + y^2$ is (0.393, 0.786, 0.772) and this minimum distance is approximately 1.56.

24. Let (x, y, z) denote a point on the cone, and s denote the distance between (x, y, z) and $(1, 2, 0)$.
$s^2 = (x-1)^2 + (y-2)^2 + z^2$ and $z^2 = x^2 + y^2$. Minimize $s^2 = f(x, y) = (x-1)^2 + (y-2)^2 + (x^2 + y^2)$, x, y in $\mathbb{R}$.

$\nabla f(x, y) = 2\langle 2x-1, 2y-2\rangle = \langle 0, 0\rangle$ at $\left(\frac{1}{2}, 1\right)$. At $\left(\frac{1}{2}, 1\right)$, $D > 0$ and $f_{xx} > 0$, so local minimum.

Conclusion: Minimum distance is $s = \sqrt{\frac{5}{2}} \approx 1.5811$.

25. $A = \left(\frac{1}{2}\right)[y + (y + 2x\sin\alpha)](x\cos\alpha)$ and

$2x + y = 12$. Maximize $A(x, \alpha) = 12x\cos\alpha - 2x^2\cos\alpha + \left(\frac{1}{2}\right)x^2\sin 2\alpha$, x in $(0, 6]$, a in $\left(0, \frac{\pi}{2}\right)$.

$A(x, \alpha) = \left\langle 12\cos\alpha - 4x\cos\alpha + 2x\sin\alpha\cos\alpha, -12x\sin\alpha + 2x^2\sin\alpha + x^2\cos 2\alpha\right\rangle = \langle 0, 0\rangle$ at $\left(4, \frac{\pi}{6}\right)$.

At $\left(4, \frac{\pi}{6}\right)$, $D > 0$ and $A_{xx} < 0$, so local maximum, and $A = 12\sqrt{3} \approx 20.78$. At the boundary point of $x = 6$, we get

$\alpha = \frac{\pi}{4}$, $A = 18$. Thus, the maximum occurs for width of turned-up sides = 4", and base angle $= \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$.

26. The lines are skew since there are no values of s and t that simultaneously satisfy $t - 1 = 3s$, $2t = s + 2$, and $t + 3 = 2s - 1$. Minimize f, the square of the distance between points on the two lines.

$f(s, t) = (3s - t + 1)^2 + (s + 2 - 2t)^2 + (2s - 1 - t - 3)^2$

Let

$\nabla f(s, t) = \langle 2(3s - t + 1)(3) + 2(s - 2t + 2)(1) + 2(2s - t - 4)(2), 2(3s - t + 1)(-1) + 2(s - 2t + 2)(-2) + 2(2s - t - 4)(-1)\rangle$

$= \langle 28s - 14t - 6, -14s + 12t - 28\rangle = \langle 0, 0\rangle$.

Solve $28s - 14t - 6 = 0$, $-14s + 12t - 2 = 0$, obtaining $s = \frac{5}{7}, t = 1$.

$D = f_{ss}f_{tt} - f_{st}^2 = (28)(12) - (-14)^2 > 0$; $f_{ss} = 28 > 0$. (local minimum)

The nature of the problem indicates the global minimum occurs here.

$f\left(\frac{5}{7}, 1\right) = \left(\frac{15}{7}\right)^2 + \left(\frac{5}{7}\right)^2 + \left(-\frac{25}{7}\right)^2 = \frac{875}{49}$

Conclusion: The minimum distance between the lines is $\sqrt{875}/7 \approx 4.2258$.

27. Let M be the maximum value of $f(x, y)$ on the polygonal region, P. Then $ax + by + (c - M) = 0$ is a line that either contains a vertex of P or divides P into two subregions. In the latter case $ax + by + (c - M)$ is positive in one of the regions and negative in the other. $ax + by + (c - M) > 0$ contradicts that M is the maximum value of $ax + by + c$ on P. (Similar argument for minimum.)

a.

x	y	$2x + 3y + 4$
–1	2	8
0	1	7
1	0	6
–3	0	–2
0	–4	–8

Maximum at $(-1, 2)$

b.

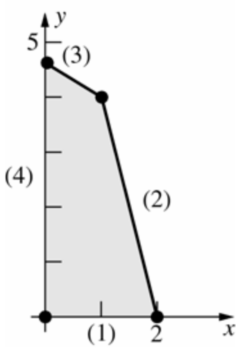
x	y	$-3x + 2y + 1$
–3	0	10
0	5	11
2	3	1
4	0	–11
1	–4	–10

Minimum at (4, 0)

28.

x	y	$2x + y$
0	0	0
2	0	4
1	4	6
0	14/3	14/3

Maximum of 6 occurs at $(1, 4)$

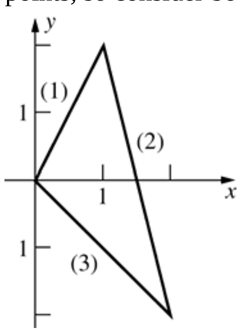


The edges of P are segments of the lines:
1. $y = 0$
2. $4x + y = 8$
3. $2x + 3y = 14$, and
4. $x = 0$

29. $z(x, y) = y^2 - x^2$

$z(x, y) = \langle -2x, 2y \rangle = \langle 0, 0 \rangle$ at (0, 0).

There are no stationary points and no singular points, so consider boundary points.



On side 1:

$y = 2x$, so $z = 4x^2 - x^2 = 3x^2$

$z'(x) = 6x = 0$ if $x = 0$.

Therefore, (0, 0) is a candidate.

On side 2:

$y = -4x + 6$, so

$z = (-4x+6)^2 - x^2 = 15x^2 - 48x + 36.$

$z'(x) = 30x - 48 = 0$ if $x = 1.6$.

Therefore, (1.6, –0.4) is a candidate.

On side 3:

$y = -x$, so $z = (-x)^2 - x^2 = 0.$

Also, all vertices are candidates.

x	y	z
0	0	0
1.6	–0.4	–2.4
2	–2	0
1	2	3

Minimum value of –2.4; maximum value of 3

30. **a.** $$\frac{\partial f}{\partial m} = \sum_{i=1}^{n} \frac{\partial}{\partial m}\left(y_i - mx_i - b\right)^2$$

$$= 2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)(-x_i)$$

$$= -2\sum_{i=1}^{n}\left(x_i y_i - mx_i^2 - bx_i\right)$$

Setting this result equal to zero yields

$$0 = -2\sum_{i=1}^{n}\left(x_i y_i - mx_i^2 - bx_i\right)$$

$$0 = \sum_{i=1}^{n}\left(x_i y_i - mx_i^2 - bx_i\right)$$

or equivalently,

$$\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i$$

$$\frac{\partial f}{\partial b} = \sum_{i=1}^{n} \frac{\partial}{\partial b}\left(y_i - mx_i - b\right)^2$$

$$= 2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)(-1)$$

$$= -2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)$$

Setting this result equal to zero yields

$$0 = -2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)$$

$$0 = \sum_{i=1}^{n}\left(y_i - mx_i - b\right)$$

or equivalently,

$$m\sum_{i=1}^{n} x_i + nb = \sum_{i=1}^{n} y_i$$

b. $$nb = \sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i$$

Therefore,

$$b = \frac{\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i}{n}$$

$$\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + \frac{\left(\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i}{n}$$

This simplifies into

$$m = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$$

c. $\frac{\partial^2 f}{\partial m^2} = 2\sum_{i=1}^{n} x_i^2$

$\frac{\partial^2 f}{\partial b^2} = 2n$

$\frac{\partial^2 f}{\partial m \partial b} = 2\sum_{i=1}^{n} x_i$

Then, by Theorem C, we have

$$D = 4n\left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right).$$

Assuming that all the x_i are not the same, we find that $D > 0$ and $\frac{\partial^2 f}{\partial m^2} > 0$.

Thus, $f(m, b)$ is minimized.

31.

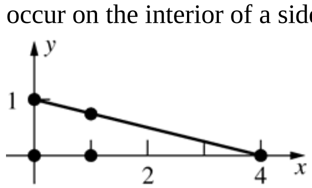
	x_i	y_i	x_i^2	$x_i y_i$
	3	2	9	6
	4	3	16	12
	5	4	25	20
	6	4	36	24
	7	5	49	35
$\Sigma_{i=1}^{5}$	25	18	135	97

$m(135) + b(25) = (97)$ and $m(25) + (5)b = (18)$.
Solve simultaneously and obtain $m = 0.7$, $b = 0.1$.
The least-squares line is $y = 0.7x + 0.1$.

32. $z = 2x^2 + y^2 - 4x - 2y + 5$, so

$\nabla z = \langle 4x - 4,\ 2y - 2\rangle = \mathbf{0}$.

$\nabla z = \mathbf{0}$ at (1, 1) which is outside the region. Therefore, extreme values occur on the boundary. Three critical points are the vertices of the triangle, (0, 0), (0, 1), and (4, 0). Others may occur on the interior of a side of the triangle.



On vertical side: $x = 0$

$z(y) = y^2 - 2y + 5$, $y = [0, 1]$. $z'(y) = 2y - 2$, so $z'(y) = 0$ if $y = 1$. Hence, no additional critical point.

On horizontal side: $y = 0$

$z(x) = 2x^2 - 4x + 5$, x in [0, 4]. $z'(x) = 4x - 4$, so $z'(x) = 0$ if $x = 1$. Hence, an additional critical point is (1, 0).

On hypotenuse: $x = 4 - 4y$

$z(y) = 2(4 - 4y)^2 + y^2 - 4(4 - 4y) - 2y + 5$

$= 33y^2 - 50y + 21$, y in [0, 1].

$z'(y) = 66y - 50$, so $z'(y) = 0$ if $y = \frac{25}{33}$.

Hence, an additional critical point is $\left(\frac{32}{33}, \frac{25}{33}\right)$.

x	y	z
0	0	5
4	0	21
0	1	4
1	0	3
32/33	25/33	2.06

Maximum value of z is 21; it occurs at (4, 0).
Minimum value of z is about 2.06; it occurs at $\left(\frac{32}{33}, \frac{25}{33}\right)$.

33. Let x and y be defined as shown in Figure 4 from Section 12.8. The total cost is given by

$C(x, y) = 400\sqrt{x^2 + 50^2} + 200(200 - x - y)$

$+300\sqrt{y^2 + 100^2}$

Taking partial derivatives and setting them equal to 0 gives

$C_x(x, y) = 200(x^2 + 50^2)^{-1/2}(2x) - 200 = 0$

$C_y(x, y) = 150(y^2 + 100^2)^{-1/2}(2y) - 200 = 0$

The solution of these equations is

$x = \frac{50}{\sqrt{3}} \approx 28.8675$ and $y = \frac{100}{\sqrt{1.25}} \approx 89.4427$

We now apply the second derivative test:

$$C_{xx}(x, y) = \frac{400\sqrt{x^2 + 50^2} - 400x^2 \Big/ \sqrt{x^2 + 50^2}}{x^2 + 50^2}$$

$$C_{yy}(x, y) = \frac{300\sqrt{y^2 + 100^2} - 300y^2 \Big/ \sqrt{y^2 + 100^2}}{y^2 + 100^2}$$

$C_{xy}(x, y) = 0$

Evaluated at $x = 50/\sqrt{3}$ and $y = 100/\sqrt{1.25}$,

$D \approx (5.196)(1.24) - 0^2 > 0$

Thus, a local minimum occurs with

$C\left(50/\sqrt{3}, 100/\sqrt{1.25}\right) \approx \$79,681$

We must also check the boundary. When x = 0,

$C_1(y) = C(0, y) = 200(200 - y) + 300\sqrt{y^2 + 100^2}$

and when y = 0,

$C_2(x) = C(x,0) = 400\sqrt{x^2+50^2} + 200(200-x)$
Using the methods from Chapter 3, we find that C_1 reaches a minimum of about $82,361 when $y = \sqrt{8000}$ and C_2 reaches a minimum of about $87,321 when $x = \sqrt{2500/3}$. Addressing the boundary x + y = 200, we find that
$C_3(x) = C(x, 200-x) = 400\sqrt{x^2+50^2}$
$+300\sqrt{(200-x)^2+100^2}$ This function reaches a minimum of about $82,214 when $x \approx 41.08$.
Thus, the minimum cost path is when
$x = 50/\sqrt{3} \approx 28.8675$ ft and
$y = 100/\sqrt{1.25} \approx 89.4427$ ft, which produces a cost of about $79,681.

34. Let x and y be defined as shown in Figure 4 from Section 12.8. The total cost is given by
$C(x,y) = 500\sqrt{x^2+50^2} + 200(200-x-y)$
$+100\sqrt{y^2+100^2}$
Taking partial derivatives and setting them equal to 0 gives
$C_x(x,y) = 500(x^2+50^2)^{-1/2}(2x) - 200 = 0$
$C_y(x,y) = 100(y^2+100^2)^{-1/2}(2y) - 200 = 0$
There is, however, no solution to $C_y(x,y) = 0$
Now we check the boundary. When x = 0,
$C_1(y) = C(0,y) = 200(200-y) + 100\sqrt{y^2+100^2}$
There is, however, no solution to $C_1'(y) = 0$.
When $y = 0$,
$C_2(x) = C(x,0) = 500\sqrt{x^2+50^2} + 200(200-x)$
$C_2'(x) = 0$ yields $x = 100/\sqrt{21}$ and
$C\left(100/\sqrt{21}, 0\right) \approx \$72,913$
On the boundary $x + y = 200$, we find that
$C_3(x) = C(x, 200-x) = 500\sqrt{x^2+50^2}$
$+100\sqrt{(200-x)^2+100^2}$ This function reaches a minimum of about $46,961 when $x \approx 9.0016$.
Thus, the minimum cost path is when
$x \approx 9.0016$ ft and $y \approx 190.9984$ ft, which produces a cost of about $46,961.

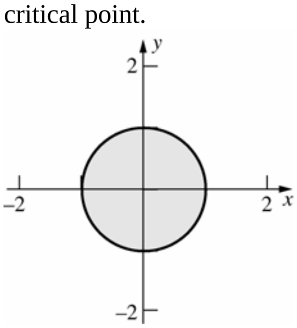
35. $f(x,y) = 10 + x + y$
$\nabla f = \langle 1,1 \rangle \neq \mathbf{0}$; thus no interior critical points exist. Letting
$x = 3\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi,$
$g(t) = f(3\cos t, 3\sin t)$ and
$g'(t) = 3\cos t - 3\sin t.$ Setting $g'(t) = 0$
yields $t = \pi/4$ or $5\pi/4$.
The critical points are $\left(3/\sqrt{2}, 3/\sqrt{2}\right)$ and
$\left(-3/\sqrt{2}, -3/\sqrt{2}\right)$.
Since $f\left(3/\sqrt{2}, 3/\sqrt{2}\right) = 10 + 6/\sqrt{2}$ and
$f\left(-3/\sqrt{2}, -3/\sqrt{2}\right) = 10 - 6/\sqrt{2}$, the minimum value of f on $x^2 + y^2 \le 3$ is
$10 - 6/\sqrt{2}$ and the maximum value of f is $10 + 6/\sqrt{2}$.

36. $f(x,y) = x^2 + y^2;\ \nabla f = \langle 2x, 2y \rangle.$
$\nabla f = \mathbf{0}$ at (0,0).
$D(0,0) = 2 \cdot 2 - 0^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0,$
Thus, $f(0,0) = 0$ is a minimum.
In order to optimize $g(t) = f(a\cos t, b\sin t)$
where $0 \le t \le 2\pi$, we find
$g'(t) = 2x(-a\sin t) + 2y(b\cos t)$
$= 2b^2 \sin t\cos t - 2a^2 \sin t \cos t$
$= (b^2 - a^2)\sin 2t.$ Setting $g'(t) = \mathbf{0}$, we have $t = 0, \pi/2, \pi,$ or $3\pi/2$. The resulting critical points are $(a,0)$, $(0,b)$, $(-a,0)$, and $(0,-b)$.
$f(a,0) = f(-a,0) = a^2;\quad f(0,b) = f(0,-b) = b^2.$
Since a > b, the maximum value of f on the given region is a^2 and the minimum value of f is 0.

37. The volume of the box can be expressed as $V(l,w,h) = lwh = 2$ and the surface area as
$S(l,w,h) = 2lh + 2wh + lw + lw$. Since $h = \frac{2}{lw}$,
$S(l,w) = \frac{4}{w} + \frac{4}{l} + lw + lw$ When cost is factored,
$C(l,w) = \dfrac{1}{w} + \dfrac{1}{l} + 0.65lw$ with $w > 0, l > 0$
$C_l(l,w) = -\dfrac{1}{l^2} + 0.65w = 0$
$C_w(l,w) = -\dfrac{1}{w^2} + 0.65l = 0$
Solving this system of equations leads to
$w = \sqrt[3]{\dfrac{0.65}{0.4225}} \approx 1.1544$ and $l = w \approx 1.1544$.
Consequently, $h \approx 1.501$. Applying the second derivative test with $C_{ll}(l,w) = \dfrac{2}{l^3}$,
$C_{ww}(l,w) = \dfrac{2}{w^3}$ and $C_{lw}(l,w) = 0.65,$
$D \approx 1.268 > 0.$ Thus, the minimum cost occurs when the length is approximately 1.1544 feet, the width is approximately 1.1544 feet and the height is approximately 1.501 feet.

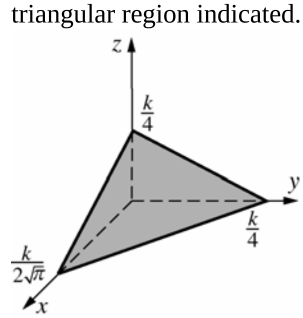
38. The cost function in three variables is
$C(l, w, h) = 4lw + 2lh + 2wh + 6l + 6w + 4h,$
where $lwh = 60$. Substituting $h = \frac{60}{lw}$ yields
$C(l, w) = 4lw + \frac{120}{w} + 6l + 6w + \frac{240}{lw}$ with
$l > 0$ and $w > 0$.
$C_l(l, w) = 4w - \frac{120}{l^2} + 6 - \frac{240}{wl^2} = 0$
$C_w(l, w) = 4l - \frac{120}{w^2} + 6 - \frac{240}{lw^2} = 0$
Multiplying both sides of the first equation by wl^2, multiplying both sides of the second equation by lw^2, and subtracting the resulting equations produces $-120w + 120l = 0$ or $l = w$.
Consequently, $4w - \frac{120}{w^2} + 6 - \frac{240}{w^3} = 0$ or
$2w^4 + 3w^3 - 60w - 120 = 0$ Using a CAS, this equation yields $w \approx 3.2134$
$C_{ll}(l, w) = \frac{240}{l^3} + \frac{480}{wl^3}$; $C_{ww}(l, w) = \frac{240}{w^3} + \frac{480}{lw^3}$;
$C_{lw}(l, w) = 4 + \frac{240}{l^2w^2}$; Using the critical point
(3.2134, 3.2134), $D \approx 131.44 > 0$
Thus, $w = l \approx 3.2$ yields a minimum. The minimum cost involved with making this box is approximately \$177.79. This minimum cost occurs when the length and width are approximately 3.2 feet and the height is approximately 5.8 feet.

39. $T(x, y) = 2x^2 + y^2 - y$
$\nabla T = \langle 4x, 2y - 1\rangle = \mathbf{0}$
If $x = 0$ and $y = \frac{1}{2}$, so $\left(0, \frac{1}{2}\right)$ is the only interior critical point.



On the boundary $x^2 = 1 - y^2$, so **T** is a function of y there.
$T(y) = 2(1 - y^2) + y^2 - y = 2 - y - y^2,$
$y = [-1, 1]$
$T'(y) = -1 - 2y = 0$ if $y = -\frac{1}{2}$, so on the boundary, critical points occur where y is
$-1, -\frac{1}{2}, 1.$
Thus, points to consider are $\left(0, \frac{1}{2}\right)$, $(0, -1)$, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and (0, 1). Substituting these into $T(x, y)$ yields that the coldest spot is $\left(0, \frac{1}{2}\right)$ where the temperature is $-\frac{1}{4}$, and there is a tie for the hottest spot at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ where the temperature is $\frac{9}{4}$.

40. Let x^2, y^2, z^2 denote the areas enclosed by the circle, and the two squares, respectively. Then the radius of the circle is $\frac{x}{\sqrt{\pi}}$, and the edges of the two squares are y and z, respectively. We wish to optimize $A(x, y, z) = x^2 + y^2 + z^2$, subject to $2\pi\left(\frac{x}{\sqrt{\pi}}\right) + 4y + 4z = k$, or equivalently $2\sqrt{\pi}x + 4y + 4z = k$, with each of x, y, and z nonnegative. Geometrically: we seek the smallest and largest of all spheres with center at the origin and some point in common with the triangular region indicated.

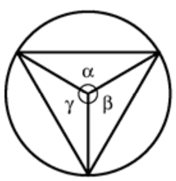


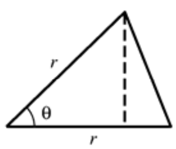
Since $\frac{k}{2\sqrt{\pi}} > \frac{k}{4}$, the largest sphere will intersect the region only at point $\left(\frac{k}{2\sqrt{\pi}}, 0, 0\right)$ and will thus have radius $\frac{k}{2\sqrt{\pi}}$. Thus A will be maximum if $x = \frac{k}{2\sqrt{\pi}}$, $y = z = 0$ (all of the wire goes into the circle). The smallest sphere will be tangent to the triangle. The point of tangency is on the normal line through the origin,

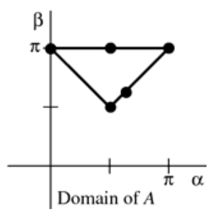
$\langle x, y, z\rangle = t\left\langle\sqrt{\pi}, 2, 2\right\rangle$. Substituting $x = \sqrt{\pi}$, $y = 2$, $z = 2$ into the equation of the plane yields the value $t = \dfrac{k}{2(\pi+8)}$, so the minimum value of A is obtained for the values of $x = \dfrac{k\sqrt{\pi}}{2(\pi+8)}$, $y = z = \dfrac{k}{\pi+8}$. Thus the circle will have radius $\dfrac{\left[\frac{k\sqrt{\pi}}{2(\pi+8)}\right]}{\sqrt{\pi}} = \dfrac{k}{2(\pi+8)}$, and the squares will each have sides $\dfrac{k}{(\pi+8)}$. Therefore, the circle will use $\dfrac{\pi k}{(\pi+8)}$ units and the squares will each use $\dfrac{4k}{(\pi+8)}$ units.

[Note: sum of the three lengths is k.]

41. Without loss of generality we will assume that $\alpha \le \beta \le \gamma$. We will consider it intuitively clear that for a triangle of maximum area the center of the circle will be inside or on the boundary of the triangle; i.e., α, β, and γ are in the interval $[0, \pi]$. Along with $\alpha + \beta + \gamma = 2\pi$, this implies that $\alpha + \beta \ge \pi$.







The area of an isosceles triangle with congruent sides of length r and included angle θ is $\frac{1}{2}r^2 \sin\theta$.

$$\begin{aligned}\text{Area}(\Delta ABC) &= \frac{1}{2}r^2 \sin\alpha + \frac{1}{2}r^2 \sin\beta + \frac{1}{2}r^2 \sin\gamma \\ &= \frac{1}{2}r^2(\sin\alpha + \sin\beta + \sin[2\pi - (\alpha+\beta)] \\ &= \frac{1}{2}r^2[\sin\alpha + \sin\beta - \sin(\alpha+\beta)]\end{aligned}$$

Area(ΔABC) will be maximum if (*) $A(\alpha, \beta) = \sin\alpha + \sin\beta - \sin(\alpha+\beta)$ is maximum.

Restrictions are $0 \le \alpha \le \beta \le \pi$, and $\alpha + \beta \ge \pi$.

Three critical points are the vertices of the triangular domain of $A: \left(\frac{\pi}{2}, \frac{\pi}{2}\right), (0, \pi)$, and (π, π). We will now search for others.

$\Delta A(\alpha, \beta) = \langle \cos\alpha - \cos(\alpha+\beta), \cos\beta - \cos(\alpha+\beta)\rangle = 0$ if

$\cos\alpha = \cos(\alpha+\beta) = \cos\beta$.

Therefore, $\cos\alpha = \cos\beta$, so $\alpha = \beta$ [due to the restrictions stated]. Then

$\cos\alpha = \cos(\alpha+\alpha) = \cos 2\alpha = 2\cos^2\alpha - 1$, so $\cos\alpha = 2\cos^2\alpha - 1$.

Solve for α: $2\cos^2\alpha - \cos\alpha - 1 = 0$; $(2\cos\alpha + 1)(\cos\alpha - 1) = 0$;

$\cos\alpha = -\dfrac{1}{2}$ or $\cos\alpha = 1$; $\alpha = \dfrac{2\pi}{3}$ or $\alpha = 0$.

(We are still in the case where $\alpha = \beta$.) $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ is a new critical point, but (0, 0) is out of the domain of A.

There are no critical points in the interior of the domain of A.

On the $\beta = \pi$ edge of the domain of A; $A(\alpha) = \sin\alpha - \sin(\alpha - \pi) = 2\sin\alpha$ so $A'(\alpha) = 2\cos\alpha$.

$A'(\alpha) = 0$ if $\alpha = \frac{\pi}{2}$. $\left(\frac{\pi}{2}, \pi\right)$ is a new critical point.

On the $\beta = \pi - \alpha$ edge of the domain of A:

$A(\alpha) = \sin\alpha + \sin(\pi - \alpha) - \sin(2\alpha - \pi) = 2\sin\alpha + \sin 2\alpha$, so

$A'(\alpha) = 2\cos\alpha + 2\cos 2\alpha = 2[\cos\alpha + (2\cos^2\alpha - 1)] = 2(2\cos\alpha - 1)(\cos\alpha + 1)$.

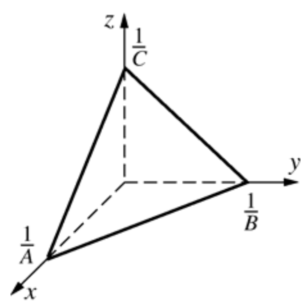
$A'(\alpha) = 0$ if $\cos\alpha = \frac{1}{2}$ or $\cos\alpha = -1$, so $\alpha = \frac{\pi}{3}$ or $\alpha = \pi$.

$\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ and $(\pi, 0)$ are outside the domain of A.

(The critical points are indicated on the graph of the domain of A.)

α	β	A	
$\frac{\pi}{2}$	$\frac{\pi}{2}$	2	
0	π	0	
π	π	0	
$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$	Maximum value of A. The triangle is equilateral.
$\frac{\pi}{2}$	π	2	

42. If the plane through (a, b, c) is expressed as $Ax + By + Cz = 1$, then the intercepts are $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}$; volume of tetrahedron is $V = \left(\frac{1}{3}\right)\left[\left(\frac{1}{2}\right)\left(\frac{1}{A}\right)\left(\frac{1}{B}\right)\right]\left(\frac{1}{C}\right) = \frac{1}{6ABC}$.



To maximize V subject to $Aa + Bb + Cc = 1$ is equivalent to maximizing $z = ABC$ subject to $Aa + Bb + Cc = 1$.

$C = \frac{1 - aA - bB}{c}$, so $z = \frac{AB(1 - aA - bB)}{c}$.

$\nabla z = \left(\frac{1}{c}\right)\left\langle B - 2aAB - bB^2, A - 2bAB - aA^2\right\rangle = \mathbf{0}$ if $A = \frac{1}{3a}, B = \frac{1}{3b}\left[C = \frac{1}{3c}\right]$.

$\left(\frac{1}{3a}, \frac{1}{3b}\right)$ is the only critical point in the first quadrant. The second partials test yields that z is maximum at this point. The plane is $\frac{1}{3a}x + \frac{1}{3b}y + \frac{1}{3c}z = 1$, or $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

The volume of the first quadrant tetrahedron formed by the plane is $\frac{1}{\left[6\left(\frac{1}{3a}\right)\left(\frac{1}{3b}\right)\left(\frac{1}{3c}\right)\right]} = \frac{9abc}{2}$.

43. Local max: $f(1.75, 0) = 1.15$
Global max: $f(-3.8, 0) = 2.30$

44. Global max: $f(0, 1) = 0.5$
Global min: $f(0, -1) = -0.5$

45. Global min: $f(0, 1) = f(0, -1) = -0.12$

46. Global max: $f(0, 0) = 1$

Global min: $f(2, -2) = f(-2, 2) = e^{-9} \approx 0.00012341$

47. Global max: $f(1.13, 0.79) = f(1.13, -0.79) = 0.53$
Global min: $f(-1.13, 0.79) = f(-1.13, -0.79) = -0.53$

48. No global maximum or global minimum

49. Global max: $f(3,3) = f(-3,3) \approx 74.9225$
Global min: $f(1.5708, 0) = f(-1.5708, 0) = -8$

50. Global max: $f(1, 43, 0) = 0.13$
Global min: $f(-1.82, 0) = -0.23$

51. Global max: $f(0.67, 0) = 5.06$
Global min: $f(-0.75, 0) = -3.54$

52. Global max: $f(-5.12, -4.92) = 1071$
Global min: $f(5.24, -4.96) = -658$

53. Global max: $f(2.1, 2.1) = 3.5$
Global min: $f(4.2, 4.2) = -3.5$

54. a.
$$\begin{aligned} k(\alpha,\beta) &= \frac{1}{2}[80\sin\alpha + 60\sin\beta + 48\sin(2\pi - \alpha - \beta)] \\ &= 40\sin\alpha + 30\sin\beta - 24\sin(\alpha+\beta) \\ L(\alpha,\beta) &= (164 - 160\cos\alpha)^{1/2} + (136 - 120\cos\beta)^{1/2} \\ &\quad + (100 - 96\cos(\alpha+\beta))^{1/2} \end{aligned}$$

b. (1.95, 2.04)

c. (2.26, 2.07)

12.9 Concepts Review

1. free; constrained

2. parallel

3. $g(x, y) = 0$

4. (2, 2)

Problem Set 12.9

1. $\langle 2x,\ 2y\rangle = \lambda\langle y,\ x\rangle$
$2x = \lambda y$, $2y = \lambda x$, $xy = 3$
Critical points are
$\left(\pm\sqrt{3}, \pm\sqrt{3}\right)$, $f\left(\pm\sqrt{3}, \pm\sqrt{3}\right) = 6.$
It is not clear whether 6 is the minimum or maximum, so take any other point on $xy = 3$, for example (1, 3). $f(1, 3) = 10$, so 6 is the minimum value.

2. $\langle y,\ x\rangle = \lambda\langle 8x,\ 18y\rangle$
$y = 8\lambda x$, $x = 18\lambda y$, $4x^2 + 9y^2 = 36$
Critical points are $\left(\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right).$
Maximum value of 3 occurs at $\left(\pm\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right).$

3. Let $\nabla f(x,\ y) = \lambda\nabla g(x,\ y)$, where
$g(x,\ y) = x^2 + y^2 - 1 = 0.$
$\langle 8x - 4y, -4x + 2y\rangle = \lambda\langle 2x,\ 2y\rangle$
1. $4x - 2y = \lambda x$
2. $-2x + y = \lambda y$
3. $x^2 + y^2 = 1$
4. $0 = \lambda x + 2\lambda y$ (From equations 1 and 2)
5. $\lambda = 0$ or $x + 2y = 0$ (4)

$\lambda = 0$: 6. $y = 2x$ (1)
7. $x = \pm\frac{1}{\sqrt{5}}$ (6, 3)
8. $y = \pm\frac{2}{\sqrt{5}}$ (7, 6)

$x + 2y = 0$: 9. $x = -2y$
10. $y = \pm\frac{1}{\sqrt{5}}$ (9, 3)
11. $x = \frac{2}{\sqrt{5}}$ (10, 9)

Critical points: $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, $\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$,
$\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$, $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
$f(x, y)$ is 0 at the first two critical points and 5 at the last two. Therefore, the maximum value of $f(x, y)$ is 5.

4. $\langle 2x+4y, 4x+2y\rangle = \lambda\langle 1,-1\rangle$

$2x + 4y = \lambda$, $4x + 2y = -\lambda$, $x - y = 6$

Critical point is (3, –3).

5. $\langle 2x, 2y, 2z\rangle = \lambda\langle 1, 3, -2\rangle$

$2x = \lambda$, $2y = 3\lambda$, $2z = -2\lambda$, $x + 3y - 2z = 12$

Critical point is $\left(\frac{6}{7}, \frac{18}{7}, -\frac{12}{7}\right)$.

$f\left(\frac{6}{7}, \frac{18}{7}, -\frac{12}{7}\right) = \frac{72}{7}$ is the minimum.

6. Let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, where

$g(x, y, z) = 2x^2 + y^2 - 3z = 0$.

$\langle 4, -2, 3\rangle = \lambda\langle 4x, 2y, -3\rangle$

1. $4 = 4\lambda x$
2. $-2 = 2\lambda y$
3. $3 = -3\lambda$
4. $2x^2 + y^2 - 3z = 0$
5. $\lambda = -1$ (3)
6. $x = -1$, $y = 1$ (5, 1, 2)
7. $z = 1$ (6, 4)

Therefore, (–1, 1, 1) is a critical point, and $f(-1, 1, 1) = -3$. (–3 is the minimum rather than maximum since other points satisfying $g = 0$ have larger values of f. For example, $g(1, 1, 1) = 0$, and $f(1, 1, 1) = 5$.)

7. Let l and w denote the dimensions of the base, h denote the depth. Maximize $V(l, w, h) = lwh$ subject to $g(l, w, h) = lw + 2lh + 2wh = 48$.

$\langle wh, lh, lw\rangle = \lambda\langle w+2h, l+2h, 2l+2w\rangle$

$wh = \lambda(w + 2h)$, $lh = \lambda(l + 2h)$, $lw = \lambda(2l + 2w)$, $lw + 2lh + 2wh = 48$

Critical point is (4, 4, 2).

$V(4, 4, 2) = 32$ is the maximum. ($V(11, 2, 1) = 22$, for example.)

8. Minimize the square of the distance to the plane, $f(x, y, z) = x^2 + y^2 + z^2$, subject to $x + 3y - 2z - 4 = 0$.

$\langle 2x, 2y, 2z\rangle = \lambda\langle 1, 3, -2\rangle$

$2x = \lambda$, $2y = 3\lambda$, $2z = -2\lambda$, $x + 3y - 2z = 4$

Critical point is $\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right)$. The nature of the problem indicates that this will give a minimum rather than a maximum. The least distance to the plane is $\left[f\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right)\right]^{1/2} = \left(\frac{8}{7}\right)^{1/2} \approx 1.0690$.

9. Let l and w denote the dimensions of the base, h the depth. Maximize $V(l, w, h) = lwh$ subject to $0.601w + 0.20(lw + 2lh + 2wh) = 12$, which simplifies to $21w + lh + wh = 30$, or $g(l, w, h) = 2lw + lh + wh - 30$.

Let $\nabla V(l, w, h) = \lambda \nabla g(l, w, h)$;

$\langle wh, lh, lw\rangle = \lambda\langle 2w+h, 2l+h, l+w\rangle$.

1. $wh = \lambda(2w + h)$
2. $lh = \lambda(2l + h)$
3. $lw = \lambda(l + w)$
4. $2lw + lh + wh = 30$
5. $(w - l)h = 2\lambda(w - l)$ (1, 2)
6. $w = l$ or $h = 2\lambda$

$w = 1$:

7. $l = 2\lambda = w$ (3) Note: $w \neq 0$, for then $V = 0$.
8. $h = 4\lambda$ (7, 2)
9. $\lambda = \frac{\sqrt{5}}{2}$ (7, 8, 4)
10. $l = w = \sqrt{5}$, $h = 2\sqrt{5}$ (9, 7, 8)

$h = 2\lambda$:

11. $\lambda = 0$ (2)
12. $l = w = h = 0$ (11, 1 – 3)

(Not possible since this does not satisfy 4.)

$\left(\sqrt{5}, \sqrt{5}, 2\sqrt{5}\right)$ is a critical point and

$V\left(\sqrt{5}, \sqrt{5}, 2\sqrt{5}\right) = 10\sqrt{5} \approx 22.36$ ft^3 is the maximum volume (rather than the minimum volume since, for example, $g(1, 1, 14) = 30$ and $V(1, 1, 14) = 14$ which is less than 22.36).

10. Minimize the square of the distance, $f(x, y, z) = x^2 + y^2 + z^2$, subject to $g(x, y, z) = x^2y - z^2 + 9 = 0$.

$\langle 2x, 2y, 2z\rangle = \lambda\left\langle 2xy, x^2, -2z\right\rangle$

$2x = 2\lambda xy$, $2y = \lambda x^2$, $2z = -2\lambda z$, $x^2y - z^2 + 9 = 0$

Critical points are $(0, 0, \pm 3)$ [case $x = 0$]; $\left(\pm\sqrt{2}, -1, \pm\sqrt{7}\right)$ [case $x \neq 0$, $\lambda = -1$]; and $\left(\pm 3\sqrt[6]{2/9}, -\sqrt[3]{9/2}, 0\right)$ [case $x \neq 0, \lambda \neq -1$].

Evaluating f at each of these eight points yields 9 (case $x = 0$), 10 (case $x \neq 0$, $\lambda = -1$), and $\frac{3}{2}\sqrt[3]{2}\left(\sqrt[3]{9}\right)^2$ (case $x \neq 0, \lambda \neq -1$). The latter is the smallest, so the least distance between the origin and the surface is $3\sqrt[6]{\frac{3}{4}} \approx 2.8596$.

11. Maximize $f(x, y, z) = xyz$, subject to

$$g(x,\ y,\ z) = b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 - a^2b^2c^2 = 0$$

$$\langle yz,\ xz,\ xy\rangle = \lambda\langle 2b^2c^2x,\ 2a^2c^2y,\ 2a^2b^2z\rangle$$

$$yz = 2b^2c^2x,\ xz = 2a^2c^2y,\ xy = 2a^2b^2z,$$

$$b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2$$

Critical point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

$V\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$, which is the maximum.

12. Maximize $V(x, y, z) = xyz$, subject to

$g(x,\ y,\ z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$. Let

$\nabla V(x,\ y,\ z) = \lambda \nabla g(x,\ y,\ z)$, so

$\langle yz,\ xz,\ xy\rangle = \lambda\left\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\rangle$. Then

$\frac{\lambda x}{a} = \frac{\lambda y}{b} = \frac{\lambda z}{c}$ (each equals xyz).

$\lambda \neq 0$ since $\lambda = 0$ would imply $x = y = z = 0$ which would not satisfy the constraint.

Thus, $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. These along with the constraints yield $x = \frac{a}{3},\ y = \frac{b}{3},\ z = \frac{c}{3}$.

The maximum value of $V = \frac{abc}{27}$.

13. Maximize $f(x,y,z) = x + y + z$ with the constraint

$g(x,y,z) = x^2 + y^2 + z^2 - 81 = 0$. Let

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, so

$\langle 1,1,1\rangle = \lambda\langle 2x,2y,2z\rangle$; Thus, $x = y = z$

and $3x^2 = 81$ or $x = y = z = \pm 3\sqrt{3}$.

The maximum value of f is $9\sqrt{3}$ when

$\langle x,y,z\rangle = \langle 3\sqrt{3}, 3\sqrt{3}, 3\sqrt{3}\rangle$

14. Minimize $d^2 = f(x,y,z) = x^2 + y^2 + z^2$ with the constraint $g(x,y,z) = 2x + 4y + 3z - 12 = 0$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle 2x, 2y, 2z\rangle = \lambda\langle 2,3,4\rangle$;

$2x = 2\lambda;\ 2y = 4\lambda;\ 2z = 3\lambda$ leads to a critical point of $\left(\frac{24}{29}, \frac{48}{29}, \frac{36}{29}\right)$ The nature of the problem indicates this will give a minimum rather than a maximum value. The minimum distance is

$$\sqrt{\frac{24}{29}^2 + \frac{48}{29}^2 + \frac{36}{29}^2} \approx 2.2283$$

15. Minimize $d^2 = f(x,y,z)$

$= (x-1)^2 + (y-2)^2 + z^2$ with the constraint

$g(x,y,z) = x^2 + y^2 - z = 0$;

$\langle 2x-2, 2y-4, 2z\rangle = \lambda\langle 2x, 2y, -1\rangle$

Setting up, solving each equation for λ, and substituting into equation $x^2 + y^2 - z = 0$ produces $\lambda \approx -1.5445$; The resulting critical point is approximately (0.393, 0.786, 0.772). The nature of this problem indicates this will give a minimum value rather than a maximum. The minimum distance is approximately 1.5616.

16. Minimize $d^2 = f(x,y,z)$

$= (x-1)^2 + (y-2)^2 + z^2$ with the constraint

$g(x,y,z) = x^2 + y^2 - z^2 = 0$

$\langle 2x-2, 2y-4, 2z\rangle = \lambda\langle 2x, 2y, -2z\rangle$

$\lambda = -1,\ x = \frac{1}{2},\ y = 1,\ z = \pm\frac{\sqrt{5}}{2}$; The critical points are $\left(\frac{1}{2}, 1, \frac{\sqrt{5}}{2}\right)$ and $\left(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2}\right)$ which both lead to a minimum distance of $\frac{\sqrt{10}}{2}$.

17. (See problem 37, section 12.8). Let the dimensions of the box be l, w, and h. Then the cost of the box is

$.25(2hl + 2hw + lw) + .4(lw)$ or

$C(l,w,h) = .5hl + .5hw + .65lw$.

We want to minimize C subject to the constraint $lhw = 2$; set $V(l,h,w) = lwh - 2$.

Now:

$\nabla C(l,w,h) = (.5h + .65w)\mathbf{i} + (.5h + .65l)\mathbf{j} + .5(l+w)\mathbf{k}$

and

$\nabla V(l,w,h) = wh\,\mathbf{i} + lh\,\mathbf{j} + lw\,\mathbf{k}$

Thus the Lagrange equations are

$$.5h + .65w = \lambda wh \qquad (1)$$

$$.5h + .65l = \lambda lh \qquad (2)$$

$$.5(l+w) = \lambda lw \qquad (3)$$

$$lwh = 2 \qquad (4)$$

Solving (4) for h and putting the result in (1) and (2), we get

$$\frac{1}{lw} + .65w = \frac{2\lambda}{l} \qquad (5)$$

$$\frac{1}{lw} + .65l = \frac{2\lambda}{w} \qquad (6)$$

Multiply (5) by l and (6) by w to get

$$\frac{1}{w} + .65lw = 2\lambda \qquad (7)$$

$$\frac{1}{l} + .65lw = 2\lambda \qquad (8)$$

from which we conclude that $l = w$. Putting this result into (3) we have

$$l = \lambda l^2 \qquad (9)$$

Since $V \neq 0$, $l \neq 0$ and (9) tells us that $l = \frac{1}{\lambda}$;

thus $l = \frac{1}{\lambda}, \quad w = l = \frac{1}{\lambda}, \quad h = \frac{2}{lw} = 2\lambda^2$.

Putting these results into equation (1), we conclude

$$.5(2\lambda^2) + .65\left(\frac{1}{\lambda}\right) = \lambda\left(\frac{1}{\lambda}\right)(2\lambda^2) \quad \text{or}$$

$$.65\left(\frac{1}{\lambda}\right) = \lambda^2.$$

Hence: $\lambda = \sqrt[3]{.65} \approx .866$, so the minimum cost is obtained when:

$$l = w = \frac{1}{\lambda} \approx 1.154 \quad \text{and} \quad h = 2\lambda^2 \approx 1.5$$

18. (See problem 38, section 12.8). Let the dimensions of the box be *l, w,* and *h*. Then the cost of the box is

$$1(2hl + 2hw) + 4(lw) + 3(2l + 2w) + 4h \quad \text{or}$$

$$C(l, w, h) = 2hl + 2hw + 4lw + 6l + 6w + 4h.$$

We want to minimize *C* subject to the constraint $lhw = 60$; set $V(l, h, w) = lwh - 60$.

Now:

$$\nabla C(l, w, h) = (2h + 4w + 6)\mathbf{i} + (2h + 4l + 6)\mathbf{j} \text{ and } + (2l + 2w + 4)\mathbf{k}$$

$$\nabla V(l, w, h) = wh\,\mathbf{i} + lh\,\mathbf{j} + lw\mathbf{k}$$

Thus the Lagrange equations are

$$2h + 4w + 6 = \lambda wh \qquad (1)$$
$$2h + 4l + 6 = \lambda lh \qquad (2)$$
$$2l + 2w + 4 = \lambda lw \qquad (3)$$
$$lwh = 60 \qquad (4)$$

Solving (4) for $h = \frac{60}{lw}$ and putting the result in (1) and (2), we get

$$\frac{120}{lw} + 4w + 6 = \frac{60\lambda}{l} \qquad (5)$$

$$\frac{120}{lw} + 4l + 6 = \frac{60\lambda}{w} \qquad (6)$$

Multiply (5) by *l* and (6) by *w* to get

$$\frac{120}{w} + 4lw + 6l = 60\lambda \qquad (7)$$

$$\frac{120}{l} + 4lw + 6w = 60\lambda \qquad (8)$$

from which we conclude that

$$\frac{120}{l} + 6w = \frac{120}{w} + 6l \quad \text{or} \quad (l - w)(lw + 20) = 0.$$

Since lw cannot be negative ($= -20$), we conclude that $l = w$; putting this result into equation (3), we have

$$2w + 2w + 4 = \lambda w^2 \quad \text{or} \quad \lambda = 4\left(\frac{w+1}{w^2}\right).$$

Therefore, from equation (1), we have

$$\frac{120}{w^2} + 4w + 6 = 4\left(\frac{w+1}{w^2}\right)w\left(\frac{60}{w^2}\right) \quad \text{or}$$

(multiplying through by w^3 and simplifying)

$$2w^4 + 3w^3 - 60w - 120 = 0$$

Using one of several techniques available to solve, we conclude that $w = l = 3.213$ and

$$h = \frac{60}{(3.213)^2} \approx 5.812.$$

19. (See problem 40, section 12.8)

Let

c = circumference of circle

p = perimeter of first square

q = perimeter of second square

Then the sum of the areas is

$$A(c, p, q) = \frac{c^2}{4\pi} + \frac{p^2}{16} + \frac{q^2}{16} = \frac{1}{4}\left[\frac{c^2}{\pi} + \frac{p^2}{4} + \frac{q^2}{4}\right]$$

so we wish to maximize and minimize

$$\tilde{A}(c, p, q) = \frac{c^2}{\pi} + \frac{p^2}{4} + \frac{q^2}{4} \text{ subject to the}$$

constraint $L(c, p, q) = c + p + q - k = 0$.

Now

$$\nabla \tilde{A}(c, p, q) = \frac{2c}{\pi}\mathbf{i} + \frac{p}{2}\mathbf{j} + \frac{q}{2}\mathbf{k}$$

$$\nabla L(c, p, q) = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

so the Lagrange equations are

$$\frac{2c}{\pi} = \lambda \qquad (1)$$

$$\frac{p}{2} = \lambda \qquad (2)$$

$$\frac{q}{2} = \lambda \qquad (3)$$

$$c + p + q = k \qquad (4)$$

Putting (1), (2) and (3) into (4) we get

$$(4 + \frac{\pi}{2})\lambda = k \quad \text{or} \quad \lambda = \frac{2k}{8+\pi}$$

Therefore:

$$c_0 = \frac{\pi k}{8 + \pi} \approx 0.282k$$

$$p_0 = \frac{4k}{8 + \pi} \approx 0.359k$$

$$q_0 = \frac{4k}{8 + \pi} \approx 0.359k$$

Now $A(c_0, p_0, q_0) \approx 0.0224k^2$ while $A(k, 0, 0) \approx .079k^2$, so we conclude that

$A(c_0, p_0, q_0)$ is a minimum value. There is also a maximum value (see problem 40, section 12.8) but our Lagrange approach does not capture this. The reason is that the maximum exists because *c*, *p*, and *q* must all be ≥ 0. Our constraint, however, does not require this and allows negative values for any or all of the variables. Under these conditions, there is no global maximum.

20. (See problem 42, section 12.8). Let P be the plane $\frac{x}{A}+\frac{y}{B}+\frac{z}{C}=1$. This plane will cross the first octant, forming a triangle, T, in P; the vertices of this triangle occur where P intersects the coordinate axes. They are: $V_x=(A,0,0),\ V_y=(0,B,0),\ V_z=(0,0,C)$.

a. Define the vectors $\mathbf{g}=\langle -A,B,0\rangle$ and $\mathbf{h}=\langle -A,0,C\rangle$. From example 3 in 11.4, we know the area of T is

$$\frac{1}{2}|\mathbf{g}\times\mathbf{h}|=\frac{1}{2}\sqrt{(BC)^2+(AC)^2+(AB)^2}\ .$$

b. The height of the tetrahedron in question is the distance is the distance between (0,0,0) and P . By example 10 in 11.3, this distance is

$$h=\frac{1}{\sqrt{\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}}}=\sqrt{\frac{(ABC)^2}{(BC)^2+(AC)^2+(AB)^2}}$$

c. Finally, the volume of the tetrahedron is $\frac{1}{3}h(\text{area of } T)$, or

$$V(A,B,C)=\frac{1}{6}\sqrt{(BC)^2+(AC)^2+(AB)^2}\cdot\left[\frac{\sqrt{(ABC)^2}}{\sqrt{(BC)^2+(AC)^2+(AB)^2}}\right]$$

That is, $V(A,B,C)=\frac{1}{6}\sqrt{(ABC)^2}=\frac{1}{6}|ABC|$.

Hence we want to minimize $\tilde{V}(A,B,C)=|ABC|$ subject to the constraint $\frac{a}{A}+\frac{b}{B}+\frac{c}{C}=1$; define

$$g(A,B,C)=\frac{a}{A}+\frac{b}{B}+\frac{c}{C}-1.$$

Now

$$\nabla\tilde{V}(A,B,C)=\frac{A}{|ABC|}\mathbf{i}+\frac{B}{|ABC|}\mathbf{j}+\frac{C}{|ABC|}\mathbf{k}$$

and $\nabla g(A,B,C)=\frac{-a}{A^2}\mathbf{i}+\frac{-b}{B^2}\mathbf{j}+\frac{-c}{C^2}\mathbf{k}$. Thus the Lagrange equations are

$$\frac{A}{|ABC|}=\frac{-\lambda a}{A^2} \qquad (1)$$

$$\frac{B}{|ABC|}=\frac{-\lambda b}{B^2} \qquad (2)$$

$$\frac{C}{|ABC|}=\frac{-\lambda c}{C^2} \qquad (3)$$

$$\frac{a}{A}+\frac{b}{B}+\frac{c}{C}=1 \qquad (4)$$

From (1) – (3) we have

$$\lambda|ABC|=\frac{-A^3}{a}=\frac{-B^3}{b}=\frac{-C^3}{c} \qquad (5)$$

Solving in pairs we get

$$B=\left(\sqrt[3]{\frac{b}{a}}\right)A,\quad C=\left(\sqrt[3]{\frac{c}{a}}\right)A \qquad (6)$$

and putting these results into (4) we obtain

$$A=a+\sqrt[3]{ab^2}+\sqrt[3]{ac^2}$$
$$=\sqrt[3]{a}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)$$

Similarly, we have

$$B=\sqrt[3]{a^2b}+b+\sqrt[3]{bc^2}$$
$$=\sqrt[3]{b}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)$$

$$C=\sqrt[3]{a^2c}+\sqrt[3]{b^2c}+c$$
$$=\sqrt[3]{c}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)$$

Finally, the volume of the tetrahedron is

$$\frac{|ABC|}{6}=\frac{\sqrt[3]{abc}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)^3}{6}.$$

21. Finding critical points on the interior first: $\frac{\partial f}{\partial x}=1\neq 0\quad \frac{\partial f}{\partial y}=1\neq 0$; There are no critical points on the interior. Finding critical points on the boundary: $\nabla f(x,y)=\lambda\nabla g(x,y)$; $\langle 1,1\rangle=\lambda\langle 2x,2y\rangle$; The solution to the system $1=\lambda\cdot 2x, 1=\lambda\cdot 2y, x^2+y^2=1$ is $\lambda=\pm\frac{1}{\sqrt{2}}$, $x=\pm\frac{1}{\sqrt{2}},\ y=\pm\frac{1}{\sqrt{2}}$ The four critical points are $\left(\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$.

$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=10+\sqrt{2}$ is the maximum value.

$f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)=10-\sqrt{2}$ is the minimum value.

22. Finding critical points on the interior:

$\frac{\partial f}{\partial x} = 1 - y = 0 \Rightarrow y = 1;$

$\frac{\partial f}{\partial y} = 1 - x = 0 \Rightarrow x = 1;$ The only critical point on the interior is $c_1 =$ (1,1). Finding critical points on the boundary: Solve the system of equations

$1 - y = \lambda \cdot 2x;\ 1 - x = \lambda \cdot 2y;\ x^2 + y^2 = 9$

Using substitution, it can be found that the critical points on the boundary are

$c_2 = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, $c_3 = \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$,

$c_4 =$ (2.56155, -1,56155),

$c_5 =$ (-1.56155, 2.56155)

The maximum value of 5 is obtained substituting either c_4 or c_5 into *f*. The minimum value of about -8.7426 is obtained by substituting c_3 into *f*.

23. Finding critical points on the interior:

$\frac{\partial f}{\partial x} = 2x + 3 - y = 0;\ \frac{\partial f}{\partial y} = 2y - x = 0$

The solution to this system is the only critical point on the interior, $c_1 =$ (-2,-1).

Critical points on the boundary will come from the solutions to the following system of equations:

$2x + 3 - y = \lambda \cdot 2x,\quad 2y - x = \lambda \cdot 2y,$

$x^2 + y^2 = 9$. From the solutions to this system, the critical points are $c_2 = (0,3)$,

$c_3 = \left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right),\ c_4 = \left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

$f(c_1) = -3,\quad f(c_2) = 9,\quad f(c_3) \approx 20.6913,$

$f(c_4) \approx -2.6913$ The max value of *f* is ≈ 20.6913 and the min value is -3.

24. $f(x, y) = \frac{x}{1 + y^2}$ on the set $S = \left\{(x, y): \frac{x^2}{4} + \frac{y^2}{9} \le 1\right\}$

We first find the max and min for f on the set $\tilde{S} = \left\{(x, y): \frac{x^2}{4} + \frac{y^2}{9} < 1\right\}$ using the methods of section 12.8:

$\nabla f(x, y) = \frac{1}{1 + y^2}\mathbf{i} + \frac{-2xy}{(1 + y^2)^2}\mathbf{j}$ so setting $\nabla f(x, y) = \mathbf{0}$ we have $\frac{1}{1 + y^2} = 0$ (impossible).

Thus f has no max or min on $\tilde{S}$.

We now look for the max and min of f on the boundary $\bar{S} = \left\{(x, y): \frac{x^2}{4} + \frac{y^2}{9} = 1\right\}$; this is done using Lagrange multipliers. Let

$g(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$; then

$\nabla f(x, y) = \frac{1}{1 + y^2}\mathbf{i} + \frac{-2xy}{(1 + y^2)^2}\mathbf{j}$ and

$\nabla g(x, y) = \frac{x}{2}\mathbf{i} + \frac{2y}{9}\mathbf{j}$

The Lagrange equations are

$$\frac{1}{1 + y^2} = \frac{\lambda x}{2} \qquad (1)$$

$$\frac{-2xy}{(1 + y^2)^2} = \frac{2\lambda y}{9} \qquad (2)$$

$$9x^2 + 4y^2 = 36 \qquad (3)$$

Putting (1) into (2) yields

$$\frac{-\lambda^2 x^3 y}{2} = \frac{2\lambda y}{9} \qquad (4)$$

One solution to (4) is $y = 0$ which yields, from (3), $x = \pm 2$. Thus $(2,0)$ and $(-2,0)$ are candidates for optimization points.

If $y \ne 0$, (4) can be reduced to

$$\frac{-\lambda^2 x^3}{2} = \frac{2\lambda}{9} \qquad (5)$$

so that $\lambda = \frac{-4}{9x^3}$. Putting this result into (1) yields $\frac{1}{1 + y^2} = -\frac{2}{9x^2}$, which has no solutions (left side always +, right side always -).

Therefore the only two candidates for max/min are $(2,0)$ and $(-2,0)$. Since $f(2,0) = 2$ and $f(-2,0) = -2$ we conclude that the max value of f on S is 2 and the min value is -2.

25. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 2(1 + x + y) = 0 \Rightarrow x + y = -1$

There is no minimum or maximum value on the interior since there are an infinite number of critical points. The critical points on the boundary will come from the solutions to the following system of equations:

$2(1 + x + y) = \lambda \cdot \frac{1}{2}x$

$2(1 + x + y) = \lambda \cdot \frac{1}{8}y$

Solving these two equations for λ leads to $y = -x - 1$ or $y = 4x$. Together with the constraint $\frac{x^2}{4} + \frac{y^2}{16} - 1 = 0$ leads to the critical

points on the boundary: $\left(\frac{-1-2\sqrt{19}}{5}, \frac{-4+2\sqrt{19}}{5}\right)$, $\left(\frac{-1+2\sqrt{19}}{5}, \frac{-4-2\sqrt{19}}{5}\right)$, $\left(-\frac{2}{\sqrt{5}}, -\frac{8}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right)$. Respectively, the maximum value is ≈ 29.9443 and the minimum value is 0.

26. It is clear that the maximum will occur for a triangle which contains the center of the circle. (With this observation in mind, there are additional constraints: $0 < \alpha < \pi$, $0 < \beta < \pi$, $0 > \gamma < \pi$.)
Note that in an isosceles triangle, the side opposite the angle θ which is between the congruent sides of length r has length $2r\sin\left(\frac{\theta}{2}\right)$. Then we wish to maximize

$$P(\alpha, \beta, \gamma) = 2r\left[\sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\beta}{2}\right) + \sin\left(\frac{\gamma}{2}\right)\right]$$

subject to $g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - 2\pi = 0 = 0$.

Let $r\left\langle \cos\left(\frac{\alpha}{2}\right), \cos\left(\frac{\beta}{2}\right), \cos\left(\frac{\gamma}{2}\right)\right\rangle = \lambda\langle 1, 1, 1\rangle$.

Then $\lambda = r\cos\left(\frac{\alpha}{2}\right) = r\cos\left(\frac{\beta}{2}\right) = r\cos\left(\frac{\gamma}{2}\right)$, so

$\alpha = \beta = \gamma$ $\left(\text{since } \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi\right)$.

$3\alpha a = 2\pi$, so $\alpha = \frac{2\pi}{3}$; then $\beta = \gamma = \frac{2\pi}{3}$.

27. Let $\alpha + \beta + \gamma = 1$, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.
Maximize $P(x, y, z) = kx^{\alpha}y^{\beta}z^{\gamma}$, subject to $g(x, y, z) = ax + by + cz - d = 0$.

Let $\nabla P(x, y, z) = \lambda\nabla g(x, y, z)$. Then $\left\langle k\alpha x^{\alpha-1}y^{\beta}z^{\gamma}, k\beta x^{\alpha}y^{\beta-1}z^{\gamma}, k\gamma x^{\alpha}y^{\beta}z^{\gamma-1}\right\rangle = \lambda\langle a, b, c\rangle$.

Therefore, $\frac{\lambda ax}{\alpha} = \frac{\lambda by}{\beta} = \frac{\lambda cz}{\gamma}$ (since each equals $kx^{\alpha}y^{\beta}z^{\gamma}$).

$\lambda \neq 0$ since $\lambda = 0$ would imply $x = y = z = 0$ which would imply $P = 0$.

Therefore, $\frac{ax}{\alpha} = \frac{by}{\beta} = \frac{cz}{\gamma}$ (*).

The constraints $ax + by + cz = d$ in the form $\alpha\left(\frac{ax}{\alpha}\right) + \beta\left(\frac{by}{\beta}\right) + \gamma\left(\frac{cz}{\gamma}\right) = d$ becomes

$\alpha\left(\frac{ax}{\alpha}\right) + \beta\left(\frac{ax}{\alpha}\right) + \gamma\left(\frac{ax}{\alpha}\right) = d$, using (*).

Then $(\alpha + \beta + \gamma)\left(\frac{ax}{\alpha}\right) = d$, or $\frac{ax}{\alpha} = d$ (since $\alpha + \beta + \gamma = 1$).

$x = \frac{\alpha d}{a}$ (**); $y = \frac{\beta d}{b}$ and $z = \frac{\gamma d}{c}$ then following using (*) and (**).

Since there is only one interior critical point, and since P is 0 on the boundary, P is maximum when $x = \frac{\alpha d}{a}$, $y = \frac{\beta d}{b}$, $z = \frac{\gamma d}{c}$.

28. Let (x, y, z) denote a point of intersection. Let $f(x, y, z)$ be the square of the distance to the origin. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g(x, y, z) = x + y + z - 8 = 0$ and $h(x, y, z) = 2x - y + 3z - 28 = 0$.
Let $\nabla f(x, y, z) = \lambda\nabla g(x, y, z) + \mu\nabla h(x, y, z)$.

$\langle 2x, 2y, 2z\rangle = \lambda\langle 1,1,1\rangle + \mu\langle 2,-1,3\rangle$

1. $2x = \lambda + 2\mu$
2. $2y = \lambda - \mu$
3. $2z = \lambda + 3\mu$
4. $x + y + z = 8$
5. $2x - y + 3z = 28$
6. $3\lambda + 4\mu = 16$ (1, 2, 3, 4)
7. $2\lambda + 7\mu = 28$ (1, 2, 3, 5)
8. $\lambda = 0$, $\mu = 4$ (6, 7)
9. $x = 4$, $y = -2$, $z = 6$ (8, 1-3)

$f(4, -2, 6) = 56$, and the nature of the problem indicates this is the minimum rather than the maximum.
Conclusion: The least distance is $\sqrt{56} \approx 7.4833$.

29. $\langle -1,2,2\rangle = \lambda\langle 2x,2y,0\rangle + \mu\langle 0,1,2\rangle$

$-1 = 2\lambda x$, $2 = 2\lambda y + \mu$, $2 = 2\mu$, $x^2 + y^2 = 2$,
$y + 2z = 1$
Critical points are (–1, 1, 0) and (1, –1, 1).
$f(-1, 1, 0) = 3$, the maximum value;
$f(1, -1, 1) = -1$, the minimum value.

30. **a.** Maximize
$w(x_1, x_2, \ldots, x_n) = x_1x_2, \ldots, x_n, (x_i > 0)$
subject to the constraint
$g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n - 1 = 0.$ Le
t $\nabla w(x_1, x_2, \ldots, x_n) = \lambda\nabla g(x_1, x_2, \ldots, x_n).$
$\langle x_2 \ldots x_n,\ x_1x_3 \ldots x_n,\ x_1 \ldots x_{n-1}\rangle = \lambda\langle 1, 1, \ldots, 1\rangle.$
Therefore, $\lambda x_1 = \lambda x_2 = \ldots = \lambda x_n$ (since each equals $x_1x_2 \ldots x_n$). Then $x_1 = x_2 = \ldots = x_n$.
(If $\lambda = 0$, some $x_i = 0$, so $w = 0$.)

Therefore, $nx_i = 1;\ x_i = \dfrac{1}{n}.$

The maximum value of w is $\left(\dfrac{1}{n}\right)^n$, and

occurs when each $x_i = \dfrac{1}{n}.$

b. From part a we have that $x_1x_2 \ldots x_n \le \left(\dfrac{1}{n}\right)^n.$

Therefore, $\sqrt[n]{x_1x_2 \ldots x_n} \le \dfrac{1}{n}.$

If $x_i = \dfrac{a_i}{a_1 + \ldots + a_n} = \dfrac{a_i}{A}$ for each i, then

$\sqrt[n]{\dfrac{a_1}{A}\dfrac{a_2}{A} \ldots \dfrac{a_n}{A}} \le \dfrac{1}{n}$, so $\sqrt[n]{a_1a_2 \ldots a_n} \le \dfrac{A}{n}$, or

$\sqrt[n]{a_1a_2 \ldots a_n} \le \dfrac{a_1 + a_2 + \ldots a_n}{n}.$

31. Let $\langle a_1, a_2, \ldots a_n\rangle = \lambda\langle 2x_1, 2x_2, \ldots, 2x_n\rangle.$
Therefore, $a_i = 2\lambda x_i$, for each $i = 1, 2, \ldots, n$
(since $\lambda = 0$ implies $a_i = 0$, contrary to the hypothesis).

$\dfrac{x_i}{a_i} = \dfrac{x_j}{a_j}$ for all i, j $\left(\text{since each equals } \dfrac{1}{2\lambda}\right).$

The constraint equation can be expressed

$$a_1^2\left(\frac{x_1}{a_1}\right)^2 + a_2^2\left(\frac{x_2}{a_2}\right)^2 + \ldots + a_n^2\left(\frac{x_n}{a_n}\right)^2 = 1.$$

Therefore, $\left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(\dfrac{x_1}{a_1}\right)^2 = 1.$

$x_1^2 = \dfrac{a_1^2}{a_1^2 + \ldots + a_n^2}$; similar for each other x_i^2.

The function to be maximized in a hyperplane with positive coefficients and constant (so intercepts on all axes are positive), and the constraint is a hypersphere of radius 1, so the maximum will occur where each x_i is positive.
There is only one such critical point, the one obtained from the above by taking the principal square root to solve for x_i.
Then the maximum value of w is

$$a_1\left(\frac{a_1}{\sqrt{A}}\right) + a_2\left(\frac{a_2}{\sqrt{A}}\right) + \ldots + a_n\left(\frac{a_n}{\sqrt{A}}\right) = \frac{A}{\sqrt{A}} = \sqrt{A}$$

where $A = a_1^2 + a_2^2 + \ldots + a_n^2$.

32. Max: $f(-0.71, 0.71) = f(-0.71, -0.71) = 0.71$

33. Min: $f(4, 0) = -4$

34. Max: $f(1.41, 1.41) = f(-1.41, -1.44) = 0.037$

35. Min: $f(0, 3) = f(0, -3) = -0.99$

12.10 Chapter Review

Concepts Test

1. True: Except for the trivial case of $z = 0$, which gives a point.

2. False: Use $f(0, 0) = 0$; $f(x, y) = \dfrac{xy}{x^2 + y^2}$ elsewhere for counterexample.

3. True: Since $g'(0) = f_x(0, 0)$

4. True: It is the limit along the path, $y = x$.

5. True: Use "Continuity of a Product" Theorem.

6. True: Straight forward calculation of partial derivatives

7. False: See Problem 25, Section 12.4.

8. False: It is perpendicular to the level curves of f. The gradient of $F(x, y, z) = f(x, y) - z$ is perpendicular to the graph of $z = f(x, y)$.

9. True: Since $\langle 0,0,-1\rangle$ is normal to the tangent plane

10. False: C^{ex}: For the cylindrical surface $f(x, y) = y^3$, $f(\mathbf{p}) = \mathbf{0}$ for every $\mathbf{p}$ on the x-axis, but $f(\mathbf{p})$ is not an extreme value.

11. True: It will point in the direction of greatest increase of heat, and at the origin, $\nabla T(0, 0) = \langle 1, 0\rangle$ is that direction.

12. True: It is nonnegative for all x, y, and it has a value of 0 at (0, 0).

13. True: Along the x-axis, $f(x, 0) \to \pm\infty$ as $x \to \pm\infty$.

14. False: $|D_{\mathbf{u}} f(x, y)| = |\langle 4, 4\rangle \cdot \mathbf{u}| \le 4\sqrt{2}$ (equality if $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\langle 1,1\rangle$)

15. True: $-D_{\mathbf{u}} f(x, y) = -[\nabla f(x, y) \cdot \mathbf{u}]$
$= \nabla f(x, y) \cdot (-\mathbf{u}) = D_{-\mathbf{u}} f(x, y)$

16. True: The set (call it S, a line segment) contains all of its boundary points because for every point P not in S (i.e., not on the line segment), there is an open neighborhood of P (i.e., a circle with P as center) that contains no point of S.

17. True: By the Min-Max Existence Theorem

18. False: (x_0, y_0) could be a singular point.

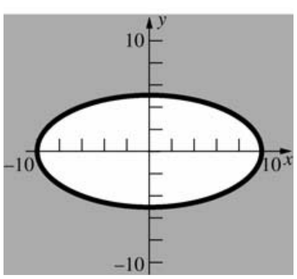
19. False: $f\left(\frac{\pi}{2}, 1\right) = \sin\left(\frac{\pi}{2}\right) = 1$, the maximum value of f, and $(\pi/2, 1)$ is in the set.

20. False: The same function used in Problem 2 provides a counterexample.

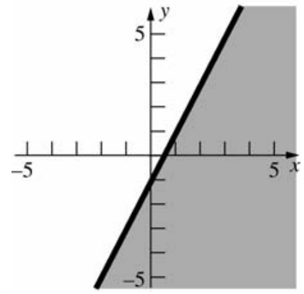
Sample Test Problems

1. a. $x^2 + 4y^2 - 100 \ge 0$

$$\frac{x^2}{100} + \frac{y^2}{25} \ge 1$$

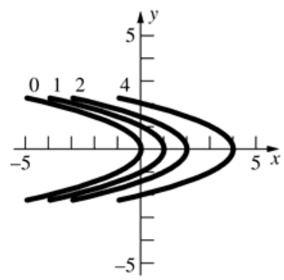


b. $2x - y - 1 \ge 0$



2. $x + y^2 = k$

$x - k = -y^2$



3. $f_x(x, y) = 12x^3y^2 + 14xy^7$

$f_{xx}(x, y) = 36x^2y^2 + 14y^7$

$f_{xy}(x, y) = 24x^3y + 98xy^6$

4. $f_x(x, y) = -2\cos x \sin x = -\sin 2x$

$f_{xx}(x, y) = -2x\cos 2x$

$f_{xy}(x, y) = 0$

5. $f_x(x, y) = e^{-y}\sec^2 x$

$f_{xx}(x, y) = 2e^{-y}\sec^2 x \tan x$

$f_{xy}(x, y) = -e^{-y}\sec^2 x$

6. $f_x(x, y) = -e^{-x}\sin y$

$f_{xx}(x, y) = e^{-x}\sin y$

$f_{xy}(x, y) = -e^{-x}\cos y$

7. $F_y(x, y) = 30x^3y^5 - 7xy^6$

$F_{yy}(x, y) = 150x^3y^4 - 42xy^5$

$F_{yyx}(x, y) = 450x^2y^4 - 42y^5$

8. $f_x(x, y, z) = y^3 - 10xyz^4$

$f_y(x, y, z) = 3xy^2 - 5x^2z^4$

$f_z(x, y, z) = -20x^2yz^3$

Therefore, $f_x(2, -1, 1) = 19$;

$f_y(2, -1, 1) = -14$; $f_z(2, -1, 1) = 80$

9. $z_y(x, y) = \frac{y}{2}$; $z_y(2, 2) = \frac{2}{2} = 1$

10. Everywhere in the plane except on the parabola $x^2 = y$.

11. No. On the path $y = x$, $\lim_{x\to 0} \frac{x - x}{x + x} = 0.$ On the path $y = 0$, $\lim_{x\to 0} \frac{x - 0}{x + 0} = 1.$

12. **a.** $\lim_{(x,\, y)\to(2,\, 2)} \frac{x^2 - 2y}{x^2 + 2y} = \frac{4 - 4}{4 + 4} = 0$

b. Does not exist since the numerator lends to 4 and the denominator to 0.

c. $\lim_{(x,\, y)\to(0,\, 0)} \frac{(x^2 + 2y^2)(x^2 - 2y^2)}{x^2 + 2y^2}$

$= \lim_{(x,\, y)\to(0,\, 0)} (x^2 - y^2) = 0$

13. **a.** $\nabla f(x, y, z) = \left\langle 2xyz^3, x^2z^3, 3x^2yz^2 \right\rangle$

$f(1, 2, -1) = \langle -4, -1, 6\rangle$

b. $\nabla f(x, y, z)$

$= \left\langle y^2 z\cos xz, 2y\sin xz, xy^2 \cos xz \right\rangle$

$\nabla f(1, 2, -1) = -4\langle \cos(1), \sin(1), -\cos(1)\rangle$

$\approx \langle -2.1612, -3.3659, 2.1612\rangle$

14. $D_{\mathbf{u}} f(x, y) = \left\langle 3y(1 + 9x^2y^2)^{-1}, 3x(1 + 9x^2y^2)^{-1} \right\rangle \cdot \mathbf{u}$

$D_{\mathbf{u}} f(4, 2) = \left\langle \frac{6}{577}, \frac{12}{577} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = \frac{\left(3\sqrt{3} - 6\right)}{577}$

≈ -0.001393

15. $z = f(x, y) = x^2 + y^2$

$\left\langle 1, -\sqrt{3}, 0 \right\rangle$ is horizontal and is normal to the vertical plane that is given. By inspection, $\left\langle \sqrt{3}, 1, 0 \right\rangle$ is also a horizontal vector and is perpendicular to $\left\langle 1, -\sqrt{3}, 0 \right\rangle$ and therefore is parallel to the vertical plane. Then $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ is the corresponding 2-dimensional unit vector.

$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$

$= \langle 2x, 2y\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \sqrt{3}x + y$

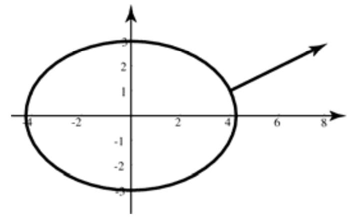
$D_{\mathbf{u}} f(1, 2) = \sqrt{3} + 2 \approx 3.7321$ is the slope of the tangent to the curve.

16. In the direction of $\nabla f(1, 2) = 4\langle 9, 4\rangle$

17. **a.** $f(4, 1) = 9$, so $\frac{x^2}{2} + y^2 = 9$, or $\frac{x^2}{18} + \frac{y^2}{9} = 1.$

b. $\nabla f(x, y) = \langle x, 2y\rangle$, so $f(4,1) = \langle 4, 2\rangle$.

c.



18. $F_x = F_u u_x + F_v v_x$

$= \frac{v}{1 + u^2v^2} \frac{y}{2\sqrt{xy}} + \frac{u}{1 + u^2v^2} \frac{1}{2\sqrt{x}}$

$= \frac{v\sqrt{y} + u}{2(1 + u^2v^2)\sqrt{x}}$

$F_y = F_u u_y + F_v v_y$

$= \frac{v}{1 + u^2v^2} \frac{x}{2\sqrt{xy}} + \frac{u}{1 + u^2v^2} \frac{-1}{2\sqrt{y}}$

$= \frac{v\sqrt{x} - u}{2(1 + u^2v^2)\sqrt{y}}$

19. $f_x = f_u u_x + f_v u_y = \left(\frac{1}{v}\right)(2x) + \left(-\frac{u}{v^2}\right)(yz)$

$= x^{-2}y^{-1}z^{-1}(x^2 + 3y - 4z)$

$f_y = f_u u_y + f_v v_y = \left(\frac{1}{v}\right)(-3) + \left(-\frac{u}{v^2}\right)(xz)$

$= -x^{-1}y^{-2}z^{-1}(x^2 + 4z)$

$f_z = f_u u_z + f_v v_z = \left(\frac{1}{v}\right)(4) + \left(-\frac{u}{v^2}\right)(xy)$

$= x^{-1}y^{-1}z^{-2}(3y - x^2)$

20. $\dfrac{dF}{dt} = \dfrac{dF}{dx}\dfrac{dx}{dt} + \dfrac{dF}{dy}\dfrac{dy}{dt}$

$= (3x^2 - y^2)(-6\sin 3t) + (-2xy - 4y^3)(3\cos t)$

$t = 0 \Rightarrow x = 2$ and $y = 0$, so $\left.\left(\dfrac{dF}{dt}\right)\right|_{t=0} = 0$.

21. $F_t = F_x x_t + F_y y_t + F_z z_t$

$= \left(\dfrac{10xy}{z^3}\right)\left(\dfrac{3t^{1/2}}{2}\right) + \left(\dfrac{5x^2}{z^3}\right)\left(\dfrac{1}{t}\right) + \left(-\dfrac{15x^2y}{z^4}\right)(3e^{3t})$

$= \dfrac{15xy\sqrt{t}}{z^3} + \dfrac{5x^2}{z^3t} - \dfrac{45x^2ye^{3t}}{z^4}$

22. $\dfrac{dc}{dt} = 3, \dfrac{db}{dt} = -2, \dfrac{d\alpha}{dt} = 0.1$

Area $= A(b, c, \alpha) = \left(\dfrac{1}{2}\right)c(b\sin\alpha)$

$\dfrac{dA}{dt} = \left[\left(\dfrac{b}{2}\right)(\sin\alpha)\left(\dfrac{dc}{dt}\right) + \left(\dfrac{c}{2}\right)(\sin\alpha)\left(\dfrac{db}{dt}\right) + \left(\dfrac{b}{2}\right)(bc\cos\alpha)\left(\dfrac{d\alpha}{dt}\right)\right]$

$\left.\left(\dfrac{dA}{dt}\right)\right|_{\left(8,\, 10,\, \frac{\pi}{6}\right)} = \dfrac{\left(7 + 4\sqrt{3}\right)}{2} \approx 6.9641\ \text{in.}^2/\text{s}$

23. Let $F(x, y, z) = 9x^2 + 4y^2 + 9z^2 - 34 = 0$

$\nabla F(x, y, z) = \langle 18x, 8y, 18z\rangle$, so $\nabla f(1, 2, -1) = 2\langle 9, 8, -9\rangle$.

Tangent plane is $9(x - 1) + 8(y - 2) - 9(z + 1) = 0$, or $9x + 8y - 9z = 34$.

24. $V = \pi r^2 h;\ \ dV = V_r\,dr + V_h\,dh = 2\pi rh\,dr + \pi r^2\,dh$

If $r = 10$, $|dr| \le 0.02$, $h = 6$, $|dh| = 0.01$, then

$|dV| \le 2\pi rh|dr| + \pi r^2|dh| \le 2\pi(10)(6)(0.02) + \pi(100)(0.01) = 3.4\pi$

$V(10, 6) = \pi(100)(6) = 600\pi$

Volume is $600\pi \pm 3.4\pi \approx 1884.96 \pm 10.68$

25. $df = y^2(1+z^2)^{-1}dx + 2xy(1+z^2)^{-1}dy - 2xy^2z(1+z^2)^{-2}dz$

If $x = 1$, $y = 2$, $z = 2$, $dx = 0.01$, $dy = -0.02$, $dz = 0.03$, then $df = -0.0272$.

Therefore, $f(1.01, 1.98, 2.03) \approx f(1, 2, 2) + df = 0.8 - 0.0272 = 0.7728$

26. $\nabla f(x, y) = \left\langle 2xy - 6x,\ x^2 - 12y\right\rangle = \langle 0, 0\rangle$

at $(0, 0)$ and $(\pm 6, 3)$.

$D = f_{xx}f_{yy} - f_{xy}^2 = (2y - 6)(-12) - (2x)^2$

$= 4(18 - 6y - x^2);\ \ f_{xx} = 2(y - 3)$

At $(0, 0)$: $D = 72 > 0$ and $f_{xx} < 0$, so local maximum at $(0, 0)$.

At $(\pm 6, 3)$: $D < 0$, so $(\pm 6, 3)$ are saddle points.

27. Let (x, y, z) denote the coordinates of the 1st octant vertex of the box. Maximize $f(x, y, z) = xyz$ subject to

$$g(x, y, z) = 36x^2 + 4y^2 + 9z^2 - 36 = 0$$

(where $x, y, z > 0$ and the box's volume is $V(x, y, z) = f(x, y, z)$.

Let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

$\langle yz, xz, xy \rangle 8 = \lambda \langle 72x, 8y, 18z \rangle$

1. $8yz = 72\lambda x$
2. $8xz = 8\lambda y$
3. $8xy = 18\lambda z$
4. $36x^2 + 4y^2 + 9z^2 = 36$
5. $\dfrac{yz}{xz} = \dfrac{72\lambda x}{8\lambda y}$, so $y^2 = 9x^2$. (1, 2)
6. $\dfrac{yz}{xz} = \dfrac{72\lambda x}{18\lambda y}$, so $z^2 = 4x^2$. (1, 3)
7. $36x^2 + 36x^2 + 36x^2 = 36$, so $x = \dfrac{1}{\sqrt{3}}$. (5, 6, 4)
8. $y = \dfrac{3}{\sqrt{3}}$, $z = \dfrac{2}{\sqrt{3}}$ (7, 5, 6)

$$V\left(\frac{1}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{16}{\sqrt{3}} \approx 9.2376$$

The nature of the problem indicates that the critical point yields a maximum value rather than a minimum value.

28. $\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$

$y = 2\lambda x$, $x = 2\lambda y$, $x^2 + y^2 = 1$

Critical points are $\left(\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{\sqrt{2}}\right)$ and $\left(-\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{\sqrt{2}}\right)$. Maximum of $\dfrac{1}{2}$ at $\left(\pm\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{\sqrt{2}}\right)$; minimum of $-\dfrac{1}{2}$ at $\left(\pm\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.

29. Maximize $V(r, h) = \pi r^2 h$, subject to

$S(r, h) = 2\pi r^2 + 2\pi rh - 24\pi = 0.$

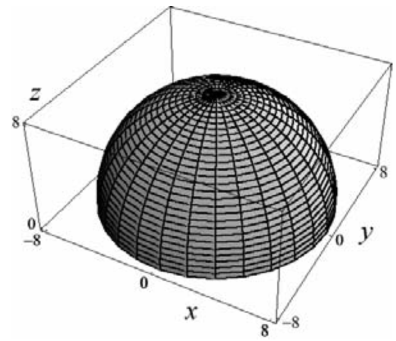
$\langle 2\pi rh, \pi r^2 \rangle = \lambda \langle 4\pi r + 2\pi h, 2\pi r \rangle$

$rh = \lambda(2r + h)$, $r = 2\lambda$, $r^2 + rh = 12$

Critical point is (2, 4). The nature of the problem indicates that the critical point yields a maximum value rather than a minimum value. Conclusion: The dimensions are radius of 2 and height of 4.

Review and Preview Problems

1.



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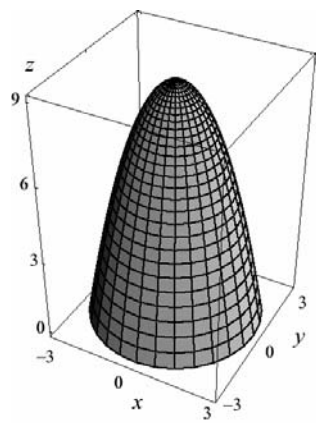

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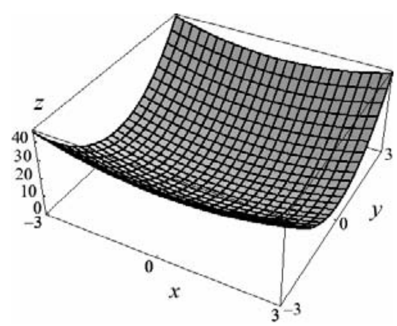

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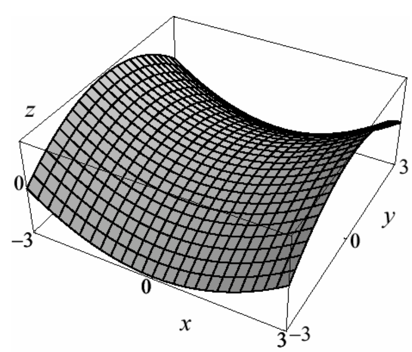

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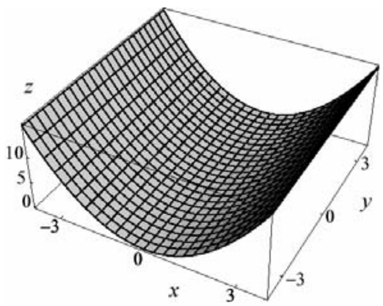
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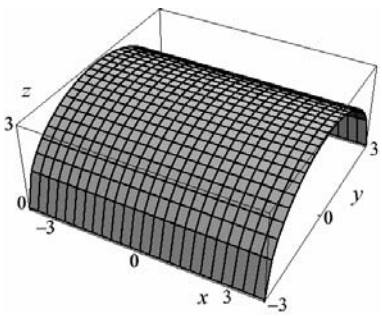
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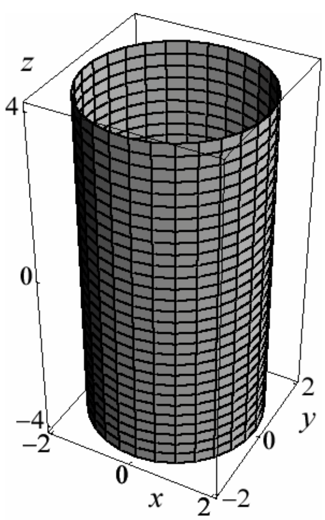
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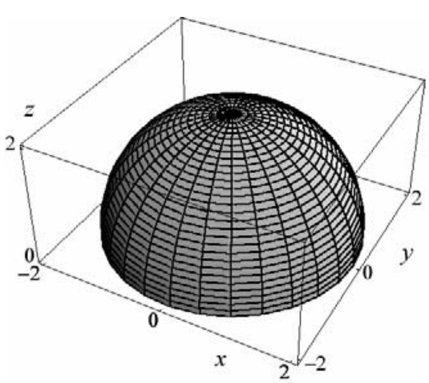
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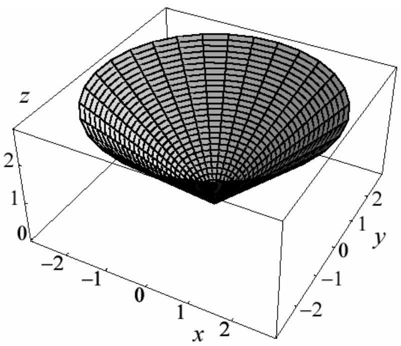
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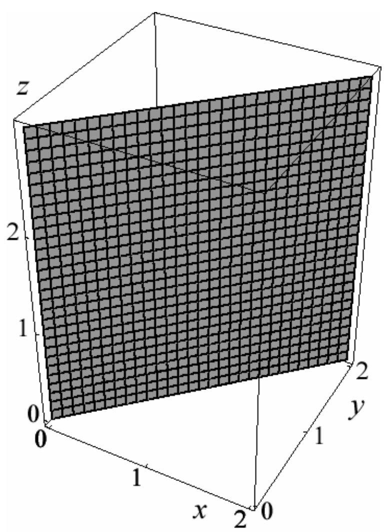
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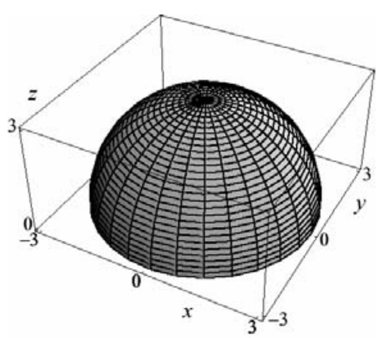
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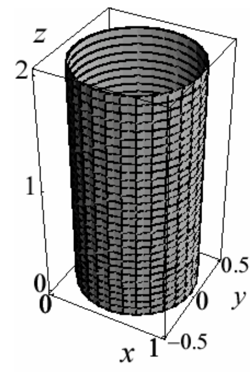
12.



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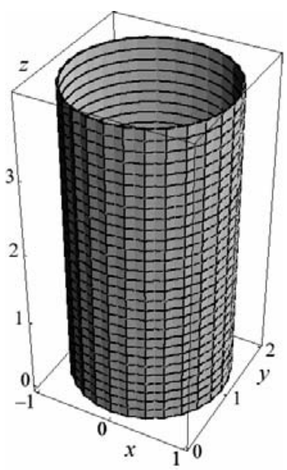

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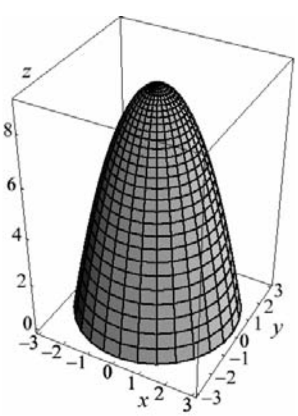


15. $\int e^{-2x}dx = -\frac{1}{2}e^{-2x} + C$

16. $\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx + C$

$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

17. $\int_{-a/2}^{a/2}\cos\left(\frac{x\pi}{a}\right)dx = \frac{a}{\pi}\sin\left(\frac{x\pi}{a}\right)_{-a/2}^{a/2} = \frac{2a}{\pi}$

18. $\int_0^2\left(a + bx + c^2x^2\right)dx = \left[ax + \frac{1}{2}bx^2 + \frac{1}{3}c^2x^3\right]_0^2$

$= 2a + 2b + \frac{8}{3}c^2$

19. $\int_0^{\pi}\sin^2 x\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}$

20. $\int_{1/4}^{3/4}\frac{1}{1-x^2}dx = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|_{1/4}^{3/4} = \frac{1}{2}\ln\left(\frac{21}{5}\right)$

21. $\int_{x=1/4}^{3/4}\frac{1}{1+u}du = \frac{1}{2}\ln\left|1+x^2\right|_{1/4}^{3/4} = \frac{1}{2}\ln 2$

22. $\int_{x=0}^{4}\frac{1}{1+u^2}du = \left[\tan^{-1}e^x\right]_0^4 \approx 0.7671$

23. $u = 4r^2 + 1;\ du = 8r\,dr$

$\int_0^3 r\sqrt{4r^2+1}\,dr = \frac{1}{8}\int_{r=0}^{3}\sqrt{u}\,du$

$= \left[\frac{1}{8}\bullet\frac{2}{3}\left(4r^2+1\right)^{3/2}\right]_0^3 = \frac{-1+37^{3/2}}{12}$

24. $u = a^2 - r^2;\ du = -2r\,dr$

$\int_0^{a/2}\frac{ar}{\sqrt{a^2-r^2}}dr = -\frac{a}{2}\int_{r=0}^{a/2}\frac{1}{\sqrt{u}}du$

$= -\frac{a}{2}\left[\frac{1}{2}\sqrt{a^2-r^2}\right]_0^{a/2} = \frac{a^2\left(2-\sqrt{3}\right)}{8}$

25. $\int_0^{\pi/2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$

$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4}$

26. $\int_0^{\pi/2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)^2 d\theta$

$= \int_0^{\pi/2}\left(\frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right)^2 d\theta$

$= \int_0^{\pi/2}\left(\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right)d\theta$

$= \left[\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{16}$

27. $2\pi\left(\sqrt{a^2-b^2} - \sqrt{a^2-c^2}\right)$

θ is not part of the integrand.

28. The area is an equilateral triangle of length $\sqrt{2}$.

$$A = \frac{1}{2}\sqrt{2}\frac{\sqrt{6}}{2} = \frac{\sqrt{3}}{2}$$

29. The solid is half of a right circular cylinder of radius 3 and height 8.

$$V = \frac{1}{2}\pi r^2 h = \frac{\pi}{2}(9)(8) = 36\pi$$

30. The solid is a sphere of radius 7.

$$V = \frac{4}{3}\pi r^3 = \frac{4\pi}{3}7^3 = \frac{1372\pi}{3} \approx 1436.8$$ **31.**

The solid looks similar to a football.

$$V = \pi\int_0^{\pi} \sin^2 x\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{2}$$

32. The solid is a right circular cylinder of radius 7 and height 100.

$$V = \pi r^2 h = 4900\pi$$

33. The solid is half an elliptic paraboloid.
In the xz-plane, we can consider rotating the graph of $z = 9 - x^2$ around the z-axis for $0 \le x \le 3$. Using the Shell Method, we would get

$$V = 2\pi\int_0^3 x\left(9 - x^2\right)dx$$

$$= 2\pi\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = 2\pi\left[\frac{81}{2} - \frac{81}{4}\right] = \frac{81\pi}{2}$$

34. The solid is half of a hollow sphere of radius 1 inside half of a solid sphere of radius 4.

$$V = \frac{1}{2}\left(\frac{4}{3}\pi 4^3 - \frac{4}{3}\pi 1^3\right) = 42\pi$$

CHAPTER 13 Multiple Integrals

13.1 Concepts Review

1. $\sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k)\Delta A_k$

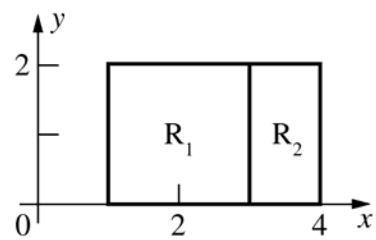
2. the volume of the solid under $z = f(x, y)$ and above R

3. continuous

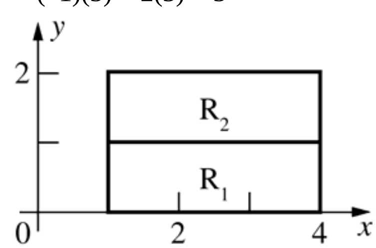
4. 12

Problem Set 13.1

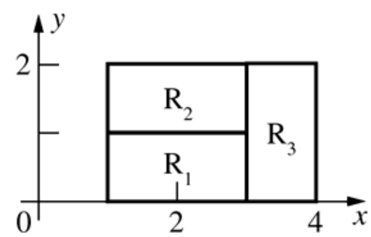
1. $\iint_{R_1} 2\,dA + \iint_{R_2} 3\,dA = 2A(R_1) + 3A(R_2)$
$= 2(4) + 3(2) = 14$



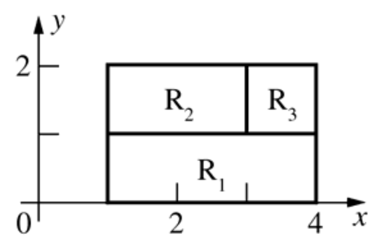
2. $\iint_{R_1} (-1)dA + \iint_{R_2} 2\,dA = (-1)A(R_1) + 2A(R_2)$
$= (-1)(3) + 2(3) = 3$



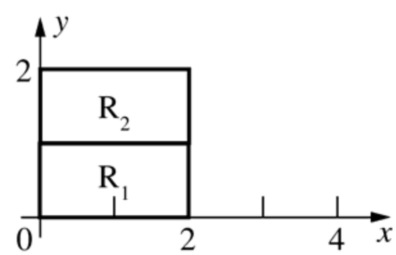
3. $\iint_R f(x, y)dA = \iint_{R_1} 2\,dA + \iint_{R_2} 1\,dA + \iint_{R_3} 3\,dA$
$= 2A(R_1) + 1A(R_2) + 3A(R_3)$
$= 2(2) + 1(2) + 3(2) = 12$



4. $\iint_{R_1} 2\,dA + \iint_{R_2} 3\,dA + \iint_{R_3} 1\,dA$
$= 2A(R_1) + 3A(R_2) + 1A(R_3)$
$= 2(3) + 3(2) + 1(1) = 13$



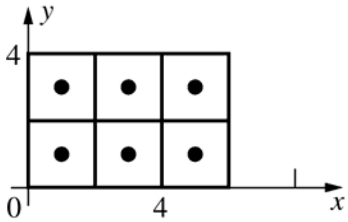
5. $3\iint_R f(x,y)dA - \iint_R g(x,y)dA = 3(3) - (5) = 4$



6. $2\iint_R f(x,y)dA + 5\iint_R g(x,y)dA$
$= 2(3) + 5(5) = 31$

7. $\iint_R g(x,y)dA - \iint_{R_1} g(x,y)dA = (5) - (2) = 3$

8. $2\iint_{R_1} g(x,y)dA + \iint_{R_1} 3\,dA = 2(2) + 3A(R_1)$
$= 4 + 3(2) = 10$

9. 

$[f(1, 1) + f(3, 1) + f(5, 1) + f(1, 3) + f(3, 3) + f(5, 3)](4) = [(10) + (8) + (6) + (8) + (6) + (4)](4) = 168$

10. $4(9 + 9 + 9 + 1 + 1 + 1) = 120$

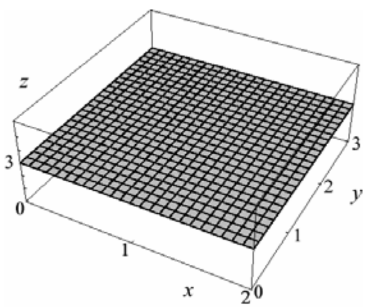
11. $4(3 + 11 + 27 + 19 + 27 + 43) = 520$

12. $\left[\left(\frac{41}{6}\right)+\left(\frac{33}{6}\right)+\left(\frac{25}{6}\right)+\left(\frac{35}{6}\right)+\left(\frac{27}{6}\right)+\left(\frac{19}{6}\right)\right](4) = 120$

13. $4\left(\sqrt{2}+\sqrt{4}+\sqrt{6}+\sqrt{4}+\sqrt{6}+\sqrt{8}\right) \approx 52.5665$

14. $4(e+e^3+e^5+e^3+e^9+e^{15}) \approx 13109247$

15.



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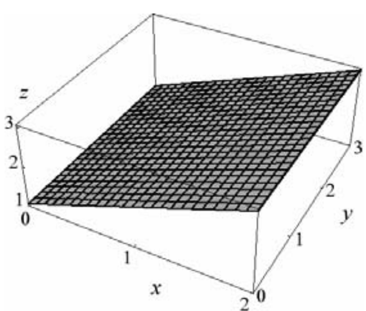

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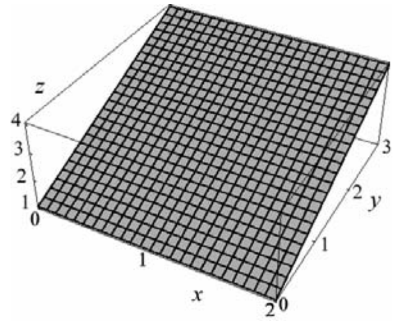

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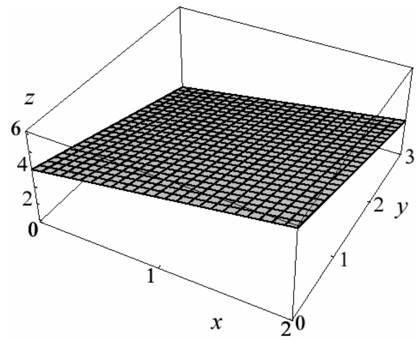

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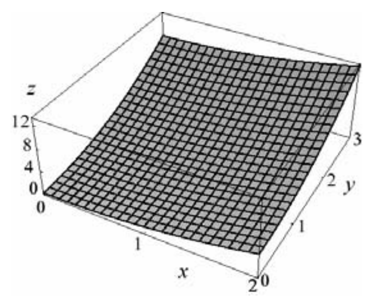

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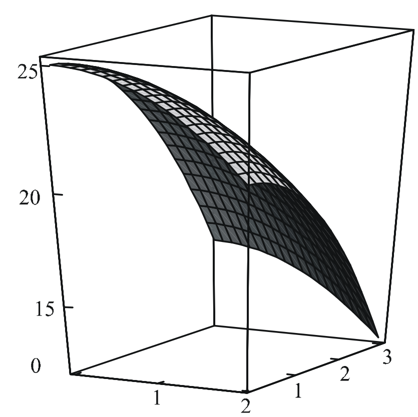
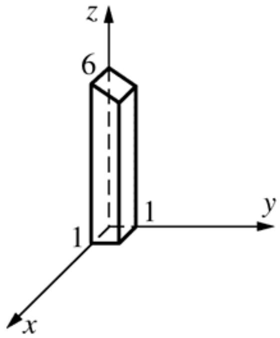


21.

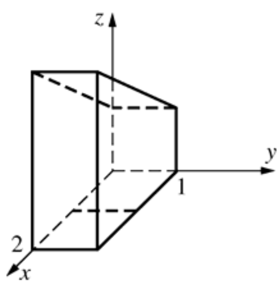


$z = 6 - y$ is a plane parallel to the x-axis. Let T be the area of the front trapezoidal face; let D be the distance between the front and back faces.

$\iint_R (6-y)dA = \text{volume of solid} = (T)(D)$

$= \left[\left(\frac{1}{2}\right)(6+5)\right](1) = 5.5$

22.



$z = 1 + x$ is a plane parallel to the y-axis.

$\iint_R (1+x)dA$ is the product of the area of a trapezoidal side face and the distance between the side faces.

$= \left[\left(\frac{1}{2}\right)(1+3)(2)\right](1) = 4$

23. $\iint_R 0\, dA = 0A(R) = 0$

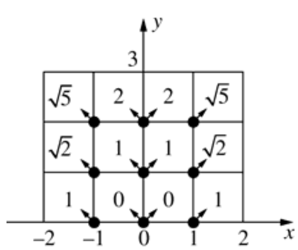
The conclusion follows.

24. $\iint_R m\, dA < \iint_R f(x,y)dA < \iint_R M\, dA$

(Comparison property)

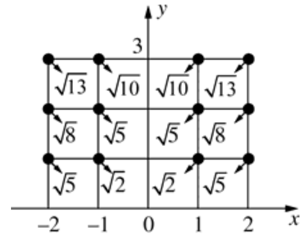
Therefore, $mA(R) < \iint_R f(x,y)dA < MA(R)$

25.



For c, take the sample point in each square to be the point of the square that is closest to the origin.

Then $c = 2\sqrt{5} + 2\sqrt{2} + 2(2) + 4(1) \approx 15.3006$



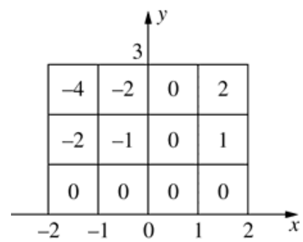
For C, take the sample point in each square to be the point of the square that is farthest from the origin. Then,

$C = 2\sqrt{13} + 2\sqrt{10} + 2\sqrt{8} + 4\sqrt{5} + 2\sqrt{2} \approx 30.9652.$

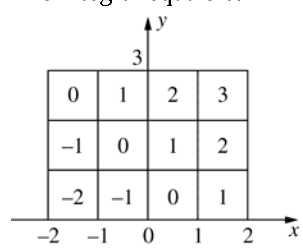
26. The integrand is symmetric with respect to the y-axis (i.e. an odd function), so the value of the integral is 0.

27. The values of $[\![x]\!][\![y]\!]$ and $[\![x]\!]+[\![y]\!]$ are indicated in the various square subregions of R. In each case the value of the integral on R is the sum of the values in the squares since the area of each square is 1.

a. The integral equals –6.



b. The integral equals 6.



28. Mass of the plate in grams

29. Total rainfall in Colorado in 2005; average rainfall in Colorado in 2005.

30. For each partition of R, each subrectangle contains some points at which $f(x, y) = 0$ and some points at which $f(x, y) = 1$. Therefore, for each partition there are sample points for which the Riemann sum is 0 and others for which the Riemann sum is (1)[Area (R)] = 12.

31. To begin, we divide the region R (we will use the outline of the contour plot) into 16 equal squares. Then we can approximate the volume by

$$V = \iint_R f(x, y)dA \approx \sum_{k=1}^{16} f(\bar{x}_k, \bar{y}_k)\Delta A_k .$$

Each square will have $\Delta A = (1\cdot 1) = 1$ and we will use the height at the center of each square as $f(\bar{x}_k, \bar{y}_k)$.

Therefore, we get

$$V \approx \sum_{k=1}^{16} f(\bar{x}_k, \bar{y}_k) = 20+21+24+29+22+23+26+32+26+27+30+35+32+33+36+42$$
$$= 458 \text{ cubic units}$$

13.2 Concepts Review

1. iterated

2. $\int_{-1}^{2}\left[\int_0^2 f(x,\ y)dy\right]dx$; $\int_0^2\left[\int_{-1}^{2} f(x,\ y)dx\right]dy$

3. signed; plus; minus

4. is below the xy-plane

Problem Set 13.2

1. $\int_0^2 [9y - xy]_0^3\, dx = \int_0^2 [27 - 3x]dx$

$= \left[27x - \frac{3}{2}x^2\right]_0^2 = 48$

2. $\int_{-2}^{2}\left[9y - yx^2\right]_0^1 dx = \int_{-2}^{2}\left[9 - x^2\right]dx$

$= \left[9x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{92}{3}$

3. $\int_0^2\left[\left(\frac{1}{2}\right)x^2y^2\right]_{y=1}^{3} dx = \int_0^2 4x^2 dx = \frac{32}{3}$

4. $\int_{-1}^{4}\left[xy + \left(\frac{1}{3}\right)y^3\right]_{y=1}^{2} dx = \int_{-1}^{4}\left(x + \frac{7}{3}\right)dx = \frac{115}{6}$

5. $\int_1^2\left[\frac{x^2y}{2} + xy^2\right]_{x=0}^{3} dx = \int_1^2\left(\frac{9y}{2} + 3y^2\right)dy$

$= \left[\frac{9y^2}{4} + y^3\right]_1^2 = 17 - \frac{13}{4} = \frac{55}{4} = 13.75$

6. $\int_{-1}^{1}\left[\left(\frac{1}{3}\right)x^3 + xy^2\right]_{x=1}^{2} dy = \int_{-1}^{1}\left(\frac{7}{3} + y^2\right)dy = \frac{16}{3}$

7. $\int_0^{\pi}\left[\left(\frac{1}{2}\right)x^2 \sin y\right]_{x=0}^{1} dy = \int_0^{\pi}\left(\frac{1}{2}\right)\sin y\, dy = 1$

8. $\int_0^{\ln 3}\int_0^{\ln 2} e^x e^y\, dy\, dx = \int_0^{\ln 3}[e^x e^y]_{y=0}^{\ln 2} dx$

$= \int_0^{\ln 3}[e^x(2) - e^x(1)]dx = \int_0^{\ln 3} e^x dx$

$= [e^x]_0^{\ln 3} = 3 - 1 = 2$

9. $\int_0^{\pi/2}[-\cos xy]_{y=0}^{1} dx = \int_0^{\pi/2}(1 - \cos x)dx$

$= \frac{\pi}{2} - 1 \approx 0.5708$

10. $\int_0^1 [e^{xy}]_{y=0}^{1} dx = \int_0^1 (e^x - 1)dx = e - 2 \approx 0.7183$

11. $\int_0^3\left[\frac{2(x^2+y)^{3/2}}{3}\right]_{x=0}^{1} dy$

$= \int_0^3 \frac{2[(1+y)^{3/2} - y^{3/2}]}{3} dy$

$= \left[\frac{4[(1+y)^{5/2} - y^{5/2}]}{15}\right]_0^3 = \frac{4\left(32 - 9\sqrt{3}\right) - 4}{15}$

$= \frac{4\left(31 - 9\sqrt{3}\right)}{15} \approx 4.1097$

12. $\int_0^1[-(xy+1)^{-1}]_{x=0}^{1} dy = \int_0^1\left(1 - \frac{1}{y+1}\right)dy$

$= 1 - \ln 2 \approx 0.3069$

13. $\int_0^{\ln 3}\left[\left(\frac{1}{2}\right)\exp(xy^2)\right]_{y=0}^{1} dx = \int_0^{\ln 3}\left(\frac{1}{2}\right)(e^x - 1)dx$

$= 1 - \left(\frac{1}{2}\right)\ln 3 \approx 0.4507$

14. $\int_0^1\left[\frac{y^2}{2(1+x^2)}\right]_{y=0}^{2} dx = \int_0^1 \frac{2}{1+x^2}dx$

$= [2\tan^{-1} x]_0^1 = 2\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{2}$

15. $\int_0^{\pi}\left[\frac{1}{2}y^2\cos^2 x\right]_0^3 dx = \int_0^{\pi}\frac{9}{2}\cos^2 x\,dx$

$= \left[\frac{9}{4}x + \frac{9}{8}\cos 2x\right]_0^{\pi} = \frac{9\pi}{4}$

16. $\frac{1}{2}\int_{-1}^{1}\left[e^{x^2}\right]_0^1 dy = \frac{1}{2}\int_{-1}^{1}(e-1)\,dy$

$= \frac{1}{2}\left[y(e-1)\right]_{-1}^{1} = e-1$

17. $\int_0^1 0\,dx = 0$ (since xy^3 defines an odd function in y).

18. $\int_{-1}^{1}\left[x^2y + \left(\frac{1}{3}\right)y^3\right]_{y=0}^{2} dx = \int_{-1}^{1}\left(2x^2 + \frac{8}{3}\right)dx$

$= \left[\left(\frac{2}{3}\right)x^3 + \left(\frac{8}{3}\right)x\right]_{-1}^{1} = \frac{20}{3}$

19. $\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dx\,dy$

$= \int_0^{\pi/2}[-\cos(x+y)]_{x=0}^{\pi/2}dy$

$= \int_0^{\pi/2}\left[-\cos\left(\frac{\pi}{2}+y\right)+\cos y\right]dy$

$= \int_0^{\pi/2}(\sin y + \cos y)dy = [-\cos y + \sin y]_0^{\pi/2}$

$= (0 + 1) - (-1 + 0) = 2$

20. $\int_1^2\left[\left(\frac{1}{3}\right)y(1+x^2)^{3/2}\right]_{x=0}^{\sqrt{3}} dy = \int_1^2\left(\frac{7}{3}\right)y\,dy$

$= \left[\left(\frac{7}{6}\right)y^2\right]_1^2 = 3.5$

21. $V = \int_0^2\int_0^3(20-x-y)\,dy\,dx$

$= \int_0^2\left(20y - xy - \frac{1}{2}y^2\right)_0^3 dx$

$= \int_0^2\left(60 - 3x - \frac{9}{2}\right)dy = 105$

22. $V = \int_0^2\int_0^3\left(25-x^2-y^2\right)dy\,dx$

$= \int_0^2\left(25y - x^2y - \frac{1}{3}y^3\right)_0^3 dx$

$= \int_0^2\left(75-3x^2-9\right)dx = 124$

23. $V = \int_0^3\int_0^4\left(1+x^2+y^2\right)dx\,dy$

$= \int_0^3\left(x + \frac{1}{3}x^3 + xy^2\right)_0^3 dy$

$= \int_0^3\left(4 + \frac{64}{3} + 4y^2\right)dy = 112$

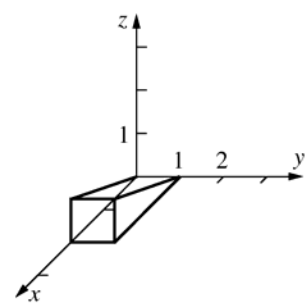
24. $V = \int_0^3\int_0^2 5xye^{-x^2}\,dx\,dy$

$= \int_0^3 5y\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)_0^2 dy$

$= \int_0^3 5y\left[-\frac{5}{4}e^{-4} - \frac{1}{4}\right]dy = \frac{45\left(e^4-1\right)}{4e^4} \approx 11.0439$

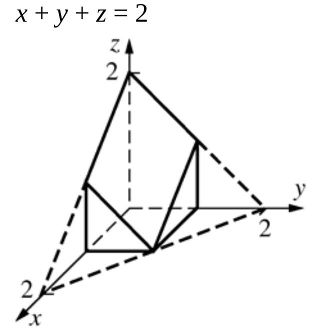
25. $z = \frac{x}{2}$ is a plane.

$x - 2z = 0$

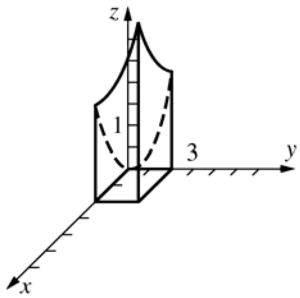


26. $z = 2 - x - y$ is a plane.

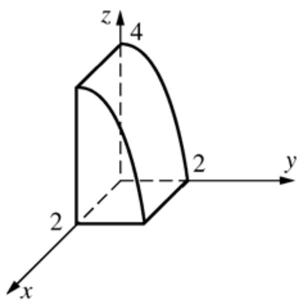
$x + y + z = 2$



27. $z = x^2 + y^2$ is a paraboloid opening upward with z-axis.



28. $z = 4 - y^2$ is a parabolic cylinder parallel to the x-axis.



29. $\int_1^3 \int_0^1 (x+y+1)dx\,dy = \int_1^3 \left[\left(\frac{1}{2}\right)x^2 + yx + x\right]_{x=0}^1 dy$

$= \int_1^3 \left(y + \frac{3}{2}\right)dy = 7$

30. $\int_1^2 \int_0^4 (2x+3y)dy\,dx = \int_1^2 \left[2xy + \left(\frac{3}{2}\right)y^2\right]_{y=0}^4 dx$

$= \int_1^2 (8x+24)dx = 36$

31. $x^2 + y^2 + 2 > 1$

$\int_{-1}^1 \int_0^1 [(x^2+y^2+2)-1]dy\,dx$

$= \int_{-1}^1 \left[x^2 y + \left(\frac{1}{3}\right)y^3 + y\right]_{y=0}^1 dx$

$= \int_{-1}^1 \left(x^2 + \frac{4}{3}\right)dx = \frac{10}{3}$

32. $\int_0^2 \int_0^2 (4-x^2)dx\,dy = \int_0^2 \left[4x - \frac{x^3}{3}\right]_0^2 dy$

$= \int_0^2 \left(\frac{16}{3}\right)dy = \left[\frac{16y}{3}\right]_0^2 = \frac{32}{3}$

33. $\int_a^b \int_c^d g(x)h(y)dy\,dx = \int_a^b g(x)\int_c^d h(y)dy\,dx$

$= \int_c^d h(y)\,dy \int_a^b g(x)dx$

(First step used linearity of integration with respect to y; second step used linearity of integration with respect to x; now commute.)

34. $\int_0^{\sqrt{\ln 2}} xe^{x^2} dx \int_0^1 y(1+y^2)^{-1} dy$

$= \left[\left(\frac{1}{2}\right)e^{x^2}\right]_0^{\sqrt{\ln 2}} \left[\left(\frac{1}{2}\right)\ln(1+y^2)\right]_0^1$

$= \left[\frac{1}{2}\right]\left[\left(\frac{1}{2}\right)\ln 2\right] = \left(\frac{1}{4}\right)\ln 2 \approx 0.1733$

35. $\int_0^1 \int_0^1 xye^{x^2} e^{y^2} dy\,dx = \left(\int_0^1 xe^{x^2} dx\right)\left(\int_0^1 ye^{y^2} dy\right)$

$= \left(\int_0^1 xe^{x^2} dx\right)^2$ (Changed the dummy variable y to the dummy variable x.)

$= \left(\left[\frac{e^{x^2}}{2}\right]_0^1\right)^2 = \left(\frac{e-1}{2}\right)^2 \approx 0.7381$

36. $V = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\cos x \cos y|\,dx\,dy$

$= \int_{-\pi}^{\pi} |\cos x|\,dx \int_{-\pi}^{\pi} |\cos y|\,dy = \left(\int_{-\pi}^{\pi} |\cos x|\,dx\right)^2$

$= \left(4\int_0^{\pi/2} |\cos x|\,dx\right)^2 = \left(4[\sin x]_0^{\pi/2}\right)^2 = 16$

37. $\int_{-2}^2 x^2 dx \int_{-1}^1 |y^3|\,dy = \left(2\int_0^2 x^2 dx\right)\left(2\int_0^1 y^3 dy\right)$

$= 2\left(\frac{8}{3}\right)2\left(\frac{1}{4}\right) = \frac{8}{3}$

38. $\int_{-2}^2 [\![x^2]\!]dx \int_{-1}^1 y^3 dy = 0$ (since the second integral equals 0).

39. $\int_{-2}^2 [\![x^2]\!]dx \int_{-1}^1 |y^3|\,dy = 2\int_0^2 [\![x^2]\!]dx\, 2\int_0^1 y^3 dy$

$= 2\left[\int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^2 3\,dx\right]\left[2\left(\frac{1}{4}\right)\right]$

$= 2\left[0 + \left(\sqrt{2}-1\right) + 2\left(\sqrt{3}-\sqrt{2}\right) + 3\left(2-\sqrt{3}\right)\right]\left[\frac{1}{2}\right]$

$= 5 - \sqrt{3} - \sqrt{2} \approx 1.8537$

40. $\int_0^1\int_0^{\sqrt{3}} 8x(x^2+y^2+1)^{-2}\,dx\,dy = \int_0^1 [-4(x^2+y^2+1)^{-1}]_{x=0}^{\sqrt{3}}\,dy \;\; = 4\int_0^1\left[\frac{-1}{4+y^2}+\frac{1}{1+y^2}\right]dy$

$= 4\left[-\frac{1}{2}\arctan\left(\frac{y}{2}\right)+\arctan(y)\right]_0^1 \;\; = 4\left[\left(-\frac{1}{2}\arctan\left(\frac{1}{2}\right)+\frac{\pi}{4}\right)-0\right] \;\; = \pi - 2\arctan\left(\frac{1}{2}\right) \approx 2.2143$

41. $0 \le \int_a^b\int_a^b [f(x)g(y)-f(y)g(x)]^2\,dx\,dy \;\; = \int_a^b\int_a^b [f^2(x)g^2(y)-2f(x)g(x)f(y)g(y)+f^2(y)g^2(x)]dx\,dy$

$= \int_a^b f^2(x)dx\int_a^b g^2(y)dy - 2\int_a^b f(x)g(x)dx\int_a^b f(y)g(y)dy + \int_a^b f^2(y)dy\int_a^b g^2(x)dx$

$= 2\int_a^b f^2(x)dx\int_a^b g^2(x)dx - 2\left[\int_a^b f(x)g(x)dx\right]^2$

Therefore, $\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx.$

42. Since f is increasing, $[y-x][f(y)-f(x)] > 0$. Therefore,

$0 < \int_a^b\int_a^b [y-x][f(y)-f(x)]dx\,dy \;\; = \int_a^b\int_a^b yf(y)dx\,dy - \int_a^b\int_a^b yf(x)dx\,dy - \int_a^b\int_a^b xf(y)dx\,dy + \int_a^b\int_a^b xf(x)dx\,dy$

$= (b-a)\int_a^b yf(y)dy - \frac{b^2-a^2}{2}\int_a^b f(x)dx - \frac{b^2-a^2}{2}\int_a^b f(y)dy + (b-a)\int_a^b xf(x)dx$

$= 2(b-a)\int_a^b xf(x)dx - (b^2-a^2)\int_a^b f(x)dx \;\; = (b-a)\left[2\int_a^b xf(x)dx - (b+a)\int_a^b f(x)dx\right]$

Therefore, $(b+a)\int_a^b f(x)dx < 2\int_a^b xf(x)dx.$ Now divide each side by the positive number $2\int_a^b f(x)dx$ to obtain the desired result.

Interpretation:
If f is increasing on $[a, b]$ and $f(x) \ge 0$, then the x-coordinate of the centroid (of the region between the graph of f and the x-axis for x in $[a, b]$) is to the right of the midpoint between a and b.

Another interpretation:
If $f(x)$ is the density at x of a wire and the density is increasing as x increases for x in $[a, b]$, then the center of mass of the wire is to the right of the midpoint of $[a, b]$.

13.3 Concepts Review

1. A rectangle containing S; 0

2. $\phi_1(x) \le y \le \phi_2(x)$

3. $\int_a^b\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy\,dx$

4. $\int_0^1\int_0^{1-x} 2x\,dy\,dx; \frac{1}{3}$

Problem Set 13.3

1. $\int_0^1 [x^2y]_{y=0}^{3x}dx = \int_0^1 3x^3dx = \frac{3}{4}$

2. $\int_1^2\left[\left(\frac{1}{2}\right)y^2\right]_{y=0}^{x-1}dx = \int_1^2\left(\frac{1}{2}\right)(x-1)^2dx = \frac{1}{6}$

3. $\int_{-1}^3\left[\frac{x^3}{3}+y^2x\right]_{x=0}^{3y}dy = \int_{-1}^3(9y^3+3y^3)dy$

$= [3y^4]_{-1}^3 = 243-3 = 240$

4. $\int_{-3}^1\left[x^2y-\left(\frac{1}{4}\right)y^4\right]_{y=0}^{x}dx = \int_{-3}^1\left[x^3-\left(\frac{1}{4}\right)x^4\right]dx$

$= -32.2$

5. $\int_1^3\left[\left(\frac{1}{2}\right)x^2\exp(y^3)\right]_{x=-y}^{2y}dy = \int_1^3\left(\frac{3}{2}\right)y^2\exp(y^3)dy$

$= \left(\frac{1}{2}\right)(e^{27}-e) \approx 2.660\times 10^{11}$

6. $\int_1^5\left[\frac{3}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_{y=0}^{x}dx=\int_1^5\frac{3}{x}\frac{\pi}{4}dx$

$=\left[\frac{3\pi\ln x}{4}\right]_1^5=\frac{3\pi\ln 5}{4}\approx 3.7921$

7. $\int_{1/2}^{1}[y\cos(\pi x^2)]_{y=0}^{2x}dx=\int_{1/2}^{1}2x\cos(\pi x^2)dx$

$=-\frac{\sqrt{2}}{2\pi}\approx -0.2251$

8. $\int_0^{\pi/4}\left[\left(\frac{1}{2}\right)r^2\right]_{r=\sqrt{2}}^{\sqrt{2}\cos\theta}d\theta=\int_0^{\pi/4}(\cos^2\theta-1)d\theta$

$=\frac{(2-\pi)}{8}\approx -0.1427$

9. $\int_0^{\pi/9}[\tan\theta]_{\theta=\pi/4}^{3r}dr=\int_0^{\pi/9}(\tan 3r-1)dr$

$=\left[-\frac{\ln|\cos 3r|}{3}-r\right]_0^{\pi/9}$

$=\left(\frac{-\ln\left(\frac{1}{2}\right)}{3}-\frac{\pi}{9}\right)-\left(-\frac{\ln(1)}{3}-0\right)$

$=\frac{3\ln 2-\pi}{9}\approx -0.1180$

10. $\int_0^2\left[ye^{-x^2}\right]_{-x}^{x}dx=\int_0^2 2xe^{-x^2}dx=\left[-e^{-x^2}\right]_0^2$

$=1-e^{-4}$

11. $\int_0^{\pi/2}[e^x\cos y]_{x=0}^{\sin y}dy=\int_0^{\pi/2}(e^{\sin y}\cos y-\cos y)dy$

$=e-2\approx 0.7183$

12. $\int_1^2\left[\frac{y^3}{3x}\right]_0^{x^2}dx=\int_1^2\frac{x^5}{3}dx=\left[\frac{1}{18}x^6\right]_1^2=3.5$

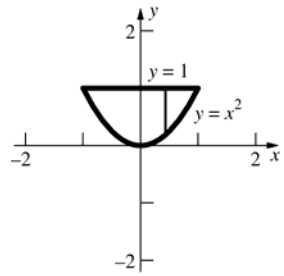
13. $\int_0^2\left[xy+\left(\frac{1}{2}\right)y^2\right]_{y=0}^{\sqrt{4-x^2}}dx$

$=\int_0^2\left[x(4-x^2)^{1/2}+2-\left(\frac{1}{2}\right)x^2\right]dx\;=\frac{16}{3}$

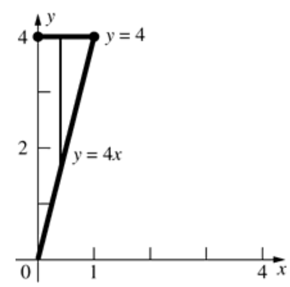
14. $\int_{\pi/6}^{\pi/2}[3r^2\cos\theta]_{r=0}^{\sin\theta}d\theta=\int_{\pi/6}^{\pi/2}3\sin^2\theta\cos\theta\,d\theta$

$=[\sin^3\theta]_{\pi/6}^{\pi/2}=\frac{7}{8}=0.875$

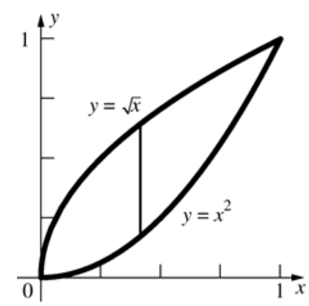
15. $\int_{-1}^{1}\int_{x^2}^{1}xy\,dy\,dx=0$



16. $\int_0^1\int_{4x}^{4}(x+y)dy\,dx=6$



17.



$\int_0^1\int_{x^2}^{\sqrt{x}}(x^2+2y)dy\,dx=\int_0^1[x^2y+y^2]_{y=x^2}^{\sqrt{x}}dx$

$=\int_0^1[(x^{5/2}+x)-(x^4+x^4)]dx$

$=\left[\frac{2x^{7/2}}{7}+\frac{x^2}{2}-\frac{2x^5}{5}\right]_0^1=\frac{2}{7}+\frac{1}{2}-\frac{2}{5}$

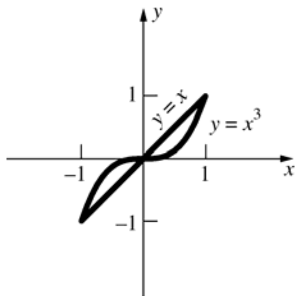
$=\frac{27}{70}\approx 0.3857$

18. $\int_0^2\int_x^{3x-x^2}(x^2-xy)dy\,dx$

$=\int_0^2\frac{-x^3\left(x^2-4x+4\right)}{2}dx=-\frac{8}{15}$

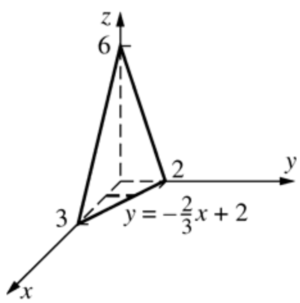
19. $\int_0^2\int_x^2 2(1+x^2)^{-1}dy\,dx=4\tan^{-1}2-\ln 5\;\approx\;2.8192$

20.



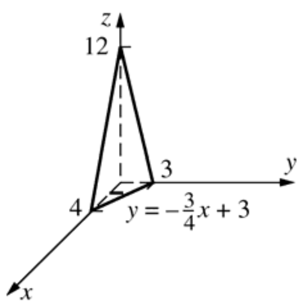
Since S is symmetric with respect to the origin and the integrand is an odd function in x, the value of the integral is 0.

21.



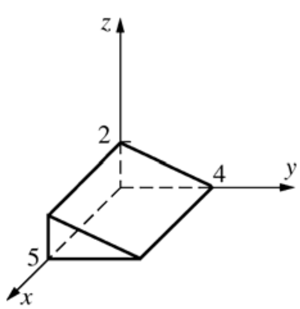
$$\int_0^3 \int_0^{(-2/3)x+2} (6-2x-3y)\,dy\,dx = 6$$

22.



$$\int_0^4 \int_0^{(-3/4)x+3} (12-3x-4y)\,dy\,dx = 24$$

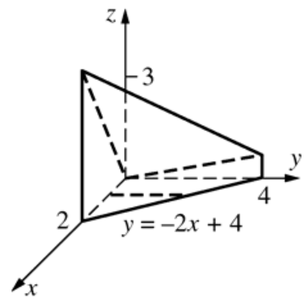
23.



$$\int_0^5 \int_0^4 \frac{4-y}{2}\,dy\,dx = \left(\int_0^5 1\,dx\right)\left(\int_0^4 \frac{4-y}{2}\,dy\right)$$

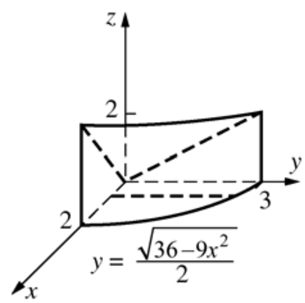
$$= 5\left[2y - \frac{y^2}{4}\right]_0^4 = 5(8-4) = 20$$

24.



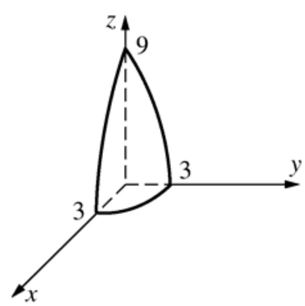
$$\int_0^2 \int_0^{-2x+4} \left[2x + \left(\frac{1}{4}\right)y\right] dy\,dx = \frac{20}{3}$$

25.



$$\int_0^2 \int_0^{(1/2)\sqrt{36-9x^2}} \left(\frac{1}{6}\right)(9x+4y)\,dy\,dx = 10$$

26.



$$\int_0^3 \int_0^{\sqrt{9-x^2}} (9-x^2-y^2)\,dy\,dx$$

$$= \int_0^3 \left[(9-x^2)y - \frac{y^3}{3}\right]_{y=0}^{\sqrt{9-x^2}} dx = \int_0^3 \frac{2(9-x^2)^{3/2}}{3}\,dx$$

$$= \int_0^{\pi/2} 18\cos^3 t\, 3\cos t\,dt = \int_0^{\pi/2} 54\cos^4 t\,dt$$

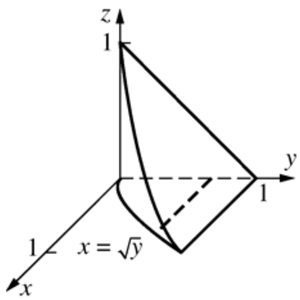
$$= \int_0^{\pi/2} \left(\frac{81}{4} + 27\cos 2t + \frac{27\cos 4t}{4}\right) dt$$

$$= \left[\frac{81t}{4} + \frac{27\sin 2t}{2} + \frac{27\sin 4t}{16}\right]_0^{\pi/2}$$

$$= \frac{81\pi}{8} \approx 31.8086$$

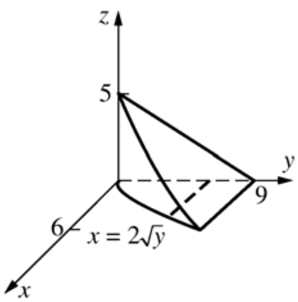
(At the third step, the substitution $x = 3\sin t$ was used. At the 5th step the identity $\cos^2 A = \left(\frac{1}{2}\right)(1+\cos 2A)$ was used a few times.)

27.

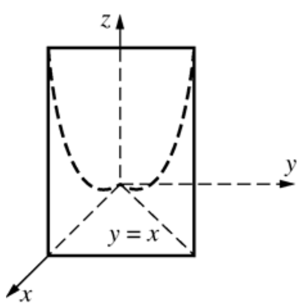


$\int_0^1 \int_0^{\sqrt{y}} (1-y)\,dx\,dy = \frac{4}{15}$

28. $\int_0^9 \int_0^{2\sqrt{y}} \left[5 - \left(\frac{5}{9}\right)y\right] dx\,dy = 72$



29.

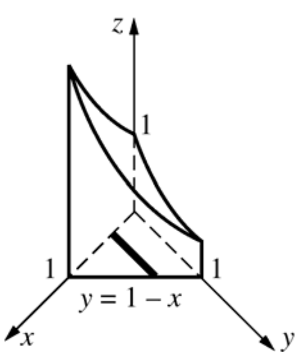


$\int_0^1 \int_0^x \tan x^2\,dy\,dx = \int_0^1 [y \tan x^2]_{y=0}^x\,dx$

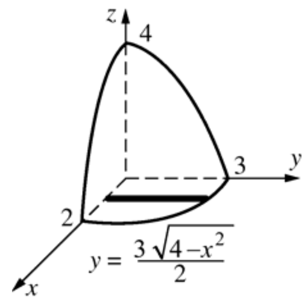
$= \int_0^1 x \tan x^2\,dx = \left[-\frac{\ln\left|\cos x^2\right|}{2}\right]_0^1 = \left(-\frac{1}{2}\right)\ln(\cos 1)$

$\approx\ 0.3078$

30. $\int_0^1 \int_0^{1-x} e^{x-y}\,dy\,dx = \left(\frac{1}{2}\right)(e + e^{-1} - 2) \approx 0.5431$



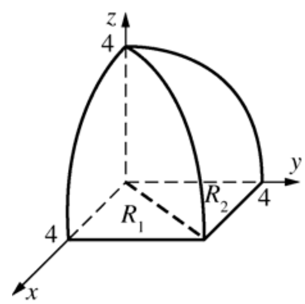
31.



$\int_0^2 \int_0^{(3/2)\sqrt{4-x^2}} \left[4 - x^2 - \left(\frac{4}{9}\right)y^2\right] dy\,dx = 3\pi$

≈ 9.4248

32.



Making use of symmetry, the volume is

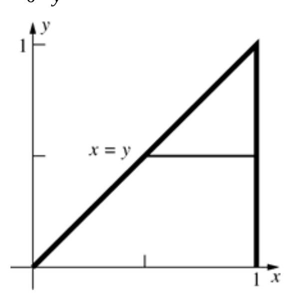
$2\iint_{R_1} (16 - x^2)^{1/2}\,dA = 2\int_0^4 \int_0^x (16 - x^2)^{1/2}\,dy\,dx$

$= 2\int_0^4 [(16 - x^2)^{1/2}\,y]_{y=0}^x\,dx$

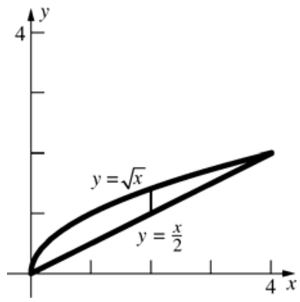
$= 2\int_0^4 (16 - x^2)^{1/2}\,x\,dx = \left[\frac{-2(16 - x^2)^{3/2}}{3}\right]_0^4$

$= 0 + \frac{2(64)}{3} = \frac{128}{3} \approx 42.6667$

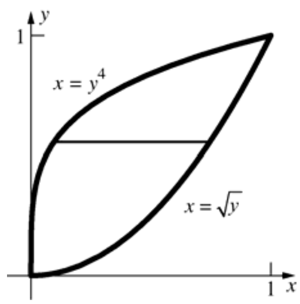
33. $\int_0^1 \int_y^1 f(x,\ y)\,dx\ dy$



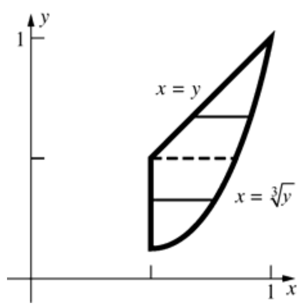
34. $\int_0^4 \int_{x/2}^{\sqrt{x}} f(x, y)\,dy\,dx$



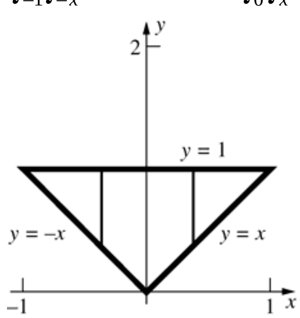
35. $\int_0^1 \int_{y^4}^{\sqrt{y}} f(x, y)\,dx\,dy$



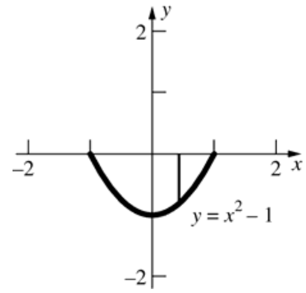
36. $\int_{1/8}^{1/2} \int_{1/2}^{y^{1/3}} f(x, y)\,dx\,dy + \int_{1/2}^{1} \int_{y}^{y^{1/3}} f(x, y)\,dx\,dy$



37. $\int_{-1}^{0} \int_{-x}^{1} f(x, y)\,dy\,dx + \int_0^1 \int_x^1 f(x, y)\,dy\,dx$

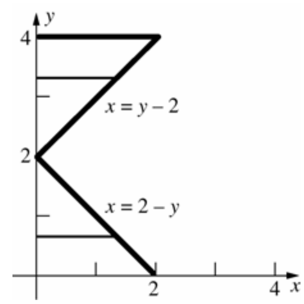


38. $\int_{-1}^{1} \int_{x^2-1}^{0} f(x, y)\,dy\,dx$



39. $\int_0^2 \int_0^{2-y} xy^2\,dx\,dy + \int_2^4 \int_0^{y-2} xy^2\,dx\,dy = \dfrac{256}{15}$

≈ 17.0667



40. $\int_{-2}^{1} \int_{x^2}^{-x+2} xy\,dy\,dx - \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1/2} xy\,dy\,dx$

$= -\dfrac{45}{8} = -5.625$

41. The integral over S of $x^4 y$ is 0 since this is an odd function of y. Therefore,

$\iint_S (x^2 + x^4 y)\,dA = \iint_S x^2\,dA$

$= 4\left(\iint_{S_1} x^2\,dA + \iint_{S_2} x^2\,dA \right)$

$= 4\left(\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x^2\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x^2\,dy\,dx \right)$

$= 4\left(\int_0^2 x^2 \sqrt{4-x^2}\,dx - \int_0^1 x^2 \sqrt{1-x^2}\,dx \right)$

$= 4\left(16\int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta - \int_0^{\pi/2} \sin^2\phi \cos^2\phi\,d\phi \right)$

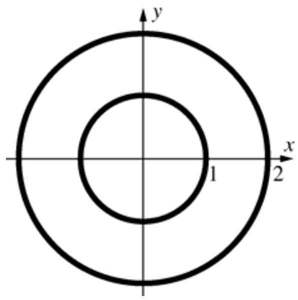
(using $x = 2\sin\theta$ in 1st integral; $x = \sin\phi$ in 2nd)

$= 60\int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta = \dfrac{15\pi}{4}$

(See work in Problem 42.)

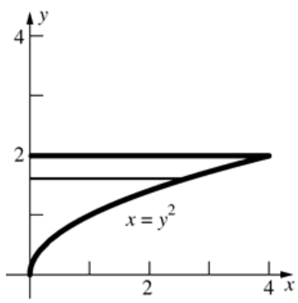
≈ 11.7810

42.



$z = f(x, y) = \sin(xy^2)$ is symmetric with respect to the x-axis, as is the annulus. Therefore, the integral equals 0.

43.

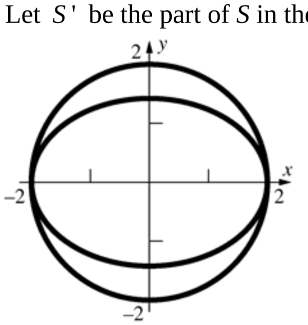


$$\int_0^2 \int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 [x\sin(y^3)]_{x=0}^{y^2}\,dy$$

$$= \int_0^2 y^2 \sin(y^3)\,dy = \left[-\frac{\cos(y^3)}{3}\right]_0^2$$

$$= \frac{1-\cos 8}{3} \approx 0.3818$$

44. Let S' be the part of S in the first quadrant.



$$\iint_{S'} x^2\,dA = \int_0^2 \int_{\sqrt{4-x^2}/\sqrt{2}}^{\sqrt{4-x^2}} x^2\,dy\,dx$$

$$= \int_0^2 x^2\left(\sqrt{4-x^2} - \frac{\sqrt{4-x^2}}{\sqrt{2}}\right)dx$$

$$= \int_0^2 x^2\sqrt{4-x^2}\left(1-\frac{1}{\sqrt{2}}\right)dx$$

$$= \frac{\left(2-\sqrt{2}\right)}{2}\int_0^2 x^2\sqrt{4-x^2}\,dx$$

Let $x = 2\sin\theta$, θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Then $dx = 2\cos\theta\,d\theta$

$x = 2 \Rightarrow \theta = \frac{\pi}{2}$

$x = 0 \Rightarrow \theta = 0$

$$= \frac{\left(2-\sqrt{2}\right)}{2}\int_0^{\pi/2}(2\sin\theta)^2(2\cos\theta)2\cos\theta\,d\theta$$

$$= 8\left(2-\sqrt{2}\right)\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$^* = 8\left(2-\sqrt{2}\right)\left(\frac{\pi}{16}\right) = \frac{\pi\left(2-\sqrt{2}\right)}{2}$$

Therefore,

$$\iint_S x^2\,dA = 4\left[\frac{\pi\left(2-\sqrt{2}\right)}{2}\right] = 2\pi\left(2-\sqrt{2}\right) \approx 3.6806$$

$$^* = \int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{1}{2}(1-\cos 2\theta)\right]\left[\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}(1-\cos^2 2\theta)\,d\theta$$

$$= \frac{1}{4}\frac{\pi}{2} - \frac{1}{4}\int_0^{\pi/2}\frac{1}{2}(1+\cos 4\theta)\,d\theta$$

$$= \frac{\pi}{8} - \frac{1}{8}\frac{\pi}{2} + \frac{1}{8}\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi}{8} - \frac{\pi}{16} + 0 = \frac{\pi}{16}$$

45. We first slice the river into eleven 100’ sections parallel to the bridge. We will assume that the cross-section of the river is roughly the shape of an isosceles triangle and that the cross-sectional area is uniform across a slice. We can then approximate the volume of the water by

$$V \approx \sum_{k=1}^{11} A_k(y_k)\Delta y = \sum_{k=1}^{11}\frac{1}{2}(w_k)(d_k)100$$

$$= 50\sum_{k=1}^{11}(w_k)(d_k)$$

where w_k is the width across the river at the left side of the kth slice, and d_k is the center depth of the river at the left side of the kth slice. This gives

$$V \approx 50[300\cdot 40 + 300\cdot 39 + 300\cdot 35 + 300\cdot 31$$
$$+290\cdot 28 + 275\cdot 26 + 250\cdot 25 + 225\cdot 24$$
$$+205\cdot 23 + 200\cdot 21 + 175\cdot 19]$$
$$= 4{,}133{,}000 \text{ ft}^3$$

46. Since f is continuous on the closed and bounded set R, it achieves a minimum m and a maximum M on R. Suppose (x_1, y_1) and (x_2, y_2) are such that $f(x_1, y_1) = m$ and $f(x_2, y_2) = M$. Then,

$$m \le f(x, y) \le M$$

$$\iint_R m\,dA \le \iint_R f(x, y)\,dA \le \iint_R M\,dA$$

$$mA(R) \le \iint_R f(x, y)\,dA \le MA(R)$$

$$m \le \frac{1}{A(R)} \iint_R f(x, y)\,dA \le M$$

Let C be a continuous curve in the plane from (x_1, y_1) to (x_2, y_2) that is parameterized by $x = x(t)$, $y = y(t)$, $c \le t \le d$. Let $h(t) = f(x(t), y(t))$. Since f is continuous, so is h. By the Intermediate Value Theorem, there exists a t_0 in (c, d) such that

$h(t_0) = \dfrac{1}{A(R)} \displaystyle\iint_R f(x, y)\,dA$. But $h(t_0) = f(x(t_0), y(t_0)) = f(a, b)$, where $a = x(t_0)$ and $b = y(t_0)$. Thus,

$$f(a, b) = \frac{1}{A(R)} \iint_R f(x, y)\,dA$$

or, $\displaystyle\iint_R f(x, y)\,dA = f(a, b) \cdot A(R)$.

13.4 Concepts Review

1. $a \le r \le b; \alpha \le \theta \le \beta$

2. $r\,dr\,d\theta$

3. $\int_0^{\pi} \int_0^2 r^3 dr\,d\theta$

4. 4π

Problem Set 13.4

1. $\displaystyle\int_0^{\pi/2} \left[\left(\frac{1}{3}\right) r^3 \sin\theta\right]_{r=0}^{\cos\theta} d\theta$

$= \displaystyle\int_0^{\pi/2} \left(\frac{1}{3}\right) \cos^3\theta \sin\theta\,d\theta$

$= \dfrac{1}{12} \approx 0.0833$

2. $\displaystyle\int_0^{\pi/2} \left[\left(\frac{1}{2}\right) r^2\right]_{r=0}^{\sin\theta} d\theta = \int_0^{\pi/2} \left(\frac{1}{2}\right) \sin^2\theta\,d\theta$

$= \dfrac{\pi}{8} \approx 0.3927$

3. $\displaystyle\int_0^{\pi} \left[\frac{r^3}{3}\right]_{r=0}^{\sin\theta} d\theta = \int_0^{\pi} \frac{\sin^3\theta}{3} d\theta$

$= \displaystyle\int_0^{\pi} \frac{(1-\cos^2\theta)\sin\theta}{3} d\theta$

$= \left[\dfrac{-\cos\theta}{3} + \dfrac{\cos^3\theta}{9}\right]_0^{\pi}$

$= \left(\dfrac{1}{3} - \dfrac{1}{9}\right) - \left(-\dfrac{1}{3} + \dfrac{1}{9}\right) = \dfrac{4}{9}$

4. $\displaystyle\int_0^{\pi} \left[\left(\frac{1}{2}\right) r^2 \sin\theta\right]_{r=0}^{1-\cos\theta} d\theta$

$= \displaystyle\int_0^{\pi} \left(\frac{1}{2}\right) (1-\cos\theta)^2 \sin\theta\,d\theta$

$= \dfrac{4}{3}$

5.

$\displaystyle\int_0^{\pi} \left[\frac{1}{2} r^2 \cos\frac{\theta}{4}\right]_0^2 d\theta = \int_0^{\pi} 2\cos\frac{\theta}{4} d\theta$

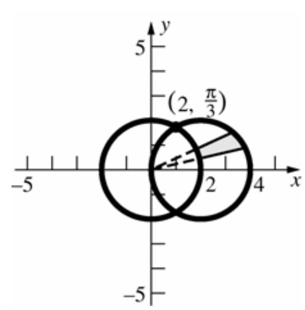
$= \left[8\sin\dfrac{\theta}{4}\right]_0^{\pi} = 4\sqrt{2}$

6.

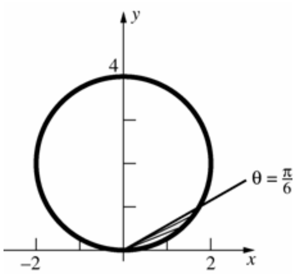
$\displaystyle\int_0^{2\pi} \left[\frac{1}{2} r^2\right]_0^{\theta} d\theta = \int_0^{2\pi} \frac{1}{2}\theta^2\,d\theta = \left[\frac{1}{6}\theta^3\right]_0^{2\pi}$

$= \dfrac{4\pi^3}{3}$

7.



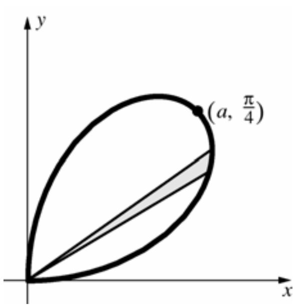
$2\displaystyle\int_0^{\pi/3} \int_2^{4\cos\theta} r\,dr\,d\theta = 2\left[\frac{2\pi}{3} + \sqrt{3}\right] \approx 7.6529$

8. 

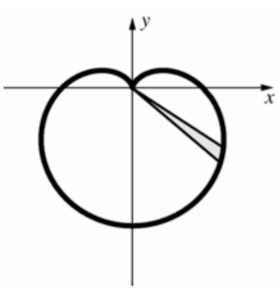
$$\int_0^{\pi/6}\int_0^{4\sin\theta} r\,dr\,d\theta = \int_0^{\pi/6}\left[\frac{r^2}{2}\right]_0^{4\sin\theta} d\theta$$

$$= \int_0^{\pi/6} 8\sin^2\theta\,d\theta = \int_0^{\pi/6} 4(1-\cos 2\theta)d\theta$$

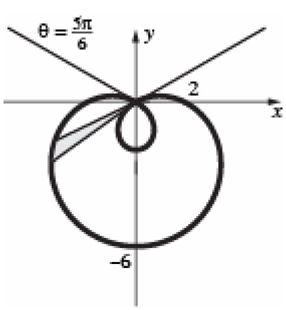
$$= [4\theta - 2\sin 2\theta]_0^{\pi/6} = \frac{2\pi}{3} - \sqrt{3} \approx 0.3623$$

9. 

$$\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\,dr\,d\theta = \frac{a^2\pi}{8}$$

10. 

$$\int_0^{2\pi}\int_0^{6-6\sin\theta} r\,dr\,d\theta = 54\pi \approx 169.6460$$

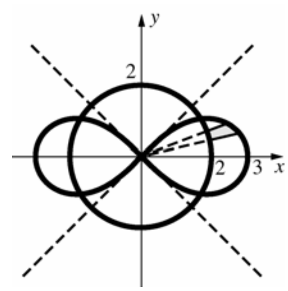
11. 

$$2\int_{5\pi/6}^{3\pi/2}\int_0^{2-4\sin\theta} r\,dr\,d\theta = 2\int_{5\pi/6}^{3\pi/2}\left[\frac{r^2}{2}\right]_0^{2-4\sin\theta} d\theta$$

$$= 2\int_{5\pi/6}^{3\pi/2}(6 - 8\sin\theta - 4\cos 2\theta)d\theta$$

$$= 2[6\theta + 8\cos\theta - 2\sin 2\theta]_{5\pi/6}^{3\pi/2}$$

$$= 2\left(4\pi + 3\sqrt{3}\right) \approx 35.525$$

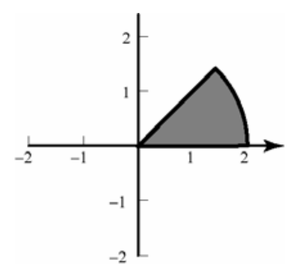
12. 

$$4\int_0^{(1/2)\cos^{-1}(4/9)}\int_2^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

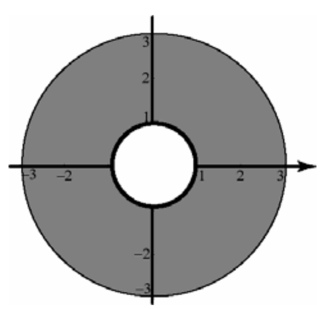
$$= \sqrt{65} - 4\cos^{-1}\left(\frac{4}{9}\right)$$

$$\approx 3.6213$$

13. $\int_0^{\pi/4}\left[\frac{1}{2}r^2\right]_0^2 d\theta = \int_0^{\pi/4} 2d\theta = \frac{\pi}{2}$

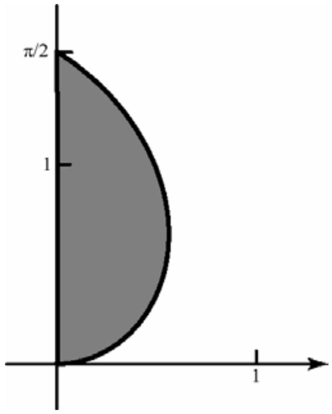


14. $\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_1^3 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi$



15. $\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_0^{\theta} d\theta = \int_0^{\pi/2}\frac{1}{2}\theta^2\, d\theta$

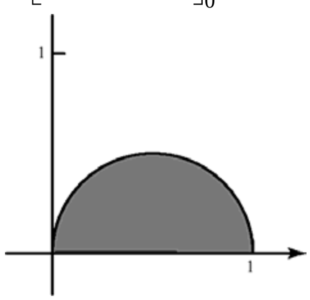
$$=\left[\frac{1}{6}\theta^3\right]_0^{\pi/2} = \frac{\pi^3}{48}$$



16. $\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_0^{\cos\theta} d\theta = \int_0^{\pi/2}\frac{1}{2}\cos^2\theta\, d\theta$

$$=\int_0^{\pi/2}\frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\cos 2\theta\right]d\theta$$

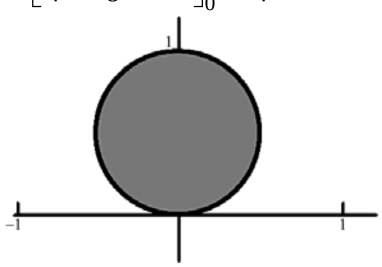
$$=\left[\frac{1}{4}\theta+\frac{1}{8}\cos 2\theta\right]_0^{\pi/2} = \frac{\pi}{8}$$



17. $\int_0^{\pi}\left[\frac{1}{2}r^2\right]_0^{\sin\theta} d\theta = \int_0^{\pi}\frac{1}{2}\sin^2\theta\, d\theta$

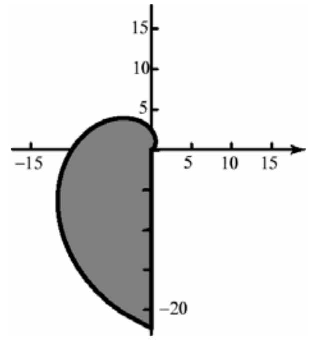
$$=\int_0^{\pi}\frac{1}{2}\left[\frac{1}{2}-\frac{1}{2}\sin 2\theta\right]d\theta$$

$$=\left[\frac{1}{4}\theta+\frac{1}{8}\cos 2\theta\right]_0^{\pi} = \frac{\pi}{4}$$

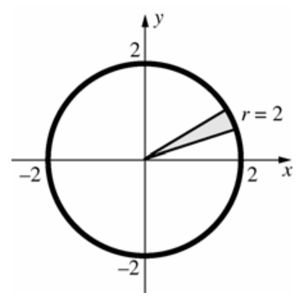


18. $\int_0^{3\pi/2}\left[\frac{1}{2}r^2\right]_0^{\theta^2} d\theta = \int_0^{3\pi/2}\frac{1}{2}\theta^4\, d\theta$

$$=\left[\frac{1}{10}\theta^5\right]_0^{3\pi/2} = \frac{243\pi^5}{320}$$

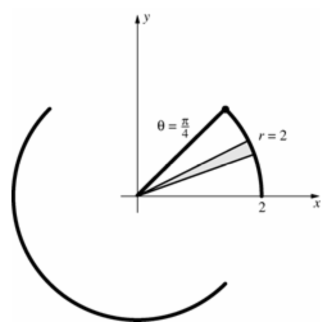


19.



$$2\int_0^{\pi}\int_0^{2} e^{r^2} r\,dr\, d\theta = \pi(e^4-1) \approx 168.3836$$

20.



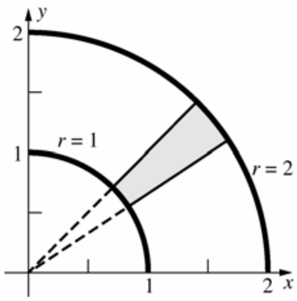
$$\int_0^{\pi/4}\int_0^{2}(4-r^2)^{1/2} r\,dr\,d\theta$$

$$=\int_0^{\pi/4}\left[\frac{(4-r^2)^{3/2}}{-3}\right]_0^2 d\theta$$

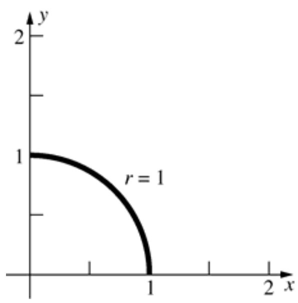
$$=\int_0^{\pi/4}\left(\frac{8}{3}\right)d\theta = \left[\frac{8\theta}{3}\right]_0^{\pi/4} = \frac{2\pi}{3} \approx 2.0944$$

21. $\int_0^{\pi/4}\int_0^{2}(4+r^2)^{-1} r\,dr\,d\theta = \left(\frac{\pi}{8}\right)\ln 2 \approx 0.2722$

22. $\int_0^{\pi/2}\int_1^2 r\sin\theta r\,dr\,d\theta = \frac{7}{3}$



23.



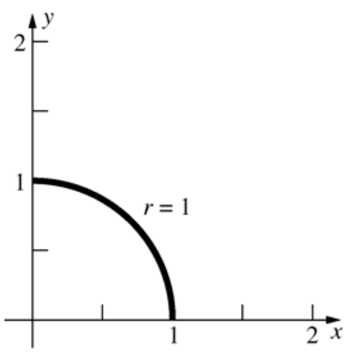
$\int_0^{\pi/2}\int_0^1 (4-r^2)^{-1/2} r\,dr\,d\theta$

$= \int_0^{\pi/2} [-(4-r^2)^{1/2}]_0^1 d\theta$

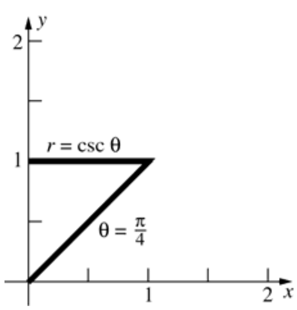
$= \int_0^{\pi/2}\left(-\sqrt{3}+2\right)d\theta = \left(-\sqrt{3}+2\right)\left(\frac{\pi}{2}\right) \approx 0.4209$

24. $\int_0^{\pi/2}\int_0^1 [\sin(r^2)] r\,dr\,d\theta = \left(\frac{\pi}{4}\right)(1-\cos 1)$

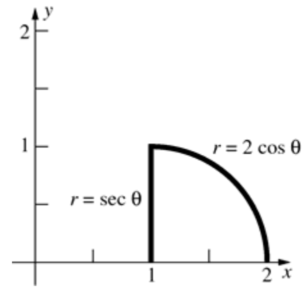
≈ 0.3610



25. $\int_{\pi/4}^{\pi/2}\int_0^{\csc\theta} r^2\cos^2\theta\, r\,dr\,d\theta = \frac{1}{12} \approx 0.0833$



26.



$\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r^{-1} r\,dr\,d\theta = \int_0^{\pi/4} [r]_{\sec\theta}^{2\cos\theta} d\theta$

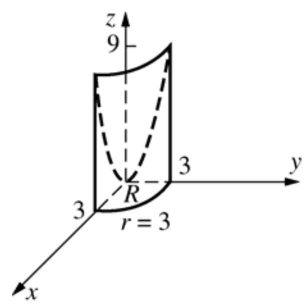
$= \int_0^{\pi/4} (2\cos\theta - \sec\theta)d\theta$

$= \left[2\sin\theta - \ln\left|\sec\theta + \tan\theta\right|\right]_0^{\pi/4}$

$= \left[\sqrt{2} - \ln\left(\sqrt{2}+1\right)\right] - [0 - \ln(1+0)]$

$= \sqrt{2} - \ln\left(\sqrt{2}+1\right) \approx 0.5328$

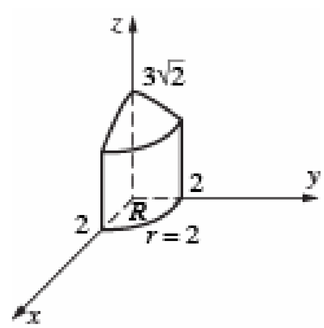
27.



$\iint_R (x^2+y^2)dA = \int_0^{\pi/2}\int_0^3 r^2 r\,dr\,d\theta$

$= \frac{81\pi}{8} \approx 31.8086$

28.

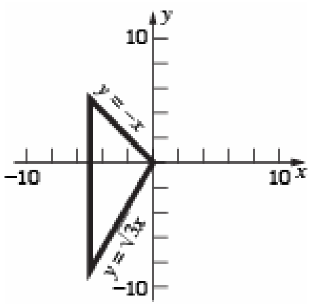


$4\iint_R (18-2x^2-2y^2)^{1/2} dA$

$= 4\int_0^{\pi/2}\int_0^2 (18-2r^2)^{1/2} r\,dr\,d\theta$

$= \left(\frac{\pi}{3}\right)(18^{3/2} - 10^{3/2}) \approx 46.8566$

29.

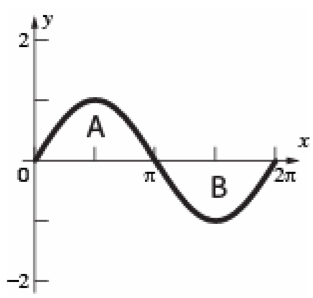


$$\int_{-5}^{0}\int_{\sqrt{3}x}^{-x}(y^2)\,dy\,dx = \int_{-5}^{0}\left[\frac{y^3}{3}\right]_{\sqrt{3}x}^{-x} dx$$

$$= \int_{-5}^{0}\frac{-1-3\sqrt{3}}{3}x^3 dx = \left[\frac{\left(-1-3\sqrt{3}\right)x^4}{12}\right]_{-5}^{0}$$

$$= \frac{\left(1+3\sqrt{3}\right)625}{12} \approx 322.7163$$

30. a. The solid bounded by the xy-plane and $z = \sin\sqrt{x^2+y^2}$ for $x^2+y^2 \le 4\pi^2$ is the solid of revolution obtained by revolving about the z-axis the region in the xz-plane that is bounded by the x-axis and the graph of $z = \sin x$ for $0 \le x \le 2\pi$.



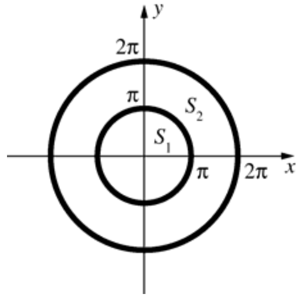
Regions A and B are congruent but region B is farther from the origin, so it generates a larger solid than region A generates. Therefore, the integral is negative.

b. $V = \int_0^{2\pi}\int_0^{2\pi}(\sin r)r\,dr\,d\theta = 2\pi\int_0^{2\pi}(\sin r)r\,dr$

Now use integration by parts.

$= 2\pi(-2\pi) = -4\pi^2 \approx -39.4784$

c.



$$W = \iint_{S_1}\sin\sqrt{x^2+y^2}\,dA + \iint_{S_2} -\sin\sqrt{x^2+y^2}\,dA$$

$$= \int_0^{2\pi}\left[\int_0^{\pi}(\sin r)r\,dr - \int_{\pi}^{2\pi}(\sin r)r\,dr\right]d\theta$$

$$= 2\pi[(\pi)-(-3\pi)] = 8\pi^2 \approx 78.9568$$

31. This can be done by the methods of this section, but an easier way to do it is to realize that the intersection is the union of two congruent segments (of one base) of the spheres, so (see Problem 20, Section 5.2, with $d = h$ and $a = r$) the volume is $2\left[\left(\frac{1}{3}\right)\pi d^2(3a-d)\right] = 2\pi d^2\,\frac{(3a-d)}{3}.$

32. $100 = \int_0^{2\pi}\int_0^{10} ke^{-r/10}r\,dr\,d\theta = 2\pi\int_0^{10} ke^{-r/10}r\,dr$

Let $u = r$ and $dv = e^{-r/10}dr$.

Then $du = dr$ and $v = -10e^{-r/10}$.

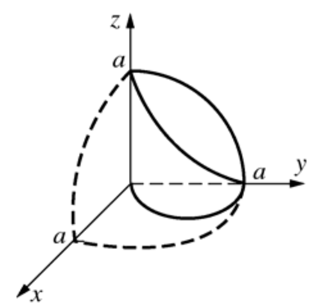
$$= 2\pi k\left(\left[-10re^{-r/10}\right]_0^{10} + \int_0^{10}10e^{-r/10}dr\right)$$

$$= 2\pi k\left(-100e^{-1} - \left[100e^{-r/10}\right]_0^{10}\right)$$

$$= 2\pi k(-100e^{-1} - 100e^{-1} + 100)$$

$$= 200\pi k(1-2e^{-1}),\ \text{so}\ k = \frac{e}{2\pi(e-2)} \approx 0.6023.$$

33.

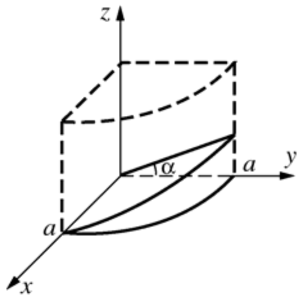


$$\text{Volume} = 4\int_0^{\pi/2}\int_0^{a\sin\theta}\sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\left(-\frac{1}{3}\right)(a^3\cos^3\theta - a^3)\right]d\theta$$

$$= \left(-\frac{4}{3}\right)a^3\left[\frac{2}{3}-\frac{\pi}{2}\right] = \left(\frac{2}{9}\right)a^3(3\pi-4)$$

34. Normal vector to plane is $\langle 0, -\sin\ a, \cos\ a \rangle$. Therefore, an equation of the plane is $(-\sin\ \alpha)y + (\cos\ \alpha)z = 0$, or $z = (\tan\ \alpha)y$, or $z = (\tan\ \alpha)(r \sin\ \theta)$.



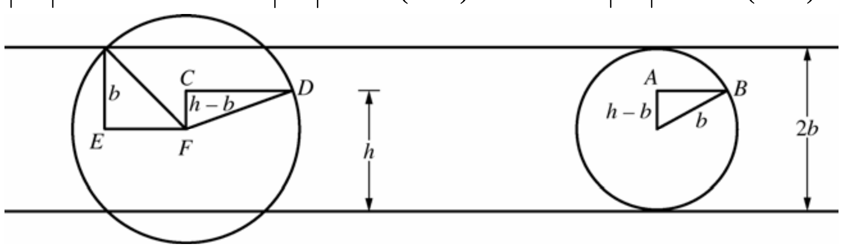
$$\text{Volume} = 2\int_0^{\pi/2}\int_0^a (\tan\alpha) r \sin\theta\, r\, dr\, d\theta = 2(\tan\alpha)\int_0^{\pi/2} \sin\theta\, d\theta \int_0^a r^2 dr = 2(\tan\alpha)[1]\left[\frac{a^3}{3}\right] = \left(\frac{2}{3}\right)a^3 \tan\alpha$$

35. Choose a coordinate system so the center of the sphere is the origin and the axis of the part removed is the z-axis.
Volume (Ring) = Volume (Sphere of radius a) – Volume (Part removed)

$$= \frac{4}{3}\pi a^3 - 2\int_0^{2\pi}\int_0^{\sqrt{a^2-b^2}} \sqrt{a^2 - r^2}\, r\, dr\, d\theta = \frac{4}{3}\pi a^3 - 2(2\pi)\int_0^{\sqrt{a^2-b^2}} (a^2 - r^2)^{1/2}\, r\, dr$$

$$= \frac{4}{3}\pi a^3 + 4\pi\left[\frac{1}{3}(a^2 - r^2)^{3/2}\right]_0^{\sqrt{a^2-b^2}} = \frac{4}{3}\pi a^3 + 4\pi\frac{1}{3}(b^3 - a^3) = \frac{4}{3}\pi b^3$$

36. $|EF|^2 = a^2 - b^2$ $\qquad$ $|CD| = a^2 - (h-b)^2$ $\qquad$ $|AB|^2 = b^2 - (h-b)^2$



Area of left cross-sectional region $= \pi[a^2 - (h-b)^2] - \pi[a^2 - b^2]$

$= \pi[b^2 - (h-b)^2] =$ area of right cross-sectional region

$$\text{Volume} = \left(\frac{4}{3}\right)\pi b^3 - \left(\frac{1}{3}\right)\pi(2b-h)^2[3b - (2b-h)] = \left(\frac{1}{3}\right)\pi h^2(3b - h)$$

Alternative: $V = \int_0^h \pi\left[b^2 - (t-b)^2\right]dt = \frac{1}{3}\pi h^2(3b - h)$

37. $$\int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b (1+r^2)^{-2} r\, dr\right]d\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\left[\left(-\frac{1}{2}\right)(1+b^2)^{-1} - \left(-\frac{1}{2}\right)\right]\right)d\theta = \int_0^{\pi/2}\left(\frac{1}{2}\right)d\theta = \frac{\pi}{4} \approx 0.7854$$

38. $$A = \frac{1}{2}r_2^2(\theta_2 - \theta_1) - \frac{1}{2}r_1^2(\theta_2 - \theta_1)$$

$$= \frac{1}{2}(\theta_1 - \theta_2)(r_2^2 - r_1^2)$$

$$= \frac{1}{2}(\theta_2 - \theta_1)(r_2 - r_1)(r_2 + r_1)$$

$$= \frac{r_1 + r_2}{2}(r_2 - r_1)(\theta_2 - \theta_1)$$

39. Using the substitution $u=\frac{x-\mu}{\sigma\sqrt{2}}$ we get $du=\frac{dx}{\sigma\sqrt{2}}$. Our integral then becomes

$$\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}dx=\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-u^2}du$$
$$=\frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-u^2}du$$

Using the result from Example 4, we see that $\int_0^{\infty}e^{-u^2}du=\frac{\sqrt{\pi}}{2}$. Thus we have

$$\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}dx$$
$$=\frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=1.$$

13.5 Concepts Review

1. $\iint_S x^2y^4\,dA$

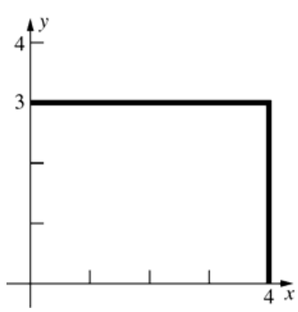
2. $\iint_S \frac{x^2y^5\,dA}{m}$

3. $\iint_S x^4y^4\,dA$

4. greater

Problem Set 13.5

1.



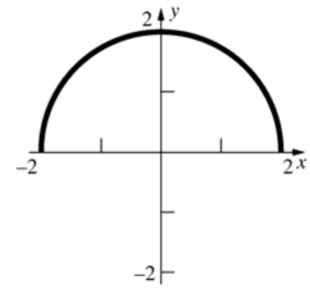
$m=\int_0^3\int_0^4(y+1)dx\,dy=30$

$M_y=\int_0^3\int_0^4 x(y+1)dx\,dy=60$

$M_x=\int_0^3\int_0^4 y(y+1)dx\,dy=54$

$(\bar{x},\ \bar{y})=(2,\ 1.8)$

2.



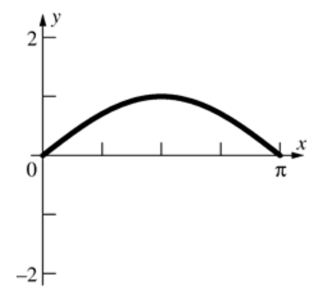
$m=\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}y\,dy\,dx=\frac{16}{3}$

$M_y=0$ (symmetry)

$M_x=\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}yy\,dx\,dy=2\pi$

$(\bar{x},\ \bar{y})=\left(0,\ \frac{3\pi}{8}\right)$

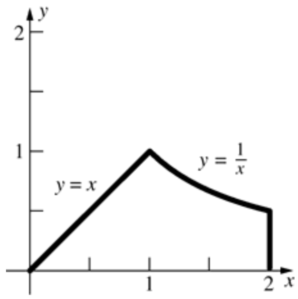
3.



$$m=\int_0^{\pi}\int_0^{\sin x}y\,dy\,dx=\int_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\sin x}dx$$
$$=\int_0^{\pi}\frac{\sin^2 x}{2}dx=\int_0^{\pi}\frac{1-\cos 2x}{4}dx$$
$$=\left[\frac{x}{4}-\frac{\sin 2x}{8}\right]_0^{\pi}=\frac{\pi}{4}$$
$$M_x=\int_0^{\pi}\int_0^{\sin x}yy\,dy\,dx=\int_0^{\pi}\left[\frac{y^3}{3}\right]_0^{\sin x}dx$$
$$=\int_0^{\pi}\frac{\sin^3 x}{3}dx=\frac{1}{3}\int_0^{\pi}(1-\cos^2 x)\sin x\,dx$$
$$=\frac{1}{3}\left[-\cos x+\frac{\cos^3 x}{3}\right]_0^{\pi}=\frac{4}{9}$$
$$\bar{y}=\frac{M_x}{m}=\frac{\frac{4}{9}}{\frac{\pi}{4}}=\frac{16}{9\pi}\approx 0.5659;$$
$$\bar{x}=\frac{\pi}{2}\text{ (by symmetry)}$$

Thus, $M_y=\bar{x}\cdot m=\frac{\pi}{4}\cdot\frac{\pi}{2}=\frac{\pi^2}{8}$

4.



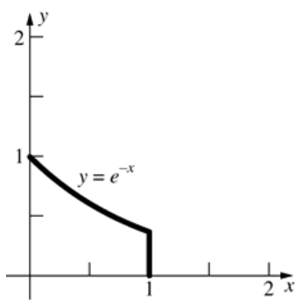
$$m = \int_0^1 \int_0^x x\,dy\,dx + \int_1^2 \int_0^{1/x} x\,dy\,dx = \frac{4}{3}$$

$$M_y = \int_0^1 \int_0^x x^2 dy\,dx + \int_1^2 \int_0^{1/x} x^2 dy\,dx = \frac{7}{4}$$

$$M_x = \int_0^1 \int_0^x xy\,dy\,dx + \int_1^2 \int_0^{1/x} xy\,dy\,dx = \left(\frac{1}{8}\right)(1 + 4\ln 2)$$

$$(\bar{x}, \bar{y}) = \left(\frac{21}{16}, \left(\frac{3}{32}\right)(1 + 4\ln 2)\right) \approx (1.3125, 0.3537)$$

5.



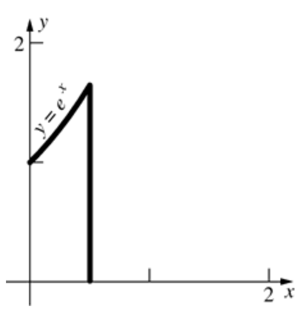
$$m = \int_0^1 \int_0^{e^{-x}} y^2 dy\,dx = \left(\frac{1}{9}\right)(1 - e^{-3})$$

$$M_x = \int_0^1 \int_0^{e^{-x}} y^3 dy\,dx = \left(\frac{1}{16}\right)(1 - e^{-4}) \approx 0.0614$$

$$M_y = \int_0^1 \int_0^{e^{-x}} xy^2 dy\,dx = \left(\frac{1}{27}\right)(1 - 4e^{-3}) \approx 0.0297$$

$$(\bar{x}, \bar{y}) = \left(\left(\frac{1}{3}\right)(e^3 - 4)(e^3 - 1)^{-1}, \left(\frac{9}{16}\right)e^{-1}(e^4 - 1)(e^3 - 1)^{-1}\right) \approx (0.2809, 0.5811)$$

6.



$$m = \int_0^1\int_0^{e^x} (2-x+y)dy\,dx = \int_0^1\left[(2-x)y+\frac{y^2}{2}\right]_{y=0}^{e^x} dx = \int_0^1\left(2e^x - xe^x + \frac{e^{2x}}{2}\right)dx$$

$$= \left[2e^x - (xe^x - e^x) + \frac{e^{2x}}{4}\right]_0^1 = \frac{e^2+8e-13}{4}$$

$$M_x = \int_0^1\int_0^{e^x} (2-x+y)y\,dy\,dx = \int_0^1\left[y^2 - \frac{xy^2}{2}+\frac{y^3}{3}\right]_{y=0}^{e^x} dx = \int_0^1\left[e^{2x} - \frac{xe^{2x}}{2}+\frac{e^{3x}}{3}\right]dx$$

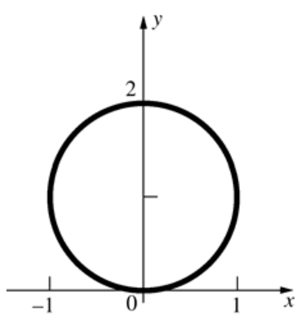
$$= \left[\frac{e^{2x}}{2} - \left(\frac{xe^{2x}}{4}-\frac{e^{2x}}{8}\right)+\frac{e^{3x}}{9}\right]_0^1 = \left(\frac{e^2}{2}-\frac{e^2}{4}+\frac{e^2}{8}+\frac{e^3}{9}\right)-\left(\frac{1}{2}-0+\frac{1}{8}+\frac{1}{9}\right) = \frac{8e^3+27e^2-53}{72}$$

$$M_y = \int_0^1\int_0^{e^x} (2-x+y)x\,dy\,dx = \int_0^1\left[2xy - x^2y+\frac{xy^2}{2}\right]_{y=0}^{e^x} dx = \int_0^1\left(2xe^x - x^2e^x+\frac{xe^{2x}}{2}\right)dx$$

$$= \left[(2xe^x - 2e^x) - (x^2e^x - 2xe^x + 2e^x) + \left(\frac{xe^{2x}}{4}-\frac{e^{2x}}{8}\right)\right]_0^1 = \frac{e^2-8e+33}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{e^2-8e+33}{2(e^2+8e-13)} \approx 0.5777;\ \bar{y} = \frac{M_x}{m} = \frac{8e^3+27e^2-53}{18(e^2+8e-13)} \approx 1.0577$$

7.



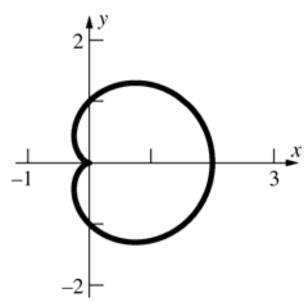
$$m = \int_0^{\pi}\int_0^{2\sin\theta} r\,r\,dr\,d\theta = \frac{32}{9}$$

$$M_x = \int_0^{\pi}\int_0^{2\sin\theta} (r\sin\theta)r\,r\,dr\,d\theta = \frac{64}{15}$$

$M_y = 0$ (symmetry)

$(\bar{x}, \bar{y}) = (0, 1.2)$

8.



$$m = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\,r\,dr\,d\theta = \frac{5\pi}{3}$$

$$M_y = 2\int_0^{\pi}\int_0^{1+\cos\theta} (r\cos\theta)r\,r\,dr\,d\theta = \frac{7\pi}{4}$$

$M_x = 0$ (symmetry)

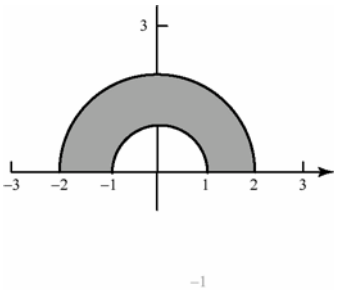
$(\bar{x}, \bar{y}) = (1.05, 0)$

9. $m = \int_0^{\pi}\int_1^2 \frac{1}{r} r\,dr\,d\theta = \int_0^{\pi}\int_1^2 dr\,d\theta = \pi$

$M_x = \int_0^{\pi}\int_1^2 \frac{1}{r} r\cos\theta\, r\,dr\,d\theta = \int_0^{\pi} \frac{3}{2}\cos\theta\,d\theta = 3$

$M_y = 0$ by symmetry

$(\bar{x}, \bar{y}) = \left(0, \frac{3}{\pi}\right)$



10. $m = \int_0^{2\pi}\int_0^{2+2\cos\theta} r\,r\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^{2+2\cos\theta} d\theta$

$= \int_0^{2\pi}\frac{1}{3}\left[8+24\cos\theta+24\cos^2\theta+8\cos^3\theta\right]d\theta$

$= \frac{1}{3}\left[0+24\pi+16\pi\right] = \frac{40\pi}{3}$

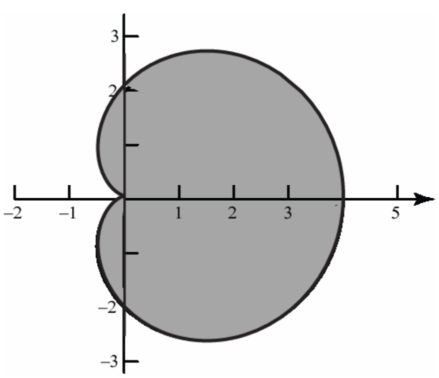
$M_x = 0$ by symmetry

$M_y = \int_0^{2\pi}\int_0^{2+2\cos\theta} r\left(r\cos\theta\right) r\,dr\,d\theta$

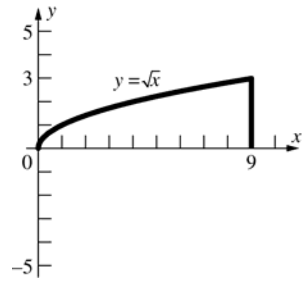
$= \int_0^{2\pi}\left[\frac{1}{4}r^4\cos\theta\right]_0^{2+2\cos\theta} d\theta$

$M_y = 28\pi$

$(\bar{x}, \bar{y}) = \left(\frac{28\pi}{40\pi/3}, 0\right) = \left(\frac{21}{10}, 0\right)$



11.



$I_x = \int_0^3\int_{y^2}^9 y^2(x+y)\,dx\,dy$

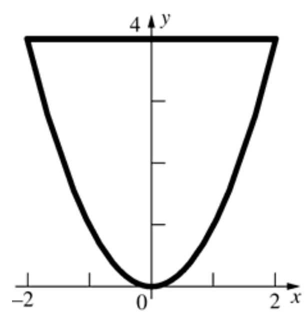
$= \int_0^3\left(\frac{81y^2}{2}+9y^3-\frac{y^6}{2}-y^5\right)dy = \frac{7533}{28} \approx 269$

$I_y = \int_0^9\int_0^{\sqrt{x}} x^2(x+y)\,dy\,dx = \int_0^9\left(x^{7/2}+\frac{x^3}{2}\right)dx$

$= \frac{41553}{8} \approx 5194$

$I_z = I_x + I_y = \frac{305937}{56} \approx 5463$

12.

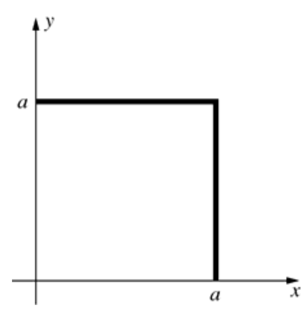


$I_x = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} y^3\,dx\,dy = \frac{2048}{9} \approx 227.56$

$I_y = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} x^2 y\,dx\,dy = \frac{512}{21} \approx 24.38$

$I_z = I_x + I_y = \frac{15872}{63} \approx 251.94$

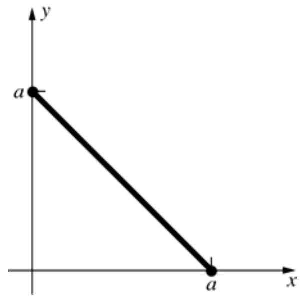
13.



$I_x = \int_0^a\int_0^a (x+y)y^2\,dx\,dy = \left(\frac{5}{12}\right)a^5$

$I_y = \left(\frac{5}{12}\right)a^5;\quad I_z = \left(\frac{5}{6}\right)a^5$

14.



$$I_x = \int_0^a \int_0^{a-y} (x^2 + y^2) y^2 \, dx \, dy$$

$$= \frac{1}{3} \int_0^a (a^3 y^2 - 3a^2 y + 6ay^2 - 4y^5) dy \quad = \frac{7a^6}{180}$$

$$I_y = \frac{7a^6}{180}; I_z = \frac{7a^6}{90} \quad \text{(Same result for } a < 0\text{)}$$

15. The density is constant , $\delta(x, y) = k$.

$$m = \int_0^2 kx \, dx = \left[\frac{k}{2} x^2 \right]_0^2 = 2k$$

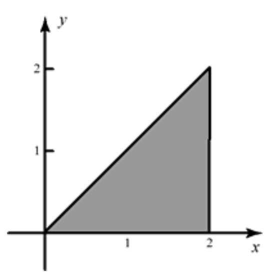
$$M_x = \int_0^2 \int_0^x ky \, dy \, dx = \int_0^2 \left[\frac{k}{2} y^2 \right]_0^x dx$$

$$= \int_0^2 \frac{k}{2} x^2 \, dx = \left[\frac{k}{6} x^3 \right]_0^2 = \frac{4k}{3}$$

$$M_y = \int_0^2 \int_0^x kx \, dy \, dx = \int_0^2 [kxy]_0^x \, dx$$

$$= \int_0^2 kx^2 \, dx = \left[\frac{k}{3} x^3 \right]_0^2 = \frac{8k}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8k/3}{2k}, \frac{4k/3}{2k} \right) = \left(\frac{4}{3}, \frac{2}{3} \right)$$



16. The density is proportional to the distance from the x-axis, $\delta(x, y) = ky$.

$$m = \int_0^1 \left[\frac{k}{2} y^2 \right]_x^1 dx = \int_0^1 \frac{k}{2} \left(1 - x^2 \right) = \frac{k}{3}$$

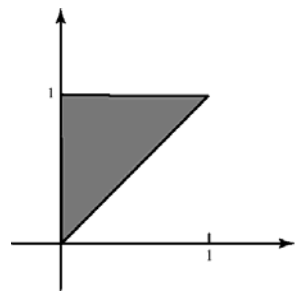
$$M_x = \int_0^1 \int_0^x ky^2 \, dy \, dx = \int_0^1 \left[\frac{k}{3} y^3 \right]_x^1 dx$$

$$= \frac{k}{3} \int_0^1 \left(1 - x^3 \right) dx = \frac{k}{4}$$

$$M_y = \int_0^1 \int_x^1 kxy \, dy \, dx = \int_0^1 \left[\frac{kxy2}{2} \right]_x^1 dx$$

$$= \frac{k}{2} \int_0^1 \left(x - x^3 \right) dx = \frac{k}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{k/8}{k/3}, \frac{k/4}{k/3} \right) = \left(\frac{3}{8}, \frac{3}{4} \right)$$



17. The density is proportional to the squared distance from the origin, $\delta(x, y) = k\left(x^2 + y^2\right)$.

$$m = \int_{-3}^3 \int_0^{9-x^2} k\left(x^2 + y^2\right) dy \, dx$$

$$= \int_{-3}^3 k \left[x^2 y + \frac{1}{3} y^3 \right]_0^{9-x^2} dx$$

$$= \int_{-3}^3 k \left[246 - 72x^2 + 8x^4 - \frac{1}{3} x^6 \right] dx$$

$$= k \left[246x - 24x^3 + \frac{8}{5} x^5 - \frac{1}{21} x^7 \right]_{-3}^3 = \frac{25596k}{35}$$

$$M_x = \int_{-3}^3 \int_0^{9-x^2} ky\left(x^2 + y^2\right) dy \, dx$$

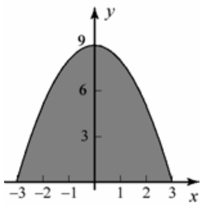
$$= \int_{-3}^3 k \left[\frac{1}{2} x^2 y^2 + \frac{1}{4} y^4 \right]_0^{9-x^2} dx$$

$$= \int_{-3}^3 k \left[\frac{6561}{4} - \frac{1377x^2}{2} + \frac{225}{2} x^4 - \frac{17x^6}{2} + \frac{x^8}{4} \right] dx$$

$$= \frac{29160k}{7}$$

$M_y = 0$ by symmetry

$$(\bar{x}, \bar{y}) = \left(0, \frac{29160k/7}{25596k/35}\right) = \left(0, \frac{450}{79}\right)$$



18. The density is constant, $\delta(x, y) = k$.

$$m = \int_{-\pi/2}^{\pi/2} k\cos x\, dx = [k\sin x]_{-\pi/2}^{\pi/2} = 2k$$

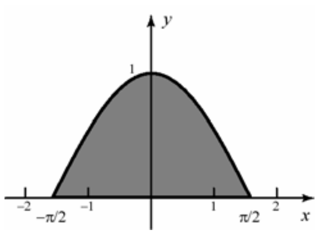
$$M_x = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} ky\, dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{k}{2}y^2\right]_0^{\cos x} dx$$

$$= \frac{k}{2}\int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = \frac{k\pi}{4}$$

$M_y = 0$ by symmetry

$$(\bar{x}, \bar{y}) = \left(0, \frac{k\pi/4}{2k}\right) = \left(0, \frac{\pi}{8}\right)$$



19. The density is proportional to the distance from the origin, $\delta(r, \theta) = k \cdot r$.

$$m = \int_0^{\pi} \int_1^3 kr^2 dr\, d\theta = \int_0^{\pi} \left[\frac{k}{3}r^3\right]_1^3 d\theta$$

$$= \int_0^{\pi} \frac{26}{3} k\, d\theta = \frac{26k\pi}{3}$$

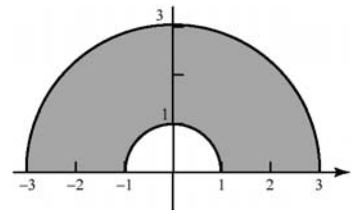
$$M_x = \int_0^{\pi} \int_1^3 kr^2\, r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi} \left[\frac{k}{4}r^4 \sin\theta\right]_1^3 d\theta = \int_0^{\pi} 20k\sin\theta\, d\theta$$

$$= [-20k\cos\theta]_0^{\pi} = 40k$$

$M_y = 0$ by symmetry

$$(\bar{x}, \bar{y}) = \left(0, \frac{40k}{26k\pi/3}\right) = \left(0, \frac{60}{13\pi}\right)$$



20. The density is constant, $\delta(r, \theta) = k$.

$$m = \int_0^{\pi/2} \int_0^{\theta} k\, r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{k}{2}r^2\right]_0^{\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{k}{2}\theta^2\, d\theta = \left[\frac{k}{6}\theta^3\right]_0^{\pi/2} = \frac{k\pi^3}{48}$$

$$M_x = \int_0^{\pi/2} \int_0^{\theta} kr^2 \sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{k}{3}r^3 \sin\theta\right]_0^{\theta} d\theta = \int_0^{\pi/2} \frac{k}{3}\theta^3 \sin\theta\, d\theta$$

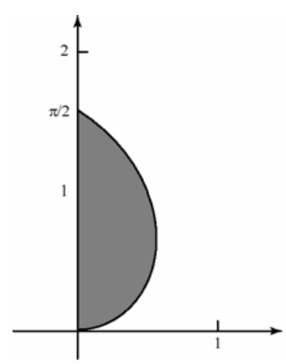
$$= \frac{k\left(\pi^2 - 8\right)}{4}$$

$$M_y = \int_0^{\pi/2} \int_0^{\theta} kr^2 \cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{k}{3}r^3 \cos\theta\right]_0^{\theta} d\theta = \int_0^{\pi/2} \frac{k}{3}\theta^3 \cos\theta\, d\theta$$

$$= \frac{k\left(\pi^3 - 24\pi + 48\right)}{24}$$

$$(\bar{x}, \bar{y}) = \left(\frac{2\left(\pi^3 - 24\pi + 48\right)}{\pi^3}, \frac{12\left(\pi^2 - 8\right)}{\pi^3}\right)$$



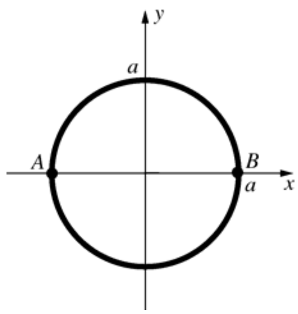
21. $m = \int_0^a \int_0^a (x+y)dx\,dy = a^3$

$\bar{r} = \left(\frac{I_x}{m}\right)^{1/2} = \left(\frac{5}{12}\right)^{1/2} a \approx 0.6455a$

22. $m = \int_0^a \int_0^{a-y} (x^2+y^2)\,dxdy = \left(\frac{1}{6}\right)a^4$

$\bar{r} = \left(\frac{I_y}{m}\right)^{1/2} = \left(\frac{7}{30}\right)^{1/2} a \approx 0.4830a$

23.



$m = \delta\pi a^2$

The moment of inertia about diameter AB is

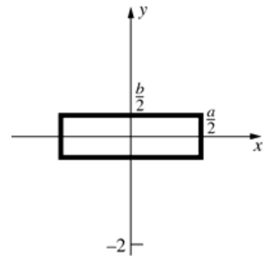
$I = I_x = \int_0^{2\pi}\int_0^a \delta r^2 \sin^2\theta\, r\,dr\,d\theta$

$= \int_0^{2\pi} \frac{\delta a^4 \sin^2\theta}{4} d\theta = \frac{\delta a^4}{8}\int_0^{2\pi}(1-\cos 2\theta)d\theta$

$= \frac{\delta a^4}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\delta a^4 \pi}{4}$

$\bar{r} = \left(\frac{I}{m}\right)^{1/2} = \left(\frac{\frac{\delta a^4\pi}{4}}{\delta\pi a^2}\right)^{1/2} = \frac{a}{2}$

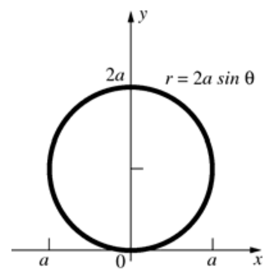
24.



$I = I_z = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(x^2+y^2)k\,dx\,dy$

$= \left(\frac{k}{12}\right)(a^3b + ab^3)$

25.



$I_x = \iint_S \delta y^2 dA$

$= 2\delta\int_0^{\pi/2}\int_0^{2a\sin\theta}(r\sin\theta)^2 r\,dr\,d\theta$

$= 2\delta\int_0^{\pi/2} 4a^4 \sin^6\theta\,d\theta$

$= 8a^4\delta\frac{(1)(3)(5)}{(2)(4)(6)}\frac{\pi}{2} = \frac{5a^4\delta\pi}{4}$

26. $\bar{x} = 0$ (by symmetry)

$M_x = \iint_S 1y\,dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)}(r\sin\theta)r\,dr\,d\theta = 2k\int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\sin\theta\right]_{r=0}^{a(1+\sin\theta)} d\theta$

$= \frac{2ka^3}{3}\int_{-\pi/2}^{\pi/2}(1+\sin\theta)^3\sin\theta\,d\theta$

$= \frac{2ka^3}{3}\int_{-\pi/2}^{\pi/2}(\sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta)d\theta$

$= \frac{4ka^3}{3}\int_0^{\pi/2}(3\sin^2\theta + \sin^4\theta)d\theta$ (using the symmetry property for odd and even functions.)

$= \frac{4ka^3}{3}\left(3\frac{1}{2}\frac{\pi}{2} + \frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2}\right) = \frac{5\pi ka^3}{4}$ (using Formula 113)

Therefore, $\bar{y} = \frac{M_x}{m} = \frac{5a}{6}$.

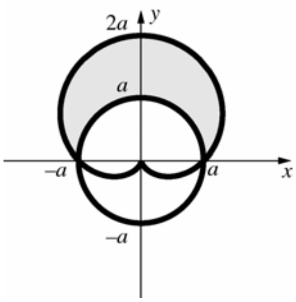
$$I_x = \iint_S ky^2 dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)}(r\sin\theta)^2 r\,dr\,d\theta = 2k\int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\sin^2\theta\right]_{r=0}^{1(1+\sin\theta)} d\theta$$

$$= \frac{ka^4}{2}\int_{-\pi/2}^{\pi/2}(1+\sin\theta)^4\sin^2\theta\,d\theta = \frac{ka^4}{2}\int_{-\pi/2}^{\pi/2}(\sin^2\theta+4\sin^3\theta+6\sin^4\theta+4\sin^5\theta+\sin^6\theta)d\theta$$

$$= ka\int_0^{\pi/2}(\sin^2\theta+6\sin^4\theta+\sin^6\theta)d\theta \text{ (symmetry property for odd and even functions)}$$

$$= ka^4\left[\frac{1}{2}\frac{\pi}{2}+6\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{\pi}{2}\right] = \frac{49\pi ka^4}{32} \text{ (using Formula 113)}$$

27.



$\bar{x} = 0$ (by symmetry)

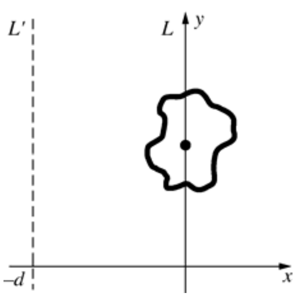
$$M_x = \iint_S ky\,dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)}(r\sin\theta)r\,dr\,d\theta = 2k\int_0^{\pi/2}\left(\frac{a^3}{3}\right)(3\sin^2\theta+3\sin^3\theta+\sin^4\theta)d\theta$$

$$= \left(\frac{2}{3}\right)ka^3\left[\frac{(15\pi+32)}{16}\right] = \left(\frac{1}{24}\right)ka^3(15\pi+32)$$

$$m = \iint_S k\,dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)} r\,dr\,d\theta = 2k\int_0^{\pi/2}\left(\frac{1}{2}\right)a^2(2\sin\theta+\sin^2\theta)d\theta = ka^2\left[\frac{(8+\pi)}{4}\right] = \left(\frac{1}{4}\right)ka^2(\pi+8)$$

Therefore, $\bar{y} = \dfrac{M_x}{m} = \dfrac{\left(\frac{1}{24}\right)ka^3(15\pi+32)}{\left(\frac{1}{4}\right)ka^2(\pi+8)} = \dfrac{a(15\pi+32)}{6(\pi+8)} \approx 1.1836a$

28.

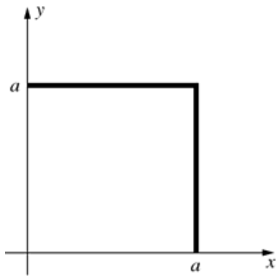


$$I' = \iint_S (x+d)^2\delta(x,\,y)dA = \iint_S (x^2+2xd+d^2)\delta(x,\,y)dA$$

$$= \iint_S x^2\delta(x,\,y)dA + \iint_S 2xd\,\delta(x,\,y)dA + \iint_S d^2\delta(x,\,y)dA$$

$$= I + M_y + d^2 m = I + 0 + d^2 m = I + d^2 m$$

29. a.

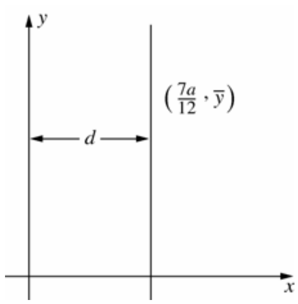


$$m = \iint_S (x+y)dA = \int_0^a \int_0^a (x+y)dx\,dy$$
$$= \int_0^a \left(\left[\frac{x^2}{2} + xy \right]_{x=0}^{a} \right) dy = \int_0^a \left(\frac{a^2}{2} + ay \right) dy$$
$$= \left[\frac{a^2 y}{2} + \frac{ay^2}{2} \right]_0^a = a^3$$

b. $$M_y = \iint_S x(x+y)dA = \int_0^a \int_0^a (x^2 + xy)dy\,dx$$
$$= \int_0^a \left(\left[x^2 y + \frac{xy^2}{2} \right]_{y=0}^{a} \right) dx = \int_0^a \left(ax^2 + \frac{a^2 x}{2} \right) dx$$
$$= \left[\frac{ax^3}{3} + \frac{a^2 x^2}{4} \right]_0^a = \frac{7a^4}{12}$$

Therefore, $\bar{x} = \frac{M_y}{m} = \frac{7a}{12}$.

c.



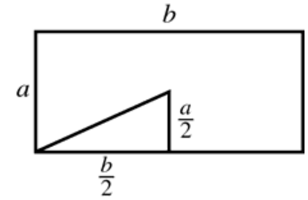
$I_y = I_L + d^2 m$, so $\frac{5a^5}{12} = I_L + \left(\frac{7a}{12}\right)^2 (a^3)$;

$I_L = \frac{11a^5}{144}$

30. $I_{25} = I_{23} + md^2 = 0.25\delta a^4 \pi + (\delta \pi a^2)a^2$
$= 1.25\delta a^4 \pi$

31. $I_x = 2[I_{23}] = \frac{ka^4\pi}{2}$; $I_y = 2[I_{23} + md^2]$
$= 2[0.25a^4\pi + (k\pi a^2)(2a)^2] = 8.5ka^4\pi$
$I_z = I_x + I_y = 9ka^4\pi$

32.



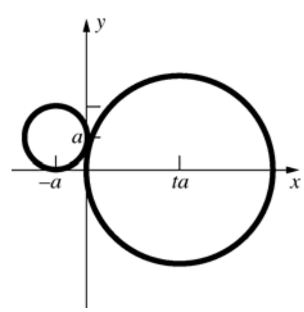
The square of the distance of the corner from the center of mass is $d^2 = \frac{a^2 + b^2}{4}$.

$I = I(\text{Prob. 16}) + md^2$
$= \frac{k(a^3 b + ab^3)}{12} + (kab)\frac{a^2 + b^2}{4} = \frac{k(a^3 b + ab^3)}{3}$

33. $M_y = \iint_{S_1 \cup S_2} x\delta(x, y)dA$
$= \iint_{S_1} x\delta(x, y)dA + \iint_{S_2} x\delta(x, y)dA$
$= \frac{m_1 \iint_{S_1} x\delta(x, y)dA}{m_1} + \frac{m_2 \iint_{S_2} x\delta(x, y)dA}{m_2}$
$= m_1 \bar{x}_1 + m_2 \bar{x}_2$

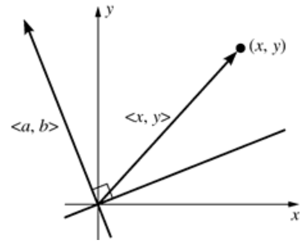
Thus, $\bar{x} = \frac{M_y}{m} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ which is equal to what we are to obtain and which is what we would obtain using the center of mass formula for two point masses. (Similar result can be obtained for $\bar{y}$.)

34.



$$\bar{x} = \frac{(-a)(\delta \pi a^2) + (ta)[\delta \pi (ta)^2]}{\delta \pi a^2 + \delta \pi (ta)^2} = \frac{a(t^3 - 1)}{t^2 + 1}$$
$$\bar{y} = \frac{(a)(\delta \pi a^2) + (0)}{\delta \pi a^2 + \delta \pi (ta)^2} = \frac{a}{t^2 + 1}$$

35.



$\langle a,b\rangle$ is perpendicular to the line $ax + by = 0$. Therefore, the (signed) distance of (x, y) to L is the scalar projection of $\langle x,y\rangle$ onto $\langle a,b\rangle$, which is $d(x, y) = \dfrac{\langle x, y\rangle \cdot \langle a, b\rangle}{|\langle a, b\rangle|} = \dfrac{ax+by}{|\langle a, b\rangle|}$.

$M_L = \iint_S d(x, y)\delta(x, y)dA$

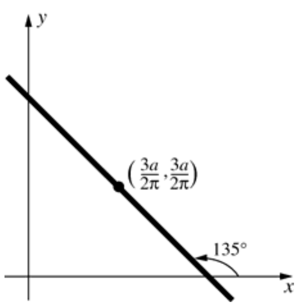
$= \iint_S \dfrac{ax+by}{|\langle a, b\rangle|}\delta(x, y)dA$

$= \dfrac{a}{|\langle a, b\rangle|}\iint_S x\delta(x, y)dA + \dfrac{b}{|\langle a, b\rangle|}\iint_S y\delta(x, y)dA$

$= \dfrac{a}{|\langle a, b\rangle|}(0) + \dfrac{b}{|\langle a, b\rangle|}(0) = 0$

[since $(\bar{x}, \bar{y}) = (0, 0)$]

36.



The equation has the form $x + y = b$.

$\dfrac{3a}{2\pi} + \dfrac{3a}{2\pi} = b$ so $b = \dfrac{3a}{\pi}$.

Therefore, the equation is $x + y = \dfrac{3a}{\pi}$, or

$\pi x + \pi y = 3a$.

13.6 Concepts Review

1. $\|\mathbf{u} \times \mathbf{v}\|$

2. $\displaystyle\iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA$

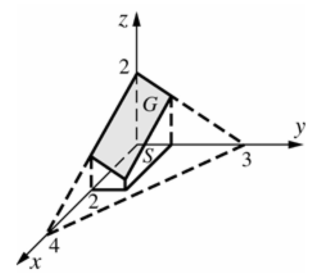
3. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\frac{a}{\sqrt{a^2 - x^2 - y^2}}\right) dy\, dx$

$\displaystyle = \int_0^{2\pi}\int_0^{a} \left(\frac{ar}{\sqrt{a^2 - r^2}}\right) dr\, d\theta;\ \ 2\pi a^2$

4. $2\pi ah$

Problem Set 13.6

1.



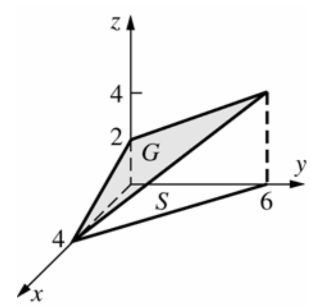
$z = 2 - \dfrac{1}{2}x - \dfrac{2}{3}y$

$f_x(x, y) = -\dfrac{1}{2};\ f_y(x, y) = -\dfrac{2}{3}$

$A(G) = \displaystyle\int_0^2\int_0^1 \sqrt{\frac{1}{4} + \frac{4}{9} + 1}\, dy\, dx$

$= \dfrac{\sqrt{61}}{3} \approx 2.6034$

2.



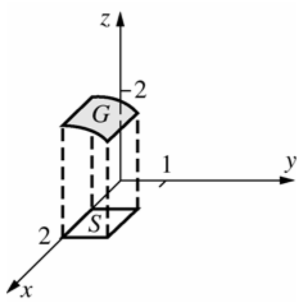
$z = 2 - \dfrac{1}{2}x + \dfrac{1}{3}y$

$f_x(x, y) = -\dfrac{1}{2};\ f_y(x, y) = \dfrac{1}{3}$

$A(G) = \displaystyle\int_0^4\int_0^{-\frac{3}{2}x+6} \sqrt{\frac{1}{4} + \frac{1}{9} + 1}\, dy dx$

$= \dfrac{7}{6}\displaystyle\int_0^4 \left(-\frac{3}{2}x + 6\right) dx = \left(\frac{7}{6}\right)(12) = 14$

3.



$z = f(x, y) = (4 - y^2)^{1/2}$; $f_x(x, y) = 0$;

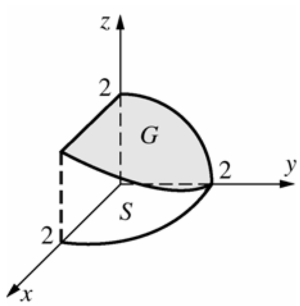
$f_y(x, y) = -y(4 - y^2)^{-1/2}$

$$A(G) = \int_0^1 \int_1^2 \sqrt{y^2(4 - y^2)^{-1} + 1}\, dx\, dy$$

$$= \int_0^1 \int_1^2 \frac{2}{\sqrt{4 - y^2}}\, dx\, dy = \int_0^1 \frac{2}{\sqrt{4 - y^2}}\, dy$$

$$= \left[2\sin^{-1}\left(\frac{y}{2}\right) \right]_0^1 = 2\left(\frac{\pi}{6}\right) - 2(0) = \frac{\pi}{3} \approx 1.0472$$

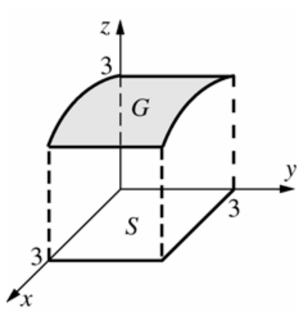
4.



$$A(G) = \int_0^2 \int_0^{\sqrt{4-y^2}} 2(4 - y^2)^{-1/2}\, dx\, dy = 4$$

(See problem 3 for the integrand.)

5.



Let $z = f(x, y) = (9 - x^2)^{1/2}$.

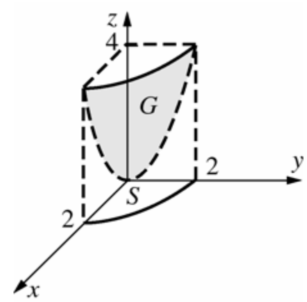
$f_x(x, y) = -x(9 - x^2)^{-1/2}$, $f_y(x, y) = 0$

$$A(G) = \int_0^2 \int_0^3 [x^2(9 - x^2)^{-1} + 1]\, dy\, dx$$

$$= \int_0^2 \int_0^3 3(9 - x^2)^{-1/2}\, dy\, dx = 9\sin^{-1}\left(\frac{2}{3}\right)$$

≈ 6.5675

6.



Let $z = f(x, y) = x^2 + y^2$; $f_x(x, y) = 2x$;

$f_y(x, y) = 2y$.

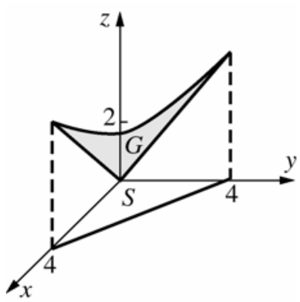
$$A(G) = 4\iint_S \sqrt{4x^2 + 4y^2 + 1}\, dy dx$$

$$= 4\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4x^2 + 4y^2 + 1}\, dy\, dx$$

$$= 4\int_0^{\pi/2} \int_0^2 (4r^2 + 1)^{1/2}\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} \left[\frac{(4r^2 + 1)^{3/2}}{12} \right]_0^2 dr = \frac{(17^{3/2} - 1)}{3} \frac{\pi}{2}$$

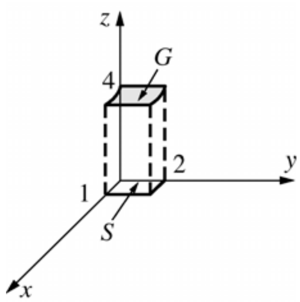
≈ 36.1769

7. 

$z = f(x, y) = (x^2 + y^2)^{1/2}$

$f_x(x, y) = x(x^2 + y^2)^{-1/2},\ f_y(x, y) = y(x^2 + y^2)^{-1/2}$

$A(G) = \int_0^4 \int_0^{4-x} [x^2(x^2 + y^2)^{-1} + y^2(x^2 + y^2)^{-1} + 1]^{1/2} dy\, dx \ = \int_0^4 \int_0^{4-x} \sqrt{2} dy\, dx = 8\sqrt{2}$

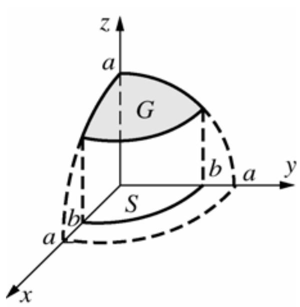
8. 

$z = f(x, y) = \left(\frac{1}{4}\right)x^2 + 4$

$f_x = (x, y) = \frac{x}{2};\ f_y(x, y) = 0$

$A(G) = \int_0^1 \int_0^2 \left[\left(\frac{1}{4}\right)x^2 + 1\right]^{1/2} dy\, dx$

$= \frac{\sqrt{5}}{2} + 2\ln\left[\frac{\left(\sqrt{5}+1\right)}{2}\right] \approx 2.0805$

9. 

$f_x(x, y) = \frac{x^2}{a^2 - x^2 - y^2};\ f_y(x, y) = \frac{y^2}{a^2 - x^2 - y^2}$

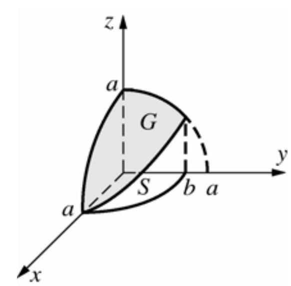
(See Example 3.)

$A(G) = 8\iint_S \frac{a}{\sqrt{a^2 - x^2 - y^2}} dA$

$= 8\int_0^{\pi/2} \int_0^b \frac{a}{\sqrt{a^2 - r^2}} r\, dr\, d\theta$

$= 8a\left(\frac{\pi}{2}\right)\int_0^b (a^2 - r^2)^{-1/2} r\, dr$

$= -4a\pi\left[(a^2 - r^2)^{1/2}\right]_0^b = 4\pi a\left(a - \sqrt{a^2 - b^2}\right)$

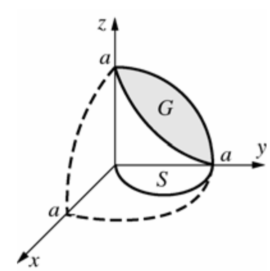
10. 

$f_x(x, y) = \frac{x^2}{a^2 - x^2 - y^2};\ f_y(x, y) = \frac{y^2}{a^2 - x^2 - y^2}$

(See Example 3.)

$A(G) = 8\int_0^a \int_0^{(b/a)\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} dy\, dx$

$= 8a\int_0^a \sin^{-1}\left(\frac{b}{a}\right) dx = 8a^2 \sin^{-1}\left(\frac{b}{a}\right)$

11. 

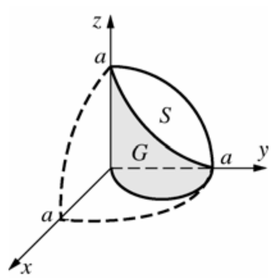
$$f_x(x,y)=\frac{x^2}{a^2-x^2-y^2};\ f_y(x,y)=\frac{y^2}{a^2-x^2-y^2}$$

(See Example 3.)

$$A(G)=4\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{a}{\sqrt{a^2-r^2}}r\,dr\,d\theta$$

$$=4a^2\int_0^{\pi/2}(1-\cos\theta)d\theta=2a^2(\pi-2)$$

12.



Following the hint, treat this as a surface $x=f(y,z)=\sqrt{ay-yz}$.

$$f_y=\frac{a-2y}{2\sqrt{y(a-y)}},\ \ f_z=0$$

$$\sqrt{f_y^2+f_z^2+1}=\frac{a}{2\sqrt{y(a-y)}}.$$

The region S in the yz-plane is a quarter circle.

$$A(G)=4\iint_S\frac{a}{2\sqrt{y(a-y)}}dzdy$$

$$=2a\int_0^a\int_0^{\sqrt{a^2-y^2}}\frac{1}{\sqrt{y(a-y)}}dzdy$$

$$=2a\int_0^a\frac{\sqrt{a^2-y^2}}{\sqrt{y(a-y)}}dy=2a\int_0^a\sqrt{\frac{a+y}{y}}dy$$

Make the substitution:

$$y=a\tan^2 u$$

$$a+y=a\left(1+\tan^2 u\right)=a\sec^2 u$$

$$\sqrt{\frac{a+y}{y}}=\csc u$$

$$dy=2a\tan u\sec^2 u\,du$$

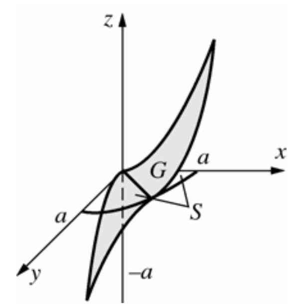
$$A(G)=2a\int_0^{\pi/4}\csc u\cdot 2a\tan u\sec^2 u\,du$$

$$=4a^2\int_0^{\pi/4}\sec^3 u\,du$$

$$=4a^2\left[\tfrac{1}{2}\sec u\tan u+\tfrac{1}{2}\ln\left|\sec u+\tan u\right|\right]_0^{\pi/4}$$

$$=2a^2\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$$

13.



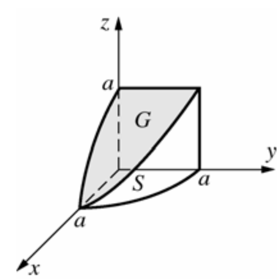
Let $f(x,y)=\dfrac{\left(x^2-y^2\right)}{a}$.

$$f_x(x,y)=\frac{2x}{a};\ f_y(x,y)=\frac{-2y}{a}$$

$$A(G)=\int_0^{2\pi}\int_0^a\frac{\sqrt{4r^2+a^2}}{a}r\,dr\,d\theta$$

$$=\frac{2\pi}{a}\int_0^a(4r^2+a^2)^{1/2}r\,dr=\frac{\pi a^2\left(5\sqrt{5}-1\right)}{6}$$

14.



$$f_x(x,y)=\frac{-x}{\sqrt{a^2-x^2}};\ f_y(x,y)=0$$

$$\sqrt{\left(f_x(x,y)\right)^2+\left(f_y(x,y)\right)+1}=\frac{a}{\sqrt{a^2-x^2}}$$

$$=\frac{a}{\sqrt{a^2-r^2\cos^2\theta}}$$

$$A(\text{all sides})=8\int_0^{\pi/2}\int_0^a\frac{a}{\sqrt{a^2-r^2\cos^2\theta}}d\theta rdr$$

$$=8\int_0^{\pi/2}\frac{a^2}{1+\sin\theta}d\theta=8a^2(1)=8a^2$$

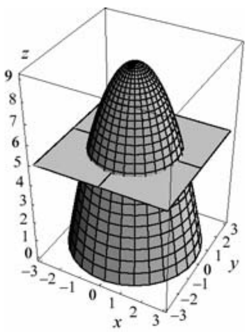
15. $f_x = -2x;\ f_y = -2y$

$$SA = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} \left[\left(4r^2 + 1\right)^{3/2} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{17^{3/2} - 1}{12}\, d\theta = \frac{\left(17^{3/2} - 1\right)\pi}{6} \approx 36.18$$



16. Using Formula 44 from the table of integrals,

$f_x = -2x;\ f_y = 0$

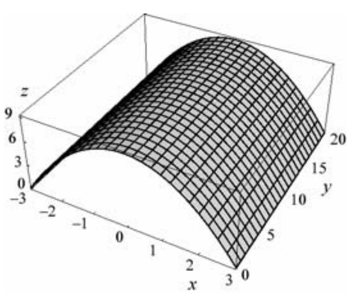
$$A(G) = \int_0^{20} \int_{-3}^{3} \sqrt{4x^2 + 1}\, dx\, dy$$

$$= 2\int_0^{20} \int_0^3 \sqrt{4x^2 + 1}\, dx\, dy$$

$$= \int_0^{20} \left[x\sqrt{4x^2 + 1} + \frac{1}{2} \ln\left| 2x + \sqrt{4x^2 + 1} \right| \right]_0^3 dy$$

$$= \int_0^{20} \left[3\sqrt{37} + \frac{1}{2} \ln\left| 6 + \sqrt{37} \right| - (0 + \ln 1) \right] dy$$

$$= 20\left[3\sqrt{37} + \frac{1}{2} \ln\left(6 + \sqrt{37}\right) \right] \approx 389.88$$



17. $z = f(x, y) = (D - Ax - By)/C$

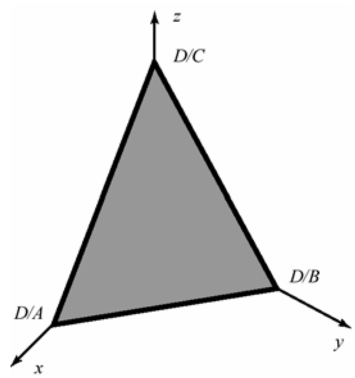
$$A(G) = \iint_R \sqrt{(A/C)^2 + (B/C)^2 + 1}\, dA$$

$$= \int_0^{D/A} \int_0^{D/B - (A/B)x} \sqrt{(A/C)^2 + (B/C)^2 + 1}\, dy\, dx$$

$$= \sqrt{(A/C)^2 + (B/C)^2 + 1} \times Area(Triangle)$$

$$= \frac{1}{2} \frac{D}{A} \frac{D}{B} \sqrt{(A/C)^2 + (B/C)^2 + 1}$$

$$= \frac{D^2}{2ABC} \sqrt{A^2 + B^2 + C^2}$$



18. Let $z = C - x$ be the equation of the plane that defines the roof, where C is a constant. Thus, $f_x = -1$ and $f_y = 0$.

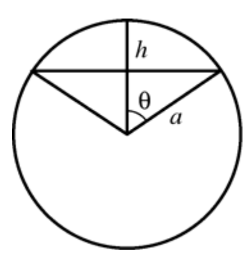
$$A(G) = \iint_R \sqrt{(-1)^2 + 0^2 + 1}\, dA$$

$$= \iint_R \sqrt{2}\, dA = \pi(18)^2 \sqrt{2} \approx 1440 \text{ sq.ft.}$$

19. $\overline{x} = \overline{y} = 0$ (by symmetry)

Let $h = \dfrac{h_1 + h_2}{2}$. Planes $z = h_1$ and $z = h$ cut out the same surface area as planes $z = h$ and $z = h_2$. Therefore, $\overline{z} = h$, the arithmetic average of h_1 and h_2.

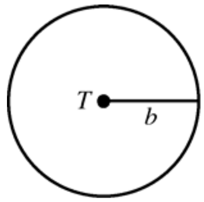
20.



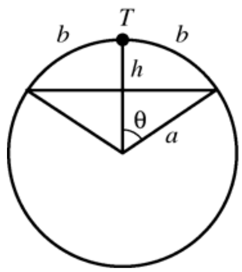
Area $= 2\pi ah$

$= 2\pi a(a - a\cos\phi) = 2\pi a^2 (1 - \cos\phi)$

21. a. $A = \pi b^2$



b.



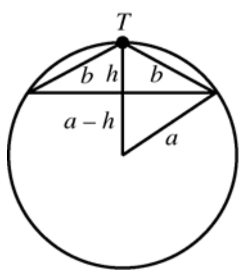
$B = 2\pi a^2(1-\cos\phi)$ (Problem 20)

$= 2\pi a^2\left[1-\cos\left(\frac{b}{a}\right)\right]$

$= 2\pi a^2\left[\frac{b^2}{2!a^2}-\frac{b^4}{4!a^4}+\frac{b^6}{6!a^4}+\ldots\right]$

$= \pi b^2\left[1-\frac{b^2}{12a^2}+\frac{b^4}{360a^4}-\ldots\right] \le \pi b^2$

c.

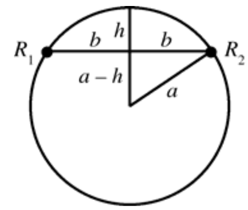


$a^2-(a-h)^2 = b^2-h^2$, so $h = \frac{b^2}{2a}$.

Thus, $C = 2\pi ah$

$= 2\pi a\left(\frac{b^2}{2a}\right) = \pi b^2.$

d.



$D = 2\pi ah$

$= 2\pi a\left(a-\sqrt{a^2-b^2}\right) = \frac{2\pi a[a^2-(a^2-b^2)]}{a+\sqrt{a^2-b^2}}$

$= \frac{2\pi ab^2}{a+\sqrt{a^2-b^2}} > \pi b^2$

Therefore, $B < A = C < D$.

22. $[A(S_{yz})]^2+[A(S_{xz})]^2+[A(S_{xy})]^2$

$= [A(S)\cos\alpha]^2+[A(S)\cos\beta]^2+[A(S)\cos\gamma]^2$

(where α, β, and γ are direction angles for a normal to S.)

$= [A(S)]^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma) = [A(S)]^2$

23. In the following, each double integral is over S_{xy}

$A(S_{xy})f(\bar{x},\bar{y}) = A(S_{xy})(a\bar{x}+b\bar{y}+c)$

$= \iint dA\left[a\frac{\iint x\,dA}{\iint dA}+b\frac{\iint y\,dA}{\iint dA}+c\right]$

$= a\iint x\,dA+b\iint y\,dA+c\iint dA$

$= \iint(ax+by+c)dA$

= Volume of solid cylinder under S_{xy}

24. Because the slopes of both roofs are the same, the area of T_m will be the same for both roofs. (Essentially we will be integrating over a constant). Therefore, the area of the roofs will be the same.

25. Let G denote the surface of that part of the plane $z = Ax+By+C$ over the region S. First, suppose that S is the rectangle $a \le x \le b$, $c \le y \le d$. Then the vectors **u** and **v** that form the edge of the parallelogram G are $\mathbf{u} = (b-a)\mathbf{i}+0\mathbf{j}+A(b-a)\mathbf{k}$ and $\mathbf{v} = 0\mathbf{i}+(d-c)\mathbf{j}+B(d-c)\mathbf{k}$. The surface area of G is thus

$|\mathbf{u}\times\mathbf{v}| =$

$|-A(b-a)(d-c)\mathbf{i}-B(b-a)(d-c)\mathbf{j}+(b-a)(d-c)\mathbf{k}|$

$= (b-a)(d-c)\sqrt{A^2+B^2+1}$

A normal vector to the plane is $\mathbf{n} = -A\mathbf{i}-B\mathbf{j}+\mathbf{k}$. Thus,

$$\cos\gamma = \frac{\mathbf{n}\cdot\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \frac{\langle -A,-B,1\rangle\cdot\langle 0,0,1\rangle}{\sqrt{A^2+B^2+1}\cdot 1}$$

$$= \frac{1}{\sqrt{A^2+B^2+1}}$$

$$\sec\gamma = \frac{1}{\cos\gamma} = \sqrt{A^2+B^2+1} = \frac{|\mathbf{u}\times\mathbf{v}|}{A(S)} = \frac{A(G)}{A(S)}.$$

If S is not a rectangle, then make a partition of S with rectangles $R_1, R_2, \ldots, R_n$. The Riemann sum will be $\sum_{m=1}^{n} A(G_m) = \sec\gamma \sum_{m=1}^{n} A(R_m)$.

As we take the limit as $|P| \to 0$ the sum converges to the area of S. Thus the surface area will be

$$A(G) = \lim_{|P|\to 0} \sec\gamma \sum_{m=1}^{n} A(R_m) = \sec\gamma\, A(S).$$

26. Let $\gamma = \gamma(x, y, f(x, y))$ be the acute angle between a unit vector $\mathbf{n}$ that is normal to the surface and makes an acute angle with the z-axis. Let $F(x, y, z) = z - f(x, y)$. Then the normal vector to the surface $F(x, y, z) = 0 = z - f(x, y)$ is parallel to the gradient $\nabla F(x, y, z) = -f_x\mathbf{i} - f_y\mathbf{j} + 1\mathbf{k}$. The unit normal vector is thus

$$\mathbf{n} = \left(-f_x\mathbf{i} - f_y\mathbf{j} + 1\mathbf{k}\right)/\sqrt{f_x^2 + f_y^2 + 1}$$

The cosine of the angle γ is thus

$$\cos\gamma = \frac{\mathbf{n}\cdot\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \frac{-f_x\mathbf{i} - f_y\mathbf{j} + 1\mathbf{k}}{\sqrt{f_x^2 + f_y^2 + 1}}\cdot\mathbf{k}$$

$$= \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$$

Hence, $\sec\gamma = \sqrt{f_x^2 + f_y^2 + 1}$.

27. a. $f_x = 2x,\ f_y = 2y$

$$A(G) = \iint_S \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$= \int_0^{\pi/2}\int_0^3 \sqrt{4r^2+1}\, r\,dr\,d\theta$$

$$= \frac{\pi}{2}\left[\frac{1}{12}\left(4r^2+1\right)^{3/2}\right]_0^3$$

$$= \frac{\pi}{24}\left(37^{3/2} - 1\right) \approx 29.3297$$

b. $f_x = 2x,\ f_y = 2y$

$$A(G) = \int_0^3\int_0^{3-x} \sqrt{4x^2 + 4y^2 + 1}\, dy\,dx$$

Parabolic rule with $n = 10$ gives $SA \approx 15.4233$

28. a. $f_x = 2x,\ f_y = -2y$

$$A(G) = \iint_S \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$= \frac{\pi}{24}\left(37^{3/2} - 1\right) \approx 29.3297$$

(same integral as problem 27a)

b. $f_x = 2x,\ f_y = -2y$

$$A(G) = \int_0^3\int_0^{3-x} \sqrt{4x^2 + 4y^2 + 1}\, dy\,dx$$

Parabolic rule with $n = 10$ gives $SA \approx 15.4233$

(same integral as problem 27b)

29. The surface area of a paraboloid and a hyperbolic paraboloid are the same over identical regions. So, the areas depend on the regions.
$E = F < A = B < C = D$

13.7 Concepts Review

1. volume

2. $\iiint\limits_S |xyz|\, dV$

3. y; $\sqrt{y}$

4. 0

Problem Set 13.7

1. $\int_{-3}^{7}\int_{0}^{2x}(x-1-y)\,dy\,dx = \int_{-3}^{7} -2x\,dx = -40$

2. $\int_0^2\int_{-1}^4 (3y+x)\,dy\,dx = \int_0^2\left(\frac{45}{2}+5x\right)dx = 55$

3. $\int_1^4\int_{z-1}^{2z}\int_0^{y+2z} dx\,dy\,dz = \int_1^4\int_{z-1}^{2z}(y+2z)\,dy\,dz$

$$= \int_1^4\left[\frac{y^2}{2}+2yz\right]_{y=z-1}^{2z} dz$$

$$= \int_1^4\left(\frac{7z^2}{2}+3z-\frac{1}{2}\right)dz$$

$$= \left[\frac{7z^3}{6}+\frac{3z^2}{2}-\frac{z}{2}\right]_1^4 = \frac{189}{2} = 94.5$$

4. $6\int_0^5 z^3 dz\int_{-2}^4 y^2 dy\int_1^2 x\,dx = \left(\frac{625}{4}\right)(72)(3)$

$$= 33{,}750$$

5. $\int_4^{24}\int_0^{24-x}\left[\frac{1}{x}\left(yz+\frac{1}{2}z^2\right)\right]_0^{24-x-y} dy\,dx$

$$= \int_4^{24}\int_0^{24-x}\left[\frac{(x+y-24)(x-y-24)}{2x}\right]dy\,dx$$

$$= \int_4^{24}\left[-\frac{y\left(y^2-3(x-24)^2\right)}{6x}\right]_0^{24-x} dx$$

$$= \int_4^{24} -\frac{(x-24)^3}{3x}\,dx$$

$$= \left[-576x+12x^2-\frac{x^3}{9}+4608\ln x\right]_4^{24} \approx 1927.54$$

6. $\int_0^5\int_0^3\left[\frac{yzx^2}{2}\right]_{z^2}^{9} dz\,dy = \int_0^5\int_0^3\left[\frac{81yz-yz^5}{2}\right]dz\,dy$

$$= \int_0^5\left[\frac{-z^2\left(z^4-243\right)y}{12}\right]_0^3 dy = \int_0^5 \frac{243y}{2}\,dy$$

$$= \left[\frac{243}{4}y^2\right]_0^5 = 1518.75$$

7. $\int_0^2\int_1^z x^2\,dx\,dz = \int_0^2\left(\frac{1}{3}\right)(z^3-1)\,dz = \frac{2}{3}$

8. $\int_0^{\pi/2}\int_0^z\int_0^y \sin(x+y+z)\,dx\,dy\,dz$

$$= \int_0^{\pi/2}\int_0^z[-\cos(2y+z)+\cos(y+z)]\,dy\,dz$$

$$= \int_0^{\pi/2}\left(-\frac{\sin 3z}{2}+\sin 2z-\frac{\sin z}{2}\right)dz = \frac{1}{3}$$

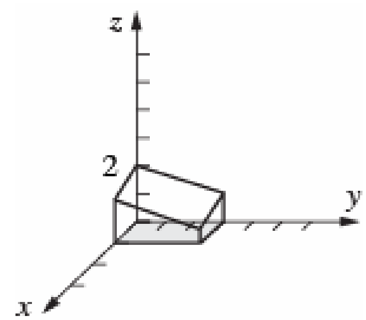
9. $\int_{-2}^4\int_{x-1}^{x+1} 3y^2\,dy\,dx = \int_{-2}^4(6x^2+2)\,dx = 156$

10. $\int_0^{\pi/2}\int_{\sin 2z}^{0} y(1-\cos 2z)\,dy\,dz$

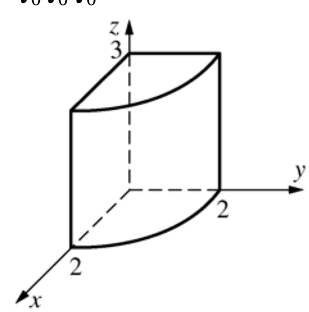
$$= \int_0^{\pi/2}\left(-\frac{1}{2}\right)(\sin^2 2z)(1-\cos 2z)$$

$$= -\frac{\pi}{8} \approx -0.3927$$

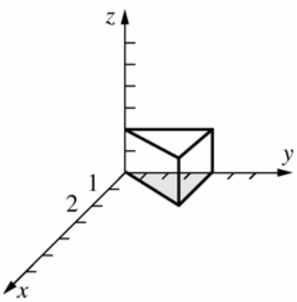
11. $\int_0^1\int_0^3\int_0^{(12-3x-2y)/6} f(x,\ y,\ z)\,dz\,dy\,dx$



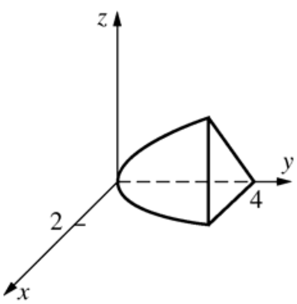
12. $\int_0^3\int_0^2\int_0^{\sqrt{4-y^2}} f(x,\ y,\ z)\,dx\,dy\,dz$



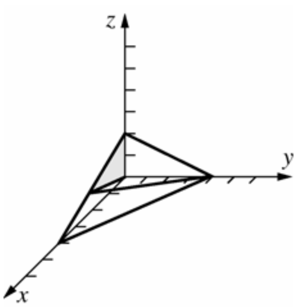
13. $\int_0^2 \int_0^4 \int_0^{y/2} f(x, y, z)\,dx\,dy\,dz$



14. $\int_0^4 \int_0^{\sqrt{y}} \int_0^{3x/2} f(x, y, z)\,dz\,dx\,dy$



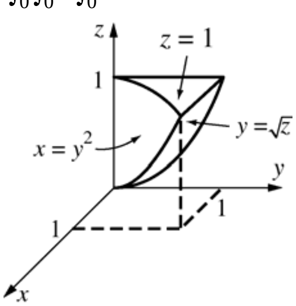
15. $\int_0^2 \int_0^{3z} \int_0^{4-x-2z} f(x, y, z)\,dy\;dx\;dz$



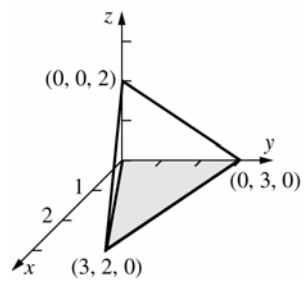
Alternate:

$\int_0^{12/5} \int_{x/3}^{(4-x)/2} \int_0^{4-x-2z} f(x, y, z)\,dy\;dz\;dx$

16. $\int_0^1 \int_0^{\sqrt{z}} \int_0^{y^2} f(x, y, z)\,dx\,dy\,dz$



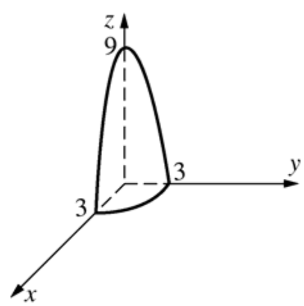
17.



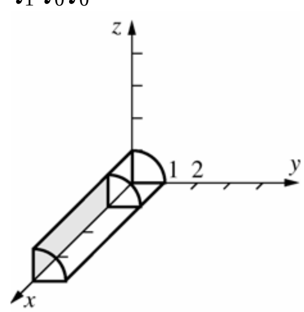
Using the cross product of vectors along edges, it is easy to show that $\langle 2,6,9 \rangle$ is normal to the upward face. Then obtain that its equation is $2x + 6y + 9z = 18$.

$\int_0^3 \int_{2x/3}^{(9-x)/3} \int_0^{(18-2x-6y)/9} f(x, y, z)\,dz\,dy\,dx$

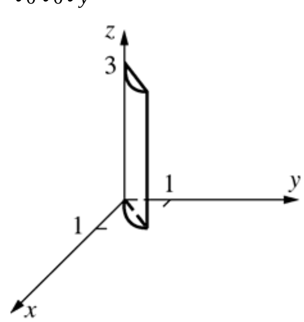
18. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} f(x, y, z)\,dz\,dy\,dx$



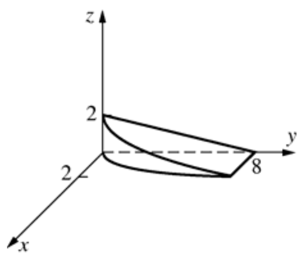
19. $\int_1^4 \int_0^1 \int_0^{\sqrt{1-y^2}} f(x, y, z)\,dz\,dy\,dx$



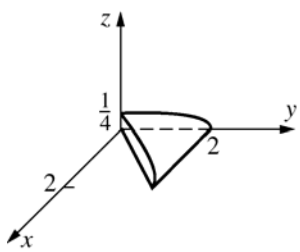
20. $\int_0^3 \int_0^1 \int_y^{\sqrt{2y-y^2}} f(x, y, z)\,dx\,dy\,dz$



21. $\int_0^2 \int_{2x^2}^8 \int_0^{2-y/4} 1\,dz\,dy\,dx = \frac{128}{15}$



22. $\int_0^2 \int_0^y \int_0^{\sqrt{4-y^2}/8} 1\,dz\,dx\,dy = \frac{1}{3}$



23. $V = 4\int_0^1 \int_0^{\sqrt{y}} \int_0^{\sqrt{y}} 1\,dz\,dx\,dy = 4\int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\,dx\,dy$

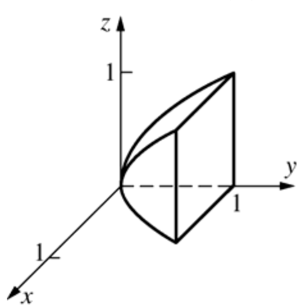
$= 4\int_0^1 \sqrt{y}\sqrt{y}\,dy = [2y^2]_0^1 = 2$

or

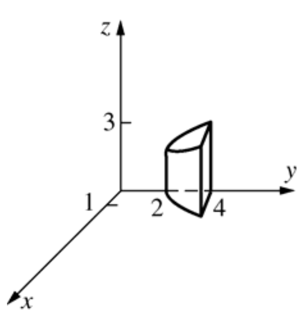
$V = 4\int_0^1 \int_{x^2}^1 \int_0^{\sqrt{y}} 1\,dz\,dy\,dx = 4\int_0^1 \int_{x^2}^1 \sqrt{y}\,dy\,dx$

$= 4\int_0^1 \left[\frac{2}{3}y^{3/2}\right]_{x^2}^1 dy = \frac{8}{3}\int_0^1 \left(1 - x^3\right) dx$

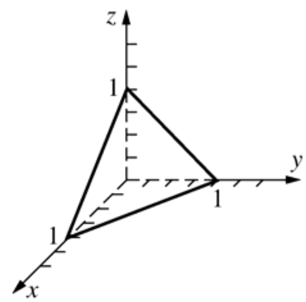
$= \frac{8}{3}\left[x - \frac{1}{4}x^4\right]_0^1 = \frac{8}{3}\left(\frac{3}{4}\right) = 2$



24. $2\int_0^{\sqrt{2}} \int_{x^2+2}^4 \int_0^{3y/4} 1\,dz\,dy\,dx = 32\frac{\sqrt{2}}{5} \approx 9.0510$



25. Let $\delta(x, y, z) = x + y + z$. (See note with next problem.)



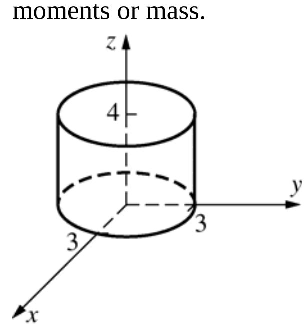
$m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z)\,dz\,dy\,dx = \frac{1}{8}$

$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x(x + y + z)\,dz\,dy\,dx = \frac{1}{30}$

$\overline{x} = \frac{4}{15}$; Then $\overline{y} = \overline{z} = \frac{4}{15}$ (symmetry).

26. $(x,\ y,\ z) = k(x^2 + y^2 + z^2)$

In evaluating the coordinates of the center of mass, k is a factor of the numerator and denominator and so may be canceled. Hence, for sake of convenience we may just let $k = 1$ when determining the center of mass. Note that this is not valid if we are concerned with values of moments or mass.



$m = 4\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^4 (x^2 + y^2 + z^2)\,dz\,dy\,dx$

$= 4\int_0^3 \int_0^{\sqrt{9-x^2}} \left[4(x^2 + y^2) + \frac{64}{3}\right] dy\,dx$

$= 4\int_0^{\pi/2} \int_0^3 \left(4r^2 + \frac{64}{3}\right) r\,dr\,d\theta$ (change to polar)

$= 4\int_0^{\pi/2} \left[r^4 + \frac{32r^2}{3}\right]_0^3 d\theta = 4\int_0^{\pi/2} 177\,d\theta = 354\pi$

$M_{xy} = 4\int_0^{\pi/2} \int_0^3 \int_0^4 z(r^2 + z^2)\,r\,dz\,dr\,d\theta$

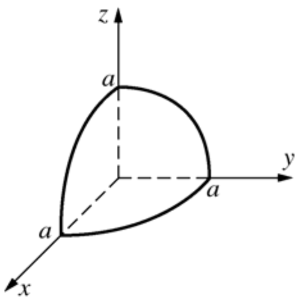
(polar coordinates)

$= 4\int_0^{\pi/2} \int_0^3 (8r^3 + 64r)\,dr\,d\theta$

$= 4\int_0^{\pi/2} 450\,d\theta = 900\pi$

$\overline{z} = \frac{900\pi}{354\pi} = \frac{150}{59} \approx 2.5425$; $\overline{x} = \overline{y} = 0$ (by symmetry)

27. Let $\delta(x, y, z) = 1$. (See note with previous problem.)



$$m = \left(\frac{1}{8}\right)(\text{volume of sphere}) = \left(\frac{\pi}{6}\right)a^3$$

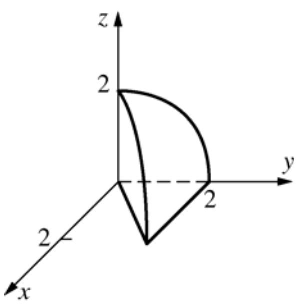
$$M_{xy} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z\,dz\,dy\,dx$$

$$= \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} zr\,dz\,dr\,d\theta = \left(\frac{\pi}{16}\right)a^4$$

$$\bar{z} = \left(\frac{3}{8}\right)a$$

$$\bar{x} = \bar{y} = \left(\frac{3}{8}\right)a \text{ (by symmetry)}$$

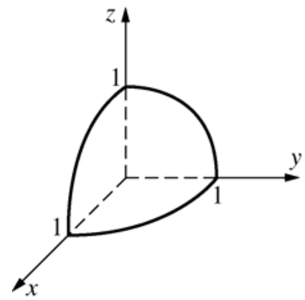
28. $y^2 + z^2$ is the distance of (x, y, z) from the x-axis.



$$I_x = \iiint_S (y^2 + z^2)\delta(x, y, z)dV$$

$$= \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^y (y^2 + z^2)z\,dx\,dy\,dz = \frac{16}{3}$$

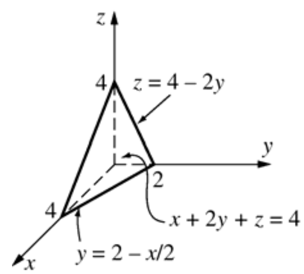
29.



The limits of integration are those for the first octant part of a sphere of radius 1.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x, y, z)dz\,dy\,dx$$

30.

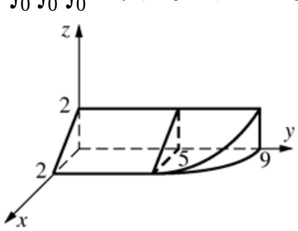


$$\int_0^4 \int_0^{2-x/2} \int_0^{4-x-2y} f(x, y, z)dz\,dy\,dx$$

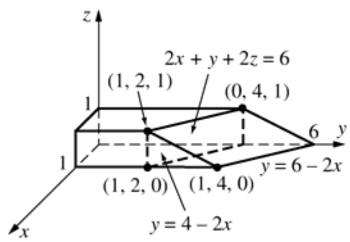
31. $\int_0^2 \int_0^{2-z} \int_0^{9-x^2} f(x, y, z)dy\,dx\,dz$

Figure is same as for Problem 32 except that the solid doesn't need to be divided into two parts.

32. $\int_0^5 \int_0^2 \int_0^{2-x} f(x, y, z)dz\,dx\,dy + \int_5^9 \int_0^{\sqrt{9-y}} \int_0^{2-x} f(x, y, z)dz\,dx\,dy$



33.



a. $\int_0^1\int_0^{4-2x}\int_0^1 dz\,dy\,dx + \int_0^1\int_{4-2x}^{6-2x}\int_0^{3-x-y/2} dz\,dy\,dx = 3+1=4$

b. $\int_0^1\int_0^1\int_0^{6-2x-2z} 1\,dy\,dx\,dz = 4$

c. $A(S_{xz})\,f(\overline{x},\,\overline{z})$ (S_{xz} is the unit square in the corner of xz-plane; and $(\overline{x},\,\overline{z}) = \left(\frac{1}{2},\frac{1}{2}\right)$ is the centroid of S_{xz}.)

$= (1)\left[6-2\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)\right]=4$

34. The moment of inertia with respect to the y-axis is the integral (over the solid) of the function which gives the square of the distance of each point in the solid from the y-axis.

$\int_0^1\int_0^1\int_0^{6-2x-2z} k(x^2+z^2)dy\,dx\,dz = \frac{7}{3}k$

35. $\int_0^1\int_0^1\int_0^{6-2x-2z}(30-z)dy\,dx\,dz = \int_0^1\int_0^1(30-z)(6-2x-2z)dx\,dz = \int_0^1([(30-z)(6x-x^2-2xz)]_{x=0}^1)dz$

$= \int_0^1(30-z)(5-2z)dz = \int_0^1(150-65z+2z^2)dz = \left[150z-\frac{65z^2}{2}+\frac{2z^3}{3}\right]_0^1 = \frac{709}{6}$

The volume of the solid is 4 (from Problem 33).

Hence, the average temperature of the solid is $\frac{\frac{709}{6}}{4} = \frac{709}{24} \approx 29.54°$.

36. $T(x,y,z) = 30 - z = 29.54$. The set of all points whose temperature is the average temperature is the plane $z = 0.46$.

37. $M_{yz} = \int_0^1\int_0^1\int_0^{6-2x-2z} x\,dy\,dx\,dz = \int_0^1\int_0^1 x(6-2x-2z)dx\,dz = \int_0^1\left[3x^2-\frac{2}{3}x^3-x^2z\right]_{x=0}^1 dz$

$= \int_0^1\left(\frac{7}{3}-z\right)dz = \left[\frac{7}{3}z-\frac{1}{2}z^2\right]_0^1 = \frac{11}{6}$

$M_{xz} = \int_0^1\int_0^1\int_0^{6-2x-2z} y\,dy\,dx\,dz = \int_0^1\int_0^1\left[\frac{1}{2}(6-2x-2z)^2\right]dx\,dz = \int_0^1\left[18x-6x^2+\frac{2}{3}x^3-12xz+2x^2z+2xz^2\right]_{x=0}^1$

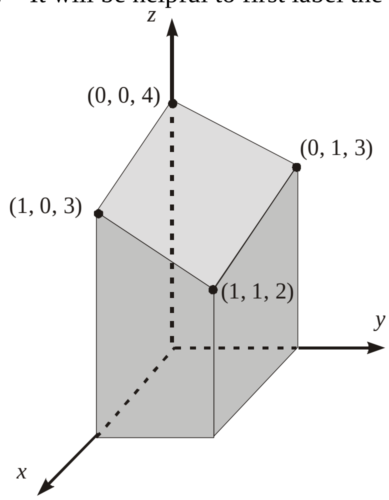
$= \int_0^1\left(\frac{38}{3}-10z+2z^2\right)dz = \frac{25}{3}$

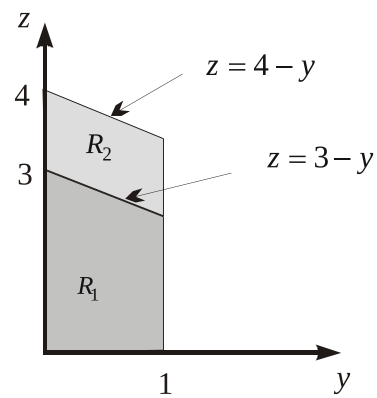
$M_{xy} = \int_0^1\int_0^1\int_0^{6-2x-2z} z\,dy\,dx\,dz = \int_0^1\int_0^1 z(6-2x-2z)dx\,dz = \int_0^1([z(6x-x^2-2xz)]_{x=0}^1)dz$

$= \int_0^1(5z-2z^2)dz = \left[\frac{5z^2}{2}-\frac{2z^3}{3}\right]_0^1 = \frac{11}{6}$

Hence, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{11/6}{4},\frac{25/3}{4},\frac{11/6}{4}\right) = \left(\frac{11}{24},\frac{25}{12},\frac{11}{24}\right)$

38. a. It will be helpful to first label the corner points at the top of the region.





Fixing z and y, we will be looking at the figure along the x-axis. The resulting projection is shown in the figure above and to the right. The possible values of x depends on where we are in the yz-plane. Therefore, we split up the solid into two parts. The volume of the solid will be the sum of these two smaller volumes. In the lower portion, x goes from 0 to 1, while in the upper portion, x goes from 0 to $4 - y - z$ (the plane that bounds the top of the square cylinder).

$$V = \int_0^1\int_0^{3-y}\int_0^1 1\,dx\,dz\,dy + \int_0^1\int_{3-y}^{4-y}\int_0^{4-y-z} 1\,dx\,dz\,dy = \frac{5}{2} + \frac{1}{2} = 3$$

b. $\int_0^1\int_0^1\int_0^{4-y-z} 1\,dx\,dy\,dz = 3$

c. $A(S_{xy})\,f(\bar{x}, \bar{y})$ (S_{xy} is the unit square in the corner of xy-plane; and $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the centroid of S_{xz}.)

$$= (1)\left[4 - \frac{1}{2} - \frac{1}{2}\right] = 3$$

39. $m = \int_0^1\int_0^1\int_0^{4-x-y} k\,dz\,dy\,dx = \int_0^1\int_0^1 k(4 - x - y)\,dy\,dx = k\int_0^1\left(\frac{7}{2} - x\right)dx = 3k$

$$M_{yz} = \int_0^1\int_0^1\int_0^{4-x-y} kx\,dz\,dy\,dx = k\int_0^1\int_0^1\left(4x - x^2 - xy\right)dy\,dx = k\int_0^1\left(\frac{7}{2}x - x^2\right)dx = \frac{17k}{12}$$

$$M_{xz} = \int_0^1\int_0^1\int_0^{4-x-y} ky\,dz\,dy\,dx = k\int_0^1\int_0^1\left(4y - xy - y^2\right)dy\,dx = k\int_0^1\left(\frac{5}{3}x - \frac{x}{2}\right)dx = \frac{17k}{12}$$

$$M_{xy} = \int_0^1\int_0^1\int_0^{4-x-y} kz\,dz\,dy\,dx = k\int_0^1\int_0^1\left(8 - 4x + \frac{x^2}{2} - 4y + xy + \frac{y^2}{2}\right)dy\,dx = k\int_0^1\left(\frac{37}{6} - \frac{7x}{2} + \frac{x^2}{2}\right)dx = \frac{55k}{12}$$

$$\bar{x} = \frac{Myz}{m} = \frac{17k/12}{3k} = \frac{17}{36} \qquad \bar{y} = \frac{Mxz}{m} = \frac{17k/12}{3k} = \frac{17}{36} \qquad \bar{z} = \frac{M_{xy}}{m} = \frac{55k/12}{3k} = \frac{55}{36}$$

40. The temperature, as a function of (x, y, z) is $T(x, y, z) = 40 + 5z$. The average temperature is

$$\frac{1}{\text{Volume}}\int_0^1\int_0^1\int_0^{4-x-y} T(x,y,z)\,dz\,dy\,dx = \frac{1}{3}\int_0^1\int_0^1\int_0^{4-x-y}(40+5z)\,dz\,dy\,dx$$

$$= \frac{1}{3}\int_0^1\int_0^1\left(200 - 60x + \frac{5}{2}x^2 - 60y + 5xy + \frac{5y^2}{2}\right)dy\,dx$$

$$= \frac{1}{3}\int_0^1\left(\frac{1025}{18} - \frac{115x}{6} + \frac{5x^2}{6}\right)dx$$

$$= \frac{1715}{36} \approx 47.64$$

41.

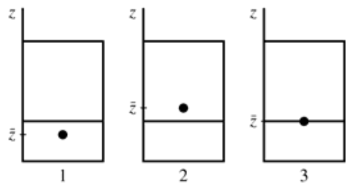


Figure 1: When the center of mass is in this position, it *will* go lower when a little more soda leaks out since mass above the center of mass is being removed.
Figure 2: When the center of mass is in this position, it *was* lower moments before since mass that was below the center of mass was removed, causing the center of mass to rise.
Therefore, the center of mass is lowest when it is at the height of the soda, as in Figure 3. The same argument would hold for a soda bottle.

42. The result obtained from a CAS is:

$$\int_0^c\int_0^{b\sqrt{1-z^2/c^2}}\int_0^{a\sqrt{1-y^2/b^2-z^2/c^2}} 8(xy+xz+yz)dxdydz = \frac{8}{15}a^2b^2c + \frac{8}{15}a^2bc^2 + \frac{8}{15}ab^2c^2 = \frac{8}{15}acb(ca+cb+ab)$$

43. **a.** $1 = \int_0^{12}\int_0^x ky\,dy\,dx = \int_0^{12}\frac{k}{2}x^2\,dx$

$$= \left[\frac{k}{6}x^3\right]_0^{12} = 288k \Rightarrow k = \frac{1}{288}$$

b. $P(Y > 4) = \int_4^{12}\int_4^x \frac{1}{288}y\,dydx$

$$= \int_4^{12}\left[\frac{1}{576}y^2\right]_4^x dx = \int_4^{12}\frac{1}{576}\left(x^2 - 16\right)dx$$

$$= \left[\frac{1}{576}\left(x^3 - 16x\right)\right]_4^{12} = \frac{20}{27}$$

c. $E[X] = \int_0^{12}\int_0^x x\frac{1}{288}y\,dy\,dx$

$$= \int_0^{12}\left[\frac{1}{576}xy^2\right]_0^x dx = \int_0^{12}\frac{1}{576}x^3\,dx$$

$$= \left[\frac{1}{2304}x^4\right]_0^{12} = 9$$

44. **a.** $1 = \int_0^2\int_0^4\int_0^y kxy\,dx\,dy\,dz$

$$= \int_0^2\int_0^4\left[\frac{k}{2}x^2y\right]_0^y dy\,dz$$

$$= \int_0^2\int_0^4\frac{k}{2}y^3dy\,dz = \int_0^2\left[\frac{k}{8}y^4\right]_0^4 dz$$

$$= \int_0^2 32k\,dz = 64k \Rightarrow k = \frac{1}{64}$$

b. $P(X > 2) = \int_0^2\int_2^4\int_x^4 \frac{1}{64}xy\,dy\,dx\,dz$

$$= \int_0^2\int_2^4\left[\frac{1}{128}xy^2\right]_x^4 dx\,dz$$

$$= \int_0^2\int_2^4\frac{1}{128}\left(16x - x^3\right)dx\,dz$$

$$= \int_0^2\frac{1}{128}\left[8x^2 - \frac{1}{4}x^4\right]_2^4 dz = \int_0^2\frac{9}{32}dz = \frac{9}{16}$$

c. $E[X]=\int_0^2\int_0^4\int_0^y \frac{1}{64}x^2 y\,dx\,dy\,dz$

$=\int_0^2\int_0^4\left[\frac{1}{192}x^3 y\right]_0^y dy\,dz$

$\int_0^2\int_0^4\left[\frac{1}{192}y^4\right]dy\,dz=\int_0^2\left[\frac{1}{960}y^5\right]_0^4 dz$

$=\int_0^2\frac{16}{15}dz=\frac{32}{15}$

45. a. $P(X>2)=\int_2^4\int_x^4\frac{3}{256}\left(x^2+y^2\right)dy\,dx$

$=\int_2^4\left[\frac{3}{256}\left(x^2y+\frac{1}{3}y^3\right)\right]_x^4 dx$

$=\int_2^4\frac{1}{64}\left(-x^3+3x^2+16\right)dx$

$=\left[\frac{1}{64}\left(-\frac{1}{4}x^4+x^3+16x\right)\right]_2^4=\frac{7}{16}$

b. $P(X+Y<4)=\int_0^2\int_x^{4-x}\frac{3}{256}\left(x^2+y^2\right)dy\,dx$

$=\int_0^2\left[\frac{3}{256}\left(x^2y+\frac{1}{3}y^3\right)\right]_x^{4-x}dx$

$=\int_0^2\frac{1}{32}\left(-x^3+3x^2-6x+8\right)dx$

$=\frac{1}{32}\left[-\frac{1}{4}x^4+x^3-3x^2+8x\right]_0^2=\frac{1}{4}$

c. $E[X+Y]$

$=\int_0^4\int_0^y(x+y)\frac{3}{256}\left(x^2+y^2\right)dx\,dy$

$=\int_0^4\left[\frac{3}{256}\left(\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3y+xy^3\right)\right]_0^y dx$

$=\int_0^4\frac{25}{1024}y^4\,dy=\left[\frac{5}{1024}y^5\right]_0^4=5$

46. a. $P(a<X<b)=\int_a^b\int_{-\infty}^{\infty}f(x,y)\,dy\,dx$

$=\int_a^b\left(\int_{-\infty}^{\infty}f(x,y)\,dy\right)dx$

$=\int_a^b f_X(x)\,dx$

b. $E[X]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xf(x,y)\,dy\,dx$

$=\int_{-\infty}^{\infty}x\left(\int_{-\infty}^{\infty}f(x,y)\,dy\right)dx$

$=\int_{-\infty}^{\infty}xf_X(x)\,dx$

47. $f_X(x)=\int_0^x\frac{1}{288}y\,dy$

$=\left[\frac{1}{576}y^2\right]_0^x=\frac{1}{576}x^2;\ 0\le x\le 12$

$E(X)=\int_0^{12}x\cdot f_X(x)\,dx=\frac{1}{576}\int_0^{12}x^3\,dx$

$=\frac{1}{576}\left[\frac{1}{4}x^4\right]_0^{12}=9$

48. $f_Y(y)=\int_c^d\int_a^b f(x,y,z)\,dx\,dz$

$=\int_0^2\int_0^y\frac{1}{64}xy\,dx\,dz=\int_0^2\left[\frac{1}{128}x^2y\right]_0^y dz$

$=\int_0^2\frac{1}{128}y^3dz=\frac{y^3}{64};\ 0\le y\le 4$

13.8 Concepts Review

1. $r\,dz\,dr\,d\theta$; $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

2. $\int_0^{\pi/2}\int_0^1\int_0^3 r^3\cos\theta\sin\theta\,dz\,dr\,d\theta$

3. $\int_0^{\pi}\int_0^{2\pi}\int_0^1\rho^4\cos^2\phi\sin\phi\,d\rho\,d\theta\,d\phi$

4. $\frac{4\pi}{15}$

Problem Set 13.8

1. The region is a right circular cylinder about the z-axis with radius 3 and height 12.

$\int_0^{2\pi}\int_0^3[rz]_0^{12}\,dr\,d\theta=\int_0^{2\pi}\int_0^3 12r\,dr\,d\theta$

$=\int_0^{2\pi}\left[6r^2\right]_0^3 d\theta=\int_0^{2\pi}54\,d\theta=108\pi$

2. The region is a hollow right circular cylinder about the z-axis with inner radius 1, outer radius 3, and height 12.

$$\int_0^{2\pi}\int_1^3 [rz]_0^{12}\,dr\,d\theta = \int_0^{2\pi}\int_1^3 12r\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[6r^2\right]_1^3 d\theta = \int_0^{2\pi} 48\,d\theta = 96\pi$$

3. The region is the region under the paraboloid $z = 9 - r^2$ above the xy-plane in that part of the first quadrant satisfying $0 \le \theta \le \dfrac{\pi}{4}$.

$$\int_0^{\pi/4}\int_0^3 \left[\frac{1}{2}rz^2\right]_0^{9-r^2} dr\,d\theta$$
$$= \int_0^{\pi/4}\int_0^3 \left[\frac{1}{2}r\left(81 - 18r + r^2\right)\right] dr\,d\theta$$
$$= \int_0^{\pi/4} \frac{1}{2}\left[\frac{81}{2}r^2 - 6r^3 + \frac{1}{4}r^4\right]_0^3 d\theta$$
$$= \int_0^{\pi/4} \frac{243}{4}\,d\theta = \frac{243\pi}{16}$$

4. The region is a right circular cylinder about the z-axis through the point $\left(0, \frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$ and height 2.

$$\int_0^{\pi}\int_0^{\sin\theta} [rz]_0^2\,dr\,d\theta = \int_0^{\pi}\int_0^{\sin\theta} 2r\,dr\,d\theta$$
$$= \int_0^{\pi}\left[r^2\right]_0^{\sin\theta} d\theta = \int_0^{\pi}\sin^2\theta\,d\theta$$
$$= \int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$
$$= \frac{1}{2}\pi$$

5. The region is a sphere centered at the origin with radius a.

$$\int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{3}\rho^3\sin\varphi\right]_0^a d\theta\,d\varphi$$
$$= \int_0^{\pi}\int_0^{2\pi}\frac{1}{3}a^3\sin\varphi\,d\theta\,d\varphi$$
$$= \int_0^{\pi}\frac{2\pi}{3}a^3\sin\varphi\,d\varphi$$
$$= \left[-\frac{2\pi}{3}a^3\cos\varphi\right]_0^{\pi} = \frac{4\pi a^3}{3}$$

6. The region is one-eighth of a sphere in the first octant of radius a, centered at the origin.

$$\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{1}{3}\rho^3\cos^2\varphi\sin\varphi\right]_0^a d\theta\,d\varphi$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{3}a^3\cos^2\varphi\sin\varphi\,d\theta\,d\varphi$$
$$= \int_0^{\pi/2}\frac{\pi}{6}a^3\cos^2\varphi\sin\varphi\,d\varphi$$
$$= \left[-\frac{\pi}{6}a^3\cos\varphi\right]_0^{\pi/2} = \frac{\pi a^3}{18}$$

7. $\displaystyle\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta = 8\pi \approx 25.1327$

8. $\displaystyle\int_0^{2\pi}\int_0^2\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = \left(\frac{2}{3}\right)\pi(27 - 5^{3/2})$

≈ 33.1326

9. $$V = \int_0^{2\pi}\int_0^3\int_4^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^3 [rz]_4^{\sqrt{25-r^2}}\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^3\left[r\sqrt{25-r^2} - 4r\right]dr\,d\theta$$
$$= \int_0^{2\pi}\left[-\frac{1}{3}\left(25-r^2\right)^{3/2} - 2r^2\right]_0^3 d\theta$$
$$= \int_0^{2\pi}\frac{7}{3}\,d\theta = \frac{14\pi}{3}$$

10. $$V = \int_0^{2\pi}\int_0^4\int_0^{4+r\sin\theta} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^4 [rz]_0^{4+r\sin\theta}\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^4\left[4r + r^2\sin\theta\right]dr\,d\theta$$
$$= \int_0^{2\pi}\left[2r^2 + \frac{1}{3}r^3\sin\theta\right]_0^4 d\theta$$
$$= \int_0^{2\pi}\left[32 + \frac{64}{3}\sin\theta\right]_0^4 d\theta$$
$$= 64\pi$$

11. $$\int_0^{2\pi}\int_0^2\int_{r^2/4}^{\sqrt{5-r^2}} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2\left[r(5-r^2)^{1/2} - \frac{r^3}{4}\right]dr\,d\theta$$
$$= \int_0^{2\pi}\frac{5^{3/2}-4}{3}\,d\theta = \frac{2\pi(5^{3/2}-4)}{3} \approx 15.0385$$

12. $2\cdot\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2\sin\theta\cos\theta} r\,dz\,dr\,d\theta = \frac{4}{3}$

13. Let $\delta(x, y, z) = 1$.
(See write-up of Problem 26, Section 13.7.)

$$m = \int_0^{2\pi}\int_0^2\int_{r^2}^{12-2r^2} r\,dz\,dr\,d\theta = 24\pi$$

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_{r^2}^{12-2r^2} zr\,dz\,dr\,d\theta = 128\pi$$

$$\bar{z} = \frac{16}{3}$$

$\bar{x} = \bar{y} = 0$ (by symmetry)

14. Let $\delta(x, y, z) = 1$.
(See comment at beginning of write-up of Problem 26 of the previous section.)

$$m = \int_0^{2\pi}\int_1^2\int_0^{12-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r(12-r^2)dr\,d\theta$$

$$= \int_0^{2\pi}\left[6r^2 - \frac{r^4}{4}\right]_1^2 d\theta = \int_0^{2\pi}\left(\frac{57}{4}\right)d\theta = \frac{57\pi}{2}$$

$$M_{xy} = \int_0^{2\pi}\int_1^2\int_0^{12-r^2} z\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \frac{(12-r^2)^2(-2r)}{-4}dr\,d\theta$$

$$= \int_0^{2\pi}\frac{11^3-8^3}{12}d\theta = \frac{273\pi}{2}$$

Therefore, $\bar{z} = \frac{\frac{273\pi}{2}}{\frac{57\pi}{2}} = \frac{91}{19} \approx 4.7895.$

$\bar{x} = \bar{y} = 0$ (by symmetry)

15. Let $\delta(x, y, z) = k\rho$

$$m = \int_0^{\pi}\int_0^{2\pi}\int_a^b k\rho\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = k\pi(b^4-a^4)$$

16. $8\int_{\pi/6}^{\pi/2}\int_0^{\pi/2}\int_{a\csc\phi}^{2a} k\rho^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$= \left(\frac{56}{5}\right)k\pi a^5\sqrt{3}$$

17. Assume that the hemisphere lies above the xy-plane.
Let $\delta(x, y, z) = \rho$.
(Letting $k = 1$ - - see comment at the beginning of the write-up of Problem 26 of the previous section.)

$$m = \int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^3\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\frac{a^4\sin\phi}{4}d\theta\,d\phi$$

$$= \int_0^{\pi/2}\frac{\pi a^4\sin\phi}{2}d\phi = \frac{\pi a^4}{2}$$

$$M_{xy} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^4\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi$$

($z = \rho\cos\phi$)

$$= \int_0^{\pi/2}\int_0^{2\pi}\frac{a^5\sin 2\phi}{10}d\theta\,d\phi$$

$$= \int_0^{\pi/2}\frac{\pi a^5\sin 2\phi}{5}d\phi = \left(\frac{\pi}{5}\right)a^5$$

$$\bar{z} = \frac{\frac{\pi a^5}{5}}{\frac{\pi a^4}{2}} = \frac{2}{5}a;\ \bar{x} = \bar{y} = 0 \text{ (by symmetry)}$$

18. Assume that the hemisphere lies above the xy-plane.
$\delta(x, y, z) = \rho\sin\phi$ (letting $k = 1$)

$$m = \int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^3\sin^2\phi\,d\rho\,d\theta\,d\phi = \left(\frac{1}{8}\right)\pi^2a^4$$

$$M_{xy} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^4\sin^2\phi\cos\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\frac{2}{15}\right)\pi a^5$$

$$\bar{z} = \frac{16a}{15\pi} \approx 0.3395a$$

$\bar{x} = \bar{y} = 0$ (by symmetry)

19. $I_z = \iint_S (x^2+y^2)k(x^2+y^2)^{1/2}dV$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^a k\rho^5\sin^4\phi\,d\rho\,d\theta\,d\phi = \left(\frac{k}{16}\right)\pi^2a^6$$

20. Volume $= \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^4 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$= \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\frac{64\sin\phi}{3}d\theta\,d\phi = \int_{\pi/4}^{\pi/2}\frac{128\pi\sin\phi}{3}d\phi$$

$$= \frac{64\sqrt{2}\pi}{3} \approx 94.7815$$

21. $\int_0^{\pi}\int_0^{\pi/6}\int_0^1 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{\pi}{9} \approx 0.3491$

22. $\int_0^{\pi}\int_0^{2\pi}\int_0^3 \rho^3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 486\pi \approx 1526.81$

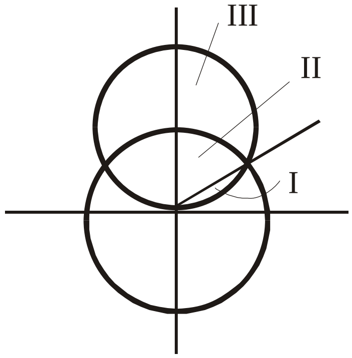
23. Volume $= \int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$

$$= \int_0^{\pi}\int_0^{\sin\theta} r(r\sin\theta - r^2)dr\,d\theta = \int_0^{\pi}\frac{\sin^4\theta}{12}d\theta$$

$$= \frac{1}{48}\int_0^{\pi}\left[1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right]d\theta$$

$$= \frac{\pi}{32} \approx 0.0982$$

24. Consider the following diagram:



Method 1: (direct, requires 2 integrals)
$V = \mathrm{I} + \mathrm{II}$

$$= \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^{2\sqrt{2}\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi + \int_0^{\pi/4}\int_0^{2\pi}\int_0^{2} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \frac{2\sqrt{2}\pi}{3} + \frac{8\left(2-\sqrt{2}\right)\pi}{3} = \frac{2\left(8-3\sqrt{2}\right)\pi}{3} \approx 7.8694$$

Method 2: (indirect, requires 1 integral)
$V = \text{upper sphere volume} - \mathrm{III}$

$$= \frac{8\sqrt{2}\pi}{3} - \int_0^{\pi/4}\int_0^{2\pi}\int_2^{2\sqrt{2}\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \frac{8\sqrt{2}\pi}{3} - \frac{2\left(7\sqrt{2}-8\right)\pi}{3} = \frac{2\left(8-3\sqrt{2}\right)\pi}{3} \approx 7.8694$$

25. a. Position the ball with its center at the origin. The distance of (x, y, z) from the origin is $(x^2+y^2+z^2)^{1/2} = \rho$.

$$\iiint_S (x^2+y^2+z^2)^{1/2} dV = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \rho(\sin\theta\rho^2) d\rho \, d\theta \, d\phi = \pi a^4$$

Then the average distance from the center is $\dfrac{\pi a^4}{\left[\left(\frac{4}{3}\right)\pi a^3\right]} = \dfrac{3a}{4}$.

b. Position the ball with its center at the origin and consider the diameter along the z-axis. The distance of (x, y, z) from the z-axis is $(x^2+y^2)^{1/2} = \rho\sin\phi$.

$$\iiint_S (x^2+y^2)^{1/2} dV = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} (\rho\sin\phi)(\rho^2\sin\theta) d\rho \, d\theta \, d\phi = \frac{a^4\pi^2}{4}$$

Then the average distance from a diameter is $\dfrac{\left[\frac{a^4\pi^2}{4}\right]}{\left[\left(\frac{4}{3}\right)\pi a^3\right]} = \dfrac{3\pi a}{16}$.

c. Position the sphere above and tangent to the xy-plane at the origin and consider the point on the boundary to be the origin. The equation of the sphere is $\rho = 2a\cos\phi$, and the distance of (x, y, z) from the origin is ρ.

$$\iiint_S (x^2+y^2+z^2)^{1/2} dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^{2a\cos\phi} \rho(\rho^2\sin\theta) d\rho \, d\theta \, d\phi = \frac{8\pi a^4}{5}$$

Then the average distance from the origin is $\dfrac{\left[\frac{8\pi a^4}{5}\right]}{\left[\left(\frac{4}{3}\right)\pi a^3\right]} = \dfrac{6a}{5}$.

26. Average value of $ax + by + cz + d$ on S is

$$\frac{\iiint_S (ax+by+cz+d)dV}{\iiint_S dV} = \frac{a\iiint_S kx\,dV + b\iiint_S ky\,dV + c\iiint_S kz\,dV + d\iiint_S k\,dV}{\iiint_S k\,dV}$$

$$= \frac{aM_{yz} + bM_{xz} + cM_{xy} + dm}{m} = a\bar{x} + b\bar{y} + c\bar{z} + d = f(\bar{x}, \bar{y}, \bar{z}).$$

27. a. $M_{yz} = \iiint_S kx\,dV \ = 4k\int_0^{\pi/2}\int_0^{\alpha}\int_0^{a}(\rho\sin\phi\cos\theta)(\rho^2\sin\phi)d\rho\,d\theta\,d\phi = ka^4\pi\frac{(\sin\alpha)}{4}$

$m = \iiint_S k\,dV = 4k\int_0^{\pi/2}\int_0^{\alpha}\int_0^{a}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{4a^3k\alpha}{3}$

Therefore, $\bar{x} = \frac{\left[\frac{ka^4\pi(\sin\alpha)}{4}\right]}{\left[\frac{4a^3k\alpha}{3}\right]} = \frac{3a\pi(\sin\alpha)}{16\alpha}.$

b. $\frac{3\pi a}{16}$ (See Problem 25b.)

28. a. $I_z = \iiint_S k[(x^2+y^2)^{1/2}]^2\,dV$

$= 8k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a}(\rho\sin\phi)^2(\rho^2\sin\phi)d\rho\,d\theta\,d\phi = \frac{8a^5\pi k}{15} = \frac{2a^2m}{5}$ (since $m = \left(\frac{4}{3}\right)\pi a^3 k$)

b. $I' = I + d^2m = \frac{2a^2m}{5} + a^2m = \frac{7a^2m}{5}$

c. $I = 2\left[\frac{2a^2m}{5} + (a+b)^2 m\right] = \frac{2m(7a^2+10ab+5b^2)}{5}$

29. Let m_1 and m_2 be the masses of the left and right balls, respectively. Then $m_1 = \frac{4}{3}\pi a^3 k$ and $m_2 = \frac{4}{3}\pi a^3(ck)$, so

$m_2 = cm_1.$

$\bar{y} = \frac{m_1(-a-b)+m_2(a+b)}{m_1+m_2}$

$= \frac{m_1(-a-b)+cm_1(a+b)}{m_1+cm_1} = \frac{-a-b+c(a+b)}{1+c}$

$= \frac{(a+b)(-1+c)}{1+c} = \frac{c-1}{c+1}(a+b)$

(Analogue) $\bar{y} = \frac{m_1\bar{y}_1 + m_2\bar{y}_2}{m_1+m_2} = \bar{y}_1\frac{m_1}{m_1+m_2} + \bar{y}_2\frac{m_2}{m_1+m_2}$

13.9 Concepts Review

1. u-curve; v-curve

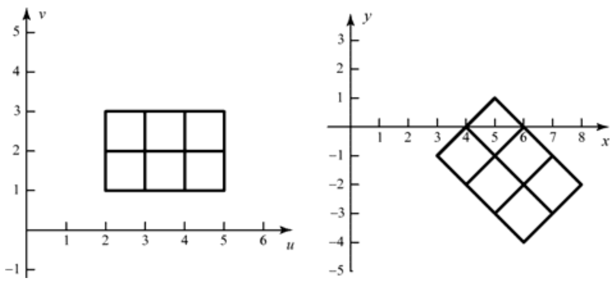
2. the integrand; the differential of $dx\,dy$; the region of integration

3. Jacobian

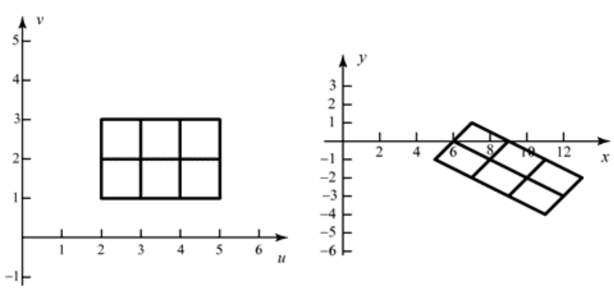
4. $\left|J(u,v)\right|$

Problem Set 13.9

1.



2.



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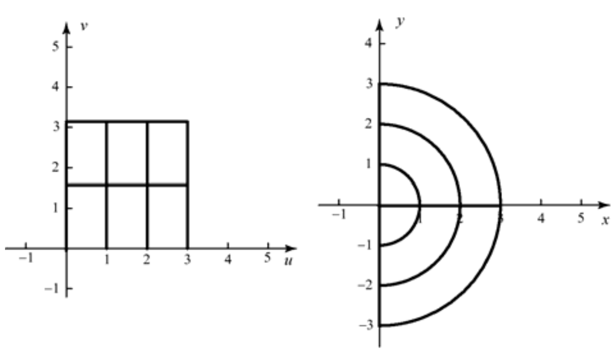
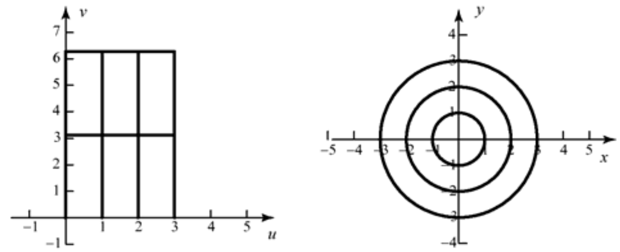
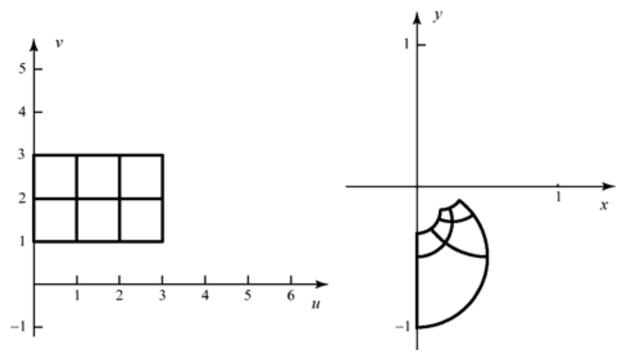


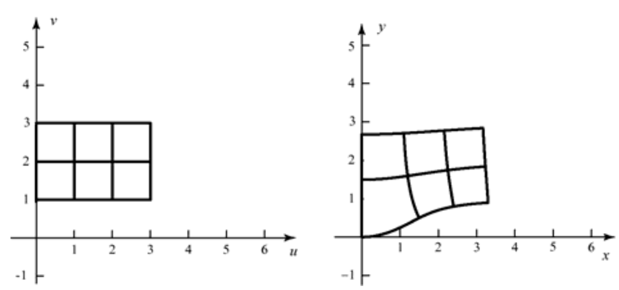
4.



5.



6.



7. $x = u + 2v; \quad y = u - 2v$

$G(0,0) = (0,0); \quad G(2,0) = (2,2)$
$G(2,1) = (4,0); \quad G(0,1) = (2,-2)$

The image is the square with corners $(0,0)$, $(2,2)$, $(4,0)$, and $(2,-2)$. The Jacobian is

$J = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4$.

8. $x = 2u + 3v; \quad y = u - v$

$G(0,0) = (0,0); \quad G(3,0) = (6,3)$
$G(3,1) = (9,2); \quad G(0,1) = (3,-1)$

The image is the parallelogram with vertices $(0,0)$, $(6,3)$, $(9,2)$, and $(3,-1)$. The Jacobian is $J = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5$

9. $x = u^2 + v^2; \quad y = v$

$G(0,0) = (0,0); \quad G(1,0) = (1,0)$
$G(1,1) = (2,1); \quad G(0,1) = (1,1)$

Solving for u and v gives
$u = \sqrt{x - y^2}$ and $v = y$
The $u = 0$ curve is $0 = \sqrt{x - y^2} \Rightarrow x = y^2$.
The $u = 1$ curve is $1 = \sqrt{x - y^2} \Rightarrow x = y^2 + 1$.
The $v = 0$ curve is $y = 0$, and the $v = 1$ curve is $y = 1$. The image is the set of (x, y) that satisfy $y^2 \le x \le y^2 + 1, 0 \le y \le 1$. The Jacobian is
$J = \begin{vmatrix} 2u & 2v \\ 0 & 1 \end{vmatrix} = 2u$

10. $x = u; \quad y = u^2 - v^2$

Solving for u and v gives
$u = x$ and $v = \sqrt{x^2 - y}$
The $u = 0$ curve is $x = 0$, and the $u = 3$ curve is $x = 3$. The $v = 0$ curve is
$0 = \sqrt{x^2 - y} \Rightarrow y = x^2$ and the $v = 1$ curve is
$1 = \sqrt{x^2 - y} \Rightarrow y = x^2 - 1$. The image is therefore the set of (x, y) that satisfy
$x^2 - 1 \le y \le x^2; 0 \le x \le 3$. The Jacobian is
$J = \begin{vmatrix} 1 & 0 \\ 2u & -2v \end{vmatrix} = -2v$.

11. $u = x + 2y; \quad v = x - 2y$ Solving for x and y gives $x = u/2 + v/2; \quad y = u/4 - v/4$. The Jacobian is $J = \begin{vmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{vmatrix} = -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}$.

12. $u = 2x - 3y; \quad v = 3x - 2y$ Solving for x and y gives $x = -\frac{2}{5}u + \frac{3}{5}v; \quad y = -\frac{3}{5}u + \frac{2}{5}v$. The Jacobian is $J = \begin{vmatrix} -\frac{2}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{vmatrix} = -\frac{4}{25} + \frac{9}{25} = \frac{1}{5}$.

13. $u = x^2 + y^2; \quad v = x$. Solving for x and y gives $x = v; \quad y = \sqrt{u - v^2}$. The Jacobian is
$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{2\sqrt{u - v^2}} & \frac{-v}{\sqrt{u - v^2}} \end{vmatrix} = -\frac{1}{2\sqrt{u - v^2}}$

14. $u = x^2 - y^2; \quad v = x + y$. Solving for x and y gives $x = \frac{v^2 + u}{2v}; \quad y = \frac{v^2 - u}{2v}$. The Jacobian is $J = \begin{vmatrix} \frac{1}{2v} & \frac{1}{2} - \frac{u}{2v^2} \\ -\frac{1}{2v} & \frac{1}{2} + \frac{u}{2v^2} \end{vmatrix} = \frac{1}{2v}$

15. $u = xy; \quad v = x$. Solving for x and y gives $x = v; \quad y = u/v$. The Jacobian is
$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}$.

16. $u = x^2; \quad v = xy$. Solving for x and y gives $x = \sqrt{u}; \quad y = \frac{v}{\sqrt{u}}$. The Jacobian is
$J = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ \frac{-v}{2u^{3/2}} & \frac{1}{\sqrt{u}} \end{vmatrix} = \frac{1}{2u}$.

17. Let $u = x + y,\ v = x - y$. Solving for x and y gives $x = u/2 + v/2$ and $y = u/2 - v/2$. The Jacobian is $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.

The region of integration gets transformed to the triangle in the *uv*-plane with vertices (1,1), (4,4), and (7,1). The integral in the *uv*-plane is more easily done by holding *v* fixed and integrating *u*. Thus,

$$\iint_R \ln\frac{x+y}{x-y}\,dA = \int_1^4 \int_v^{8-v} \ln\frac{u}{v}\left|\frac{1}{2}\right| du\,dv$$

$$= \frac{1}{2}\int_1^4 \left[-u + u\ln\frac{u}{v}\right]_v^{8-v} dv$$

$$= \frac{1}{2}\int_1^4 \left[-8 + 2v + (8-v)\ln\frac{8-v}{v}\right] dv$$

$$= \frac{1}{2}\left[3 - 64\ln 4 + \frac{49}{2}\ln 7 + 16\ln 16\right]$$

$$\approx 3.15669$$

18. Let $u = x + y,\ v = x - y$. Solving for x and y gives $x = u/2 + v/2$ and $y = u/2 - v/2$. The Jacobian is $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.

The region of integration gets transformed to the triangle in the *uv*-plane with vertices (1,1), (4,4), and (7,1). The integral in the *uv*-plane is more easily done by holding *v* fixed and integrating *u*. Thus, with the help of a CAS for the outer integral, we have

$$\iint_R \sqrt{\frac{x+y}{x-y}}\,dA = \int_1^4 \int_v^{8-v} \sqrt{\frac{u}{v}}\left|\frac{1}{2}\right| du\,dv$$

$$= \int_1^4 \left[\frac{1}{3}\frac{u^{3/2}}{v^{1/2}}\right]_v^{8-v} dv$$

$$= \frac{1}{3}\int_1^4 \left[\frac{(8-v)^{3/2}}{v^{1/2}} - \frac{v^{3/2}}{v^{1/2}}\right] dv$$

$$= \frac{49}{6} - \frac{19\sqrt{7}}{6} - 4\pi + 16\tan^{-1}\sqrt{7}$$

$$\approx 6.57295$$

19. Let $u = 2x - y$ and $v = y$. Then $x = u/2 + v/2$ and $y = v$. The Jacobian is $J = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$. Thus,

$$\iint_R \sin\left(\pi(2x-y)\right)\cos\left(\pi(y-2x)\right)dA$$

$$= \int_0^3 \int_{v+2}^{8-v} \sin\left(\pi u\right)\cos\left(\pi(-u)\right)\left|\frac{1}{2}\right| du\,dv$$

$$= \frac{1}{2}\int_0^3 \int_{v+2}^{8-v} \frac{1}{2}2\sin\left(\pi u\right)\cos\left(\pi u\right) du\,dv$$

$$= \frac{1}{4}\int_0^3 \int_{v+2}^{8-v} \sin\left(2\pi u\right) du\,dv$$

$$= \frac{1}{4}\int_0^3 \left[-\frac{\cos 2\pi u}{2\pi}\right]_{v+2}^{8-v} dv$$

$$= \frac{1}{8\pi}\int_0^3 \left[\cos\left(2\pi(v+2)\right) - \cos\left(2\pi(8-v)\right)\right] dv$$

$$= \frac{1}{8\pi}\int_0^3 \left[\cos\left(2\pi v + 4\pi\right) - \cos\left(16\pi - 2\pi v\right)\right] dv$$

$$= \frac{1}{8\pi}\int_0^3 \left[\cos\left(2\pi v\right) - \cos\left(2\pi v\right)\right] dv = 0$$

20. Let $u = 2x - y$ and $v = y$. Then $x = u/2 + v/2$ and $y = v$. The Jacobian is $J = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$. Thus,

$$\iint_R (2x-y)\cos(y-2x)\,dA$$

$$= \int_0^3 \int_{v+2}^{8-v} u\cos\left(-u\right)\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_0^3 \left[u\sin u + \cos u\right]_{v+2}^{8-v} dv$$

$$= \frac{1}{2}(-2\cos 2 + 10\cos 5 - 8\cos 8$$
$$+ 2\sin 2 - 4\sin 5 + 2\sin 8)$$

$$\approx 6.23296$$

21. The transformation to spherical coordinates is

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - 0 \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta \end{vmatrix} + (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(-\rho^2 \sin\phi\sin\theta\cos\phi\sin\theta - \rho^2 \cos\phi\sin\phi\cos^2\theta\right) - \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= -\rho^2 \cos^2\phi \sin\phi\left(\sin^2\theta + \cos^2\theta\right) - \rho^2 \sin^3\phi$$

$$= -\rho^2 \sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$= -\rho^2 \sin\phi$$

22. Let $x = ua,\ y = vb,\ z = wc$. Then the Jacobian is $J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and the region of integration becomes the sphere with radius 1 centered at the origin. Thus,

$$V = \iiint\limits_{\text{ellipsoid}} 1\,dV = \iiint\limits_{\text{ellipsoid}} 1\,dz\,dy\,dx = \iiint\limits_{\substack{\text{unit}\\\text{sphere}}} 1\,abc\,dw\,dv\,du = \frac{4}{3}\pi abc$$

The moment of inertia about the z-axis is

$$M_z = \iiint\limits_{\text{ellipsoid}} \left(x^2 + y^2\right) dz\,dy\,dx$$

$$= \iiint\limits_{\substack{\text{unit}\\\text{sphere}}} \left(a^2u^2 + b^2v^2\right) abc\,dw\,dv\,du$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \left(a^2\rho^2 \sin^2\phi\cos^2\theta + b^2\rho^2 \sin^2\phi\sin^2\theta\right)\rho\sin^2\phi\, abc\, d\rho\, d\phi\, d\theta$$

$$= \frac{abc}{4}\int_0^{2\pi}\int_0^{\pi} \left(a^2\cos^2\theta\sin^4\phi + b^2\sin^2\theta\sin^4\phi\right) d\phi\, d\theta$$

$$= \frac{abc}{4}\int_0^{2\pi}\int_0^{\pi} \sin^4\phi\left(a^2\cos^2\theta + b^2\sin^2\theta\right) d\phi\, d\theta$$

$$= \frac{abc}{32}\int_0^{2\pi} \left(3a^2\pi\cos^2\theta + 3b^2\sin^2\theta\right) d\theta$$

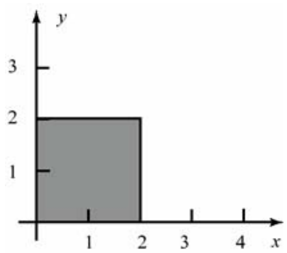
$$= \frac{3abc\pi^2}{32}\left(a^2 + b^2\right)$$

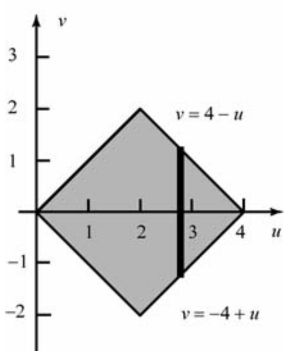
23. Let $X = x(U,V)$ and $Y = y(U,V)$. If R is a region in the xy-plane with preimage S in the uv-plane, then
$P\big((X,Y)\in R\big) = P\big((U,V)\in S\big)$

Writing each of these as a double integral over the appropriate PDF and region gives
$$\iint_R f(x,y)\,dy\,dx = \iint_S g(u,v)\,dv\,du$$
Now, make the change of variable $x = x(u,v)$ and $y = y(u,v)$ in the integral on the left. Therefore,
$$\iint_S g(u,v)\,dv\,du = \iint_R f(x,y)\,dy\,dx = \iint_S f\big(x(u,v),y(u,v)\big)\left|J(u,v)\right|dv\,du$$
Thus probabilities involving (U,V) can be obtained by integrating $f\big(x(u,v),y(u,v)\big)\left|J(u,v)\right|$. Thus, $f\big(x(u,v),y(u,v)\big)\left|J(u,v)\right|$ is the joint PDF of (U,V).

24. Let $u = x+y,\ v = x-y$.

a. Solving for x and y gives $x = u/2+v/2$ and $y = u/2-v/2$. The Jacobian is
$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$$
The joint PDF for (U,V) is therefore
$$g(u,v) = f\left(\frac{u}{2}+\frac{v}{2},\frac{u}{2}-\frac{v}{2}\right)$$
$$= \begin{cases} \frac{1}{8}, & \text{if } 0\le (u+v)/2\le 2, 0\le (u-v)/2\le 2 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{1}{8}, & \text{if } 0\le u+v\le 4, 0\le u-v\le 4 \\ 0, & \text{otherwise} \end{cases}$$





To find the marginal of U, we fix a u and integrate over all possible v. For $0\le v\le 2$,
$g_U(u) = \int_{-u}^{u}\frac{1}{8}dv = \frac{u}{4}$ and for $2 < u \le 4$, $g_U(u) = \int_{-4+u}^{4-u}\frac{1}{8}dv = \frac{8-2u}{8} = 1-\frac{u}{4}$.

Therefore,
$$g_U(u) = \begin{cases} u/4, & \text{if } 0\le u\le 2 \\ 1-u/4, & \text{if } 2<u\le 4 \\ 0 & \text{otherwise} \end{cases}$$

25. **a.** Let $u = x + y,\ v = x$. Solving for x and y gives $x = v,\ y = u - v$. The Jacobian is

$J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$

Thus,

$g(u,v) = f(v, u-v)|-1|$

$= \begin{cases} e^{-v-(u-v)}, & \text{if } 0 \le v, 0 \le u - v \\ 0, & \text{otherwise} \end{cases}$

$= \begin{cases} e^{-u}, & \text{if } 0 \le v \le u \\ 0, & \text{otherwise} \end{cases}$

b. The marginal PDF for U is obtained by integrating over all possible v for a fixed u. If $u \ge 0$, then

$g_U(u) = \int_0^u e^{-u}\, dv = ue^{-u}$

Thus, $g_U(u) = \begin{cases} ue^{-u}, & \text{if } 0 \le u \\ 0, & \text{otherwise} \end{cases}$

13.10 Chapter Review

Concepts Test

1. True: Use result of Problem 33, Section 13.2, and then change dummy variable *y* to dummy variable *x*.

2. False: Let *f*(*x*, *y*) = *x*. 1st integral is $\frac{1}{3}$; 2nd is $\frac{1}{6}$.

3. True: Inside integral is 0 since $\sin(x^3y^3)$ is an odd function in *x*.

4. True: Use Problem 33, Section 13.2. Each integrand, e^{x^2} and e^{2y^2}, determines and even function.

5. True: It is less than or equal to $\int_1^2\int_0^2 1\,dx\,dy$ which equals 2.

6. True: $f(x, y) \ge \frac{f(x_0, y_0)}{2}$ in some neighborhood *N* of (x_0, y_0) due to the continuity. Then

$\iint_R f(x, y)dA \ge \iint_N \left(\frac{1}{2}\right) f(x_0, y_0)dA$

$= \left(\frac{1}{2}\right) f(x_0, y_0)(\text{Area } N) > 0.$

7. False: Let $f(x, y) = x$, $g(x, y) = x^2$, R = {(*x*, *y*): *x* in [0, 2], *y* in [0, 1]}. The inequality holds for the integrals but *f*(0.5, 0) > *g*(0.5, 0).

8. False: Let *f*(0, 0) = 1, *f*(*x*, *y*) = 0 elsewhere for $x^2 + y^2 \le 1$.

9. True: See the write-up of Problem 26, Section 13.7.

10. True: For each *x*, the density increases as *y* increases, so the top half of *R* is more dense than the bottom half. For each *y*, the density decreases as the *x* increases, so the right half of *R* is less dense than the left half.

11. True: The integral is the volume between concentric spheres of radii 4 and 1. That volume is 84π.

12. True: See Section 13.6.
A(*T*) = (Area of base)(sec 30°)

$= \pi(1)^2\left(\frac{2}{\sqrt{3}}\right) = \frac{2\sqrt{3}\pi}{3}$

13. False: There are 6.

14. False: The integrand should be *r*.

15. True: $\sqrt{f_x^2 + f_y^2 + 1} \le 9 = 3$

16. True: $J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

17. False: $J(u,v) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

Sample Test Problems

1. $\int_0^1 \left(\frac{1}{2}\right)(x^2 - x^3)dx = \frac{1}{24} \approx 0.0417$

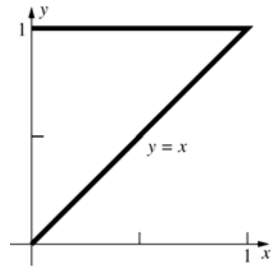
2. $\int_{-2}^2 0\,dy = 0$ (Integrand determines an odd function in *x*.)

3. $\int_0^{\pi/2}\left[\frac{r^2\cos\theta}{2}\right]_{r=0}^{2\sin\theta} d\theta = \int_0^{\pi/2} 2\sin^2\theta\cos\theta\,d\theta$

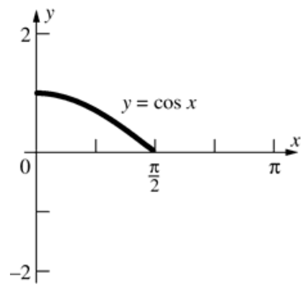
$= \left[\frac{2\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{2}{3}$

4. $\int_1^2\int_3^x \left(\frac{\pi}{3}\right)dy\,dx = \int_1^2 \left(\frac{\pi}{3}\right)(x-3)dx = -\frac{\pi}{2}$

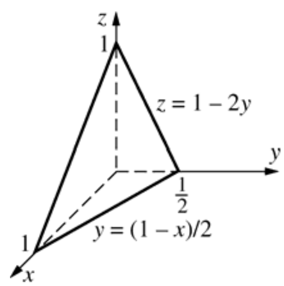
5. $\int_0^1\int_0^y f(x, y)dx\,dy$



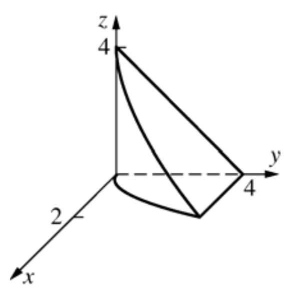
6. $\int_0^{\pi/2}\int_0^{\cos x} f(x, y)dy\,dx$



7. $\int_0^{1/2}\int_0^{1-2y}\int_0^{1-2y-z} f(x, y, z)dx\,dz\,dy$



8. $\int_0^4\int_0^{4-z}\int_0^{\sqrt{y}} f(x, y, z)dx\,dy\,dz$

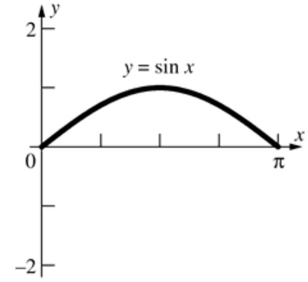


9. **a.** $8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$

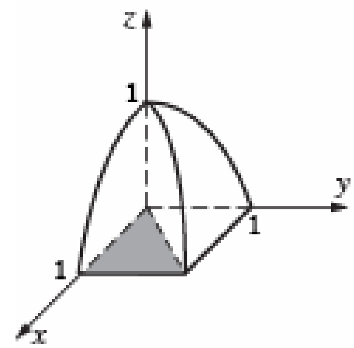
b. $8\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

c. $8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

10. $\int_0^\pi\int_0^{\sin x}(x+y)dy\,dx = \frac{5\pi}{4} \approx 3.9270$



11. $8\int_0^1\int_0^x\int_0^{1-x^2} z^2dz\,dy\,dx = \frac{1}{3}$



12. $\int_0^{2\pi}\int_2^3(r^{-2})r\,dr\,d\theta = \int_0^{2\pi}[\ln r]_2^3 d\theta$

$= \int_0^{2\pi}\ln\left(\frac{3}{2}\right)d\theta = 2\pi\ln\left(\frac{3}{2}\right) \approx 2.5476$

13. $m = \int_0^2\int_1^3 xy^2dx\,dy = \frac{32}{3}$

$M_x = \int_0^2\int_1^3 xy^3dx\,dy = 16$

$M_y = \int_0^2\int_1^3 x^2y^2dx\,dy = \frac{208}{9}$

$(\overline{x}, \overline{y}) = \left(\frac{13}{6}, \frac{3}{2}\right)$

14. $I_x = \int_0^2\int_1^3 xy^4dx\,dy = \frac{128}{5} = 25.6$

15. $z = f(x, y) = (9-y^2)^{1/2}$; $f_x(x, y) = 0$;

$f_y(x, y) = -y(9-y^2)^{-1/2}$

Area $= \int_0^3\int_{y/3}^y \sqrt{y^2(9-y^2)^{-1}+1}\,dx\,dy$

$= \int_0^3\int_{y/3}^y 3(9-y^2)^{-1/2}dx\,dy$

$= \int_0^3(9-y^2)^{-1/2}(2y)dy = [-2(9-y^2)^{1/2}]_0^3 = 6$

16. a. $\int_0^{\pi/2}\int_0^2\int_0^3 r\,r\,dr\,dz\,d\theta = 9\pi \approx 28.2743$

b. $\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} z(4-r^2)^{1/2}\,r\,dz\,dr\,d\theta$

$=\left(\frac{8}{5}\right)\pi \approx 5.0265$

17. $\delta(x, y, z) = k\rho$

$m=\int_0^{\pi}\int_0^{2\pi}\int_1^3 k\rho\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = 80\pi k$

18. $m=\iint_R 1\,dA=\int_0^{2\pi}\int_0^{4(1+\sin\theta)} r\,dr\,d\theta$

$=8\int_0^{2\pi}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta = 24\pi$

$M_x=\iint_R y\,dA=\int_0^{2\pi}\int_0^{4(1+\sin\theta)}(r\sin\theta)r\,dr\,d\theta$

$=80\pi$

$\bar{y}=\frac{80\pi}{24\pi}=\frac{10}{3}; \bar{x}=0$ (by symmetry)

19. $m=\int_0^a\int_0^{(b/a)(a-x)}\int_0^{(c/ab)(ab-bx-ay)} kx\,dz\,dy\,dx$

$=\left(\frac{k}{24}\right)a^2bc$

20. $\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r\,dz\,dr\,d\theta=\frac{3\pi}{2}\approx 4.7124$

21. Let $x=\frac{u+v}{2}$ and $y=\frac{v-u}{2}$. Then we have,

$\sin(x-y)\cos(x+y)=\sin u\cos v$ and

$J(u,v)=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{1}{4}-\left(-\frac{1}{4}\right)=\frac{1}{2}.$

Thus,

$\iint_R \sin(x-y)\cos(x+y)\,dxdy$

$=\int_0^{\pi}\int_0^{\pi}\frac{1}{2}\sin u\cos v\,du\,dv$

$=\frac{1}{2}\int_0^{\pi}\cos v\left[-\cos u\right]_0^{\pi}dv$

$=\int_0^{\pi}\cos v\,dv=0$

Review and Preview Problems

1. Answers may vary. One solution is
$x=3\cos t,\quad y=3\sin t,\qquad 0\le t<2\pi$
Then: $x^2+y^2=9\cos^2 t+9\sin^2 t=9$ as required.

2. Answers may vary. One solution is
$x=\cos t+2,\quad y=\sin t+1,\qquad 0\le t<2\pi$
Then: $(x-2)^2+(y-1)^2=\cos^2 t+\sin^2 t=1$ as required.

3. Answers may vary. One solution for the circle is
$x=2\cos t,\quad y=2\sin t$
To have the semicircle where $y>0$, we need $\sin t>0$, so we restrict the domain of t to $0<t<\pi$.

4. Answers may vary. Consider
$x=a\cos(-t),\quad y=a\sin(-t),\qquad 0\le t\le\pi$**;** this is a semicircle.

a. Since $\sin(-t)=-\sin t$, and since $\sin t\ge 0$ on $[0,\pi]$, $y\le 0$ on $[0,\pi]$.

b. As t goes from 0 to π, $-t$ goes from 0 to $-\pi$ so the orientation is clockwise.

5. Answers may vary. One solution is
$x=-2+5t,\ y=2$ for $t\in[0,1]$.

6. Note that $x+y=9$, so a simple parameterization is to let one variable be t and the other be $9-t$. Since we are restricted to the first quadrant, we must have $t>0$ and $9-t>0$; hence the domain is $t\in(0,9)$. Finally, since orientation is to be down and to the right, we want y to decrease and x to increase as t increases. Thus we use $x=t$ and $y=9-t$.

7. Note that $x+y=9$, so a simple parameterization is to let one variable be t and the other be $9-t$. Since we are restricted to the first quadrant, we must have $t>0$ and $9-t>0$; hence the domain is $t\in(0,9)$. Finally, since orientation is to be up and to the left, we want y to increase and x to decrease as t increases. Thus we use $x=9-t$ and $y=t$.

8. Since we are restricting the parabola to the points where $y>0$, a simple parameterization is

$x=t,\quad y=9-t^2,\qquad t\in[-3,3]$

Note that the orientation is left to right.

9. Since we are restricting the parabola to the points where $y > 0$, a simple parameterization is

$x = -t, \quad y = 9 - t^2, \qquad t \in [-3, 3]$

Note that the orientation is right to left.

10. $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. Using the parameterization in problem 6,

$x = t,\ y = 9 - t,\ a = 0,\ b = 9,\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = -1$ and

so $L = \int_0^9 \sqrt{1^2 + 1^2}\, dt = \left[\sqrt{2}\, t\right]_0^9 = 9\sqrt{2}$.

(Note: this can be verified by finding the distance between the points $(0,9)$ and $(9,0)$).

11. $\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$. Now if

$f(x,y) = x\sin x + y\cos y$, then

$f_x(x,y) = x\cos x + \sin x, \quad f_y(x,y) = \cos y - y\sin y$

Thus:

$\nabla f(x,y) = (x\cos x + \sin x)\,\mathbf{i} + (\cos y - y\sin y)\,\mathbf{j}$

12. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$. Now if

$f(x,y) = xe^{-xy} + ye^{xy}$, then

$$\begin{aligned} f_x(x,y) &= (-yxe^{-xy} + e^{-xy}) + (y^2 e^{xy}) \\ &= (1 - xy)e^{-xy} + y^2 e^{xy} \end{aligned}$$

and

$$\begin{aligned} f_y(x,y) &= -(x^2 e^{-xy}) + (xye^{xy} + e^{xy}) \\ &= -x^2 e^{-xy} + (1 + xy)e^{xy} \end{aligned}$$

Thus,

$$\begin{aligned} \nabla f(x,y) = {}& [(1 - xy)e^{-xy} + y^2 e^{xy}]\,\mathbf{i} \\ & + [-x^2 e^{-xy} + (1 + xy)e^{xy}]\,\mathbf{j} \end{aligned}$$

13. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = x^2 + y^2 + z^2$, then

$f_x(x,y,z) = 2x,\ f_y(x,y,z) = 2y,\ f_z(x,y,z) = 2z$

so

$\nabla f(x,y,z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$

14. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = \frac{1}{x^2 + y^2 + z^2}$, then

$f_x = \frac{-2x}{(x^2 + y^2 + z^2)^2}, \quad f_y = \frac{-2y}{(x^2 + y^2 + z^2)^2},$

and $f_z = \frac{-2z}{(x^2 + y^2 + z^2)^2}$,

so

$$\nabla f(x,y,z) = \frac{-2x}{(x^2 + y^2 + z^2)^2}\,\mathbf{i} + \frac{-2y}{(x^2 + y^2 + z^2)^2}\,\mathbf{j} + \frac{-2z}{(x^2 + y^2 + z^2)^2}\,\mathbf{k}$$

15. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = xy + xz + yz$, then $f_x(x,y,z) = y + z$,

$f_y(x,y,z) = x + z$, and $f_z(x,y,z) = x + y$ so

$\nabla f(x,y,z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

16. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, then

$f_x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}},$

$f_y = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}},$ and

$f_z = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}},$

so

$$\nabla f(x,y,z) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

17.
$$\begin{aligned} \int_0^\pi \sin^2 t\, dt &= \int_0^\pi \frac{1 - \cos 2t}{2}\, dt \\ &= \frac{1}{2}\int_0^\pi (1 - \cos 2t)\, dt \\ &= \frac{1}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^\pi \\ &= \frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(2\cdot 0)\right)\right] \\ &= \frac{\pi}{2} \end{aligned}$$

18.
$$\begin{aligned} \int_0^\pi \sin t\cos t\, dt &= \int_0^\pi \tfrac{1}{2}\sin(2t)\, dt \\ &= \left[-\tfrac{1}{4}\cos(2t)\right]_0^\pi \\ &= -\frac{1}{4}\left(\cos(2\pi) - \cos(0)\right) \\ &= -\frac{1}{4}(1 - 1) = 0 \end{aligned}$$

19. $$\int_0^1\int_1^2 xy\;dy\,dx=\int_0^1\left[\int_1^2 xy\,dy\right]dx=\int_0^1\left[\frac{xy^2}{2}\right]_{y=1}^{y=2}dx$$
$$=\int_0^1\left(\frac{3x}{2}\right)dx=\left[\frac{3x^2}{4}\right]_0^1=\frac{3}{4}$$

20. $$\int_{-1}^1\int_1^4(x^2+2y)\;dy\,dx=\int_{-1}^1\left[\int_1^4(x^2+2y)\,dy\right]dx=$$
$$\int_{-1}^1\left[x^2y+y^2\right]_{y=1}^{y=4}dx=\int_{-1}^1\left(3x^2+15\right)dx=$$
$$\left[x^3+15x\right]_{-1}^1=16+16=32$$

21. $$\int_0^{2\pi}\int_1^2 r^2\,dr\,d\theta=\int_0^{2\pi}\left[\int_1^2 r^2\,dr\right]d\theta=\int_0^{2\pi}\left[\frac{r^3}{3}\right]_{r=1}^{r=2}d\theta=$$
$$\int_0^{2\pi}\frac{7}{3}d\theta=\frac{7}{3}\theta\Big|_0^{2\pi}=\frac{14\pi}{3}\approx 14.66$$

22. $$\int_0^{2\pi}\int_0^{\pi}\int_1^2\rho^2\;\sin\phi\,d\rho\,d\phi\,d\theta=$$
$$\int_0^{2\pi}\int_0^{\pi}\left[\int_1^2\rho^2\;\sin\phi\,d\rho\right]d\phi\,d\theta=$$
$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\sin\phi\right]_{\rho=1}^{\rho=2}d\phi\,d\theta=$$
$$\int_0^{2\pi}\left[\int_0^{\pi}\frac{7}{3}\sin\phi\,d\phi\right]d\theta=\int_0^{2\pi}\left[-\frac{7}{3}\cos\phi\right]_{\phi=0}^{\phi=\pi}d\theta=$$
$$\int_0^{2\pi}\frac{14}{3}d\theta=\frac{28\pi}{3}\approx 29.32$$

23. Note that
$$\int_0^{2\pi}\int_0^{\pi}\int_r^R\rho^2\;\sin\phi\,d\rho\,d\phi\,d\theta=$$
$$\int_0^{2\pi}\int_0^{\pi}\left[\int_r^R\rho^2\;\sin\phi\,d\rho\right]d\phi\,d\theta=$$
$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\sin\phi\right]_{\rho=r}^{\rho=R}d\phi\,d\theta=$$
$$\int_0^{2\pi}\left[\int_0^{\pi}\frac{1}{3}(R^3-r^3)\sin\phi\,d\phi\right]d\theta=$$
$$\int_0^{2\pi}\left[-\frac{1}{3}(R^3-r^3)\cos\phi\right]_{\phi=0}^{\phi=\pi}d\theta=$$
$$\int_0^{2\pi}\left[\frac{2}{3}(R^3-r^3)\right]d\theta=\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3$$
which we recognize as the difference between the volume of a sphere with radius R and the volume of a sphere with radius r. Thus the volume in problem 22 is that of a spherical shell with center at (0,0,0) and, in this case, outer radius = 2 and inner radius = 1 .

24. Let $f(x,y)=z=144-x^2-y^2$; then:
$f_x(x,y)=-2x \qquad f_y(x,y)=-2y$

We note that $z=36$ when $x^2+y^2=108$; thus the surface area we seek projects onto the circular region inside $S=\left\{(x,y)\mid x^2+y^2=108\right\}$

Hence $A=\iint_S\sqrt{4x^2+4y^2+1}\;dA$; or, converting to polar coordinates $(r^2=x^2+y^2$ and $r=6\sqrt{3}$ when $r^2=108)$,
$$A=\int_0^{2\pi}\int_0^{6\sqrt{3}}(\sqrt{4r^2+1})\;r\,dr\,d\theta=$$
$$\int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_{r=0}^{r=6\sqrt{3}}d\theta=$$
$$\frac{1}{12}\int_0^{2\pi}[433^{3/2}-1]\,d\theta\approx 1501.7\pi\approx 4717.7$$

25. This will be the unit vector, at the point $(3,4,12)$, in the direction of ∇F, where $F(x,y,z) = x^2 + y^2 + z^2$.

Now $\nabla F(x,y,z) = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ so that $\nabla F(3,4,12) = \langle 6,8,24\rangle$ and the unit vector in the same direction is

$$\frac{\nabla F}{\|\nabla F\|} = \left\langle \frac{6}{26}, \frac{8}{26}, \frac{24}{26} \right\rangle = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle.$$

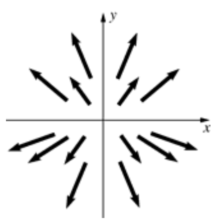
This agrees with our geometric intuition, since $x^2 + y^2 + z^2 = 169$ is the surface of a sphere with center at $O = (0,0,0)$ and radius = 13. Now the plane tangent to a sphere (center at (0,0,0) and radius r) at any point $P = (a,b,c)$ is perpendicular to the radius at that point; so it would follow that the vector $\overrightarrow{OP} = \langle a,b,c\rangle$ is perpendicular to the tangent plane and hence normal to the surface. The unit normal in the direction of $\overrightarrow{OP}$ is simply $\frac{1}{r}\langle a,b,c\rangle$, or in our case $\frac{1}{13}\langle 3,4,12\rangle$.

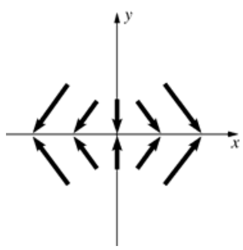
CHAPTER 14 Vector Calculus

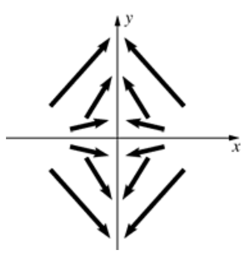
14.1 Concepts Review

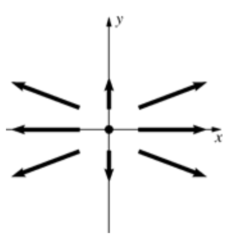
1. vector-valued function of three real variables or a vector field
2. gradient field
3. gravitational fields; electric fields
4. $\nabla \cdot \mathbf{F}, \nabla \times \mathbf{F}$

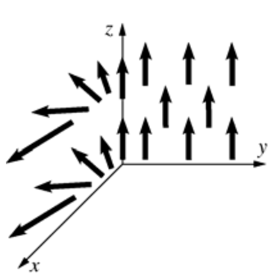
Problem Set 14.1

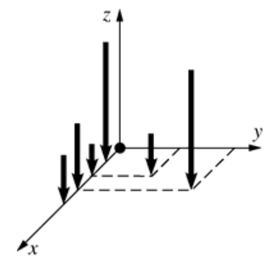
1. 

2. 

3. 

4. 

5. 

6. 

7. $\langle 2x-3y, -3x, 2\rangle$

8. $(\cos\ xyz)\langle yz, xz, xy\rangle$

9. $f(x,\ y,\ z) = \ln|x| + \ln|y| + \ln|z|$;
$\nabla f(x,\ y,\ z) = \left\langle x^{-1},\ y^{-1},\ z^{-1}\right\rangle$

10. $\langle x, y, z\rangle$

11. $e^y\langle \cos z,\ x\cos z,\ -x\sin z\rangle$

12. $\nabla f(x,\ y,\ z) = \left\langle 0,\ 2ye^{-2z},\ -2y^2e^{-2z}\right\rangle$
$= 2ye^{-2z}\langle 0, 1, -y\rangle$

13. div $\mathbf{F} = 2x - 2x + 2yz = 2yz$
curl $\mathbf{F} = \left\langle z^2,\ 0,\ -2y\right\rangle$

14. div $\mathbf{F} = 2x + 2y + 2z$
curl $\mathbf{F} = \langle 0,\ 0,\ 0\rangle = \mathbf{0}$

15. div $\mathbf{F} = \nabla \cdot \mathbf{F} = 0+0+0 = 0$
curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle x-x,\ y-y,\ z-z\rangle = \mathbf{0}$

16. div $\mathbf{F} = -\sin x + \cos y + 0$
curl $\mathbf{F} = \langle 0,0,0\rangle = \mathbf{0}$

17. div $\mathbf{F} = e^x\cos y + e^x\cos y + 1 = 2e^x\cos y + 1$
curl $\mathbf{F} = \left\langle 0,\ 0,\ 2e^x\sin y\right\rangle$

18. div $\mathbf{F} = \nabla \cdot \mathbf{F} = 0+0+0 = 0$
curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle 1-1, 1-1, 1-1\rangle = \mathbf{0}$

19. **a.** meaningless

b. vector field

c. vector field

d. scalar field

e. vector field

f. vector field

g. vector field

h. meaningless

i. meaningless

j. scalar field

k. meaningless

20. **a.** $\text{div}(\text{curl } \mathbf{F}) = \text{div} \cdot \left\langle P_y - N_z, M_z - P_x, N_x - M_y \right\rangle = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz}) = 0$

b. $\text{curl}(\text{grad } f) = \text{curl}\left\langle f_x, f_y, f_z \right\rangle = \left\langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \right\rangle = \mathbf{0}$

c. $\text{div}(f\mathbf{F}) = \text{div}\langle fM, fN, fP\rangle = (fM_x + f_x M) + (fN_y + f_y N) + (fP_z + f_z P)$
$= (f)(M_x + N_y + P_z) + (f_x M + f_y N + f_z P) = (f)(\text{div } \mathbf{F}) + (\text{grad } f) \cdot \mathbf{F}$

d. $\text{curl}(f\mathbf{F}) = \text{curl}\langle fM, fN, fP\rangle$
$= \left\langle (fP_y + f_y P) - (fN_z + f_z N), (fM_z + f_z M) - (fP_x + f_x P), (fN_x + f_x N) - (fM_y + f_y M) \right\rangle$
$= (f)\left\langle P_y - N_z, M_z - P_x, N_x - M_y \right\rangle + \left\langle f_x, f_y, f_z \right\rangle \times \langle M, N, P\rangle = (f)(\text{curl } \mathbf{F}) + (\text{grad } f) \times \mathbf{F}$

21. Let $f(x, y, z) = -c|\mathbf{r}|^{-3}$, so

$\text{grad}(f) = 3c|\mathbf{r}|^{-4} \dfrac{\mathbf{r}}{|\mathbf{r}|} = 3c|\mathbf{r}|^{-5}\,\mathbf{r}.$

Then $\text{curl } \mathbf{F} = \text{curl}\left[\left(-c|\mathbf{r}|^{-3}\right)\mathbf{r}\right]$

$= \left(-c|\mathbf{r}|^{-3}\right)(\text{curl } \mathbf{r}) + \left(3c|\mathbf{r}|^{-5}\, r\right) \times \mathbf{r}$ (by 20d)

$= \left(-c|\mathbf{r}|^{-3}\right)(\mathbf{0}) + \left(3c|\mathbf{r}|^{-5}\right)(\mathbf{r} \times \mathbf{r}) = 0 + 0 = 0$

$\text{div } \mathbf{F} = \text{div}\left[\left(-c|\mathbf{r}|^{-3}\right)\mathbf{r}\right]$

$= \left(-c|\mathbf{r}|^{-3}\right)(\text{div } \mathbf{r}) + \left(3c|\mathbf{r}|^{-5}\,\mathbf{r}\right) \cdot \mathbf{r}$ (by 20c)

$= \left(-c|\mathbf{r}|^{-3}\right)(1+1+1) + \left(3c|\mathbf{r}|^{-5}\right)|\mathbf{r}|^2$

$= \left(-3c|\mathbf{r}|^{-3}\right) + 3c|\mathbf{r}|^3 = 0$

22. $\text{curl}\left[-c|\mathbf{r}|^{-m}\,\mathbf{r}\right] = \left(-c|\mathbf{r}|^{-m}\right)(0) + mc|\mathbf{r}|^{-m-2}\,(\mathbf{0})$
$= \mathbf{0}$

$\text{div}\left[-c|\mathbf{r}|^{-m}\,\mathbf{r}\right] = \left(-c|\mathbf{r}|^{-m}\right)(3) + mc|\mathbf{r}|^{-m-2}\,|\mathbf{r}|^2$

$= (m-3)c|\mathbf{r}|^{-m}$

23. $\text{grad } f = \left\langle f'(r)xr^{-1/2}, f'(r)yr^{-1/2}, f'(r)zr^{-1/2} \right\rangle$
(if $r \neq 0$)
$= f'(r)r^{-1/2}\langle x, y, z\rangle = f'(r)r^{-1/2}\mathbf{r}$
$\text{curl } \mathbf{F} = [f(r)][\text{curl } \mathbf{r}] + [f'(r)r^{-1/2}\mathbf{r}] \times \mathbf{r}$
$= [f(r)][\text{curl } \mathbf{r}] + [f'(r)r^{-1/2}\mathbf{r}] \times \mathbf{r}$
$= \mathbf{0} + \mathbf{0} = \mathbf{0}$

24. $\text{div } \mathbf{F} = \text{div}[f(r)\mathbf{r}] = [f(r)](\text{div } \mathbf{r}) + \text{grad}[f(r)] \cdot \mathbf{r}$
$= [f(r)](\text{div } \mathbf{r}) + [f'(r)r^{-1}\mathbf{r}] \cdot \mathbf{r}$
$= [f(r)](3) + [f'(r)r^{-1}](\mathbf{r} \cdot \mathbf{r})$
$= 3f(r) + [f'(r)r^{-1}](r^2) = 3f(r) + rf'(r)$
Now if div $\mathbf{F} = 0$, and we let $y = f(r)$, we have the differential equation $3y + r\dfrac{dy}{dr} = 0$, which can be solved as follows:
$\dfrac{dy}{y} = -3\dfrac{dr}{r};\ \ln|y| = -3\ln|r| + \ln|C| = \ln\left|Cr^{-3}\right|,$
for each $C \neq 0$. Then $y = Cr^{-3}$, or
$f(r) = Cr^{-3}$, is a solution (even for $C = 0$).

25. **a.** Let $P = (x_0,\ y_0)$.

div **F** = div **H** = 0 since there is no tendency toward P except along the line $x = x_0$, and along that line the tendencies toward and away from P are balanced; div **G** < 0 since there is no tendency toward P except along the line $x = x_0$, and along that line there is more tendency toward than away from P; div **L** > 0 since the tendency away from P is greater than the tendency toward P.

b. No rotation for **F**, **G**, **L**; clockwise rotation for **H** since the magnitudes of the forces to the right of P are less than those to the left.

c. div **F** = 0; curl **F** = **0**

div $\mathbf{G} = -2ye^{-y^2} < 0$ since $y > 0$ at P; curl $\mathbf{G} = \mathbf{0}$

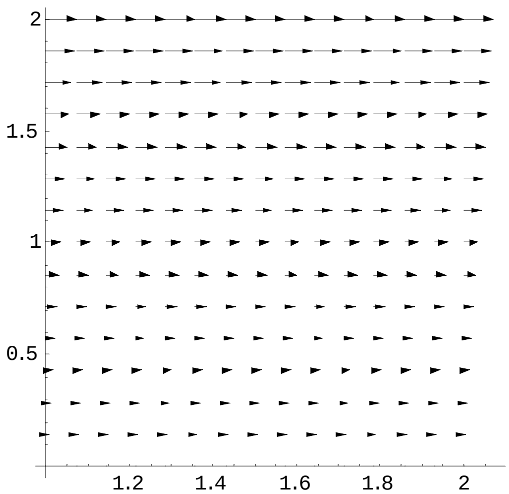
div $\mathbf{L} = (x^2 + y^2)^{-1/2}$; curl $\mathbf{L} = \mathbf{0}$

div $\mathbf{H} = 0$; curl $\mathbf{H} = \left\langle 0,\ 0,\ -2xe^{-x^2} \right\rangle$ which points downward at P, so the rotation is clockwise in a right-hand system.

26. $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$,

where

$M(x, y, z) = y,\ N(x, y, z) = 0,\ P(x, y, z) = 0$



a. Since the velocity into (1, 1, 0) equals the velocity out, there is no tendency to diverge from or accumulate to the point. Geometrically, it appears that $\operatorname{div}\mathbf{F}(1,1,0) = 0$. Calculating,

$$\operatorname{div}\mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 0 + 0 + 0 = 0$$

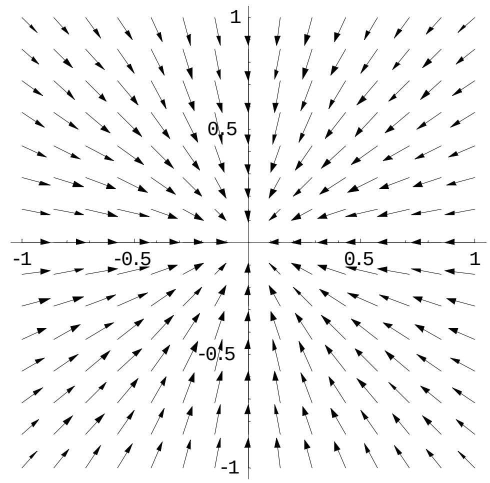
b. If a paddle wheel is placed at the point (with its axis perpendicular to the plane), the velocities over the top half of the wheel will exceed those over the bottom, resulting in a net *clockwise* motion. Using the right-hand rule, we would expect $\operatorname{curl}\mathbf{F}$ to point into the plane (negative z). By calculating

$\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-1)\,\mathbf{k} = -\mathbf{k}$

27. $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, where

$$M = -\frac{x}{(1 + x^2 + y^2)^{3/2}},\ N = -\frac{y}{(1 + x^2 + y^2)^{3/2}},$$

$P = 0$



a. Since all the vectors are directed toward the origin, we would expect accumulation at that point; thus $\operatorname{div}\mathbf{F}(0,0,0)$ .should be negative. Calculating,

$$\operatorname{div}\mathbf{F}(x, y, z) = \frac{3(x^2 + y^2)}{(1 + x^2 + y^2)^{5/2}} - \frac{2}{(1 + x^2 + y^2)^{3/2}} + 0$$

so that $\operatorname{div}\mathbf{F}(0,0,0) = -2$

b. If a paddle wheel is placed at the origin (with its axis perpendicular to the plane), the force vectors all act radially along the wheel and so will have no component acting tangentially along the wheel. Thus the wheel will not turn at all, and we would expect $\operatorname{curl}\mathbf{F} = \mathbf{0}$. By calculating

$$\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + \left(\frac{3yx}{(1 + x^2 + y^2)^{3/2}} - \frac{3xy}{(1 + x^2 + y^2)^{3/2}}\right)\mathbf{k} = \mathbf{0}$$

28. div $\mathbf{v} = 0 + 0 + 0 = 0$;
curl $\mathbf{v} = \langle 0, 0, w + w \rangle = 2\omega\mathbf{k}$

29. $\nabla f(x, y, z) = \frac{1}{2} m\omega^2 \langle 2x, 2y, 2z \rangle = m\omega^2 \langle x, y, z \rangle$
$= \mathbf{F}(x, y, z)$

30. $\nabla^2 f = \text{div}(\text{grad } f) = \text{div}\left\langle f_x, f_y, f_z \right\rangle$
$= f_{xx} + f_{yy} + f_{zz}$

a. $\nabla^2 f = 4 - 2 - 2 = 0$

b. $\nabla^2 f = 0 + 0 + 0 = 0$

c. $\nabla^2 f = 6x - 6x + 0 = 0$

d. $\nabla^2 f = \text{div}(\text{grad } f) = \text{div}\left(\text{grad } |\mathbf{r}|^{-1}\right)$
$= \text{div}\left(-|\mathbf{r}|^{-3}\mathbf{r}\right) = 0$ (by problem 21)

Hence, each is harmonic.

31. a. $\mathbf{F} \times \mathbf{G} = (f_y g_z - f_z g_y)\mathbf{i} - (f_x g_z - f_z g_x)\mathbf{j} + (f_x g_z - f_z g_x)\mathbf{k}$. Therefore,

$$div(\mathbf{F} \times \mathbf{G}) = \frac{\partial}{\partial x}(f_y g_z - f_z g_y) - \frac{\partial}{\partial y}(f_x g_z - f_z g_x) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x).$$

Using the product rule for partials and some algebra gives

$$div(\mathbf{F} \times \mathbf{G}) = g_x\left[\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right] + g_y\left[\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right] + g_z\left[\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right]$$
$$+ f_x\left[\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right] + f_y\left[\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right] + f_z\left[\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right]$$
$$= \mathbf{G} \cdot curl(\mathbf{F}) - \mathbf{F} \cdot curl(\mathbf{G})$$

b. $\nabla f \times \nabla g = \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\mathbf{k}$

Therefore, $div(\nabla f \times \nabla g) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$

Using the product rule for partials and some algebra will yield the result
$div(\nabla f \times \nabla g) = 0$

32. $\lim_{(x, y, z) \to (a, b, c)} F(x, y, z) = \mathbf{L}$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < \left|\langle x, y, z \rangle - \langle a, b, c \rangle\right| < \delta$ implies that $\left|\mathbf{F}(x, y, z) - \mathbf{L}\right| < \varepsilon$.

F is continuous at (a, b, c) if and only if $\lim_{(x, y, z) \to (a, b, c)} = \mathbf{F}(a, b, c)$.

14.2 Concepts Review

1. Increasing values of t

2. $\sum_{i=1}^{n} f(\overline{x}_i, \overline{y}_i)\Delta s_i$

3. $f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}$

4. $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$

Problem Set 14.2

1. $\int_0^1 (27t^3 + t^3)(9 + 9t^4)^{1/2}\,dt = 14\left(2\sqrt{2} - 1\right) \approx 25.5980$

2. $\int_0^1 \left(\frac{t}{2}\right)(t)\left(\frac{1}{4} + \frac{25t^3}{4}\right)^{1/2} dt = \left(\frac{1}{450}\right)(26^{3/2} - 1) \approx 0.2924$

3. Let $x = t$, $y = 2t$, t in $[0, \pi]$.

Then $\int_C (\sin x + \cos y)ds = \int_0^{\pi} (\sin t + \cos 2t)\sqrt{1+4}dt = 2\sqrt{5} \approx 4.4721$

4. Vector equation of the segment is

$\langle x, y\rangle = \langle -1, 2\rangle + t\langle 2, -1\rangle$, t in [0, 1].

$\int_0^1 (-1 + 2t)e^{2-t}(4+1)^{1/2}\,dt = \sqrt{5}e^2(1 - 3e^{-1}) \approx -1.7124$

5. $\int_0^1 (2t + 9t^3)(1 + 4t^2 + 9t^4)^{1/2}\,dt = \left(\frac{1}{6}\right)(14^{3/2} - 1) \approx 8.5639$

6. $\int_0^{2\pi} (16\cos^2 t + 16\sin^2 t + 9t^2)(16\sin t^2 + 16\cos^2 t + 9)^{1/2}\,dt = \int_0^{2\pi}(16 + 9t^2)(5)dt$

$= \left[80t + 15t^3\right]_0^{2\pi} = 160\pi + 120\pi^3 \approx 4878.11$

7. $\int_0^2 [(t^2 - 1)(2) + (4t^2)(2t)]dt = \frac{100}{3}$

8. $\int_0^4 (-1)dx + \int_{-1}^3 (4)^2\,dy = 60$

9. $\int_C y^3dx + x^3dy = \int_{C_1} y^3dx + x^3dy + \int_{C_2} y^3dx + x^3dy = \int_1^{-2}(-4)^3\,dy + \int_{-4}^2 (-2)^3\,dx = 192 + (-48) = 144$

10. $\int_{-2}^1 [(t^2 - 3)^3(2) + (2t)^3(2t)]dt = \frac{828}{35} \approx 23.6571$

11. $y = -x + 2$

$\int_1^3 ([x + 2(-x+2)](1) + [x - 2(-x+2)](-1))dx = 0$

12. $\int_0^1 [x^2 + (x)2x]dx = \int_0^1 3x^2\,dx = 1$

(letting x be the parameter; i.e., $x = x$, $y = x^2$)

13. $\langle x, y, z\rangle = \langle 1, 2, 1\rangle + t\langle 1, -1, 0\rangle$

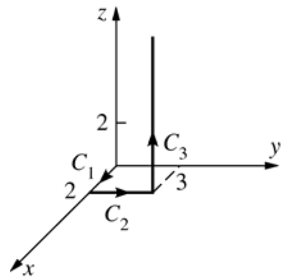
$\int_0^1 [(4-t)(1) + (1+t)(-1) - (2 - 3t + t^2)(-1)]dt = \frac{17}{6} \approx 2.8333$

14 $\int_0^1 [(e^{3t})(e^t) + (e^{-t} + e^{2t})(-3^{-t}) + (e^t)(2e^{2t})]dt = \left(\frac{1}{4}\right)e^4 + \left(\frac{2}{3}\right)e^3 - e + \left(\frac{1}{2}\right)e^{-2} - \frac{5}{12} \approx 23.9726$

15. On $C_1: y = z = dy = dz = 0$

On $C_2: x = 2, z = dx = dz = 0$

On $C_3: x = 2, y = 3, dx = dy = 0$



$$\int_0^2 x\,dx + \int_0^3 (2-2y)dy + \int_0^4 (4+3-z)dz = \left[\frac{x^2}{2}\right]_0^2 + [2y - y^2]_0^3 + \left[7z - \frac{z^2}{2}\right]_0^4 = 2 + (-3) + 20 = 19$$

16. $\langle x, y, z\rangle = t\langle 2,3,4\rangle$, t in [0, 1].

$$\int_0^1 [(9t)(2) + (8t)(3) + (3t)(4)]dt = 27$$

17. $m = \int_C k|x|ds = \int_{-2}^2 k|x|(1+4x^2)^{1/2} dx = \left(\frac{k}{6}\right)(17^{3/2} - 1) \approx 11.6821k$

18. Let $\delta(x, y, z) = k$ (a constant).

$$m = k\int_C 1ds = k\int_0^{3\pi} 1(a^2 \sin^2 t + a^2 \cos^2 t + b^2)^{1/2} dt = 3\pi k(a^2 + b^2)^{1/2}$$

$$M_{xy} = k\int_C z\,ds = k(a^2+b^2)^{1/2}\int_0^{3\pi} bt\,dt = \frac{9\pi^2 bk(a^2+b^2)^{1/2}}{2}$$

$$M_{xz} = k\int_C y\,ds = k(a^2+b^2)^{1/2}\int_0^{3\pi} a\sin t\,dt = ak(a^2+b^2)^{1/2}(2) = 2ak(a^2+b^2)^{1/2}$$

$$M_{yz} = k\int_C x\,ds = k(a^2+b^2)^{1/2}\int_0^{3\pi} a\cos t\,dt = ak(a^2+b^2)^{1/2}(0) = 0$$

Therefore, $\overline{x} = \frac{M_{yz}}{m} = 0;\ \overline{y} = \frac{M_{xz}}{m} = \frac{2a}{3\pi};\quad \overline{z} = \frac{M_{xy}}{m} = \frac{3\pi b}{2}.$

19 $\int_C (x^3 - y^3)dx + xy^2 dy = \int_{-1}^0 [(t^6 - t^9)(2t) + (t^2)(t^6)(3t^2)]dt = -\frac{7}{44} \approx -0.1591$

20. $\int_C e^x dx - e^{-y} dy = \int_1^5 \left[(t^3)\left(\frac{3}{t}\right) - \left(\frac{1}{2t}\right)\left(\frac{1}{t}\right)\right]dt = 123.6$

21. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (x+y)dx + (x-y)dy = \int_0^{\pi/2} [(a\cos t + b\sin t)(-a\sin t) + (a\cos t - b\sin t)(b\cos t)]dt$

$$= \int_0^{\pi/2} [-(a^2+b^2)\sin t\cos t + ab(\cos^2 t - \sin^2 t)]dt = \int_0^{\pi/2} \frac{-(a^2+b^2)\sin 2t}{2} + ab\cos 2t\,dt$$

$$= \left[\frac{(a^2+b^2)\cos 2t}{4} + \frac{ab\sin 2t}{2}\right]_0^{\pi/2} = \frac{a^2+b^2}{-2}$$

22. $\langle x, y, z\rangle = t\langle 1,1,1\rangle$, t in [0, 1].

$$\int_C (2x - y)dx + 2z\,dy + (y - z)dz = \int_0^1 (t + 2t + 0)dt = 1.5$$

23. $\int_0^{\pi}\left[\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)+\pi t\cos\left(\frac{\pi t}{2}\right)+\sin\left(\frac{\pi t}{2}\right)-t\right]dt = 2-\frac{2}{\pi}\approx 1.3634$

24. $W=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C y\,dx+z\,dy+x\,dz = \int_0^2[(t^2)(1)+(t^3)(2t)+(t)(3t^2)]dt = \int_0^2(2t^4+3t^3+t^2)dt$

$=\frac{64}{5}+12+\frac{8}{3}=\frac{412}{15}\approx 27.4667$

25. The line integral $\int_{C_i}\mathbf{F}\bullet d\mathbf{r}$ represents the *work* done in moving a particle through the force field $\mathbf{F}$ along the curve C_i, $i=1,2,3,$

a. In the first quadrant, the tangential component to C_1 of each force vector is in the positive *y*-direction , the same direction as the object moves along C_1. Thus the line integral (work) should be positive.

b. The force vector at each point on C_2 appears to be tangential to the curve, but in the opposite direction as the object moves along C_2. Thus the line integral (work) should be negative.

c. The force vector at each point on C_3 appears to be perpendicular to the curve, and hence has no component in the direction the object is moving. Thus the line integral (work) should be zero

26. The line integral $\int_{C_i}\mathbf{F}\bullet d\mathbf{r}$ represents the *work* done in moving a particle through the force field $\mathbf{F}$ along the curve C_i, $i=1,2,3,$

a. In the first quadrant, the tangential component to C_1 of each force vector is in the positive *y*-direction , the same direction as the object moves along C_1. Thus the line integral (work) should be positive.

b. The force vector at each point on C_2 appears to be perpendicular to the curve, and hence has no component in the direction the object is moving. Thus the line integral (work) should be zero

c. The force vector at each point on C_3 is along the curve, and in the same direction as the movement of the object. Thus the line integral (work) should be positive.

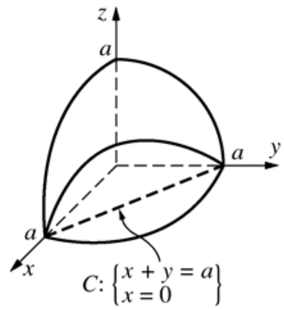
27. $\int_C\left(1+\frac{y}{3}\right)ds=\int_0^2(1+10\sin^3 t)[(-90\cos^2 t\sin t)^2+(90\sin^2 t\cos t)^2]^{1/2}dt = 225$

Christy needs $\frac{450}{200}=2.25$ gal of paint.

28. $\int_C\langle 0,\,0,\,1.2\rangle\cdot\langle dx,\,dy,\,dz\rangle=\int_C 1.2\,dz = \int_0^{8\pi}1.2(4)dt = 38.4\pi\approx 120.64$ ft-lb

Trivial way: The squirrel ends up 32π ft immediately above where it started.
(32π ft)(1.2 lb) ≈ 120.64 ft-lb

29. C: $x + y = a$
Let $x = t$, $y = a - t$, t in $[0, a]$.
Cylinder: $x + y = a$; $(x+y)^2=a^2$; $x^2+2xy+y^2=a^2$
Sphere: $x^2+y^2+z^2=a^2$
The curve of intersection satisfies: $z^2=2xy$; $z=\sqrt{2xy}$.



Area $= 8\int_C \sqrt{2xy}\,ds = 8\int_0^a \sqrt{2t(a-t)}\sqrt{(1)^2+(-1)^2}\,dt = 16\int_0^a \sqrt{at-t^2}\,dt$

$$= 16\left[\frac{t-\frac{a}{2}}{2}\sqrt{at-t^2} + \frac{\left(\frac{a}{2}\right)^2}{2}\sin^{-1}\left(\frac{t-\frac{a}{2}}{\frac{a}{2}}\right)\right]_0^a = 16\left(\left[0+\left(\frac{a^2}{8}\right)\left(\frac{\pi}{2}\right)\right]-\left[0+\left(\frac{a^2}{8}\right)\left(\frac{-\pi}{2}\right)\right]\right)$$

$$= 2a^2\pi$$

Trivial way: Each side of the cylinder is part of a plane that intersects the sphere in a circle. The radius of each circle is the value of z in $z = \sqrt{2xy}$ when $x = y = \frac{a}{2}$. That is, the radius is $\sqrt{2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)} = \frac{a\sqrt{2}}{2}$. Therefore, the total area of the part cut out is $r\left[\pi\left(\frac{a\sqrt{2}}{2}\right)^2\right] = 2a^2\pi.$

30. $I_y = \int_c kx^2\,ds = 4k\int_0^a t^2\sqrt{2}\,dt = 4\sqrt{2}\,\frac{ka^3}{3}$

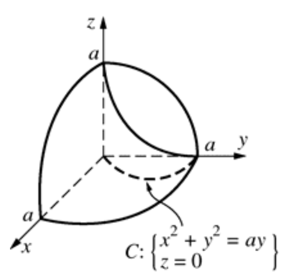
(using same parametric equations as in Problem 29)

$I_x = I_y$ (symmetry)

$I_z = I_x + I_y = 8\sqrt{2}\,\frac{ka^3}{3}$

31. C: $x^2 + y^2 = a^2$

Let $x = a\cos\theta$, $y = a\sin\theta$, θ in $\left[0, \frac{\pi}{2}\right]$.



Area $= 8\int_C \sqrt{a^2 - x^2}\,ds$

$= 8\int_0^{\pi/2}(a\sin\theta)\sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2}\,d\theta$

$= 8\int_0^{\pi/2}(a\sin\theta)\sqrt{a^2}\,d\theta = 8a^2[-\cos\theta]_0^{\pi/2}$

$= 8a^2$

32. Note that $r = a\cos\theta$ along C.

Then $(a^2 - x^2 - y^2)^{1/2} = (a^2 - r^2)^{1/2} = a\cos\theta.$

Let $\left\{\begin{matrix} x = r\cos\theta = (a\sin\theta)\cos\theta \\ y = r\sin\theta = (a\sin\theta)\sin\theta \end{matrix}\right\}$, θ in $\left[0, \frac{\pi}{2}\right]$.

Therefore, $x'(\theta) = a\cos 2\theta$; $y'(\theta) = a\sin 2\theta$.

Then Area $= 4\int_C (a^2 - x^2 - y^2)^{1/2}\,ds = 4\int_0^{\pi/2}(a\cos\theta)[(a\sin 2\theta)^2 + (a\cos 2\theta)^2]^{1/2}\,d\theta = 4a^2$.

33. a. $\int_C x^2 y\,ds = \int_0^{\pi/2}(3\sin t)^2(3\cos t)[(3\cos t)^2 + (-3\sin t)^2]^{1/2}\,dt = 81\int_0^{\pi/2}\sin^2 t\cos t\,dt = 81\left[\left(\frac{1}{3}\right)\sin^3 t\right]_0^{\pi/2} = 27$

b. $\int_{C_4} xy^2\,dx + xy^2\,dy = \int_0^3 (3-t)(5-t)^2(-1)dt + \int_0^3 (3-t)(5-t)^2(-1)dt = 2\int_0^3 (t^3 - 13t^2 + 55t - 75)dt = -148.5$

14.3 Concepts Review

1. $f(\mathbf{b}) - f(\mathbf{a})$

2. gradient; $\nabla f(\mathbf{r})$

3. 0; 0

4. $\mathbf{F}$ is conservative.

Problem Set 14.3

1. $M_y = -7 = N_x$, so $\mathbf{F}$ is conservative.

$f(x, y) = 5x^2 - 7xy + y^2 + C$

2. $M_y = 6y + 5 = N_x$, so $\mathbf{F}$ is conservative.

$f(x, y) = 4x^3 + 3xy^2 + 5xy - y^3 + C$

3. $M_y = 90x^4 y - 36y^5 \neq N_x$ since

$N_x = 90x^4 y - 12y^5$, so $\mathbf{F}$ is not conservative.

4. $M_y = -12x^2 y^3 + 9y^8 = N_x$, so $\mathbf{F}$ is conservative.

$f(x, y) = 7x^5 - x^3 y^4 + xy^9 + C$

5. $M_y = \left(-\frac{12}{5}\right)x^2 y^{-3} = N_x$, so $\mathbf{F}$ is conservative.

$f(x, y) = \left(\frac{2}{5}\right)x^3 y^{-2} + C$

6. $M_y = (4y^2)(-2xy\sin xy^2) + (8y)(\cos xy^2) \neq N_x$

since $N_x = (8x)(-y^2\sin xy^2) + (8)(\cos xy^2)$, so $\mathbf{F}$ is not conservative.

7. $M_y = 2e^y - e^x = N_x$ so $\mathbf{F}$ is conservative.

$f(x, y) = 2xe^y - ye^x + C$

8. $M_y = -e^{-x}y^{-1} = N_x$, so $\mathbf{F}$ is conservative.

$f(x, y) = e^{-x}\ln y + C$

9. $M_y = 0 = N_x, M_z = 0 = P_x$, and $N_z = 0 = P_y$, so $\mathbf{F}$ is conservative. f satisfies

$f_x(x, y, z) = 3x^2$, $f_y(x, y, z) = 6y^2$, and

$f_z(x, y, z) = 9z^2$.

Therefore, f satisfies

1. $f(x, y, z) = x^3 + C_1(y, z)$,

2. $f(x, y, z) = 2y^3 + C_2(x, z)$, and

3. $f(x, y, z) = 3z^3 + C_3(x, y)$.

A function with an arbitrary constant that satisfies 1, 2, and 3 is

$f(x, y, z) = x^3 + 2y^3 + 3z^3 + C.$

10. $M_y = 2x = N_y, M_z = 2z = P_x$, and $N_z = 0 = P_y$, so $\mathbf{F}$ is conservative.

$f(x, y, z) = x^2 y + xz^2 + \sin\pi z + C$

11. Writing $\mathbf{F}$ in the form
$\mathbf{F}(x,y,z) = M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k}$

we have $M(x,y,z) = \dfrac{-2x}{(x^2+z^2)}$, $N(x,y,z) = 0$,

$$P(x,y,z) = \frac{-2z}{(x^2+z^2)}$$

so that

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}, \frac{\partial M}{\partial z} = \frac{4xz}{(x^2+z^2)^2} = \frac{\partial P}{\partial x},$$

$\dfrac{\partial N}{\partial z} = 0 = \dfrac{\partial P}{\partial y}$. Thus $\mathbf{F}$ is conservative by Thm. D.

We must now find a function $f(x,y,z)$ such that

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2+z^2)}, \ \frac{\partial f}{\partial y} = 0, \ \frac{\partial f}{\partial z} = \frac{-2z}{(x^2+z^2)}.$$

Note that $\mathbf{F}$ is a function of x and z alone so f will be a function of x and z alone.

a. Applying the first condition gives

$$f(x,y,z) = \int \frac{-2x}{x^2+z^2}\,dx$$
$$= \ln\left(\frac{1}{x^2+z^2}\right) + C_1(z)$$

b. Applying the second condition,

$$\frac{-2z}{(x^2+z^2)} = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\ln\left(\frac{1}{x^2+z^2}\right) + \frac{\partial C_1}{\partial z} =$$

$\dfrac{-2z}{(x^2+z^2)} + \dfrac{\partial C_1}{\partial z}$ which requires $\dfrac{\partial C_1}{\partial z} = 0$

Hence $f(x,y,z) = \ln\left(\dfrac{1}{x^2+z^2}\right) + C$

12. Writing $\mathbf{F}$ in the form
$\mathbf{F}(x,y,z) = M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k}$

we have $M(x,y,z) = 0$, $N(x,y,z) = 1 + 2yz^2$,

$$P(x,y,z) = 1 + 2y^2 z$$

so that

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial z} = 4yz = \frac{\partial P}{\partial y}.$$

Thus $\mathbf{F}$ is conservative by Thm. D.
We must now find a function $f(x,y,z)$ such that

$$\frac{\partial f}{\partial y} = 1 + 2yz^2, \ \frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial z} = 1 + 2yz^2.$$

Note that f is a function of y and z only.

a. Applying the first condition gives

$$f(x,y,z) = \int (1+2yz^2)\,dy = y + y^2z^2 + C_1(z)$$

b. Applying the second condition,

$$1 + 2y^2 z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(y + y^2z^2) + \frac{\partial C_1}{\partial z}$$
$$= 2y^2 z + \frac{\partial C_1}{\partial z}$$

which requires $\dfrac{\partial C_1}{\partial z} = 1$ or $C_1(z) = z + c$.

Hence $f(x,y,z) = y + z + y^2z^2 + C$

13. $M_y = 2y + 2x = N_x$, so the integral is independent of the path. $f(x,\ y) = xy^2 + x^2y$

$$\int_{(-1,2)}^{(3,1)} (y^2 + 2xy)dx + (x^2 + 2xy)dy$$
$$= [xy^2 + x^2y]_{(-1,2)}^{(3,1)} = 14$$

14. $M_y = e^x \cos y = N_x$, so the line integral is independent of the path.
Let $f_x(x,\ y) = e^x \sin y$ and $f_y(x,\ y) = e^x \cos y$.

Then $f(x,\ y) = e^x \sin y + C_1(y)$ and

$f(x,\ y) = e^x \sin y + C_2(x)$.

Choose $f(x,\ y) = e^x \sin y$.
By Theorem A,

$$\int_{(0,0)}^{(1,\pi/2)} e^x \sin y\,dx + e^x \cos y\,dy$$
$$= [e^x \sin y]_{(0,0)}^{(1,\pi/2)} = e$$

(Or use line segments (0, 0) to (1, 0), then (1, 0) to $\left(1, \dfrac{\pi}{2}\right)$.)

15. For this problem, we will restrict our consideration to the set
$D = \{(x,y)\,|\,x>0, y>0\}$
(that is, the first quadrant), which is open and simply connected.

a. Now, $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ where

$$M(x,y) = \frac{x^3}{(x^4+y^4)^2}, \ N(x,y) = \frac{y^3}{(x^4+y^4)^2};$$

thus $\dfrac{\partial M}{\partial y} = \dfrac{-8x^3y^3}{(x^4+y^4)^3} = \dfrac{\partial N}{\partial x}$ so $\mathbf{F}$ is

conservative by Thm. D, and hence $\int_{\mathbf{C}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is independent of path in D by Theorem C.

b. Since $\mathbf{F}$ is conservative, we can find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = \frac{x^3}{(x^4+y^4)^2},\ \frac{\partial f}{\partial y} = \frac{y^3}{(x^4+y^4)^2}.$$

Applying the first condition gives

$$f(x,y) = \int \frac{x^3}{(x^4+y^4)^2}\ dx = \frac{-1}{4(x^4+y^4)} + C(y)$$

Appling the second condition gives

$$\frac{y^3}{(x^4+y^4)^2} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-1}{4(x^4+y^4)}\right) + \frac{\partial C}{\partial y} =$$

$$\frac{y^3}{(x^4+y^4)^2} + \frac{\partial C}{\partial y};$$

hence $\frac{\partial C}{\partial y} = 0$ and $C(y) = \textit{constant}$. Therefore

$f(x,y) = \frac{-1}{4(x^4+y^4)} + c$, and so, by Thm. A,

$$\int_{(2,1)}^{(6,3)} \frac{x^3}{(x^4+y^4)^2}dx + \frac{y^3}{(x^4+y^4)^2}dy$$

$$= f(6,3) - f(2,1) = \left(-\frac{1}{5508}+c\right) - \left(-\frac{1}{68}+c\right)$$

$$= \frac{20}{1377}$$

c. Consider the linear path $C: y = \frac{x}{2},\ 2 \le x \le 6$ in D which connects the points (2, 1) and (6, 3); then $dy = \frac{1}{2}dx$. Thus

$$\int_{(2,1)}^{(6,3)} \frac{x^3}{(x^4+y^4)^2}dx + \frac{y^3}{(x^4+y^4)^2}dy =$$

$$\int_2^6 \frac{x^3}{(x^4+(\frac{x}{2})^4)^2}dx + \frac{(\frac{x}{2})^3}{(x^4+(\frac{x}{2})^4)^2}(\tfrac{1}{2}dx) =$$

$$\int_2^6 \frac{16}{17x^5}dx = \left[\frac{-4}{17x^4}\right]_2^6 = -\frac{1}{5508} + \frac{1}{68} =$$

$$\frac{20}{1377}$$

16. For this problem, we can use the whole real plane as D.

a. $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ where

$M(x,y) = 3x^2 - 2xy - y^2$,

$N(x,y) = 3y^2 - 2xy - x^2$

thus $\frac{\partial M}{\partial y} = -2x - 2y = \frac{\partial N}{\partial x}$ so $\mathbf{F}$ is conservative by Thm. D, and hence $\int_{\mathbf{C}} \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r}$ is independent of path in D by Theorem C.

b. Since $\mathbf{F}$ is conservative, we can find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 3x^2 - 2xy - y^2,\ \frac{\partial f}{\partial y} = 3y^2 - 2xy - x^2.$$

Applying the first condition gives

$$f(x,y) = \int 3x^2 - 2xy - y^2\ dx$$

$$= x^3 - x^2y - xy^2 + C(y)$$

Appling the second condition gives

$$3y^2 - 2xy - x^2 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3 - x^2y - xy^2\right) + \frac{\partial C}{\partial y}$$

$$= -x^2 - 2xy + \frac{\partial C}{\partial y};$$

hence $\frac{\partial C}{\partial y} = 3y^2$ and $C(y) = y^3 + c$. Therefore

$f(x,y) = x^3 - x^2y - xy^2 + y^3 + c$, and so, by Thm. A,

$$\int_{(-1,1)}^{(4,2)} (3x^2 - 2xy - y^2)\,dx + (3x^2 - 2xy - y^2)\,dy$$

$$= f(4,2) - f(-1,1) = (24+c) - (0+c) = 24$$

c. Consider the simple linear path $C: x = 5y - 6,\ 1 \le y \le 2$ in D which connects the points $(-1,1)$ and (4, 2); then $dx = 5dy$.

Thus

$$\int_{(-1,1)}^{(4,2)} (3x^2 - 2xy - y^2)\,dx + (3y^2 - 2xy - x^2)\,dy =$$

$$\int_1^2 (64y^2 - 168y + 108)(5\,dy) + (-32y^2 + 72y - 36)\ dy$$

$$= \int_1^2 (288y^2 - 768y + 504)\,dy$$

$$= \left[96y^3 - 384y^2 + 504y\right]_1^2$$

$$= 240 + (-216) = 24$$

17. $M_y = 18xy^2 = N_x, M_z = 4x = P_x,\ \ N_z = 0 = P_y.$
By paths (0, 0, 0) to (1, 0, 0); (1, 0, 0) to (1, 1, 0); (1, 1, 0) to (1, 1, 1)
$\int_0^1 0\,dx + \int_0^1 9y^2 dy + \int_0^1 (4z+1)dz = 6$
(Or use $f(x,\ y) = 3x^2y^3 + 2xz^2 + z.$)

18. $M_y = z = N_x, M_z = y = P_x,\ \ N_z = x = P_y.$
$f(x, y) = xyz + x + y + z$
Thus, the integral equals
$[xyz + x + y + z]_{(0.1,\ 0)}^{(1,\ 1,\ 1)} = 3.$

19. $M_y = 1 = N_x, M_z = 1 = P_x,\ \ N_z = 1 = P_y$ (so path independent). From inspection observe that $f(x,\ y,\ z) = xy + xz + yz$ satisfies $f\ \ = \langle y+z, x+z, x+y \rangle$, so the integral equals $[xy + xz + yz]_{(0,\ 0,\ 0)}^{(-1,\ 0,\ \pi)} = -\pi.$ (Or use line segments $(0,1,\ 0)$ to $(1,1,0)$, then $(1,1,0)$ to $(1,1,1)$.)

20. $M_y = 2z = N_x, M_z = 2y = P_x,\ \ N_z = 2x = P_y$ by paths $(0,0,0)$ to $(\pi,0,0)$, $(\pi,0,0)$ to $(\pi,\pi,0)$.
$\int_0^\pi \cos x\,dx + \int_0^\pi \sin y\,dy = 2$
Or use $f(x,\ y,\ z) = \sin x + 2xyz - \cos y + \dfrac{z^2}{2}.$

21. $f_x = M,\ f_y = N,\ f_z = P$
$f_{xy} = M_y$, and $f_{yx} = N_x$, so $M_y = N_x$.
$f_{xz} = M_z$ and $f_{zx} = P_x$, so $M_z = P_x$.
$f_{yz} = N_z$ and $f_{zy} = P_y$, so $N_z = P_y$.

22. $f_x(x,\ y,\ z) = \dfrac{-kx}{x^2+y^2+z^2}$, so
$f(x,\ y,\ z) = \dfrac{-k}{2}\ln(x^2+y^2+z^2) + C_1(y,\ z).$
Similarly,
$f(x,\ y,\ z) = \dfrac{-k}{2}\ln(x^2+y^2+z^2) + C_2(y,\ z),$
using f_y; and
$f(x,\ y,\ z) = \dfrac{-k}{2}\ln(x^2+y^2+z^2) + C_3(y,\ z),$
using f_z.
Thus, one potential function for **F** is
$f(x,\ y,\ z) = \dfrac{-k}{2}\ln(x^2+y^2+z^2).$

23. $\mathbf{F}(x,\ y,\ z) = k|\mathbf{r}|\dfrac{\mathbf{r}}{|\mathbf{r}|} = k\mathbf{r} = k\langle x,\ y,\ z\rangle$
$f(x,\ y,\ z) = \left(\dfrac{k}{2}\right)(x^2+y^2+z^2)$ works.

24. Let $f = \left(\dfrac{1}{2}\right)h(u)$ where $u = x^2+y^2+z^2.$
Then $f_x = \left(\dfrac{1}{2}\right)h'(u)u_x = \left(\dfrac{1}{2}\right)g(u)(2x) = xg(u).$
Similarly, $f_y = yg(u)$ and $f_z = zg(u).$
Therefore, $f(x,\ y,\ z) = g(u)\langle x,\ y,\ z\rangle$
$= g(x^2+y^2+z^2)\langle x,\ y,\ z\rangle = \mathbf{F}(x,\ y,\ z).$

25. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (m\mathbf{r}''\cdot\mathbf{r}')dt$
$= m\int_a^b (x''x' + y''y' + z''z')dt$
$= m\left[\dfrac{(x')^2}{2} + \dfrac{(y')^2}{2} + \dfrac{(z')^2}{2}\right]_a^b$
$= \dfrac{m}{2}\left[|\mathbf{r}'(t)|^2\right]_a^b = \dfrac{m}{2}\left[|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2\right]$

26. The force exerted by Matt is not the only force acting on the object. There is also an equal but opposite force due to friction. The work done by the sum of the (equal but opposite) forces is zero since the sum of the forces is zero.

27. $f(x,y,z) = -gmz$ satisfies $\nabla f(x,\ y,\ z) = \langle 0,\ 0,\ -gm\rangle = \mathbf{F}.$ Then, assuming the path is piecewise smooth,
$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = [-gmz]_{(x_1,\ y_1,\ z_1)}^{(x_2,\ y_2,\ z_2)}$
$= -gm(z_2 - z_1) = gm(z_1 - z_2).$

28. **a.** Place the earth at the origin.
$GMm \approx 7.92(10^{44})$
$f(\mathbf{r}) = \dfrac{-GMm}{|\mathbf{r}|}$ is a potential function of $\mathbf{F}(\mathbf{r})$. (See Example 1.)
$\text{Work} = \int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \left[\dfrac{-GMm}{|\mathbf{r}|}\right]_{|\mathbf{r}|=152.1(10^9)}^{147.1(10^9)}$
$\approx -1.77(10^{32})$ joules

b. Zero

29. a. $M = \frac{y}{(x^2+y^2)}; M_y = \frac{(x^2-y^2)}{(x^2+y^2)^2}$

$N = -\frac{x}{(x^2+y^2)}; N_x = \frac{(x^2-y^2)}{(x^2+y^2)^2}$

b. $M = \frac{y}{(x^2+y^2)} = \frac{(\sin t)}{(\cos^2 t+\sin^2 t)} = \sin t$

$N = -\frac{x}{(x^2+y^2)} = \frac{(-\cos t)}{(\cos^2 t+\sin^2 t)} = -\cos t$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M dx + N dy$

$= \int_0^{2\pi}[(\sin t)(-\sin t)+(-\cos t)(\cos t)]dt$

$= -\int_0^{2\pi} 1 dt = -2\pi \neq 0$

30. f is not continuously differentiable on C since f is undefined at two points of C (where x is 0).

31. Assume the basic hypotheses of Theorem C are satisfied and assume $\int_C \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} = 0$ for every closed path in D. Choose any two distinct points A and B in D and let C_1, C_2 be arbitrary positively oriented paths *from A to B* in *D*. We must show that

$\int_{C_1} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} = \int_{C_2} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r}$

Let $-C_2$ be the curve C_2 with opposite orientation; then $-C_2$ is a positively oriented path *from B to A* in *D*. Thus the curve $C = C_1 \cup -C_2$ is a closed path (in *D*) between *A* and *B* and so, by our assumption,

$0 = \int_C \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} + \int_{-C_2} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} =$

$\int_{C_1} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} - \int_{C_2} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r}$

Thus we have $\int_{C_1} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} = \int_{C_2} \mathbf{F}(\mathbf{r})\bullet d\mathbf{r}$ which proves independence of path.

14.4 Concepts Review

1. $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

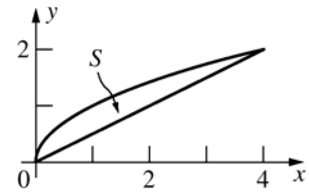
2. –2; –2

3. source; sink

4. rotate; irrotational

Problem Set 14.4

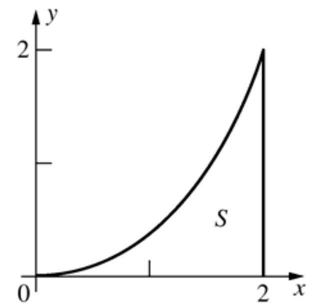
1.



$\oint_C 2xy\,dx + y^2 dy = \iint_S (0-2x)dA$

$= \int_0^2 \int_{y^2}^{2y} -2x\,dx\,dy = -\frac{64}{15} \approx -4.2667$

2.

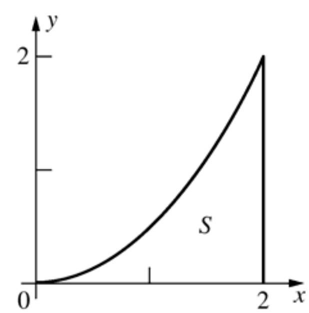


$\oint_C \sqrt{y}\,dx + \sqrt{x}\,dy = \iint_S \frac{1}{2}\left(x^{-1/2} - y^{-1/2}\right)dA$

$= \left(\frac{1}{2}\right)\int_0^2 \int_0^{x^2/2} (x^{-1/2} - y^{-1/2})dy\,dx$

$= -\frac{3\sqrt{2}}{5} \approx -0.8485$

3.

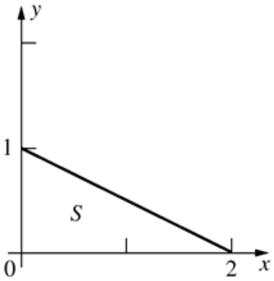


$\oint_C (2x+y^2)dx + (x^2+2y)dy = \iint_S (2x-2y)dA$

$= \int_0^2 \int_0^{x^3/4} (2x-2y)dy\,dx = \int_0^2 \left[\frac{x^4}{2} - \frac{x^6}{16}\right]dx$

$= \frac{16}{5} - \frac{8}{7} = \frac{72}{35} \approx 2.0571$

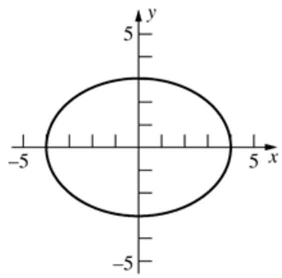
4.



$$\oint_C xy\,dx + (x+y)dy = \iint_S (1-x)dA$$

$$= \int_0^1 \int_0^{-2y+2} (1-x)dx\,dy = \frac{1}{3}$$

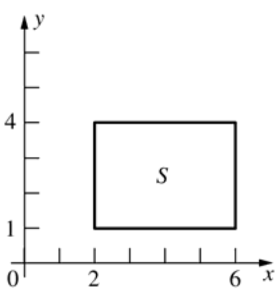
5.



$$\oint_C (x^2 + 4xy)dx + (2x^2 + 3y)dy = \iint_S (4x - 4x)dA$$

$$= 0$$

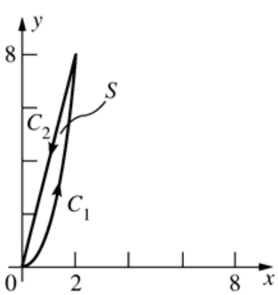
6.



$$\oint_C (e^{3x} + 2y)dx + (x^2 + \sin y)dy = \iint_S (2x-2)dA$$

$$\int_1^4 \int_2^6 (2x-2)dx\,dy = \int_1^4 24dy = 24(3) = 72$$

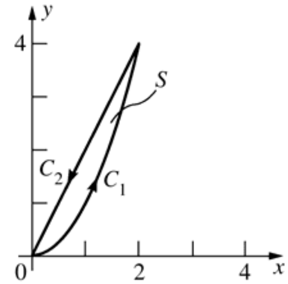
7.



$$A(S) = \left(\frac{1}{2}\right)\oint_C x\,dy - y\,dx$$

$$= \left(\frac{1}{2}\right)\int_0^2 [4x^2 - 2x^2]dx + \left(\frac{1}{2}\right)\int_2^0 [4x - 4x]dx = \frac{8}{3}$$

8.

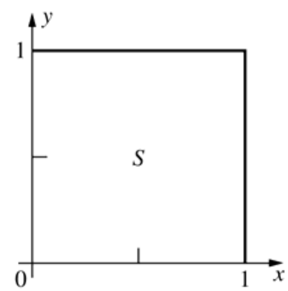


$$A(S) = \left(\frac{1}{2}\right)\oint_C x\,dy - y\,dx$$

$$= \left(\frac{1}{2}\right)\int_0^2 \left[\left(\frac{3}{2}\right)x^3 - \left(\frac{1}{2}\right)x^3\right]dx - \left(\frac{1}{2}\right)\int_2^0 [2x^2 - x^2]dx$$

$$= \frac{2}{3}$$

9.



a. $\iint_S \operatorname{div} \mathbf{F}\,dA = \iint_S (M_x + N_y)dA$

$= \iint_S (0+0)dA = 0$

b. $\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\,dA = \iint_S (N_x - M_y)dA$

$= \iint_S (2x - 2y)dA = \int_0^1 \int_0^1 (2x-2y)dx\,dy$

$= \int_0^1 (1-2y)dy = 0$

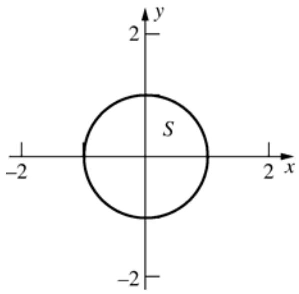
10. **a.** $\iint_S (0+0)dA = 0$

b. $\iint_S (b-a)dA = \int_0^1 \int_0^1 (b-a)dx\,dy = b - a$

11. **a.** $\iint_S (0+0)dA = 0$

b. $\iint_S (3x^2 - 3y^2)dA = 0,$ since for the integrand, $f(y, x) = -f(x, y)$.

12.



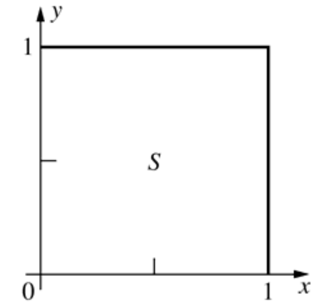
a. $\iint_S \text{div}\,\mathbf{F}\,dA = \iint_S (M_x + N_y)dA$
$= \iint_S (1+1)dA = 2[A(S)] = 2\pi$

b. $\iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{k}\,dA = \iint_S (N_x - M_y)dA$
$= \iint_S (0-0)dA = 0$

13. $\iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{k}dA = \int_{C_1}\mathbf{F}\cdot\mathbf{T}ds - \int_{C_2}\mathbf{F}\cdot\mathbf{T}ds$
= 30 – (–20) = 50

14. $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S (2x+2x)dA = \int_0^1\int_0^1 4x\,dx\,dy = 2$

15.

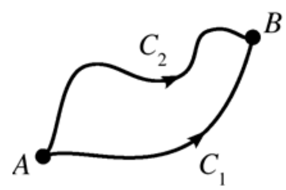


$W = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (N_x - M_y)dA$
$= \iint_S (-2y-2y)dA = \int_0^1\int_0^1 -4y\,dx\,dy$
$= \int_0^1 -4y\,dy = -2$

16. $\oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_S (2y-2y)dA = 0$

17. **F** is a constant, so $N_x = M_y = 0$.
$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_S (N_x - M_y)dA = 0$

18. $\oint M\,dx + N\,dy = \iint_S (N_x - M_y)dA = 0$



Therefore, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path since
$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} - \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} = 0$
(Where C is the loop C_1 followed by $-C_2$.)
Therefore, $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$, so **F** is conservative.

19. **a.** Each equals $(x^2-y^2)(x^2+y^2)^{-2}$.

b. $\oint_C y(x^2+y^2)^{-1}dx - x(x^2+y^2)^{-1}dy = \int_0^{2\pi}(-\sin^2 t - \cos^2 t)dt = \int_0^{2\pi} -1\,dt$

c. M and N are discontinuous at (0, 0).

20. **a.** Parameterization of the ellipse: $x = 3\cos t$, $y = 2\sin t$, t in $[0, 2\pi]$.

$$\int_0^{2\pi}\left[\frac{2\sin t}{9\cos^2 t + 4\sin^2 t}(-3\sin t) - \frac{3\cos t}{9\cos^2 t + 4\sin^2 t}(2\cos t)\right]dt = -2\pi$$

b. $\int_{-1}^{1} -(1+y^2)^{-1}dy + \int_{1}^{-1}(x^2+1)^{-1}dx + \int_{1}^{-1}(1+y^2)^{-1}dy + \int_{-1}^{1} -(x^2+1)^{-1}dx = -2\pi$

c. Green's Theorem applies here. The integral is 0 since $N_x - M_y$.

21. Use Green's Theorem with $M(x, y) = -y$ and $N(x, y) = 0$.
$\oint_C (-y)dx = \iint_S [0-(-1)]dA = A(S)$
Now use Green's Theorem with $M(x, y) = 0$ and $N(x, y) = x$.
$\oint_C x\,dy = \iint_S (1-0)dA = A(S)$

22. $\oint_C \left(-\frac{1}{2}\right)y^2dx = \iint_S (0+y)dA = M_x \cdot \oint_C \left(\frac{1}{2}\right)x^2dy = \iint_S (x-0)dA = M_y$

23. $A(S) = \left(\frac{1}{2}\right)\oint_C x\,dy - y\,dx = \left(\frac{1}{2}\right)\int_0^{2\pi}[(a\cos^3 t)(3a\sin^2 t)(\cos t) - (a\sin^3 t)(3a\cos^2 t)(-\sin t)]dt = \left(\frac{3}{8}\right)a^2\pi$

24. $W = \oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{k}dA = \iint_S (N_x - M_y)dA = \iint_S (-3-2)dA = -5[A(S)] = -5\left(\frac{3a^2\pi}{8}\right) = -\frac{15a^2\pi}{15}$, using the result of Problem 23.

25. **a.** $\mathbf{F}\cdot\mathbf{n} = \frac{x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{a}$

Therefore, $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \frac{1}{a}\int_C 1ds = \frac{1}{a}(2\pi a) = 2\pi.$

b. $\text{div }\mathbf{F} = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1)-(y)(2y)}{(x^2+y^2)^2} = 0$

c. $M = \frac{x}{(x^2+y^2)}$ is not defined at (0, 0) which is inside C.

d. If origin is outside C, then $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S \text{div }\mathbf{F}\,dA = \iint_S 0\,dA = 0.$

If origin is inside C, let C' be a circle (centered at the origin) inside C and oriented clockwise. Let S be the region between C and C'. Then $0 = \iint_S \text{div }\mathbf{F}\,dA$ (by "origin outide C" case)

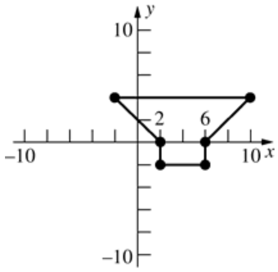
$= \int_C \mathbf{F}\cdot\mathbf{n}\,ds - \int_{C'} \mathbf{F}\cdot\mathbf{n}\,ds$ (by Green's Theorem) $= \int_C \mathbf{F}\cdot\mathbf{n}\,ds - 2\pi$ (by part a), so $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = 2\pi.$

26. **a.** Equation of C:
$\langle x, y\rangle = \langle x_0, y_0\rangle + t\langle x_1 - x_0, y_1 - y_0\rangle$,
t in [0, 1].
Thus;
$\int_C x\,dy = \int_0^1 [x_0 + t(x_1 - x_0)](y_1 - y_0)dt,$
which equals the desired result.

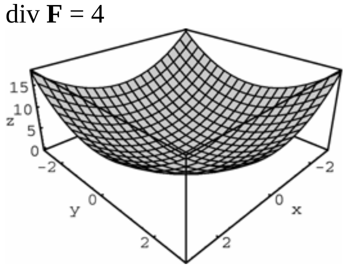
b. $\text{Area}(P) = \int_C x\,dy$ where
$C = C_1 \cup C_2 \cup \ldots \cup C_n$ and C_i is the ith edge. (by Problem 21)
$= \int_{C_1} x\,dy + \int_{C_2} x\,dy + \ldots + \int_{C_n} x\,dy$
$= \sum_{i=1} \frac{(x_i - x_{i-1})(y_i - y_{i-1})}{2}$ (by part a)

c. Immediate result of part b if each x_i and each y_i is an integer.

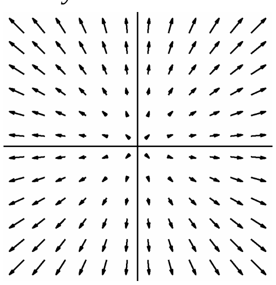
d. Formula gives 40 which is correct for the polygon in the figure below.



27. a. div **F** = 4

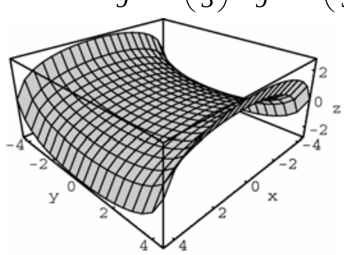


$x^2 + y^2$

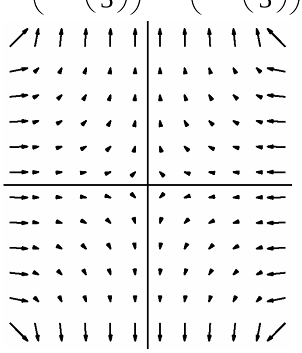


b. 4(36) = 144

28. a. div $\mathbf{F} = -\frac{1}{9}\sec^2\left(\frac{x}{3}\right) + \frac{1}{9}\sec^2\left(\frac{y}{3}\right)$



$\ln\left(\cos\left(\frac{x}{3}\right)\right) - \ln\left(\cos\left(\frac{y}{3}\right)\right)$

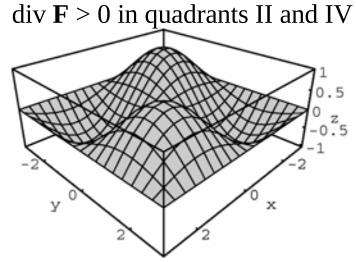


b. $\frac{1}{9}\int_{-3}^{3}\int_{-3}^{3}\left[-\sec^2\left(\frac{x}{3}\right) + \sec^2\left(\frac{y}{3}\right)\right] dy\, dx = 0$

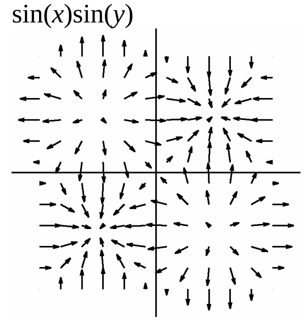
29. a. div **F** = –2 sin x sin y

div **F** < 0 in quadrants I and III

div **F** > 0 in quadrants II and IV



sin(x)sin(y)

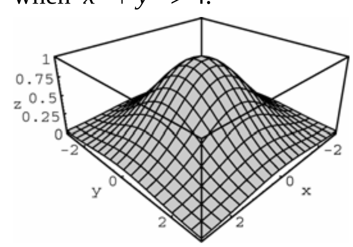


b. Flux across boundary of S is 0.

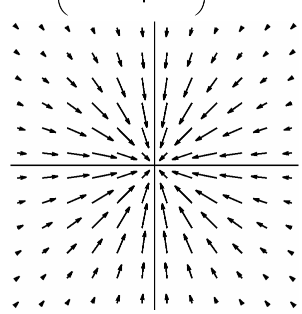
Flux across boundary T is $-2(1 - \cos 3)^2$.

30. div $\mathbf{F} = \frac{1}{4}e^{-(x^2+y^2)/4}(x^2 + y^2 - 4)$

so div **F** < 0 when $x^2 + y^2 < 4$ and div **F** > 0 when $x^2 + y^2 > 4$.



$\exp\left(-\frac{(x^2 + y^2)}{4}\right)$



14.5 Concepts Review

1. surface integral

2. $\sum_{i=1}^{n} g(\overline{x}_i, \overline{y}_i, \overline{z}_i)\Delta S_i$

3. $\sqrt{f_x^2 + f_y^2 + 1}$

4. 2; 18π

Problem Set 14.5

1. $\iint_R [x^2 + y^2 + (x+y+1)](1+1+1)^{1/2}\,dA$

$= \int_0^1\int_0^1 \sqrt{3}(x^2 + y^2 + x + y + 1)\,dx\,dy = \frac{8\sqrt{3}}{3}$

2. $\iint_R x\left(\frac{1}{4} + \frac{1}{4} + 1\right)^{1/2} dA = \int_0^1\int_0^1 \left(\frac{\sqrt{6}}{2}\right) x\,dx\,dy$

$= \frac{\sqrt{6}}{4}$

3 $\iint_R (x+y)\sqrt{[-x(4-x^2)^{-1/2}]^2 + 0 + 1}\,dA$

$= \int_0^{\sqrt{3}}\int_0^1 \frac{2(x+y)}{(4-x^2)^{1/2}}\,dy\,dx$

$= \int_0^{\sqrt{3}} \frac{2x+1}{(4-x^2)^{1/2}}\,dx$

$= \left[-2(4-x^2)^{1/2} + \sin^{-1}\left(\frac{x}{2}\right)\right]_0^{\sqrt{3}}$

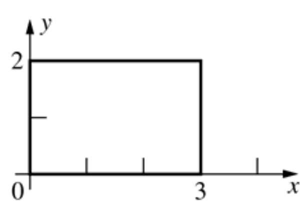
$= \frac{\pi+6}{3} = 2 + \frac{\pi}{3} \approx 3.0472$

4. $\int_0^{2\pi}\int_0^1 r^2(4r^2+1)^{1/2} r\,dr\,d\theta = \left(\frac{\pi}{60}\right)(25\sqrt{5}+1)$

$\approx$ 2.9794

5. $\int_0^{\pi}\int_0^{\sin\theta} (4r^2+1)r\,dr\,d\theta = \left(\frac{5}{8}\right)\pi \approx 1.9635$

6.



$\iint_R y(4y^2+1)^{1/2}\,dA = \int_0^3\int_0^2 (4y^2+1)^{1/2}\,y\,dy\,dx$

$= \int_0^3 \frac{(17^{3/2}-1)}{12}\,dx = \frac{17^{3/2}-1}{4} \approx 17.2732$

7. $\iint_R (x+y)(0+0+1)^{1/2}\,dA$

Bottom (z = 0): $\int_0^1\int_0^1 (x+y)\,dx\,dy = 1$

Top (z = 1): Same integral

Left side (y = 0): $\int_0^1\int_0^1 (x+0)\,dx\,dz = \frac{1}{2}$

Right side (y = 1): $\int_0^1\int_0^1 (x+1)\,dx\,dz = \frac{3}{2}$

Back (x = 0): $\int_0^1\int_0^1 (0+y)\,dy\,dz = \frac{1}{2}$

Front (x = 1): $\int_0^1\int_0^1 (1+y)\,dy\,dz = \frac{3}{2}$

Therefore, the integral equals

$1 + 1 + \frac{1}{2} + \frac{3}{2} + \frac{1}{2} + \frac{3}{2} = 6.$

8. Bottom (z = 0): The integrand is 0 so the integral is 0.

Left face (y = 0): $\int_0^4\int_0^{8-2x} z\sqrt{1}\,dz\,dx = \frac{128}{3}$

Right face ($z = 8 - 2x - 4y$):

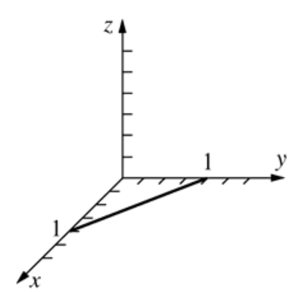
$\int_0^2\int_0^{4-2y} (8-2x-4y)(4+16+1)^{1/2}\,dx\,dy$

$= \left(\frac{32}{3}\right)\sqrt{21}$

Back face (x = 0): $\int_0^2\int_0^{8-4y} z\sqrt{1}\,dz\,dy = \frac{64}{3}$

Therefore, integral $= 64 + \left(\frac{32}{3}\right)\sqrt{21} \approx 112.88.$

9.



$\iint_G \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R (-Mf_x - Nf_y + P)\,dA$

$= \int_0^1\int_0^{1-y} (8y + 4x + 0)\,dx\,dy$

$= \int_0^1 [8(1-y)y + 2(1-y)^2]\,dy$

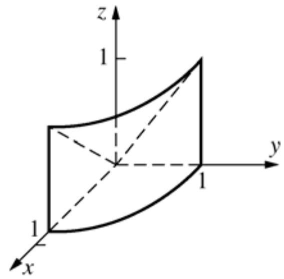
$= \int_0^1 (-6y^2 + 4y + 2)\,dy = 2$

10. $\int_0^3 \int_0^{(6-2x)/3} (x^2-9)\left(-\frac{1}{2}\right) dy\, dx = 11.25$

11. $\int_0^5 \int_{-1}^1 [-xy(1-y^2)^{-1/2} + 2] dy\, dx = 20$

(In the inside integral, note that the first term is odd in y.)

12.



$$\iint_R [-Mf_x - Nf_y + P] dA$$

$$= \iint_R [-2x(x^2+y^2)^{-1/2} - 5y(x^2+y^2)^{-1/2} + 3] dA$$

$$= \int_0^{2\pi} \int_0^1 [(-2r\cos\theta - 5r\sin\theta)r^{-1} + 3] r\, dr\, d\theta$$

$$= \int_0^{2\pi} (-2\cos\theta - 5\sin\theta + 3) d\theta \int_0^1 r\, dr$$

$$= (6\pi)\left(\frac{1}{2}\right) = 3\pi \approx 9.4248$$

13. $m = \iint_G kx^2 ds = \iint_R kx^2 \sqrt{3} dA$

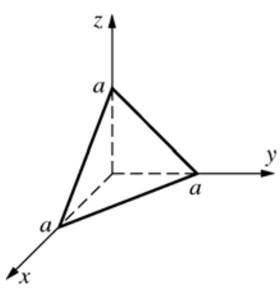
$$= \sqrt{3}k \int_0^a \int_0^{a-x} x^2 dy\, dx = \left(\frac{\sqrt{3}k}{12}\right) a^4$$

14. $m = \iint_G kxy\, ds = \iint_R kxy(x^2+y^2+1)^{1/2} dA$

$$= \int_0^1 \int_0^1 kxy(x^2+y^2+1)^{1/2} dx\, dy$$

$$= \left(\frac{k}{15}\right)(9\sqrt{3} - 8\sqrt{2} + 1) \approx 0.3516k$$

15.



Let $\delta = 1$.

$$m = \iint_S 1\, ds = \iint_R (1+1+1)^{1/2} dA$$

$$= \sqrt{3} \int_0^a \int_0^{a-y} dx\, dy = \sqrt{3} \int_0^a (a-y) dy \;= \frac{a^2\sqrt{3}}{2}$$

$$M_{xy} = \iint_S z\, ds = \iint_R (a-x-y)\sqrt{3}\, dA$$

$$= \sqrt{3} \int_0^a \int_0^{a-y} (a-x-y) dx\, dy$$

$$= \sqrt{3} \int_0^a \left[a(a-y) - \frac{(a-y)^2}{2} - y(a-y) \right] dy$$

$$= \sqrt{3} \int_0^a \left(\frac{a^2}{2} - ay + \frac{y^2}{2} \right) dy = \frac{a^3\sqrt{3}}{6}$$

$\bar{z} = \frac{M_{xy}}{m} = \frac{a}{3}$; then $\bar{x} = \bar{y} = \frac{a}{3}$ (by symmetry).

16. By using the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ we can conclude that the triangular surface is a portion of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, or

$z = f(x,y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$, over the region

$R_{xy} = \{(x,y) \mid 0 \le x \le a,\ 0 \le y \le b\left(1 - \frac{x}{a}\right)\}$. Since we are assuming a homogeneous surface, we will assume $\delta(x,y,z) = 1$.

a. $m = \iint\limits_S 1\, dS = \iint\limits_R \sqrt{f_x^2 + f_y^2 + 1}\, dA$

$$= \iint\limits_R \sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{c^2}{c^2}}\ dA$$

$$= \frac{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}{ab} \iint\limits_R 1\ dA$$

$$= \left(\frac{ab}{2}\right) \frac{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}{ab}$$

Let $w = \frac{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}{ab}$; then

$$m = \frac{abw}{2}$$

b. $M_{xy} = \iint\limits_S z\, dS = \iint\limits_R zw\, dA$

$$= w \int_0^a \int_0^{v_x} \left(c - \frac{c}{a}x - \frac{c}{b}y \right) dy\, dx$$

$$v_x = \frac{ab - x}{a}$$

$$= w \int_0^a \left(\frac{cb}{2} - \frac{cb}{a}x + \frac{cbx^2}{2a^2} \right) dx = \frac{wabc}{6}$$

c. Thus $\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{2(wabc)}{6(abw)} = \dfrac{c}{3}.$ In a like manner, using $y = g(x,z) = b\left(1 - \dfrac{x}{a} - \dfrac{z}{c}\right)$ over the region

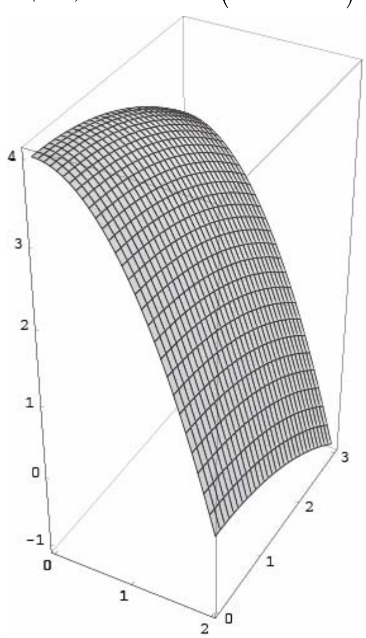
$R_{xz} = \{(x,z) \mid 0 \le x \le a,\ 0 \le z \le c\left(1 - \frac{x}{a}\right)\}$ and

$x = h(y,z) = a\left(1 - \dfrac{y}{b} - \dfrac{z}{c}\right)$ over the region

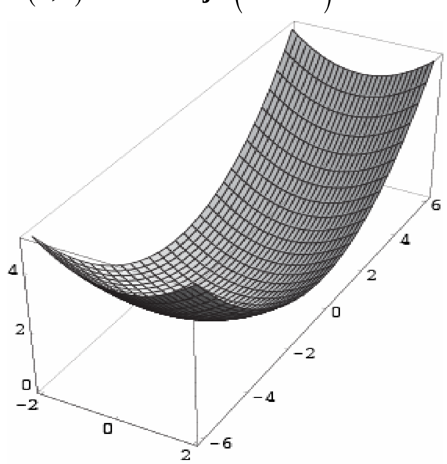
$R_{yz} = \{(y,z) \mid 0 \le y \le b,\ 0 \le z \le c\left(1 - \frac{y}{b}\right)\}$,

we can show $\bar{x} = \dfrac{a}{3}$ and $\bar{y} = \dfrac{b}{3}$ so the center of mass is $\left(\dfrac{a}{3}, \dfrac{b}{3}, \dfrac{c}{3}\right)$.

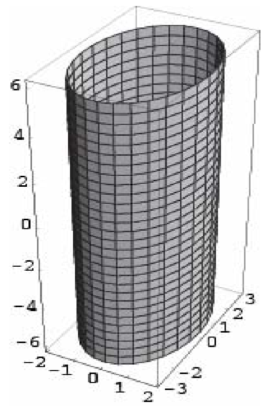
17. $\mathbf{r}(u,v) = u\,\mathbf{i} + 3v\,\mathbf{j} + \left(4 - u^2 - v^2\right)\mathbf{k}$



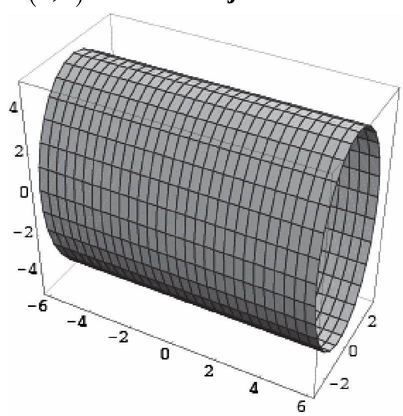
18. $\mathbf{r}(u,v) = 2u\,\mathbf{i} + 3v\,\mathbf{j} + \left(u^2 + v^2\right)\mathbf{k}$



19. $\mathbf{r}(u,v) = 2\cos v\,\mathbf{i} + 3\sin v\,\mathbf{j} + u\,\mathbf{k}$



20. $\mathbf{r}(u,v) = u\,\mathbf{i} + 3\sin v\,\mathbf{j} + 5\cos v\,\mathbf{k}$



21. $\mathbf{r_u}(u,v) = \sin v\,\mathbf{i} + \cos v\,\mathbf{j} + 0\mathbf{k}$,

$\mathbf{r_v}(u,v) = u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + 1\mathbf{k}$

$\mathbf{r_u} \times \mathbf{r_v} = \cos v\,\mathbf{i} - \sin v\,\mathbf{j} - u\,\mathbf{k}$

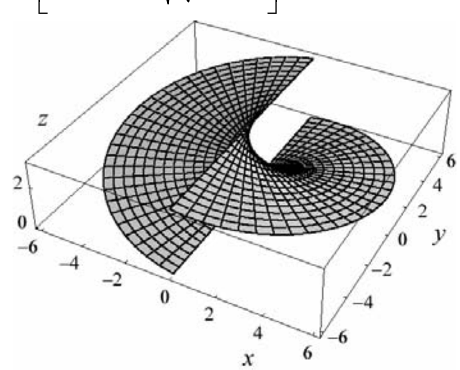
$\|\mathbf{r_u} \times \mathbf{r_v}\| = \sqrt{\cos^2 v + \sin^2 v + u^2} = \sqrt{1 + u^2}$

Using integration formula 44 in the back of the book we get

$$A = \int_{-6}^{6}\int_{0}^{\pi} \sqrt{u^2+1}\,dv\,du = \pi\int_{-6}^{6}\sqrt{u^2+1}\,du =$$

$$\pi\left[\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln\left|u + \sqrt{u^2+1}\right|\right]_{-6}^{6} =$$

$$\pi\left[6\sqrt{37} + \ln\sqrt{\frac{\sqrt{37}+6}{\sqrt{37}-6}}\right] \approx 122.49$$



22. $\mathbf{r_u}(u,v) = \cos u \sin v\,\mathbf{i} - \sin u \sin v\,\mathbf{j} + 0\mathbf{k}$,

$\mathbf{r_v}(u,v) = \sin u \cos v\,\mathbf{i} + \cos u \cos v\,\mathbf{j} + \cos v\,\mathbf{k}$

$$\mathbf{r_u}\times\mathbf{r_v} = -\sin u \sin v \cos v\,\mathbf{i} - \cos u \sin v \cos v\,\mathbf{j} + \sin v \cos v\,\mathbf{k}$$
$$= \sin v \cos v[-\sin u\,\mathbf{i} - \cos u\,\mathbf{j} + 1\mathbf{k}]$$

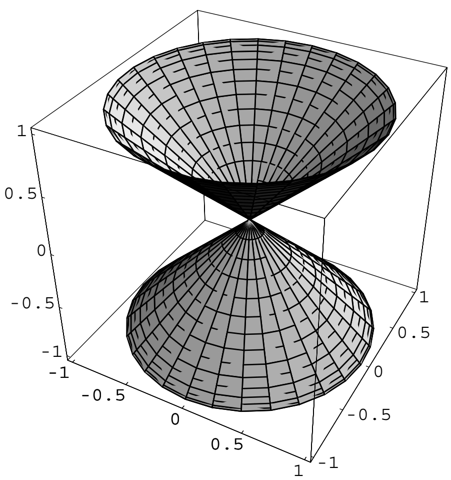
$$\|\mathbf{r_u}\times\mathbf{r_v}\| = |\sin v \cos v|\sqrt{\sin^2 u + \cos^2 u + 1}$$
$$= \sqrt{2}|\sin v \cos v| = \frac{\sqrt{2}}{2}|\sin 2v|$$

Thus

$$A = \frac{\sqrt{2}}{2}\int_0^{2\pi}\int_0^{2\pi}|\sin 2v|\,du\,dv = \sqrt{2}\pi\int_0^{2\pi}|\sin 2v|\,dv =$$

$$\sqrt{2}\pi\left[4\int_0^{\pi/2}\sin 2v\,dv\right] = 2\sqrt{2}\pi[-\cos 2v]_0^{\pi/2} =$$

$$4\sqrt{2}\pi \approx 17.77$$



23. $\mathbf{r_u}(u,v) = 2u\cos v\,\mathbf{i} + 2u \sin v\,\mathbf{j} + 5\mathbf{k}$,

$\mathbf{r_v}(u,v) = -u^2 \sin v\,\mathbf{i} + u^2 \cos v\,\mathbf{j} + 0\mathbf{k}$

$$\mathbf{r_u}\times\mathbf{r_v} = -5u^2\cos v\,\mathbf{i} - 5u^2 \sin v\,\mathbf{j} + 2u^3\,\mathbf{k}$$
$$= -u^2[5\cos v\,\mathbf{i} + 5\sin v\,\mathbf{j} - 2u\,\mathbf{k}]$$

$$\|\mathbf{r_u}\times\mathbf{r_v}\| = u^2\sqrt{25+4u^2}$$

Thus

$$A = \int_0^{2\pi}\int_0^{2\pi} u^2\sqrt{4u^2+25}\,dv\,du$$

$$= 2\pi\int_0^{2\pi} u^2\sqrt{4u^2+25}\,du$$

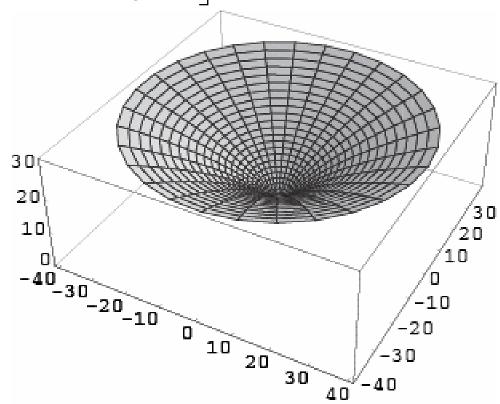
$$\underset{\substack{w=2u\\dw=2\,du}}{=} 2\pi\int_0^{4\pi}\frac{w^2}{4}\sqrt{w^2+(5)^2}\,\left(\tfrac{1}{2}dw\right) =$$

$$\frac{\pi}{4}\int_0^{4\pi} w^2\sqrt{w^2+(5)^2}\,dw.$$

Using integration formula 48 in the back of the book we get

$$A = \frac{\pi}{4}\left[\frac{w}{8}\left(2w^2+25\right)\sqrt{w^2+25} - \frac{625}{8}\ln\left|w+\sqrt{w^2+25}\right|\right]_0^{4\pi}$$

$$= \frac{\pi}{4}\left[\frac{\pi}{2}\left(32\pi^2+25\right)\sqrt{16\pi^2+25} - \frac{625}{8}\ln\left|4\pi+\sqrt{16\pi^2+25}\right| + \frac{625}{8}\ln 5\right] \approx 5585.42$$



24. $\mathbf{r_u}(u,v) = -\sin u \cos v\,\mathbf{i} - \sin u \sin v\,\mathbf{j} - \sin u\,\mathbf{k}$,

$\mathbf{r_v}(u,v) = -\cos u \sin v\,\mathbf{i} + \cos u \cos v\,\mathbf{j} + 0\mathbf{k}$

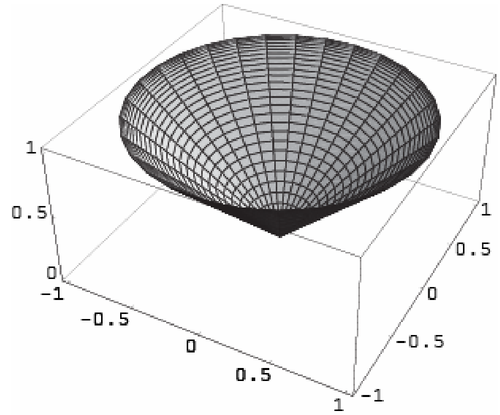
$$\mathbf{r_u}\times\mathbf{r_v} = \sin u \cos u \cos v\,\mathbf{i} + \sin u \cos u \sin v\,\mathbf{j} - \sin u \cos u\,\mathbf{k}$$
$$= \sin u \cos u[\cos v\,\mathbf{i} + \sin v\,\mathbf{j} - 1\mathbf{k}]$$

$$\|\mathbf{r_u}\times\mathbf{r_v}\| = |\sin u \cos u|\sqrt{\cos^2 v + \sin^2 v + 1}$$
$$= \sqrt{2}|\sin u \cos u| = \frac{\sqrt{2}}{2}|\sin 2u|$$

Thus (see problem 22)

$$A = \frac{\sqrt{2}}{2}\int_0^{\pi/2}\int_0^{2\pi}|\sin 2u|\,dv\,du = \sqrt{2}\pi\int_0^{\pi/2}\sin 2u\,du$$

$$= \frac{\sqrt{2}}{2}\pi[-\cos 2u]_0^{\pi/2} = \sqrt{2}\pi \approx 4.443$$



25. $\delta(x,y,z) = k|z| = 5ku \ (k>0)$. Thus

$$m = \int_0^{2\pi}\int_0^{2\pi}(5ku)\left(u^2\sqrt{4u^2+25}\right)dv\,du =$$

$$5k\int_0^{2\pi} u^3\sqrt{4u^2+25}\ du \underset{\substack{t=4u^2+25\\ dt=8u\,du}}{=}$$

$$\frac{5k}{8}\int_{25}^{16\pi^2+25}\left(\frac{t-25}{4}\right)\sqrt{t}\ dt =$$

$$\frac{5k}{32}\left[\frac{2}{5}t^{5/2}-\frac{50}{3}t^{3/2}\right]_{25}^{16\pi^2+25} \approx$$

$$\frac{5k}{32}[139760+833] \approx 21968k$$

26. a. $\delta(x,y,z) = k\sqrt{x^2+y^2} = k|\cos u|$

Thus

$$A = \int_0^{\pi/2}\int_0^{2\pi}(k\cos u)(\sqrt{2}\sin u\cos u)\ dv\,du$$

$$= 2\sqrt{2}\pi k\int_0^{\pi/2}\sin u\ \cos^2 u\ du$$

$$\underset{\substack{t=\cos u\\ dt=-\sin u\ du}}{=} -2\sqrt{2}\pi k\int_1^0 t^2\ dt$$

$$= \frac{2\sqrt{2}\pi k}{3} \approx 2.962k$$

b. $\delta(x,y,z) = k|z| = k|\cos u|$

Thus the density function is the same as in part a. and hence so is the mass: $\approx 2.962k$

27. $\mathbf{r_u} = -5\sin u\sin v\,\mathbf{i} + 5\cos u\sin v\,\mathbf{j} + 0\mathbf{k}$ and $\mathbf{r_v} = 5\cos u\cos v\,\mathbf{i} + 5\sin u\cos v\,\mathbf{j} + -5\sin v\,\mathbf{k}$.

Thus,

$$\mathbf{r_u}\times\mathbf{r_v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5\sin u\sin v & 5\cos u\sin v & 0 \\ 5\cos u\cos v & 5\sin u\cos v & -5\sin v \end{vmatrix}$$

$$= (-25\cos u\sin^2 v)\mathbf{i} + (-25\sin u\sin^2 v)\mathbf{j} +$$
$$(-25\sin^2 u\sin v\cos v - 25\cos^2 u\sin v\cos v)\mathbf{k}$$
$$= (-25\cos u\sin^2 v)\mathbf{i} + (-25\sin u\sin^2 v)\mathbf{j} +$$
$$+ (-25\sin v\cos v)\mathbf{k}$$
$$= (-25\sin v)[\cos u\sin v\,\mathbf{i} + \sin u\sin v\,\mathbf{j} + \cos v\,\mathbf{k}]$$

Thus:

$$\|\mathbf{r_u}\times\mathbf{r_v}\|$$
$$= |-25\sin v|\sqrt{(\cos u\sin v)^2 + (\sin u\sin v)^2 + \cos^2 v}$$
$$= 25|\sin v|\sqrt{[(\cos^2 u + \sin^2 u)\sin^2 v + \cos^2 v}$$
$$= 25|\sin v|\sqrt{\sin^2 v + \cos^2 v} = 25|\sin v|$$

28. $m = \iint_G z\,ds = \iint_R 3\,dA$

$(= 3A(R) = 3\pi(3)^2 = 27\pi$, ignoring the subtlety)

$$= \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{3-\varepsilon} 3r\,dr\,d\theta = \lim_{\varepsilon\to 0} 3(3-\varepsilon)^2\pi = 27\pi$$

29. a. 0 (By symmetry, since $g(x, y, -z) = -g(x, y, z)$.)

b. 0 (By symmetry, since $g(x, y, -z) = -g(x, y, z)$.)

c. $\iint_G (x^2+y^2+z^2)dS = \iint_G a^2 dS$

$= a^2\text{Area}(G) = a^2(4\pi a^2) = 4\pi a^4$

d. Note:

$\iint_G (x^2+y^2+z^2)dS = \iint_G x^2 dS + \iint_G y^2 dS$

$= \iint_G z^2 dS = 3\iint_G x^2 dS$

(due to symmetry of the sphere with respect to the origin.)

Therefore,

$$\iint_G x^2 dS = \left(\frac{1}{3}\right)\iint_G (x^2+y^2+z^2)dS$$

$$= \left(\frac{1}{3}\right)4\pi a^4 = \frac{4\pi a^4}{3}.$$

e. $\iint_G (x^2+y^2)dS = \left(\frac{2}{3}\right)4\pi a^4 = \dfrac{8\pi a^4}{3}$

30. a. Let the diameter be along the z-axis.

$I_z = \iint_G k(x^2+y^2)dS$

1. $\iint_G x^2 dS = \iint_G y^2 dS = \iint_G z^2 dS$ (by symmetry of the sphere)

2. $\iint_G (x^2+y^2+z^2)dS = \iint_G a^2 dS$

$= a^2$(Area of sphere)$=a^2(4\pi a^2) = 4\pi a^4$

Thus,

$$I_z = \iint_G k(x^2+y^2)dS = \frac{2}{3}k(4\pi a^4) = \frac{8\pi a^4 k}{3}.$$

(using 1 and 2)

b. Let the tangent line be parallel to the z-axis.

Then $I = I_z + ma^2 = \dfrac{8\pi a^4 k}{3} + [k(4\pi a^2)]a^2$

$= \dfrac{20\pi a^4 k}{3}.$

31. a. Place center of sphere at the origin.

$F = \iint_G k(a-z)dS = ka\iint_G 1\,dS - k\iint_G z\,dS$

$= ka(4\pi a^2) - 0 = 4\pi a^3 k$

b. Place hemisphere above xy-plane with center at origin and circular base in xy-plane.
F = Force on hemisphere + Force on circular base

$= \iint_G k(a-z)dS + ka(\pi a^2)$

$= ka\iint_G 1\,dS - k\iint_G z\,dS + \pi a^3 k$

$= ka(2\pi a^2) - k\iint_R z\sqrt{\dfrac{a^2}{a^2-x^2-y^2}}\,dA + \pi a^3 k$

$= 3\pi a^3 k - k\iint_R z\dfrac{a}{z}\,dA$

$= 3\pi a^3 k - ka(\pi a^2) = 2\pi a^3 k$

c. Place the cylinder above xy-plane with circular base in xy-plane with the center at the origin.
F = Force on top + Force on cylindrical side + Force on base

$= 0 + \iint_G k(h-z)dS + kh(\pi a^2)$

$= kh\iint_G 1\,dS - k\iint_G z\,dS + \pi a^2 hk$

$= kh(2\pi ah) - 4k\iint_R z\sqrt{\dfrac{a^2}{a^2-y^2} + 0 + 1}\,dA + \pi a^2 hk$

(where R is a region in the yz-plane: $0 \le y = a, 0 \le z \le h$)

$= 2\pi ah^2 k + \pi a^2 hk - 4k\int_0^a \int_0^h \dfrac{az}{\sqrt{a^2-y^2}}\,dz\,dy$

$= 2\pi ah^2 k + \pi a^2 hk - \pi kah^2$

$= \pi ah^2 k + \pi a^2 hk = \pi ahk(h+a)$

32. $\bar{x} = \bar{y} = 0$

Now let G' be the 1st octant part of G.

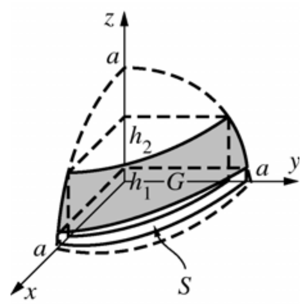
$M_{xy} = \iint_G k\,dS = 4\iint_{G'} kz\,dS = 4k\iint_{R'} z\left(\dfrac{a}{z}\right)dA$

(See Problem 19b.)

$= 4ak\ [\text{Area }(R')\ ]$

$= 4ak\pi\left[\dfrac{(a^2-h_1^2)}{4} - \dfrac{(a^2-h_2^2)}{4}\right]$

$= ak\pi(h_2^2 - h_1^2)$



$m(G) = \iint_G k\,dS = k[\text{Area}(G)]$

$= k[2\pi a(h_2 - h_1)] = 2\pi ak(h_2 - h_1)$

Therefore, $\bar{z} = \dfrac{\pi ak(h_2^2 - h_1^2)}{2\pi ak(h_2 - h_1)} = \dfrac{h_1 + h_2}{2}.$

14.6 Concepts Review

1. boundary; ∂S

2. $\mathbf{F} \cdot \mathbf{n}$

3. div $\mathbf{F}$

4. flux; the shape

Problem Set 14.6

1. $\iiint_S (0+0+0)dV = 0$

2. $\iiint_S (1+2+3)dV = 6V(S) = 6$

3. $\iint_{\partial S} \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_S (M_x + N_y + P_z)dV$

$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(0+1+0)dx\,dy\,dz = 8$

4. $\iint_{\partial S} \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_S (M_x + N_y + P_z)dV$

$= 3\iiint_S (x^2 + y^2 + z^2)\,dx\,dy\,dz$

Converting to spherical coordinates we have

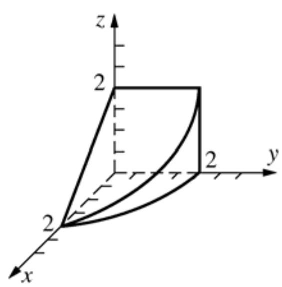
$$3\iiint_S (x^2+y^2+z^2)\,dx\,dy\,dz$$
$$=3\iiint_S \rho^2(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$
$$=3\int_0^{\pi}\int_0^{2\pi}\int_0^{a}(\rho^4\sin\phi)\,d\rho\,d\theta\,d\phi$$
$$=\frac{3a^5}{5}\int_0^{\pi}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi$$
$$=\frac{6\pi a^5}{5}\int_0^{\pi}\sin\phi\;d\phi=\frac{12\pi a^5}{5}$$

5. $\iint_{\partial S}\mathbf{F}\cdot\mathbf{n}\,dS=\iiint_S (M_x+N_y+P_z)dV\ =\int_0^c\int_0^b\int_0^a(2xyz+2xyz+2xyz)dx\,dy\,dz\ =\int_0^c\int_0^b 3a^2yz\,dy\,dz=\int_0^c\frac{3a^2b^2z}{2}dz$
$=\dfrac{3a^2b^2c^2}{4}$

6. $\iiint_S(3-2+4)dV=5V(S)=5\left[\left(\frac{4}{3}\right)\pi(3)^3\right]=180\pi=565.49$

7. $2\iiint_S(x+y+z)dV\ =2\int_0^{2\pi}\int_0^2\int_0^{4-r^2}(r\cos\theta+r\sin\theta+z)r\,dz\,dr\,d\theta\ =\dfrac{64\pi}{3}\approx 67.02$

8.



$\iiint_S(M_x+N_y+P_z)dV=\iiint_S(2x+1+2z)dV\ =\int_0^{2\pi}\int_0^2\int_0^{2-r\cos\theta}(2r\cos\theta+1+2z)r\,dz\,dr\,d\theta$
$=\int_0^{2\pi}\int_0^2[(2r^2\cos\theta+r)(2-r\cos\theta)+r(2-r\cos\theta)^2]dr\,d\theta\ =\int_0^{2\pi}\int_0^2(6r-r^3\cos^2\theta-r^2\cos\theta)dr\,d\theta$
$=\int_0^{2\pi}\left(12-4\cos^2\theta-\dfrac{8\cos\theta}{3}\right)d\theta=20\pi$

9. $\iiint_S(1+1+0)dV=2(\text{volume of cylinder})\ =2\pi(1)^2(2)=4\pi\approx 12.5664$

10. $\iiint_S(2x+2y+2z)dV=\int_0^4\int_0^{4-x}\int_0^{4-x-y}(2x+2y+2z)dz\,dy\,dx\ =64$

11. $\iiint_S(M_x+N_y+P_z)dV=\iiint_S(2+3+4)dV$
$=9(\text{Volume of spherical shell})$
$=9\left(\dfrac{4\pi}{3}\right)(5^3-3^3)=1176\pi\approx 3694.51$

12. $\iiint_S(0+0+2z)dV=\int_0^{2\pi}\int_1^2\int_0^2 2zr\,dz\,dr\,d\theta$
$=12\pi\approx 37.6991$

13. $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S (M_x + N_y + P_z)\,dS =$

$\iiint_S (0+2y+0)\,dV = 2\iiint_S y\,dx\,dy\,dz$

Using the change of variable (from (x,y,z) to (r,y,θ)) defined by $x = r\cos\theta,\ y = y,\ z = r\sin\theta$ yields the Jacobian

$$J(r,y,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -r\sin\theta & 0 & r\cos\theta \end{vmatrix} =$$

$r\cos^2\theta + r\sin^2\theta = r$. Further, the region S is now defined by $r^2 \le 1,\ 0 \le y \le 10$. Hence, by the change of variable formula in Section 13.9,

$$2\iiint_S y\,dx\,dy\,dz = 2\int_0^{2\pi}\int_0^{10}\int_0^1 yr\,dr\,dy\,d\theta =$$

$$\int_0^{2\pi}\int_0^{10} y\,dy\,d\theta = \int_0^{2\pi} 50\,d\theta = 100\pi \approx 314.16$$

14. $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S (M_x + N_y + P_z)\,dV =$

$3\iiint_S (x^2+y^2+z^2)\,dx\,dy\,dz$

Use the change of variable (basically spherical coordinates with the role of z and y interchanged and maintaining a right handed system):
$x = \rho\sin\phi\sin\theta,\ y = \rho\cos\phi,\ z = \rho\sin\phi\cos\theta$

Then the region S becomes

$\rho^2 \le 1 \quad (x^2+y^2+z^2 \le 1)$

$0 \le \phi \le \frac{\pi}{2} \quad (y \ge 0)$

$\sin^2\phi \le \frac{1}{2} \quad (x^2+z^2 \le y^2)$ so that

$\rho \in [0,1],\ \ \phi \in [0,\frac{\pi}{4}],\ \ \theta \in [0,2\pi]$. The Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\sin\theta & \cos\phi & \sin\phi\cos\theta \\ \rho\sin\phi\cos\theta & 0 & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & -\rho\sin\phi & \rho\cos\phi\cos\theta \end{vmatrix}$$

$$= -\rho^2\sin^3\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\cos^2\theta - \rho^2\cos^2\phi\sin\phi\sin^2\theta - \rho^2\sin^3\phi\cos^2\theta =$$

$$-\rho^2\sin\phi\left[\sin^2\phi\left(\sin^2\theta+\cos^2\theta\right)+\cos^2\phi\left(\sin^2\theta+\cos^2\theta\right)\right]$$

$= -\rho^2\sin\phi$.

Thus,

$$3\iiint_S (x^2+y^2+z^2)\,dx\,dy\,dz$$

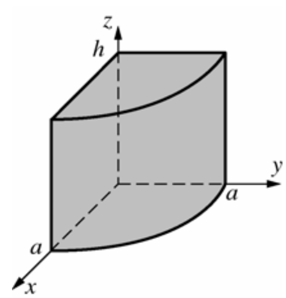
$$= 3\int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^2(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$

$$= \frac{3}{5}\int_0^{\pi/4}\int_0^{2\pi} \sin\phi\ d\theta\,d\phi = \frac{6\pi}{5}\int_0^{\pi/4} \sin\phi\,d\phi$$

$$= \frac{6\pi}{5}\left(\frac{2-\sqrt{2}}{2}\right) \approx 1.104$$

15. $\left(\frac{1}{3}\right)\iiint_S (1+1+1)dV = V(S)$

16.



$V(S) = \frac{1}{3}\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS$ for $\mathbf{F} = \langle x,y,z\rangle$

$$= \frac{1}{3}\iiint_S 3\,dV = \int_0^{2\pi}\int_0^a\int_0^h r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^a rh\,dr\,d\theta = \int_0^{2\pi}\frac{a^2h}{2}d\theta = 2\pi\frac{a^2h}{2}$$

$= \pi a^2 h$

17. Note:

1. $\iint_R (ax+by+cz)dS = \iint_R d\,dS = dD$ (R is the slanted face.)

2. $\mathbf{n} = \dfrac{\langle a, b, c\rangle}{(a^2+b^2+c^2)^{1/2}}$ (for slanted face)

3. $\mathbf{F}\cdot\mathbf{n} = 0$ on each coordinate-plane face.

Volume $= \left(\dfrac{1}{3}\right)\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ (where $\mathbf{F} = \langle x, y, z\rangle$).

$= \left(\dfrac{1}{3}\right)\iint_R \mathbf{F}\cdot\mathbf{n}\,dS$ (by Note 3)

$= \left(\dfrac{1}{3}\right)\iint_R \dfrac{(ax+by+cz)}{\sqrt{a^2+b^2+c^2}}dS$

$= \dfrac{dD}{3\sqrt{a^2+b^2+c^2}}$

18. $\iiint_S \operatorname{div}\mathbf{F}\,dV = \iiint_S 0\,dV = 0$ ("Nice" if there is an outer normal vector at each point of ∂S.)

19. **a.** div **F** = 2 + 3 + 2*z* = 5 + 2*z*

$\iint_{\partial S}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S (5+2z)dV \;= \iiint_S 5\,dV + 2\iiint_S z\,dV \;= 5\text{ (Volume of }S) + 2M_{xy}$

$= 5\left(\dfrac{4\pi}{3}\right) + 2z\text{ (Volume of }S)\; = \dfrac{20\pi}{3} + 2(0)(\text{Volume}) = \dfrac{20\pi}{3}$

b. $\mathbf{F}\cdot\mathbf{n} = (x^2+y^2+z^2)^{3/2}\langle x, y, z\rangle\cdot\dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}} \;= (x^2+y^2+z^2)^2 = 1$ on ∂S.

$\iint_{\partial S}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S 1\,dV = 4\pi(1)^2 = 4\pi$

c. div **F** = 2*x* + 2*y* + 2*z*

$\iint_{\partial S}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S 2(x+y+z)dV$

$= 2\iiint_S x\,dV$ (Since $\bar{x} = \bar{z} = 0$ as in a.)

$= 2M_{yz} = 2(\bar{x})(\text{Volume of }S) = 2(2)\left(\dfrac{4\pi}{3}\right) = \dfrac{16\pi}{3}$

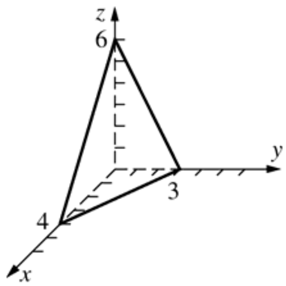
d. $\mathbf{F}\cdot\mathbf{n} = 0$ on each face except the face R in the plane $x = 1$.

$\iint_{\partial S}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,dS \;= \iint_R \langle 1, 0, 0\rangle\cdot\langle 1, 0, 0\rangle\,dS = \iint_R 1\,dS \;= (1)^2 = 1$

e. div **F** = 1 + 1 + 1 = 3

$\iint_{\partial S}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S 3\,dV = 3(\text{Volume of }S) = 3\left(\dfrac{1}{3}\left[\dfrac{1}{2}(4)(3)\right](6)\right) = 36$

f.



$\text{div}\,\mathbf{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3$ on ∂S.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = 3\iiint_S (x^2 + y^2 + z^2)dV = 3\left(\frac{3}{2}\frac{8\pi}{15}\right) = \frac{12\pi}{5}$

(That answer can be obtained by making use of symmetry and a change to spherical coordinates. Or you could go to the solution for Problem 22, Section 13.9, and realize that the value of the integral in this problem is $\frac{3}{2}$.

g. $\mathbf{F}\cdot\mathbf{n} = [\ln(x^2 + y^2)]\langle x,\ y,\ 0\rangle \cdot \langle 0,\ 0,\ 1\rangle = 0$ on top and bottom.

$\mathbf{F}\cdot\mathbf{n} = (\ln 4)\langle x,\ y,\ 0\rangle \cdot \frac{\langle x,\ y,\ 0\rangle}{\sqrt{x^2 + y^2}} = (\ln 4)\sqrt{x^2 + y^2} = (\ln 4)\sqrt{4} = 2\ln 4 = 4\ln 2$ on the side.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R 4\ln 2\,dS = (4\ln 2)[2\pi(2)(2)] = 32\pi \ln 2$

20. **a.** $\text{div}\,\mathbf{F} = 0$ (See Problem 21, Section 14.1.)
Therefore, $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_S \text{div}\,\mathbf{F}\,dV = 0.$

b. $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi$ (by Gauss's law with $-cM = 1$ as in Example 5).

c. $\mathbf{F}\cdot\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{|\mathbf{r}|} = \frac{1}{a}$ on ∂S.

Thus, $\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = \left(\frac{1}{a}\right)\iint_{\partial S} 1\,dS$

$= \left(\frac{1}{a}\right)$(Surface area of sphere) $= \left(\frac{1}{2}\right)(4\pi a^2) = 4\pi a$.

d. $\mathbf{F}\cdot\mathbf{n} = f\left(|r|\right) r \cdot \frac{r}{|r|} = |r|\, f\left(|r|\right) = af(a)$ on ∂S.

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = af(a)\iint_{\partial S} 1\,dS = [af(a)](4\pi a^2) = 4\pi a^3 f(a)$

e. The sphere is above the *xy*-plane, is tangent to the *xy*-plane at the origin, and has radius $\frac{a}{2}$.

$\text{div}\,\mathbf{F} = |\mathbf{r}|^n\,\text{div}\,\mathbf{r} + \left(\text{grad}\,|f|^n\right)\cdot\mathbf{r}$ (See Problem 20c, Section 14.1.)

$= |\mathbf{r}|^n (1+1+1) + n|f|^{n-1}\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{r} = 3|\mathbf{r}|^n + n|\mathbf{r}|^n = (3+n)|\mathbf{r}|^n$

$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\,dS = (3+n)\iiint_S |\mathbf{r}|^n\,dV = (3+n)\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a\cos\phi} \rho^n(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \frac{2\pi a^{n+3}}{n+4}$

21. $\iint_{\partial S} D_{\mathbf{n}} f\,dS = \iint_{\partial S} \nabla f\cdot\mathbf{n}\,dS = \iiint_S \text{div}(\nabla f)dV = \iiint_S \nabla^2 f\,dV$ (See next problem.)

22. $\iint_{\partial S} f(\nabla f \cdot \mathbf{n})dS = \iint_{\partial S} (f\nabla f)\cdot \mathbf{n}\, dS \;\; = \iiint_S \text{div}(f\nabla f)dV$

$= \iiint_S \text{div}(\nabla f) + (\nabla f)\cdot(\nabla f)dV$ (See Problem 20c, Section 14.1.)

$= \iiint_S \left[(f_{xx} + f_{yy} + f_{zz}) + |\nabla f|^2 \right] dV$

$= \iiint_S [(\nabla^2 f) + |\nabla f|^2] dV = \iiint_S |\nabla f|^2\, dV$ (Since it is given that $\nabla^2 f = 0$ on S.)

23. $\iint_{\partial S} fD_{\mathbf{n}} g\, dS = \iint_{\partial S} f(\nabla g \cdot \mathbf{n})dS \;\; = \iint_{\partial S} (f\nabla g)\cdot \mathbf{n}\, dS = \iiint_S \text{div}(f\nabla g)dV$ (Gauss)

$= \iiint_S [f(\text{div}\,\nabla g) + (\nabla f)\cdot(\nabla g)]dV \;\; = \iiint_S (f\nabla^2 g + \nabla f \cdot \nabla g)dV$ (See Problem 20c, Section 14.1.)

24. $\iint_{\partial S} (fD_{\mathbf{n}} g - gD_{\mathbf{n}} f)dS \; = \iint_{\partial S} fD_{\mathbf{n}} g\, dS - \iint_{\partial S} gD_{\mathbf{n}} f\, dS$

$= \iiint_S (f\nabla^2 g + \nabla f \cdot \nabla g)dV - \iiint_S (g\nabla^2 f + \nabla g \cdot \nabla f)dV$ (by Green's 1st identity)

$= \iiint_S (f\nabla^2 g - g\nabla^2 f)dV$

14.7 Concepts Review

1. $(\text{curl}\,\mathbf{F})\cdot\mathbf{n}$

2. Möbius band

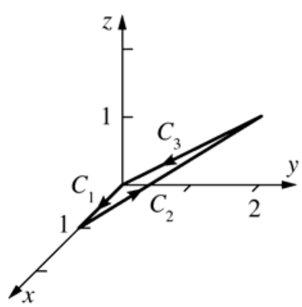
3. $\iint\limits_{S} (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\, dS$

4. curl $\mathbf{F}$

Problem Set 14.7

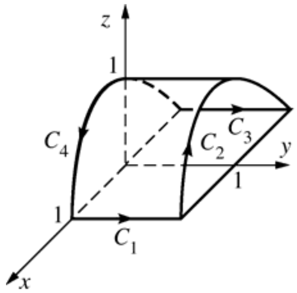
1. $\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\, dS = \iint_R (N_x - M_y)dA = \iint_R 0\, dA = 0$

2.



$\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}\, ds = \int_{C_1} 0\, dx + \int_{C_2} xy\, dx + yz\, dy + xz\, dz + \int_{C_3} yz\, dy \; = \int_0^1 (t^2 + 7t - 4)dt = -\frac{1}{6}$

3.



$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds \;\; = \oint_{\partial S} (y+z)dx + (x^2+z^2)dy + y\, dz$

$= \int_0^1 1\, dt + \int_0^{\pi} [(1+\sin t)(-\sin t) + \cos t]dt + \int_0^1 -1\, dt + \int_0^{\pi} \sin^2 t\, dt$ (*)

$= \int_0^{\pi} (-\sin t + \cos t)dt = -2$

The result at (*) was obtained by integrating along S by doing so along C_1, C_2, C_3, C_4 in that order.

Along C_1: $x = 1, y = t, z = 0, dx = dz = 0, dy = dt, t$ in $[0, 1]$

Along C_2: $x = \cos t, y = 1, z = \sin t, dx = -\sin t\, dt, dy = 0, dz = \cos t\, dt, t$ in $[0, \pi]$

Along C_3: $x = -1, y = 1 - t, z = 0, dx = dz = 0, dy = dt, t$ in $[0, 1]$

Along C_4: $x = -\cos t, y = 0, z = \sin t, dx = \sin t\, dt, dy = 0, dz = \cos t\, dt, t$ in $[0, \pi]$

4. ∂S is the circle $x^2 + y^2 = 1,\ z = 0$ (in the xy-plane).

$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_{\partial S} xy^2 dx + x^3 dy = (\cos xz)dz$

$= \oint_S x^3 dy = \int_0^{2\pi} (\cos^3 t)(-\cos t)dt = \left(-\frac{3}{4}\right)\pi$

≈ -2.3562

5. ∂S is the circle $x^2 + y^2 = 12,\ z = 2$.

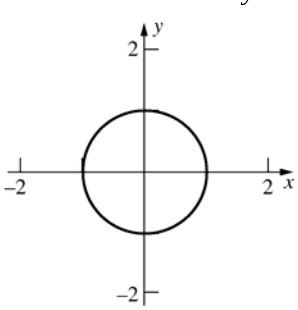
Parameterization of circle:

$x = \sqrt{12}\sin t, y = \sqrt{12}\cos t,\ z = 2,\ t$ in $[0, 2\pi]$

$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_{\partial S} yz\, dx + 3xz\, dy + z^2 dz$

$= \int_0^{2\pi} (24\sin^2 t - 72\cos^2 t)dt = -48\pi \approx -150.80$

6. ∂S is the circle $x^2 + y^2 = 1,\ z = 0$



$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds$

$= \oint_S (z-y)dx + (z+x)dy + (-x-y)dz$

$x = \cos\ t;\ \ y = \sin t;\ z = 0;\ t$ in $[0, 2\pi]$

$= \int_0^{2\pi} [(-\sin t)(-\sin t) + (\cos t)(\cos t)]dt$

$= 2\pi \approx 6.2832$

7. $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle 3, 2, 1\rangle \cdot \left\langle \left(\frac{1}{\sqrt{2}}\right)1, 0, -1\right\rangle = \sqrt{2}$

$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \sqrt{2}dS = \sqrt{2}A(S)$

$= \sqrt{2}[\sec(45°)]$(Area of a circle)$=8\pi \approx 25.1327$

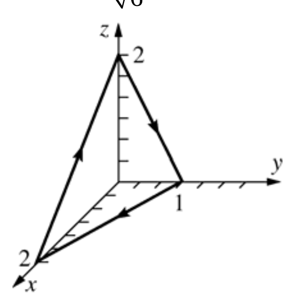
8. $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle -1, -1, -1\rangle \cdot \left[\left(\frac{1}{\sqrt{2}}\right)\langle 0, 1, -1\rangle\right] = 0,$

so the integral is 0.

9. $(\text{curl } \mathbf{F}) = \langle -1+1, 0-1, 1-1\rangle = \langle 0, -1, 0\rangle$

The unit normal vector that is needed to apply Stokes' Theorem points downward. It is

$n = \frac{\langle -1, -2, -1\rangle}{\sqrt{6}}.$



$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$= \iint_S \left(\frac{2}{\sqrt{6}}\right) dS = \iint_R \left(\frac{2}{\sqrt{6}}\right)(1+4+1)^{1/2} dA$

$= \iint_R 2\, dA = 2$(Area of triangle in $xy-$plane)

$= 2(1) = 2$

10. $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \left\langle 0, 0, -4x^2 - 4y^2 \right\rangle \cdot \left[\left(\frac{1}{\sqrt{2}} \right) \left\langle -1, 0, 1 \right\rangle \right]$

$= -2\sqrt{2}(x^2 + y^2)$

$\iint_S -\frac{2}{\sqrt{2}}(x^2 + y^2)dS$

$= -4\int_0^1 \int_0^1 (x^2 + y^2) dx\, dy = -\frac{8}{3}$

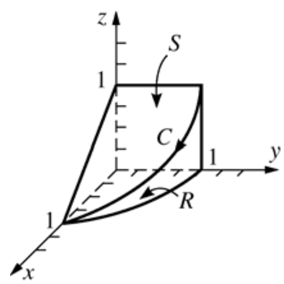
11. $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle 0, 0, 1 \rangle \cdot \langle x, y, z \rangle = z$

$\iint_S z\, dS = \iint_R 1 dA = \text{Area of } R$

$\pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4} \approx 0.7854$

12. $(\text{curl } \mathbf{F}) = \langle -1-1, -1-1, -1-1 \rangle = -2\langle 1, 1, 1 \rangle$,

$\mathbf{n} = \frac{\langle 1, 0, 1 \rangle}{\sqrt{2}}$, so $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = -2\sqrt{2}$.



$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$= -2\sqrt{2} \iint_S 1 dS = -2\sqrt{2} \iint_R (\sec 45°) dA$

$= -\frac{2}{\sqrt{2}\sqrt{2}}[A(R)] = -4\pi$

13. Let $H(x, y, z) = z - g(x, y) = 0$.

Then $\mathbf{n} = \frac{\nabla H}{|\nabla H|} = \frac{\langle -g_x, g_y, 1 \rangle}{\sqrt{1 + g_x^2 + g_y^2}}$ points upward.

Thus,

$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \sec \gamma\, dA$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{g_x^2 + g_y^2 + 1}} \sqrt{g_x^2 + g_y^2 + 1}\, dA$

(Theorem A, Section 14.5)

$= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA$

14. $\text{curl } \mathbf{F} = \langle z^2, 0, -2y \rangle$

$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$ (Stoke's Theorem)

$= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle\, dA$ (Problem 13)

$= \iint_{S_{xy}} \langle z^2, 0, -2y \rangle \cdot \langle -y, -x, 1 \rangle dA$

(where $z = xy$)

$= \int_0^1 \int_0^1 (-x^2y^3 - 2y) dx\, dy = -\frac{13}{12}$

15. curl $F = \langle 0 - x, 0 - 0, z - 0 \rangle = \langle -x, 0, z \rangle$

$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA$ where

$z = g(x, y) = xy^2$. (Problem 13)

$= \iint_{S_{xy}} \langle -x, 0, z \rangle \cdot \langle -y^2, -2xy, 1 \rangle dA$

$= \int_0^1 \int_0^1 (xy^2 + 0 + xy^2) dx\, dy$

$= \int_0^1 \left([x^2y^2]_{x=0}^1 \right) dy = \int_0^1 y^2 dy = \frac{1}{3}$

16. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA$

$= \iint_{S_{xy}} \langle -x, 0, z \rangle \cdot \langle -2xy^2, -2x^2y, 1 \rangle dA$

(where $z = x^2y^2$)

$= \iint_{S_{xy}} 3x^2y^2 dA$

$= 12\int_0^{\pi/2} \int_0^a (r\cos\theta)^2 (r\sin\theta)^2 r\, dr\, d\theta = \frac{\pi a^6}{8}$

17. $\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$= \iint_{S_{xy}} (\text{curl } \mathbf{F}) \cdot \langle -g_x, -g_y, 1 \rangle dA$

$= \iint_{S_{xy}} \langle 2, 2, 0 \rangle \cdot \left[\frac{\langle x, y, (a^2 - x^2 - y^2)^{-1/2} \rangle}{(a^2 - x^2 - y^2)^{-1/2}} \right] dA$

$= 2\iint_{S_{xy}} (x + y)(a^2 - x^2 - y^2)^{-1/2} dA$

$= 2\iint_{S_{xy}} y(a^2 - x^2 - y^2)^{-1/2} dA$

$= 4\int_0^{\pi/2} \int_0^{a\sin\theta} (r\sin\theta)(a^2 - r^2)^{-1/2} dr\, d\theta$

$= \frac{4a^2}{3}$ joules

18. curl $\mathbf{F} = 0$ by Problem 23, Section 14.1. The result then follows from Stokes' Theorem since the left-hand side of the equation in the theorem is the work and the integrand of the right-hand side equals 0.

19. **a.** Let C be any piecewise smooth simple closed oriented curve C that separates the "nice" surface into two "nice" surfaces, S_1 and S_2.

$$\iint_{\partial S} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$
$$= \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$
$$= \oint_C \mathbf{F} \cdot \mathbf{T}\, ds + \oint_{-C} \mathbf{F} \cdot \mathbf{T}\, ds = 0$$ ($-C$ is C with opposite orientation.)

b. div(curl **F**) = 0 (See Problem 20, Section 14.1.) Result follows.

20. $$\oint_S (f\nabla g) \cdot \mathbf{T} ds = \iint_S \text{curl}(f\nabla g) \cdot \mathbf{n}\, dS$$
$$= \iint_S [f(\text{curl } \nabla g) + (\nabla f \times \nabla g)] \cdot \mathbf{n}\, dS$$
$$= \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, dS, \text{ since curl } \nabla g = 0.$$
(See 20b, Section 14.1.)

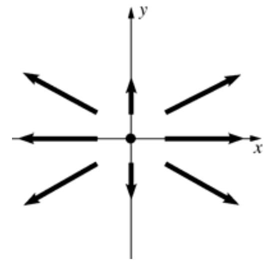
14.8 Chapter Review

Concepts Test

1. True: See Example 4, Section 14.1
2. False: It is a scalar field.
3. False: grad(curl **F**) is not defined since curl **F** is not a scalar field.
4. True: See Problem 20b, Section 14.1.
5. True: See the three equivalent conditions in Section 14.3.
6. True: See the three equivalent conditions in Section 14.3.
7. False: $N_z = 0 \neq z^2 = P_y$
8. True: See discussion on text page 750.
9. True: It is the case in which the surface is in a plane.
10. False: See the Mobius band in Figure 6, Section 14.5.
11. True: See discussion on text page 752.
12. True: div **F** = 0, so by Gauss's Divergence Theorem, the integral given equals $\iiint_D 0\, dV$ where D is the solid sphere for which $S = \partial D$.

Sample Test Problems

1.



2. div $\mathbf{F} = 2yz - 6y + 2y^2$
curl $\mathbf{F} = \langle 4yz, 2xy, -2xz \rangle$
grad(div $\mathbf{F}$) = $\langle 0, 2z - 6 + 4y, 2y \rangle$
div(curl **F**) = 0
(See 20a, Section 14.1.)

3. $\text{curl}(f\nabla f) = (f)(\text{curl } \nabla f) + (\nabla f \times \nabla f)$
$= (f)(\mathbf{0}) + \mathbf{0} = \mathbf{0}$

4. **a.** $f(x, y) = x^2 y + xy + \sin y + C$

b. $f(x, y, z) = xyz + e^{-x} + e^y + C$

5. **a.** Parameterization is $x = \sin t$, $y = -\cos t$, t in $\left[0, \frac{\pi}{2}\right]$.
$$\int_0^{\pi/2} (1 - \cos^2 t)(\sin^2 t + \cos^2 t)^{1/2} dt = \frac{\pi}{4}$$
≈ 0.7854

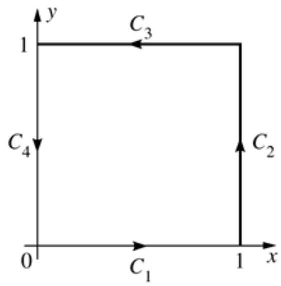
b $$\int_0^{\pi/2} [t \cos t - \sin^2 t \cos t + \sin t \cos t] dt$$
$$= \frac{(3\pi - 5)}{6} \approx 0.7375$$

6. $M_x = 2y = N_y$ so the integral is independent of the path. Find any function $f(x, y)$ such that $f_x(x, y) = y^2$ and $f_y(x, y) = 2xy$.
$f(x, y) = xy^2 + C_1(y)$ and
$f(x, y) = xy^2 + C_2(x)$, so let $f(x, y) = xy^2$.
Then the given integral equals $[xy^2]_{(0, 0)}^{(1, 2)} = 4$.

7. $[xy^2]_{(1, 1)}^{(3, 4)} = 47$

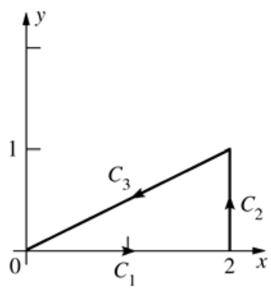
8. $[xyz + e^{-x} + e^y]_{(0, 0, 0)}^{(1, 1, 4)} = 2 + e^{-1} + e \approx 5.0862$

9. **a.**



$$\int_0^1 0\,dx + \int_0^1 (1+y^2)\,dy + \int_1^0 x\,dx + \int_1^0 y^2\,dy = 0 + \frac{4}{3} - \frac{1}{2} - \frac{1}{3} = \frac{1}{2}$$

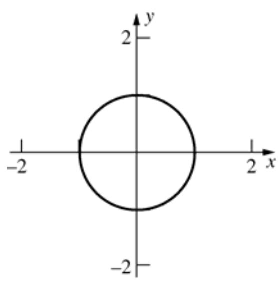
b.



A vector equation of C_3 is $\langle x, y\rangle = \langle 2,1\rangle + t\langle -2,-1\rangle$ for t in [0, 1], so let $x = 2 - 2t$, $y = 1 - t$ for t in [0, 1] be parametric equations of C_3.

$$\int_0^2 0\,dx + \int_0^1 (4+y^2)\,dy + \int_0^1 [2(1-t)^2(-2) + 5(1-t)^2(-1)]\,dt = 0 + \frac{13}{3} - 3 = \frac{4}{3}$$

c.



$x = \cos t$
$y = -\sin t$
t in $[0, 2\pi]$

$$\int_0^{2\pi} [(\cos t)(\sin t)(-\sin t) + (\cos^2 t + \sin^2 t)(\cos t)]\,dt = \int_0^{2\pi} (1 - \sin^2 t)\cos t\,dt = \left[\sin t - \frac{\sin^3 t}{3}\right]_0^{2\pi} = 0$$

10. $\iint_S \operatorname{div} \mathbf{F}\,dA = \iint_S 2\,dA = 2A(S) = 8$

11. Let $f(x, y) = (1 - x^2 - y^2)$ and $g(x, y) = -(1 - x^2 - y^2)$, the upper and lower hemispheres.

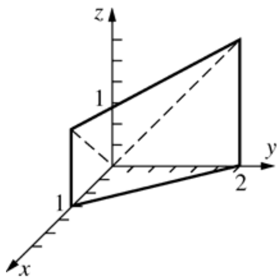
Then Flux $= \iint_G \mathbf{F}\cdot\mathbf{n}\,dS$

$= \iint_R [-Mf_x - Nf_y + P]\,dA + \iint_R [-Mg_x - Ng_y + P]\,dA = \iint_R 2P\,dA$ (since $f_x = -g_x$ and $f_y = -g_y$)

$= \iint_R 6\,dA = 6$ (Area of R, the circle $x^2 + y^2 = 1$, $z = 0$)

$= 6\pi \approx 18.8496$

12.



$$\iint_G xyz\,dS = \iint_R xy(x+y)(\sec)dA = \sqrt{3}\int_0^1\int_0^{-2x+2}(x^2y+xy^2)dy\,dx \quad = \sqrt{3}\int_0^1 \frac{4x^2(1-x)^2}{2}+\frac{8x(1-x)^3}{3}dx$$

$$= -\frac{2\sqrt{3}}{3}\int_0^1(x^4-6x^3+9x^2-4x)dx \quad = -\frac{2\sqrt{3}}{3}\frac{1}{5}-\frac{3}{2}+3-2=\frac{3}{5}\approx 0.3464$$

$$\cos\nu = \frac{\langle -1,-1,1\rangle\cdot\langle 0,0,1\rangle}{\sqrt{3}}$$

Therefore, $\sec\nu = \sqrt{3}.$

13. ∂S is the circle $x^2+y^2=1,\ z=1.$
A parameterization of the circle is $x = \cos t$,
$y = \sin t$, $z = 1$, t in $[0,2\pi]$.

$$\oint_{\partial S}\mathbf{F}\cdot\mathbf{T}\,ds = \oint_{\partial S}\left[x^3y\,dx+e^y\,dy+z\tan\left(\frac{xyz}{4}\right)dz\right] \quad = \oint_{\partial S}(x^3y+e^y\,dy)$$

$$= \int_0^{2\pi}[(\cos t)^3(\sin t)(-\sin t)+(e^{\sin t})(\cos t)]dt \quad = 0$$

14. $\iiint_S \operatorname{div}\mathbf{F}\,dv = \iiint_S[(\cos x)+(1-\cos x)+(4)]dV \quad = \iiint_S 5\,dV = 5V(S) = 5\left(\frac{1}{2}\right)\left[\left(\frac{4}{3}\right)\pi(3)^3\right]$

$= 90\pi \approx 282.7433$

15. curl $\mathbf{F}$ = $\langle 3-0,0-0,-1-1\rangle$ = $\langle 3,0,-2\rangle$

$$\mathbf{n} = \frac{\langle a,b,1\rangle}{\sqrt{a^2+b^2+1}}$$

$$\oint_C\mathbf{F}\cdot\mathbf{T}\,ds = \iint_S(\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS \quad = \iint_S\frac{3a-2}{\sqrt{a^2+b^2+1}}dS = \frac{3a-2}{\sqrt{a^2+b^2+1}}[A(S)]$$

$= \dfrac{3a-2}{\sqrt{a^2+b^2+1}}(9\pi)$ (S is a circle of radius 3.)

$$= \frac{9\pi(3a-2)}{\sqrt{a^2+b^2+1}}$$

CHAPTER 15 Differential Equations

15.1 Concepts Review

1. $r^2 + a_1 r + a_2 = 0$; complex conjugate roots

2. $C_1 e^{-x} + C_2 e^x$

3. $(C_1 + C_2 x)e^x$

4. $C_1 \cos x + C_2 \sin x$

Problem Set 15.1

1. Roots are 2 and 3. General solution is $y = C_1 e^{2x} + C_2 e^{3x}$.

2. Roots are –6 and 1. General solution is $y = C_1 e^{-6x} + C_2 e^x$.

3. Auxiliary equation: $r^2 + 6r - 7 = 0$, $(r + 7)(r - 1) = 0$ has roots –7, 1.
General solution: $y = C_1 e^{-7x} + C_2 e^x$
$y' = -7C_1 e^{-7x} + C_2 e^x$
If $x = 0$, $y = 0$, $y' = 4$, then $0 = C_1 + C_2$ and $4 = -7C_1 + C_2$, so $C_1 = -\frac{1}{2}$ and $C_2 = \frac{1}{2}$.
Therefore, $y = \dfrac{e^x - e^{-7x}}{2}$.

4. Roots are –2 and 5. General solution is $y = C_1 e^{-2x} + C_2 e^{5x}$. Particular solution is $y = \left(\frac{12}{7}\right)e^{5x} - \left(\frac{5}{7}\right)e^{-2x}$.

5. Repeated root 2. General solution is $y = (C_1 + C_2 x)e^{2x}$.

6. Auxiliary equation: $r^2 + 10r + 25 = 0, (r + 5)^2 = 0$ has one repeated root -5.
General solution: $y = C_1 e^{-5x} + C_2 x e^{-5x}$ or $y = (C_1 + C_2 x)e^{-5x}$

7. Roots are $2 \pm \sqrt{3}$. General solution is $y = e^{2x}(C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x})$.

8. Roots are $-3 \pm \sqrt{11}$. General solution is $y = e^{-3x}\left(C_1 e^{\sqrt{11}x} + C_2 e^{-\sqrt{11}x}\right)$.

9. Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.
General solution: $y = C_1 \cos 2x + C_2 \sin 2x$
If $x = 0$ and $y = 2$, then $2 = C_1$; if $x = \frac{\pi}{4}$ and $y = 3$, then $3 = C_2$.
Therefore, $y = 2\cos 2x + 3\sin 2x$.

10. Roots are $\pm 3i$. General solution is $y = (C_1 \cos 3x + C_2 \sin 3x)$. Particular solution is $y = -\sin 3x - 3\cos 3x$.

11. Roots are $-1 \pm i$. General solution is $y = e^{-x}(C_1 \cos x + C_2 \sin x)$.

12. Auxiliary equation: $r^2 + r + 1 = 0$ has roots $\frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$.
General solution:
$$y = C_1 e^{(-1/2)x} \cos\left(\frac{\sqrt{3}}{2}\right)x + C_2 e^{(-1/2)x} \sin\left(\frac{\sqrt{3}}{2}\right)x$$
$$y = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}\right)x + C_2 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$$

13. Roots are 0, 0, –4, 1.
General solution is $y = C_1 + C_2 x + C_3 e^{-4x} + C_4 e^x$.

14. Roots are –1, 1, $\pm i$. General solution is $y = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x$.

15. Auxiliary equation: $r^4 + 3r^2 - 4 = 0$, $(r + 1)(r - 1)(r^2 + 4) = 0$ has roots –1, 1, $\pm 2i$.
General solution:
$y = C_1 e^{-x} + C_2 e^x + C_3 \cos 2x + C_4 \sin 2x$

16. Roots are –2, 3, $\pm i$. General solution is $y = C_1e^{-2x} + C_2e^{3x} + C_3\cos x + C_4\sin x$.

17. Roots are –2, 2. General solution is $y = C_1e^{-2x} + C_2e^{2x}$.

$y = C_1(\cosh 2x - \sinh 2x) + C_2(\sinh 2x + \cosh 2x) = (-C_1 + C_2)\sinh 2x + (C_1 + C_2)\cosh 2x$

$= D_1\sinh 2x + D_2\cosh 2x$

18. $e^u = \cosh u + \sinh u$ and $e^{-u} = \cosh u - \sinh u$.

Auxiliary equation: $r^2 - 2br - c^2 = 0$

Roots of auxiliary equation: $\dfrac{2b \pm \sqrt{4b^2 + 4c^2}}{2} = b \pm \sqrt{b^2 + c^2}$

General solution: $y = C_1e^{(b+\sqrt{b^2+c^2})x} + C_2e^{(b-\sqrt{b^2+c^2})x}$

$$= e^{bx}\left[C_1\left(\cosh\left(\sqrt{b^2+c^2}\,x\right) + \sinh\left(\sqrt{b^2+c^2}\,x\right)\right) + C_2\left(\cosh\left(\sqrt{b^2+c^2}\,x\right) - \sinh\left(\sqrt{b^2+c^2}\,x\right)\right)\right]$$

$$= e^{bx}\left[(C_1 + C_2)\cosh\left(\sqrt{b^2+c^2}\,x\right) + (C_1 + C_2)\sin\left(\sqrt{b^2+c^2}\,x\right)\right] = e^{bx}\left[D_1\cosh\left(\sqrt{b^2+c^2}\,x\right) + D_2\sinh\left(\sqrt{b^2+c^2}\,x\right)\right]$$

19. Repeated roots $\left(-\dfrac{1}{2}\right) \pm \left(\dfrac{\sqrt{3}}{2}\right)i$.

General solution is $y = e^{-x/2}\left[(C_1 + C_2x)\cos\left(\dfrac{\sqrt{3}}{2}\right)x + (C_3 + C_4x)\sin\left(\dfrac{\sqrt{3}}{2}\right)x\right]$.

20. Roots $1 \pm i$. General solution is

$y = e^x(C_1\cos x + C_2\sin x)$

$= e^x(c\sin\gamma\cos x + c\cos\gamma\sin x) = ce^x\sin(x + \gamma)$.

21. $(*)x^2y'' + 5xy' + 4y = 0$

Let $x = e^z$. Then $z = \ln x$;

$y' = \dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{dy}{dz}\dfrac{1}{x}$;

$y'' = \dfrac{dy'}{dx} = \dfrac{d}{dx}\left(\dfrac{dy}{dz}\dfrac{1}{x}\right) = \dfrac{dy}{dz}\dfrac{-1}{x^2} + \dfrac{1}{x}\dfrac{d^2y}{dz^2}\dfrac{dz}{dx}$

$= \dfrac{dy}{dz}\dfrac{-1}{x^2} + \dfrac{1}{x}\dfrac{d^2y}{dz^2}\dfrac{1}{x}$

$\left(-\dfrac{dy}{dz} + \dfrac{d^2y}{dz^2}\right) + \left(5\dfrac{dy}{dz}\right) + 4y = 0$

(Substituting y' and y'' into (*))

$\dfrac{d^2y}{dz^2} + 4\dfrac{dy}{dz} + 4y = 0$

Auxiliary equation: $r^2 + 4r + 4 = 0$, $(r + 2)^2 = 0$ has roots –2, –2.

General solution: $y = (C_1 + C_2z)e^{-2z}$,

$y = (C_1 + C_2\ln x)e^{-2\ln x}$

$y = (C_1 + C_2\ln x)x^{-2}$

22. As done in Problem 21,

$\left[-a\left(\dfrac{dy}{dz}\right) + a\left(\dfrac{d^2y}{dx^2}\right)\right] + b\left(\dfrac{dy}{dz}\right) + cy = 0.$

Therefore, $a\left(\dfrac{d^2y}{dz^2}\right) + (b - a)\left(\dfrac{dy}{dz}\right) + cy = 0.$

23. We need to show that $y'' + a_1y' + a_2y = 0$ if r_1 and r_2 are distinct real roots of the auxiliary equation.

We have,

$y' = C_1r_1e^{r_1x} + C_2r_2e^{r_2x}$

$y'' = C_1r_1^2e^{r_1x} + C_2r_2^2e^{r_2x}$.

When put into the differential equation, we obtain

$$y'' + a_1y' + a_2y = C_1r_1^2e^{r_1x} + C_2r_2^2e^{r_2x} + a_1\left(C_1r_1e^{r_1x} + C_2r_2e^{r_2x}\right) + a_2\left(C_1e^{r_1x} + C_2e^{r_2x}\right) \quad (*)$$

The solutions to the auxiliary equation are given by

$r_1 = \dfrac{1}{2}\left(-a_1 - \sqrt{a_1^2 - 4a_2}\right)$ and

$r_2 = \dfrac{1}{2}\left(-a_1 + \sqrt{a_1^2 - 4a_2}\right)$.

Putting these values into (*) and simplifying yields the desired result: $y'' + a_1y' + a_2y = 0$.

24. We need to show that $y''+a_1y'+a_2y=0$ if $\alpha\pm\beta i$ are complex conjugate roots of the auxiliary equation. We have,

$$y'=e^{\alpha x}\left((\alpha C_1+\beta C_2)\cos(\beta x)+(\alpha C_2-\beta C_1)\sin(\beta x)\right)$$
$$y''=e^{\alpha x}\left(\left(\alpha^2C_1-\beta^2C_1+2\alpha\beta C_2\right)\cos(\beta x)+\left(\alpha^2C_2-\beta^2C_2-2\alpha\beta C_1\right)\sin(\beta x)\right).$$

When put into the differential equation, we obtain

$$y''+a_1y'+a_2y=e^{\alpha x}\left(\left(\alpha^2C_1-\beta^2C_1+2\alpha\beta C_2\right)\cos(\beta x)+\left(\alpha^2C_2-\beta^2C_2-2\alpha\beta C_1\right)\sin(\beta x)\right)$$
$$+a_1e^{\alpha x}\left((\alpha C_1+\beta C_2)\cos(\beta x)+(\alpha C_2-\beta C_1)\sin(\beta x)\right)+a_2\left(C_1e^{\alpha x}\cos(\beta x)+C_2e^{\alpha x}\sin(\beta x)\right)\ (*)$$

From the solutions to the auxiliary equation, we find that

$\alpha=\frac{-a_1}{2}$ and $\beta=-\frac{1}{2}i\sqrt{a_1^{\,2}-4a_2}$.

Putting these values into (*) and simplifying yields the desired result: $y''+a_1y'+a_2y=0$.

25. **a.** $e^{bi}=1+(bi)+\frac{(bi)^2}{2!}+\frac{(bi)^3}{3!}+\frac{(bi)^4}{4!}+\frac{(bi)^5}{5!}+\ldots=\left(1-\frac{b^2}{2!}+\frac{b^4}{4!}-\frac{b^6}{6!}+\right)\ldots+i\left(b-\frac{b^3}{3!}+\frac{b^5}{5!}-\frac{b^7}{7!}+\ldots\right)$

$=\cos(b)+i\sin(b)$

b. $e^{a+bi}=e^ae^{bi}=e^a[\cos(b)+i\sin(b)]$

c. $D_x\left[e^{(\alpha+\beta i)x}\right]=D_x[e^{\alpha x}(\cos\beta x+i\sin\beta x)]=\alpha e^{\alpha x}(\cos\beta x+i\sin\beta x)+e^{\alpha x}(-i\beta\sin\beta x+i\beta\cos\beta x)$

$=e^{\alpha x}[(\alpha+\beta i)\cos\beta x+(\alpha i-\beta)\sin\beta x]$

$(\alpha+\beta)e^{(\alpha+\beta i)x}=(\alpha+\beta i)[e^{\alpha x}(\cos\beta x+i\sin\beta x)]=e^{\alpha x}[(\alpha+\beta i)\cos\beta x+(\alpha i-\beta)\sin\beta x]$

Therefore, $D_x[e^{(\alpha+\beta i)x}]=(\alpha+i\beta)e^{(\alpha+\beta i)x}$

26. $c_1e^{(\alpha+\beta i)x}+c_2e^{(\alpha+\beta i)x}$ [c_1 and c_2 are complex constants.]

$=c_1e^{\alpha x}[\cos\beta x+i\sin\beta x]+c_2e^{\alpha x}[\cos(-\beta x)+i\sin(-\beta x)]=e^{\alpha x}[(c_1+c_2)\cos\beta x+(c_1-c_2)i\sin\beta x]$

$=e^{\alpha x}[C_1\cos\beta x+C_2\sin\beta x]$, where $C_1=c_1+c_2$, and $C_2=c_1-c_2$.

Note: If c_1 and c_2 are complex conjugates, then C_1 and C_2 are real.

27. $y=0.5e^{5.16228x}+0.5e^{-1.162278x}$

28. $y=3.5xe^{-2.5x}+2e^{-2.5x}$

29. $y=1.29099e^{-0.25x}\sin(0.968246x)$

30. $y=e^{0.333333x}[2.5\cos(0.471405x)-4.94975\sin(0.471405x)]$

15.2 Concepts Review

1. particular solution to the nonhomogeneous equation; homogeneous equation

2. $-6+C_1e^{-2x}+C_2e^{3x}$

3. $y=Ax^2+Bx+C$

4. $y=Bxe^{\frac{1}{3}x}$

Problem Set 15.2

1. $y_h=C_1e^{-3x}+C_2e^{3x}$

   $y_p=\left(-\frac{1}{9}\right)x+0$

   $y=\left(-\frac{1}{9}\right)x+C_1e^{-3x}+C_2e^{3x}$

2. $y_h=C_1e^{-3x}+C_2e^{2x}$

   $y_p=\left(-\frac{1}{3}\right)x^2+\left(-\frac{1}{9}\right)x+\left(-\frac{7}{54}\right)$

   $y=\left(-\frac{1}{3}\right)x^2-\left(\frac{1}{9}\right)x-\left(\frac{7}{54}\right)+C_1e^{-3x}+C_2e^{2x}$

3. Auxiliary equation: $r^2-2r+1=0$ has roots 1, 1.

   $y_h=(C_1+C_2x)e^x$

   Let $y_p=Ax^2+Bx+C$; $y_p'=2Ax+B$;

   $y_p''=2A$.

   Then $(2A)-2(2Ax+B)+(Ax^2+Bx+C)=x^2+x$.

   $Ax^2+(-4A+B)x+(2A-2B+C)=x^2+x$

   Thus, A = 1, $-4A$ + B = 1, $2A$ – $2B$ + C = 0, so
   A = 1, B = 5, C = 8.

   General solution: $y=x^2+5x+8+(C_1+C_2x)e^x$

4. $y_h=C_1e^{-x}+C_2\cdot y_p=2x^2+(-4)x$

   $y=2x^2-4x+C_1e^{-x}+C_2$

5. $y_h=C_1e^{2x}+C_2e^{3x}\cdot y_p=\left(\frac{1}{2}\right)e^x\cdot y$

   $=\left(\frac{1}{2}\right)e^x+C_1e^{2x}+C_2e^{3x}$

6. Auxiliary equation: $r^2 + 6r + 9 = 0, (r+3)^2 = 0$ has roots –3, –3.

$y_h = (C_1 + C_2 x)e^{-3x}$

Let $y_p = Be^{-x}; y_p' = -Be^{-x}; y_p'' = Be^{-x}$.

Then $(Be^{-x}) + 6(-Be^{-x}) + 9(Be^{-x}) = 2e^{-x};\ \ 4Be^{-x} = 2e^{-x}; B = \frac{1}{2}$

General solution: $y = \left(\frac{1}{2}\right)e^{-x} + (C_1 + C_2 x)e^{-3x}$

7. $y_h = C_1 e^{-3x} + C_2 e^{-x}$

$y_p = \left(-\frac{1}{2}\right)xe^{-3x}$

$y = \left(-\frac{1}{2}\right)xe^{-3x} + C_1 e^{-3x} + C_2 e^{-x}$

8. $y_h = e^{-x}(C_1 \cos x + C_2 \sin x)$

$y_p = \left(\frac{3}{2}\right)e^{-2x}$

$y = \left(\frac{3}{2}\right)e^{-2x} + e^{-x}(C_1 \cos x + C_2 \sin x)$

9. Auxiliary equation: $r^2 - r - 2 = 0$,
$(r + 1)(r - 2) = 0$ has roots –1, 2.

$y_h = C_1 e^{-x} + C_2 e^{2x}$

Let $y_p = B\cos x + C\sin x;\ \ y_p' = -B\sin x + C\cos x;\ \ y_p'' = -B\cos x - C\sin x.$

Then $(-B\cos x - C\sin x) - (-B\sin x + C\cos x)$
$-2(B\cos x + C\sin x) = 2\sin x.$

$(-3B - C)\cos x + (B - 3C)\sin x = 2\sin x$, so $-3B - C = 0$ so $-3B - C = 0$ and $B - 3C = 2$; $B = \frac{1}{5}; C = \frac{-3}{5}.$

General solution: $\left(\frac{1}{5}\right)\cos x - \left(\frac{3}{5}\right)\sin x + C_1 e^{2x} + C_2 e^{-x}$

10. $y_h = C_1 e^{-4x} + C_2$

$y_p = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x$

$y = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x + C_1 e^{-4x} + C_2$

11. $y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = (0)x\cos 2x + \left(\frac{1}{2}\right)x\sin 2x$

$y = \left(\frac{1}{2}\right)x\sin 2x + C_1 \cos 2x + C_2 \sin 2x$

12. Auxiliary equation: $r^2+9=0$ has roots $\pm 3i$, so $y_h = C_1\cos 3x + C_2 \sin 3x$.

Let $y_p = Bx\cos 3x + Cx\sin 3x$; $y_p' = (-3bx+C)\sin 3x + (B+3Cx)\cos 3x$;

$y_p'' = (-9Bx+6C)\cos 3x + (-9Cx-6B)\sin 3x$.

Then substituting into the original equation and simplifying, obtain $6C\cos 3x - 6B\sin 3x = \sin 3x$, so $C = 0$ and $B = -\frac{1}{6}$.

General solution: $y = \left(-\frac{1}{6}\right)x\cos 3x + C_1\cos 3x + C_2\sin 3x$

13. $y_h = C_1\cos 3x + C_2\sin 3x$

$y_p = (0)\cos x + \left(\frac{1}{8}\right)\sin x + \left(\frac{1}{13}\right)e^{2x}$

$y = \left(\frac{1}{8}\right)\sin x + \left(\frac{1}{13}\right)e^{2x} + C_1\cos 3x + C_2\sin 3x$

14. $y_h = C_1e^{-x} + C_2$

$y_p = \left(\frac{1}{2}\right)e^x + \left(\frac{3}{2}\right)x^2 + (-3)x$

$y = \left(\frac{1}{2}\right)e^x + \left(\frac{3}{2}\right)x^2 - 3x + C_1e^{-x} + C_2$

15. Auxiliary equation: $r^2 - 5r + 6 = 0$ has roots 2 and 3, so $y_h = C_1e^{2x} + C_2e^{3x}$.

Let $y_p = Be^x$; $y_p' = Be^x$; $y_p'' = Be^x$.

Then $(Be^x) - 5(Be^x) + 6(Be^x) = 2e^x$; $2Be^x = 2e^x$; $B = 1$.

General solution: $y = e^x + C_1e^{2x} + C_2e^{3x}$

$y' = e^x + 2C_1e^{2x} + 3C_2e^{3x}$

If $x = 0$, $y = 1$, $y' = 0$, then $1 = 1 + C_1 + C_2$ and $0 = 1 + 2C_1 + 3C_2$; $C_1 = 1, C_2 = -1$.

Therefore, $y = e^x + e^{2x} - e^{3x}$.

16. $y_h = C_1e^{-2x} + C_2e^{2x}$

$y_p = (0)\cos x + \left(-\frac{4}{5}\right)\sin x$

$y = \left(-\frac{4}{5}\right)\sin x + C_1e^{-2x} + C_2e^{2x}$

$y = \left(-\frac{4}{5}\right)\sin x + \left(\frac{9}{5}\right)e^{-2x} + \left(\frac{11}{5}\right)e^{2x}$ satisfies the conditions.

17. $y_h = C_1e^x + C_2e^{2x}$

$y_p = \left(\frac{1}{4}\right)(10x+19)$

$y = \left(\frac{1}{4}\right)(10x+19) + C_1e^x + C_2e^{2x}$

18. Auxiliary equation: $r^2 - 4 = 0$ has roots 2, –2, so $y_h = C_1e^{2x} + C_2e^{-2x}$.

Let $y_p = v_1e^{2x} + v_2e^{-2x}$, subject to $v_1'e^{2x} + v_2'e^{-2x} = 0$, and $v_1'(2e^{2x}) + v_2'(-2e^{-2x}) = e^{2x}$.

Then $v_1'(4e^{2x}) = e^{2x}$ and $v_2'(-4e^{-2x}) = e^{2x}$; $v_1' = \dfrac{1}{4}$ and $v_2' = -e^{4x/4}$; $v_1 = \dfrac{x}{4}$ and $v_2 = -\dfrac{e^{4x}}{16}$.

General solution: $y = \dfrac{xe^{2x}}{4} - \dfrac{e^{2x}}{16} + C_1e^{2x} + C_2e^{-2x}$

19. $y_h = C_1 \cos x + C_2 \sin x$

$y_p = -\cos \ln|\sin x| - \cos x - x \sin x$

$y = -\cos x \ln|\sin x| - x \sin x + C_3 \cos x + C_2 \sin x$ (combined cos x terms)

20. $y_h = C_1 \cos x + C_2 \sin x$

$y_p = -\sin x \ln|\csc x + \cot x|$

$y = -\sin x \ln|\csc x + \cot x| + C_1 \cos x + C_2 \sin x$

21. Auxiliary equation: $r^2 - 3r + 2 = 0$ has roots 1, 2, so $y_h = C_1e^x + C_2e^{2x}$.

Let $y_p = v_1e^x + v_2e^{2x}$ subject to $v_1'e^x + v_2'e^{2x} = 0$, and $v_1'(e^x) + v_2'(2e^{2x}) = e^x(ex + 1)^{-1}$.

Then $v_1' = \dfrac{-e^x}{e^x(e^x + 1)}$ so $v_1 = \displaystyle\int \frac{-e^x}{e^x(e^x + 1)}dx = \int \frac{-1}{u(u + 1)}du$

$= \displaystyle\int \left(\frac{-1}{u} + \frac{1}{u + 1}\right)du = -\ln u + \ln(u + 1) = \ln\left(\frac{u + 1}{u}\right) = \ln\frac{e^x + 1}{e^x} = \ln(1 + e^{-x})$

$v_2' = \dfrac{e^x}{e^{2x}(e^x + 1)}$ so $v_2 = -e^{-x} + \ln(1 + e^{-x})$

(similar to finding v_1)

General solution: $y = e^x \ln(1 + e^{-x}) - e^x + e^{2x} \ln(1 + e^{-x}) + C_1e^x + C_2e^{2x}$

$y = (e^x + e^{2x})\ln(1 + e^{-x}) + D_1e^x + D_2e^{2x}$

22. $y_h = C_1e^{2x} + C_2e^{3x}$; $y_p = e^x$

$y = e^x + C_1e^{2x} + C_2e^{3x}$

23. $L(y_p) = (v_1u_1 + v_2u_2)'' + b(v_1u_1 + v_2u_2)' + c(v_1u_1 + v_2u_2)$

$= (v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2') + b(v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2') + c(v_1u_1 + v_2u_2)$

$= (v_1''u_1 + v_1'u_1' + v_1'u_1' + v_1u_1'' + v_2''u_2 + v_2'u_2' + v_2'u_2' + v_2u_2'') + b(v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2') + c(v_1u_1 + v_2u_2)$

$= v_1(u_1'' + bu_1' + cu_1) + v_2(u_2'' + bu_2' + cu_2) + b(v_1'u_1 + v_2'u_2) + (v_1''u_1 + v_1'u_1' + v_2''u_2 + v_2'u_2) + (v_1'u_1' + v_2'u_2')$

$= v_1(u_1'' + bu_1' + cu_1) + v_2(u_2'' + bu_2' + cu_2) + b(v_1'u_1 + v_2'u_2) + (v_1'u_1 + v_2'u_2)' + (v_1'u_1' + v_2'u_2')$

$= v_1(0) + v_2(0) + b(0) + (0) + k(x) = k(x)$

24. Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.

$y_h = C_1 \cos 2x + C_2 \sin 2x$

Now write $\sin^3 x$ in a form involving sin βx's or cos βx's.

$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$

(C.R.C. Standard Mathematical Tables, or derive it using half-angle and product identities.)

Let $y_p = A \sin x + B \cos x + C \sin 3x + D \cos 3x$;

$y_p' = A \cos x - B \sin x + 3C \cos 3x - 3D \sin 3x$;

$y_p'' = -A \sin x - B \cos x - 9C \sin 3x - 9D \cos 3x$.

Then

$y_p'' + 4y_p = 3A \sin x + 3B \cos x - 5C \sin 3x - 5D \cos 3x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$, so

$A = \frac{1}{4}$, $B = 0$, $C = \frac{1}{20}$, $D = 0$.

General solution: $y = \frac{1}{4} \sin x + \frac{1}{20} \sin 3x + C_1 \cos 2x + C_2 \sin 2x$

15.3 Concepts Review

1. 3; π

2. π ; decreases

3. 0

4. electric circuit

Problem Set 15.3

1. $k = 250, m = 10, B^2 = k / m = 250/10 = 25, B = 5$

(the problem gives the mass as $m = 10\,\text{kg}$)

Thus, $y'' = -25y$. The general solution is $y = C_1 \cos 5t + C_2 \sin 5t$. Apply the initial condition to get $y = 0.1 \cos 5t$.

The period is $\frac{2\pi}{5}$ seconds.

2. k = 100 lb/ft, w = 1 lb, g = 32 ft/s^2, $y_0 = \frac{1}{12}$ ft,

$B = 40\sqrt{2}$. Then $y = \left(\frac{1}{12}\right) \cos(40\sqrt{2})t$.

Amplitude is $\frac{1}{12}$ ft $= 1$ in.

Period is $\frac{2\pi}{40\sqrt{2}} \approx 0.1111$ s.

3. $y = 0.1 \cos 5t = 0$ whenever $5t = \frac{\pi}{2} + \pi k$ or $t = \frac{\pi}{10} + \frac{\pi}{5} k$.

$\left| y'\left(\frac{\pi}{10} + \frac{\pi}{5} k\right) \right| = 0.5 \left| \sin 5\left(\frac{\pi}{10} + \frac{\pi}{5} k\right) \right| = 0.5 \left| \sin\left(\frac{\pi}{2} + \pi k\right) \right| = 0.5$ meters per second

4. $|10| = k\left(\frac{1}{3}\right)$, so $k = 30$ lb/ft, $w = 20$ lb,

$g = 32$ ft/s^2, $y_0 = -1$ ft, $v_0 = 2$ ft/s, $B = 4\sqrt{3}$

Then $y = C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t)$.

$y = \cos(4\sqrt{3}t) + \left(\frac{\sqrt{3}t}{6}\right)\sin(4\sqrt{3}t)$ satisfies the initial conditions.

5. $k = 20$ lb/ft; $w = 10$ lb; $y_0 = 1$ ft; $q = \frac{1}{10}$ s-lb/ft, $B = 8$, $E = 0.32$

$E^2 - 4B^2 < 0,$ so there is damped motion. Roots of auxiliary equation are approximately $-0.16 \pm 8i$.

General solution is $y \approx e^{-0.16t}(C_1 \cos 8t + C_2 \sin 8t)$. $y \approx e^{-0.16t}(\cos 8t + 0.02 \sin 8t)$ satisfies the initial conditions.

6. $k = 20$ lb/ft; $w = 10$ lb; $y_0 = 1$ ft; $q = 4$ s-lb/ft

$B = \sqrt{\frac{(20)(32)}{10}} = 8;$ $E = \frac{(4)(32)}{10} = 12.8;$ $E^2 - 4B^2 < 0,$ so damped motion.

Roots of auxiliary equation are $\frac{-E \pm \sqrt{E^2 - 4B^2}}{2} = -6.4 \pm 4.8i.$

General solution is $y = e^{-6.4t}(C_1 \cos 4.8t + C_2 \sin 4.8t).$

$y' = e^{-6.4t}(-4.8C_1 \sin 4.8t + 4.8C_2 \cos 4.8t) - 6.4e^{-6.4t}(C_1 \cos 4.8t + C_2 \sin 4.8t)$

If $t = 0$, $y = 1$, $y' = 0$, then $1 = C_1$ and $0 = 4.8C_2 - 6.4C_1$, so $C_1 = 1$ and $C_2 = \frac{4}{3}$.

Therefore, $y = e^{-6.4t}\left[\cos 4.8t + \left(\frac{4}{3}\right)\sin 4.8t\right]$.

7. Original amplitude is 1 ft. Considering the contribution of the sine term to be negligible due to the 0.02 coefficient, the amplitude is approximately $e^{-0.16t}$.

$e^{-0.16t} \approx 0.1$ if $t \approx 14.39$, so amplitude will be about one-tenth of original in about 14.4 s.

8. $C_1 = 1$ and $C_2 = -0.105$, so $y = e^{-0.16t}(\cos 8t + 0.105 \sin 8t)$.

9. $LQ'' + RQ' + \frac{Q}{C} = E(t);$ $10^6 Q' + 10^6 Q = 1;$ $Q' + Q = 10^{-6}$

Integrating factor: e^t

$D[Qe^t] = 10^{-6}e^t; Qe^t = 10^{-6}e^t + C;$

$Q = 10^{-6} + Ce^{-t}$

If $t = 0$, $Q = 0$, then $C = -10^{-6}$.

Therefore, $Q(t) = 10^{-6} - 10^{-6}e^{-t} = 10^{-6}(1 - e^{-t})$.

10. Same as Problem 9, except $C = 4 - 10^{-6}$, so $Q(t) = 10^{-6} + (4 - 10^{-6})e^{-t}$.

Then $I(t) = Q'(t) = -(4 - 10^{-6})e^{-t}$.

11. $\frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

a. $Q(t) = 0.00024 \sin 377t$

b. $I(t) = Q'(t) = 0.09048 \cos 377t$

12. $LQ'' + RQ' + \frac{Q}{C} = E;\ 10^{-2}Q'' + \frac{Q}{10^{-7}} = 20;\ Q'' + 10^9 Q = 2000$

The auxiliary equation, $r^2 + 10^9 = 0,$ has roots $\pm 10^{9/2} i.$

$Q_h = C_1 \cos 10^{9/2} t + C_2 \sin 10^{9/2} t$

$Q_p = 2000(10^{-9}) = 2(10^{-6})$ is a particular solution (by inspection).

General solution: $Q(t) = 2(10^{-6}) + C_1 \cos 10^{9/2} t + C_2 \sin 10^{9/2} t$

Then $I(t) = Q'(t) = -10^{9/2} C_1 \sin 10^{9/2} t + 10^{9/2} C_2 \cos 10^{9/2} t.$

If $t = 0$, $Q = 0$, $I = 0$, then $0 = 2(10^{-6}) + C_1$ and $0 = C_2.$

Therefore, $I(t) = -10^{9/2}(-2[10^{-6}]) \sin 10^{9/2} t = 2(10^{-3/2}) \sin 10^{9/2} t.$

13. $3.5Q'' + 1000Q + \frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

(Values are approximated to 6 significant figures for the remainder of the problem.)

$Q'' + 285.714Q' + 142857Q = 34.2857 \sin 377t$

Roots of the auxiliary equation are
$-142.857 \pm 349.927i.$

$Q_h = e^{-142.857t}(C_1 \cos 349.927t + C_2 \sin 349.927t)$

$Q_p = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t$

Then, $Q = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t + Q_h.$

$I = Q' = 0.119995 \sin 377t + 0.000810998 \cos 377t + Q_h'$

0.000888 cos $377t$ is small and $Q_h' \to 0$ as $t \to \infty,$ so the steady-state current is $I \approx 0.12 \sin 377t$.

14. **a.** Roots of the auxiliary equation are $\pm Bi$.

$y_h = C_1 \cos Bt + C_2 \sin Bt.$

$y_p = \left[\frac{c}{(B^2 - A^2)}\right] \sin At$

The desired result follows.

b. $y_p = \left(-\frac{c}{2B}\right) t \cos Bt,$ so

$y = C_1 \cos Bt + C_2 \sin Bt - \left(\frac{c}{2B}\right) t \cos Bt.$

c. Due to the t factor in the last term, it increases without bound.

15. $A \sin(\beta t + \gamma) = A(\sin \beta t \cos \gamma + \cos \beta t \sin \gamma)$

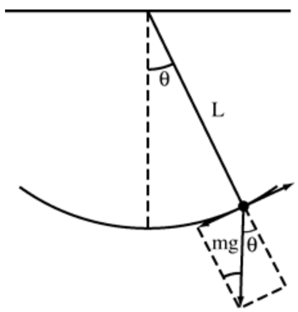
$= (A \cos \gamma) \sin \beta t + (A \sin \gamma) \cos \beta t$

$= C_1 \sin \beta t + C_2 \cos \beta t,$ where $C_1 = A \cos \gamma$ and $C_2 = A \sin \gamma.$

[Note that $C_1^2 + C_2^2 = A^2 \cos^2 \gamma + A^2 \sin^2 \gamma = A^2.$)

16. The first two terms have period $\frac{2\pi}{B}$ and the last has period $\frac{2\pi}{A}$. Then the sum of the three terms is periodic if $m\left(\frac{2\pi}{B}\right) = n\left(\frac{2\pi}{B}\right)$ for some integers m, n; equivalently, if $\frac{B}{A} = \frac{m}{n},$ a rational number.

17. The magnitudes of the tangential components of the forces acting on the pendulum bob must be equal.



Therefore, $-m\frac{d^2s}{dt^2} = mg\sin\theta$.

$s = L\theta$, so $\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2}$.

Therefore, $-mL\frac{d^2\theta}{dt^2} = mg\sin\theta$.

Hence, $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$.

18. a. Since the roots of the auxiliary equation are $\pm\sqrt{\frac{g}{L}}i$, the solution of $\theta''(t)+\left(\frac{g}{L}\right)\theta = 0$ is $\theta = C_1\cos\sqrt{\frac{g}{L}}t + C_2\sin\sqrt{\frac{g}{L}}t$, which can be written as $\theta = C\left(\sqrt{\frac{g}{L}}t+\gamma\right)$

(by Problem 15).
The period of this function is

$$\frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi\frac{L}{\sqrt{G}} = 2\pi\sqrt{\frac{LR^2}{GM}} = 2\pi R\sqrt{\frac{L}{GM}}.$$

Therefore, $\frac{p_1}{p_2} = \frac{2\pi R_1\sqrt{\frac{L_1}{GM}}}{2\pi R_2\sqrt{\frac{L_2}{GM}}} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}$.

b. To keep perfect time at both places, require $p_1 = p_2$. Then $1 = \frac{R_2\sqrt{80.85}}{3960\sqrt{81}}$, so

$R_2 \approx 3963.67$.

The height of the mountain is about $3963.67 - 3960 = 3.67$ mi (about 19,387 ft).

15.4 Chapter Review

Concepts Test

1. False: y^2 is not linear in y.

2. True: y and y'' are linear in y and y'', respectively.

3. True:

$y' = \sec^2 x + \sec x\tan x$

$2y' - y^2 = (2\sec^2 x + 2\sec x\tan x)$

$\quad -(\tan^2 x + 2\sec x\tan x + \sec^2 x)$

$= \sec^2 x - \tan^2 x = 1$

4. False: It should involve 6.

5. True: D^2 adheres to the conditions for linear operators.

$D^2(kf) = kD^2(f)$

$D^2(f+g) = D^2 f + D^2 g$

6. False: Replacing y by $C_1u_1(x)+C_2u_2(x)$ would yield, on the left side,

$C_1 f(x) + C_2 f(x) = (C_1 + C_2) f(x)$

which is $f(x)$ only if $C_1 + C_2 = 1$ or $f(x) = 0$.

7. True: –1 is a repeated root, with multiplicity 3, of the auxiliary equation.

8. True:

$L(u_1 - u_2) = L(u_1) - L(u_2)$

$= f(x) - f(x) = 0$

9. False: That is the form of y_h. y_p should have the form $Bx\cos 3x + Cx\sin 3x$.

10. True: See Problem 15, Section 15.3.

Sample Test Problems

1. $u' + 3u = e^x$. Integrating factor is e^{3x}.

$D[ue^{3x}] = e^{4x}$

$y = \left(\frac{1}{4}\right)e^x + C_1e^{-3x}$

$y' = \left(\frac{1}{4}\right)e^x + C_1e^{-3x}$

$y = \left(\frac{1}{4}\right)e^x + C_3e^{-3x} + C_2$

2. Roots are –1, 1. $y = C_1e^{-x} + C_2e^x$

3. (Second order homogeneous)
The auxiliary equation, $r^2 - 3r + 2 = 0$, has roots 1, 2. The general solution is $y = C_1e^x + C_2e^{2x}$.
$y' = C_1e^x + 2C_2e^{2x}$
If $x = 0, y = 0, y' = 3$, then $0 = C_1 + C_2$ and $3 = C_1 + 2C_2$, so $C_1 = -3, C_2 = 3$.
Therefore, $y = -3e^x + 3e^{2x}$.

4. Repeated root $-\frac{3}{2}$. $y = (C_1 + C_2x)e^{(-3/2)x}$

5. $y_h = C_1e^{-x} + C_2e^x$ (Problem 2)
$y_p = -1 + C_1e^{-x} + C_2e^x$

6. (Second-order nonhomogeneous) The auxiliary equation, $r^2 + 4r + 4 = 0$, has roots –2, –2.
$y_h = C_1e^{-2x} + C_2xe^{-2x} = (C_1 + C_2x)e^{-2x}$
Let $y_p = Be^x; y_p' = Be^x; y_p'' = Be^x$.
$(Be^x) + 4(Be^x) + 4(Be^x) = 3e^x$, so $B = \frac{1}{3}$.
General solution: $y = \frac{e^x}{3} + (C_1 + C_2x)e^{-2x}$

7. $y_h = (C_1 + C_2x)e^{-2x}$ (Problem 12)
$y_p = \left(\frac{1}{2}\right)x^2e^{-2x}$
$y = \left[\left(\frac{1}{2}\right)x^2 + C_1 + C_2x\right]e^{-2x}$

8. Roots are $\pm 2i$.
$y = C_1 \cos 2x + C_2 \sin 2x$
$y = \sin 2x$ satisfies the conditions.

9. (Second-order homogeneous)
The auxiliary equation, $r^2 + 6r + 25 = 0$, has roots $-3 \pm 4i$. General solution:
$y = e^{-3x}(C_1 \cos 4x + C_2 \sin 4x)$

10. Roots are $\pm i$. $y_h = C_1 \cos x + C_2 \sin x$
$y_p = x \cos x - \sin x + \sin x \ln|\cos x|$
$y = x \cos x - \sin x \ln|\cos x| + C_1 \cos x + C_3 \sin x$
(combining the sine terms)

11. Roots are –4, 0, 2. $y = C_1e^{-4x} + C_2 + C_3e^{2x}$

12. (Fourth-order homogeneous)
The auxiliary equation, $r^4 - 3r^2 - 10 = 0$ or $(r^2 - 5)(r^2 + 2) = 0$, has roots $-\sqrt{5}, \sqrt{5}, \pm\sqrt{2}i$.
General solution:
$y = C_1e^{\sqrt{5}x} + C_2e^{-\sqrt{5}x} + C_3 \cos\sqrt{2}x + C_4 \sin\sqrt{2}x$

13. Repeated roots $\pm\sqrt{2}$
$y = (C_1 + C_2x)e^{-\sqrt{2}x} + (C_3 + C_4x)e^{\sqrt{2}x}$

14. a. $Q'(t) = 3 - 0.02Q$

b. $Q'(t) + 0.02Q = 3$
Integrating factor is $e^{0.02t}$
$D[Qe^{0.02t}] = 3e^{0.02t}$
$Q(t) = 150 + Ce^{-0.02t}$
$Q(t) = 150 - 30e^{-0.02t}$ goes through (0, 120).

c. $Q \to 150$ g, as $t \to \infty$.

15. (Simple harmonic motion)
$k = 5$; $w = 10$; $y_0 = -1$
$B = \sqrt{\frac{(5)(32)}{10}} = 4$
Then the equation of motion is $y = -\cos 4t$.
The amplitude is $|-1| = 1$; the period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

16. It is at equilibrium when $y = 0$ or $-\cos 4t = 0$, or
$t = \frac{\pi}{8}, \frac{3\pi}{8}, \ldots.$
$y'(t) = 4 \sin 4t$, so at equilibrium $|y'| = |\pm 4| = 4$.

17. $Q'' + 2Q' + 2Q = 1$
Roots are $-1 \pm i$.
$Q_h = e^{-t}(C_1 \cos t + C_2 \sin t)$ and $Q_p = \frac{1}{2}$;
$Q = e^{-t}(C_1 \cos t + C_2 \sin t) + \frac{1}{2}$
$I(t) = Q'(t) = -e^{-t}[(C_1 - C_2)\cos t + (C_1 + C_2)\sin t]$
$I(t) = e^{-t} \sin t$ satisfies the initial conditions.